Scaling laws for dynamos in rotating spherical shells

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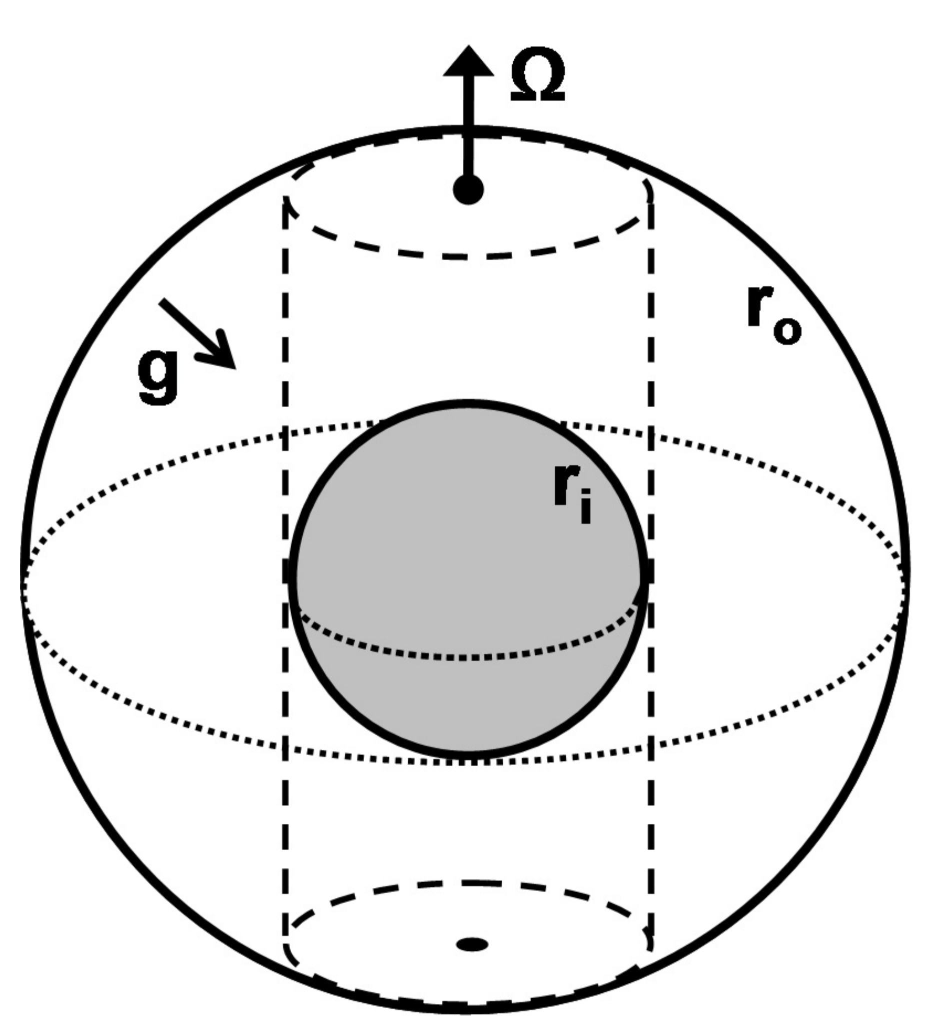
Introduction

- Geodynamo models successfully reproduced first-order properties of the geomagnetic field
- This seems surprising, because several control parameters are far from Earth values (viscosity and thermal diffusivity too large, rotation too slow)
- Pessimistic view: Models give right answer for wrong reasons
- Optimistic assumption: Perhaps current models approach a regime in which diffusive processes do not play a first-order role in the force balance

→ Use many case studies, covering decent range of control parameters, to derive scaling laws

Outline of dynamo models

- Boussinesq equations for convection-driven MHD flow
- Rigid inner and outer boundary
- $r_i / r_o = 0.35$
- Fixed temperature contrast, no internal heat sources



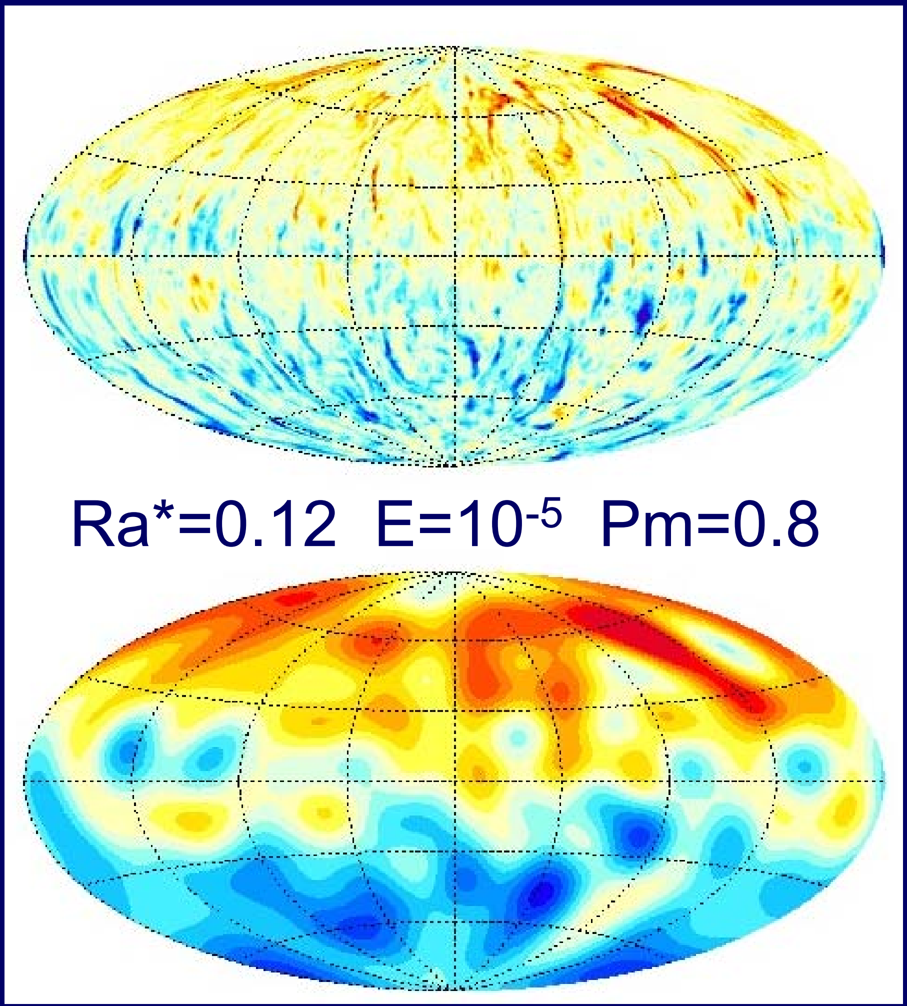
Control parameters

• Ekman number	$E = \nu/(\Omega D^2)$	10^{-6} .... 10^{-3} (*)
• Prandtl number	$Pr = \nu/\kappa$	0.1 10
• Magnetic Prandtl #	$Pm = \nu/\lambda$	0.06 20
• Modified Rayleigh #	$Ra^* = \alpha g_o \Delta T/\Omega^2 D$	5 .... 100 × Ra_{crit}

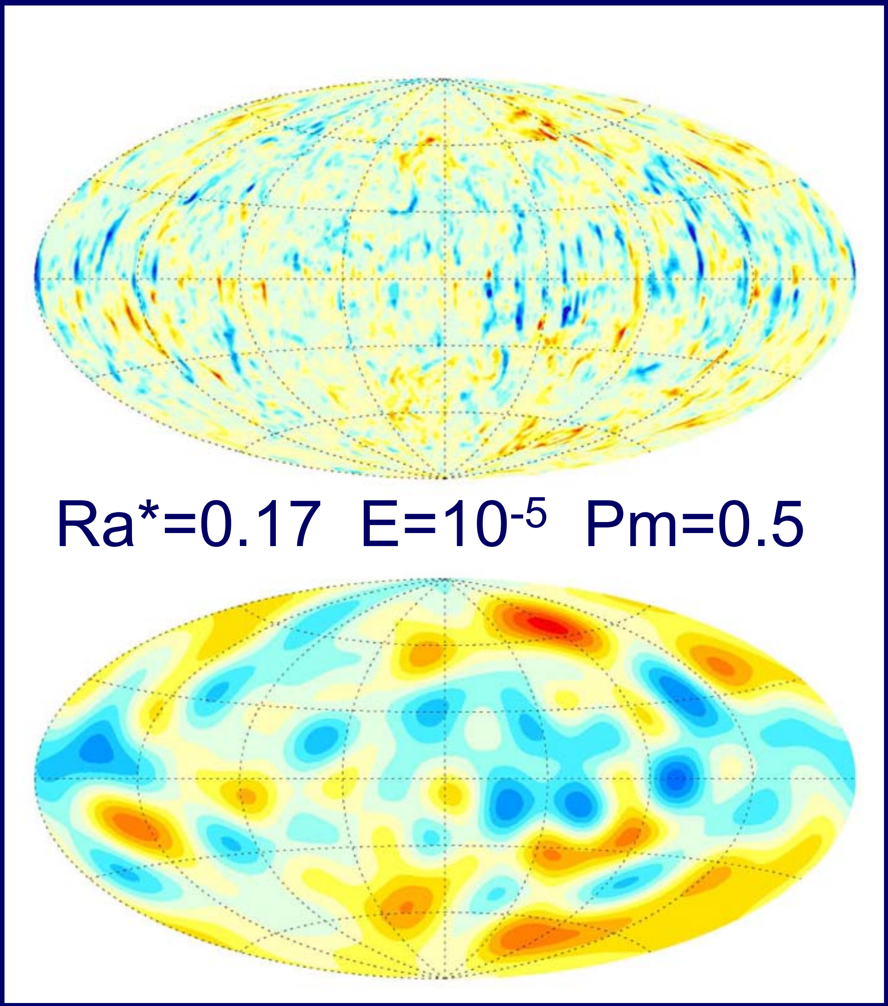
Modified Rayleigh number does not depend on diffusivity.

(*) With definition of Kono & Roberts [$E = \nu/(2\Omega r_o^2)$], the range is 2×10^{-7}... 2×10^{-4} .

Dynamo classes

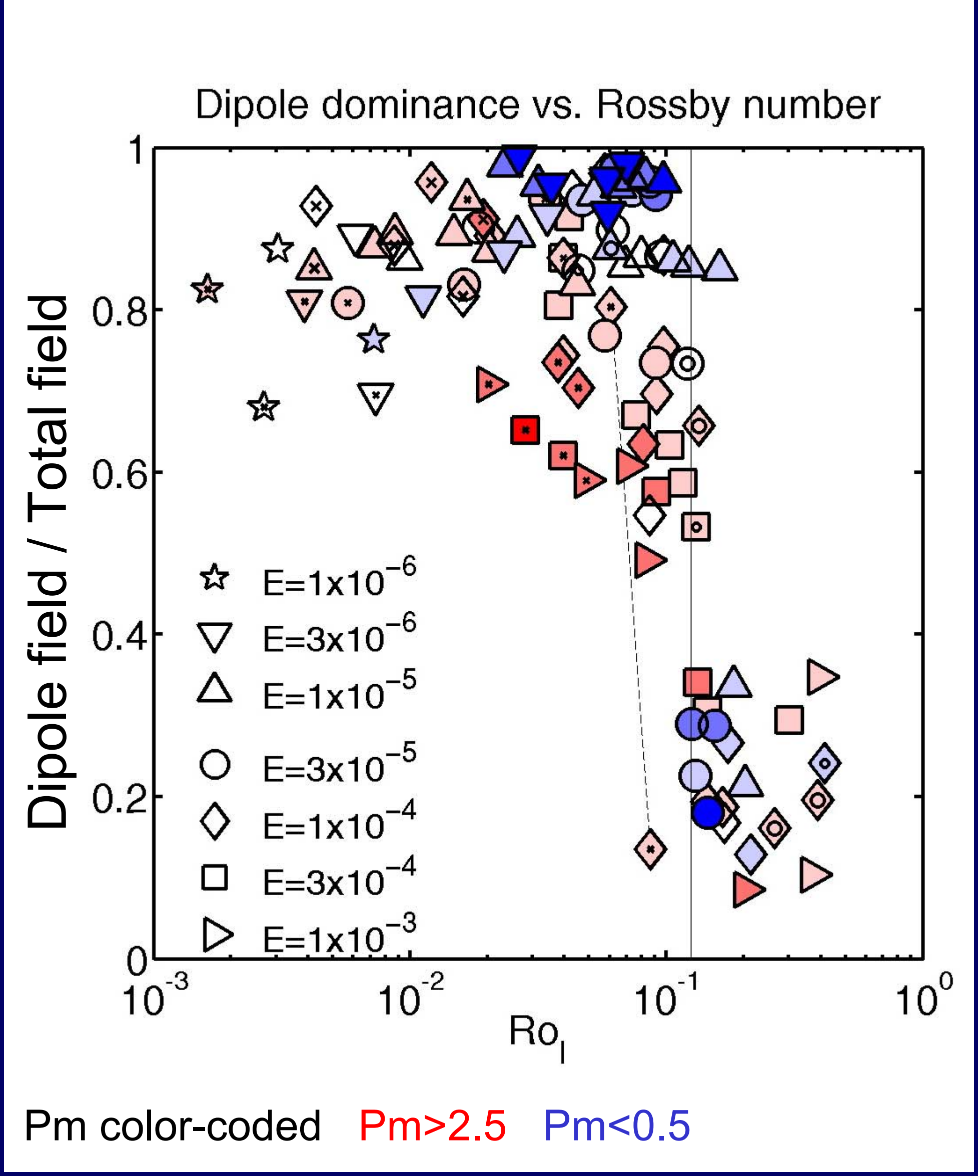


Dipolar



Multipolar

Dynamo regimes



Inertial vs. Coriolis force:

Rossby number Ro_ℓ calculated with mean length scale ℓ in the kinetic energy spectrum

$$Ro_\ell = U/\Omega\ell$$

Regime boundary at $Ro_\ell \approx 0.12$ (depends on heating mode, b.c.)

Scaling analysis: data basis and case selection

- ~160 model cases (125 dynamos, 35 failed dynamos)
- Each run for at least 50 advection times (some much longer)
- Symmetry in longitude assumed for $E \leq 10^{-5}$

Selection criteria for scaling analysis:

- Self-sustained dynamo
- Dipole-dominated magnetic field ($f_{dip} > 0.35$)
- Fully developed convection ($Nu > 2.0$)

→ 87 model cases pass these criteria

Scaling and parameters II

- Ro Rossby number $Ro = U/\Omega D$
- Lo Lorentz number $Lo = B/(\rho\mu)^{1/2}\Omega D$
- Nu* Modified Nusselt number $Nu^* = Q_{adv}/(4\pi r_o r_i \rho c_p \Delta T \Omega D)$

Relation to conventional parameters

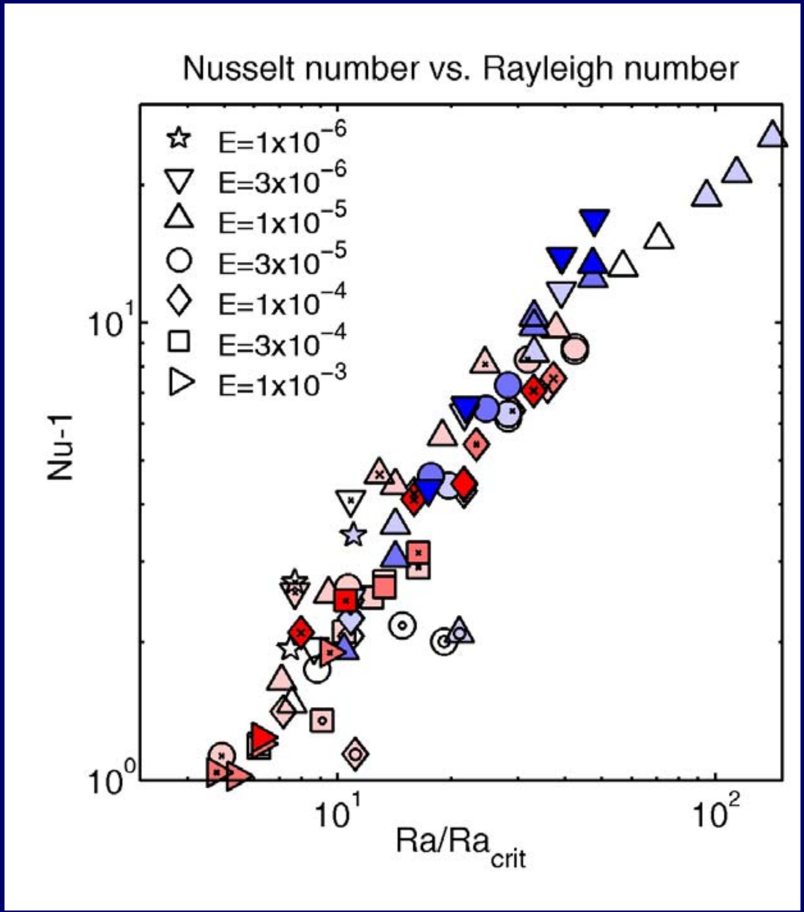
- Rayleigh number Ra $Ra^* = Ra\, E^2\, Pr^{-1}$
- Nusselt number Nu $Nu^* = (Nu-1)\, E\, Pr^{-1}$
- Elsasser number Λ $Lo = (\Lambda\, E\, Pr^{-1})^{1/2}$

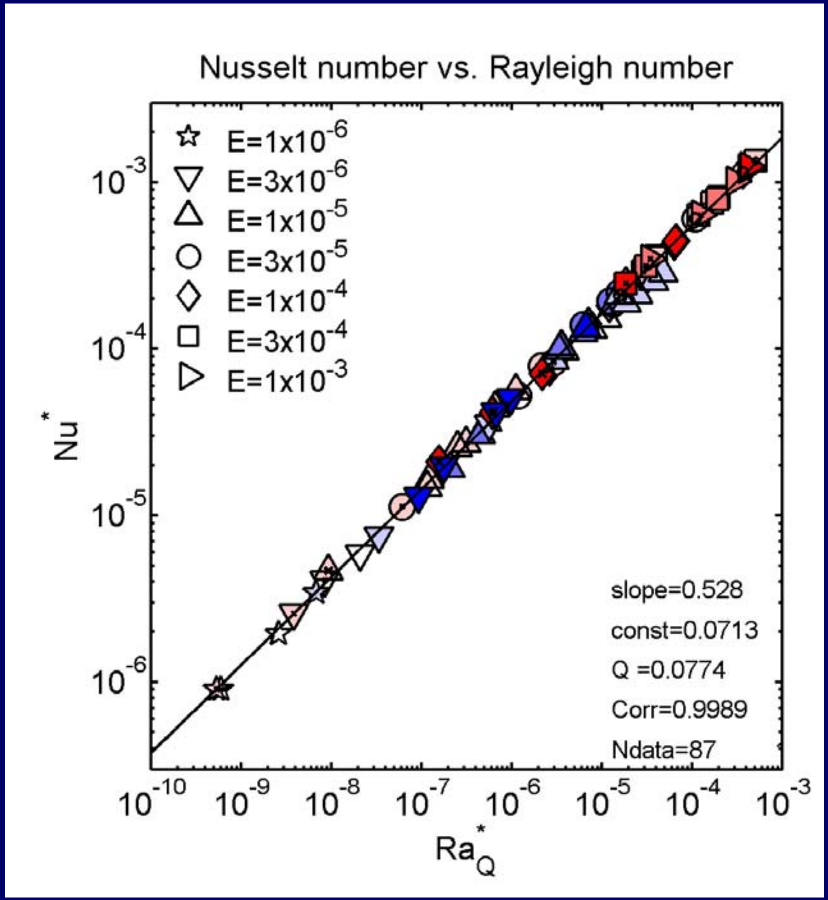
Flux-based modified Rayleigh number

$$Ra^*_Q = Ra^* Nu^* = Ra\, E^{-3}\, Pr^{-2}$$

Ra^*_Q is a measure for the work done by buoyancy forces

Scaling of Nusselt number

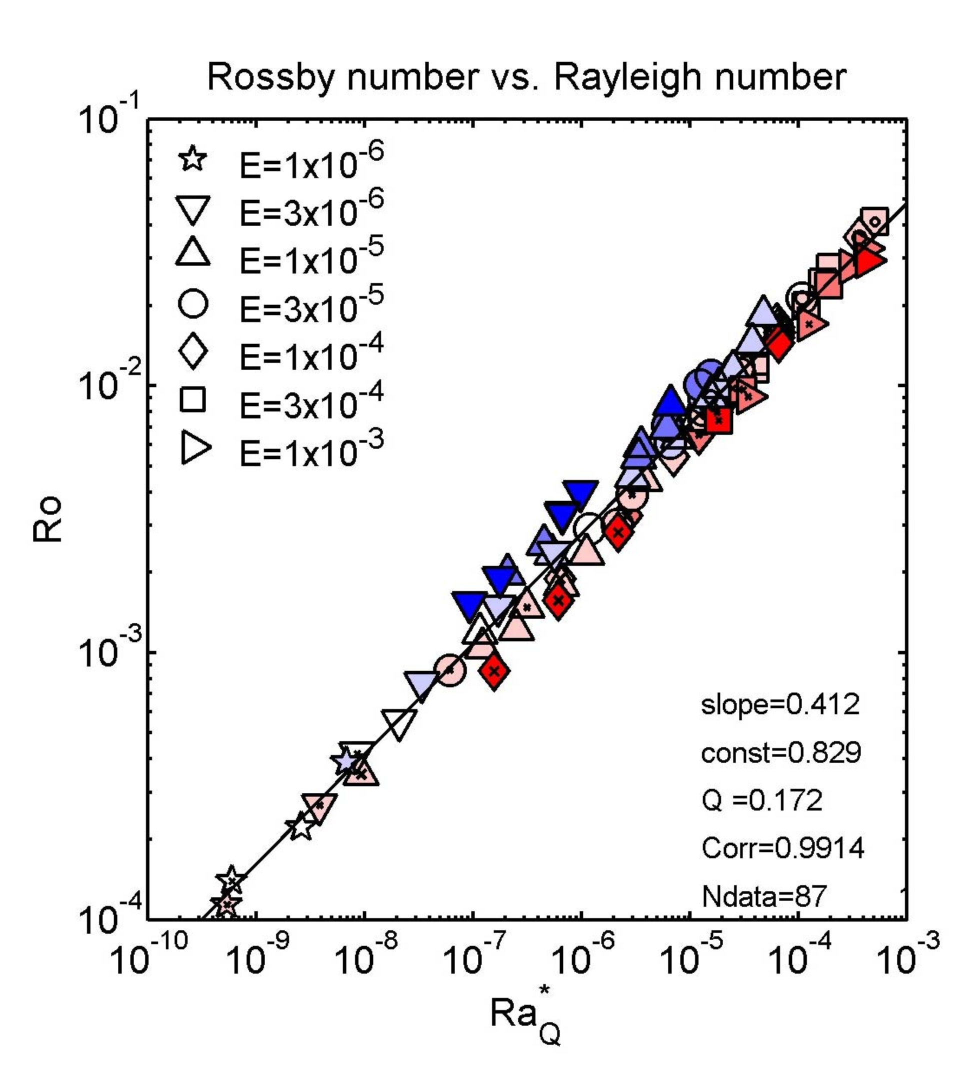




Use of modified „diffusionless" parameters allows to collapse the data and express the dependence by a single power-law.

Compared to non-rotating convection, the exponent is very large (≈ 0.53).

Velocity Scaling

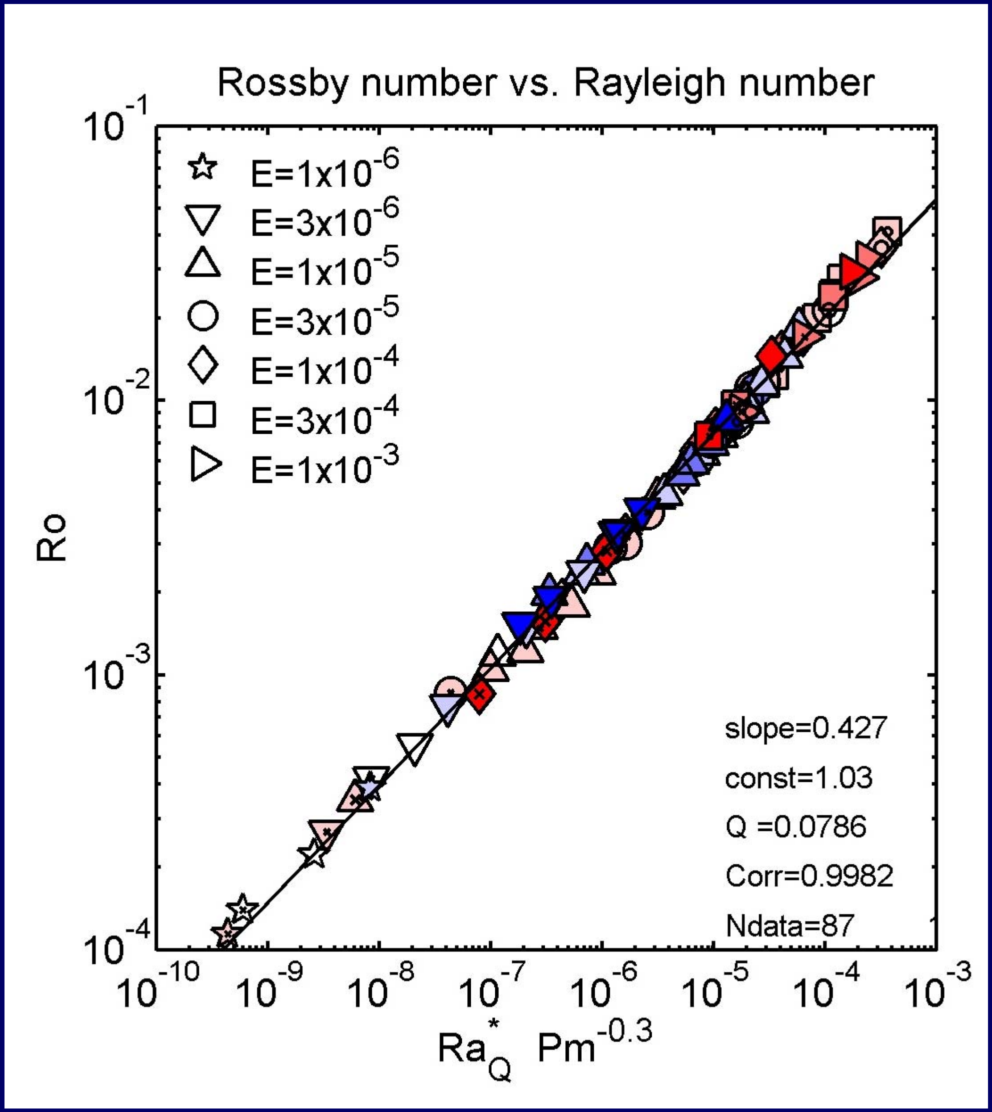


$$Ro \sim Ra^{*}_{Q}{}^{0.41}$$

Exponent 0.4 is predicted from a triple balance Coriolis – Inertia - Buoyancy

Dependence on magnetic Prandtl number ?

Two-parameter fit

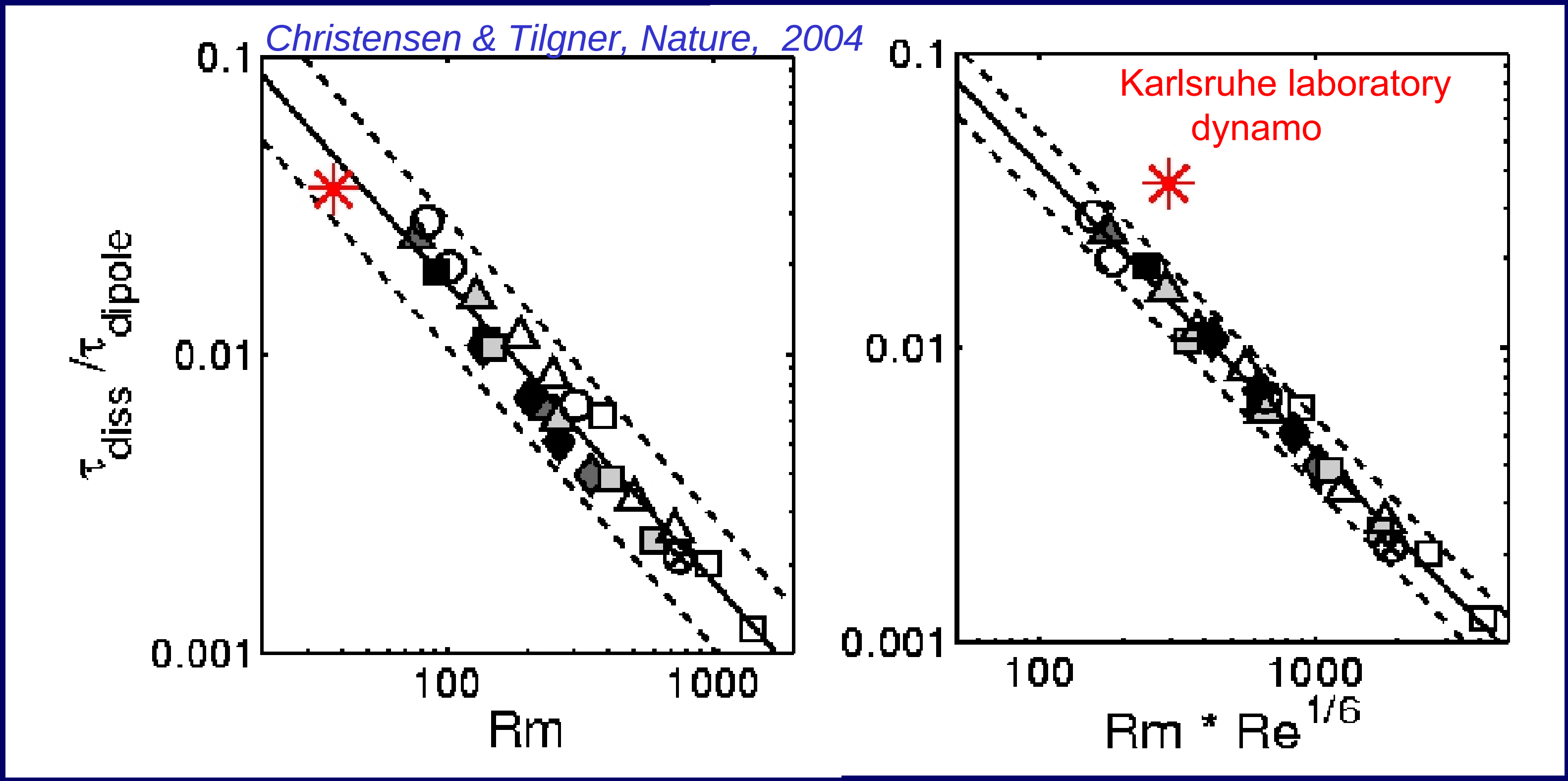


$$Ro \sim Ra^*_Q{}^{0.43} Pm^{-0.13}$$

Two-parameter regression reduces misfit by factor two

Scaling of magnetic dissipation time

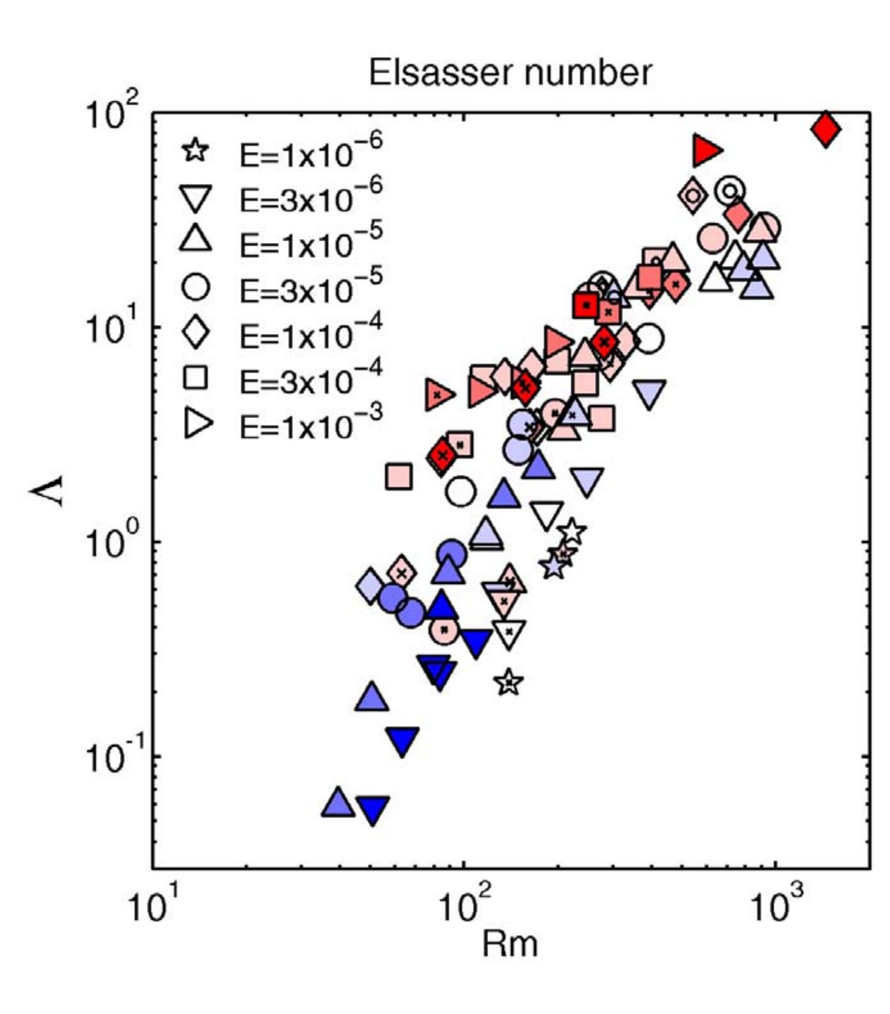
$\tau_{diss} = E_{mag} / D_{ohm}$ (Magnetic energy / Ohmic Dissipation)



$\tau_{diss} \sim Rm^{-1}$

$\tau_{diss} \sim Rm^{-7/6} Pm^{1/6}$

What controls the strength of the magnetic field?



The presence of magnetostrophic balance is often associated with an Elsasser number $\Lambda = B^2/\mu\eta\rho\Omega$ of order one.

In the numerical models, the Elsasser number varies in the range 0.06 – 100.

→ Either the force balance is not magnetostrophic, or Λ not a good measure for magnetostrophy.

Alternative scaling: Magnetic field strength based on available power ?

Power-limited magnetic field strength

$$\tfrac{1}{2}\, Lo^2 = E_{mag} = D_{ohm}\, \tau_{diss} \sim D_{ohm}\, Ro^{-1}\, E\, Pm^{-1}$$

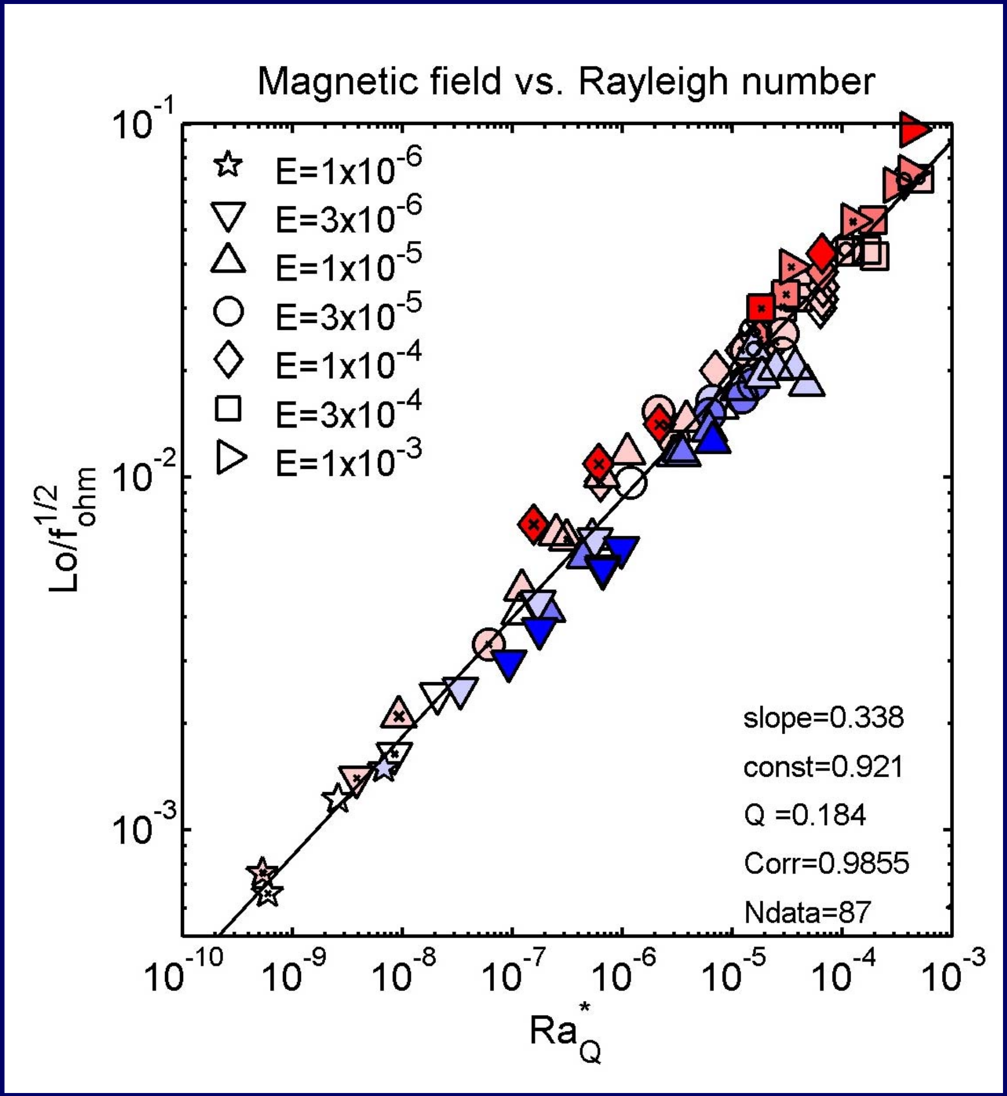
$$Ra^*_Q \sim \text{Power} = D_{ohm} / f_{ohm}$$

$$Ro \sim Ra^{*\,2/5}_Q$$

Prediction: $Lo \sim \sqrt{f_{ohm}}\; Ra^{*\,3/10}_Q$

f_{ohm}: *fraction of ohmic dissipation*

Magnetic Field Scaling



$Lo \sim Ra^*_Q{}^{1/3}$

Empirical fit is close to predicted dependence

Scaling in dimensional form

$$B \sim \mu^{1/2} \rho^{1/6} \left(\frac{P}{4\pi R^2} \right)^{1/3}$$

μ : magnetic permeability

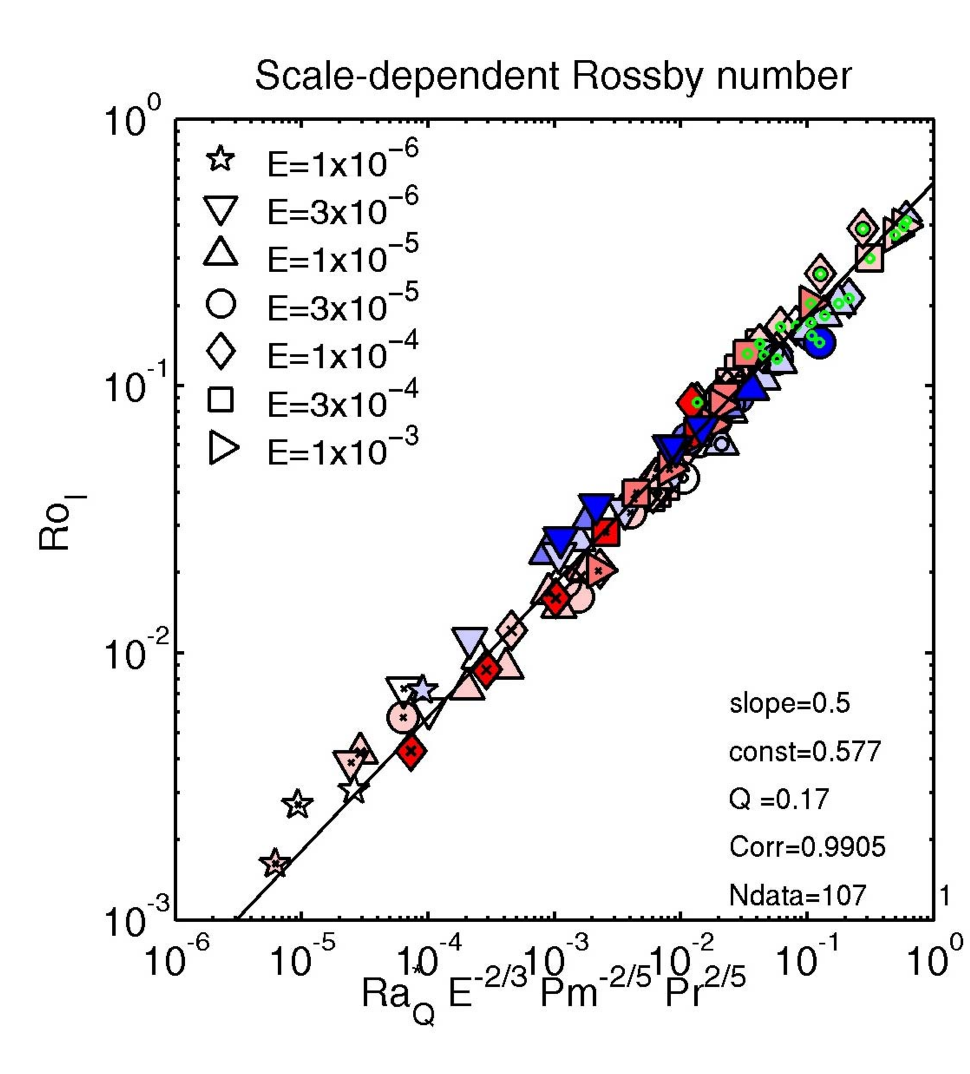
ρ: density

P: power

R: radius of dynamo layer

B independent of conductivity and rotation rate !

Ro_ℓ vs. control parameters

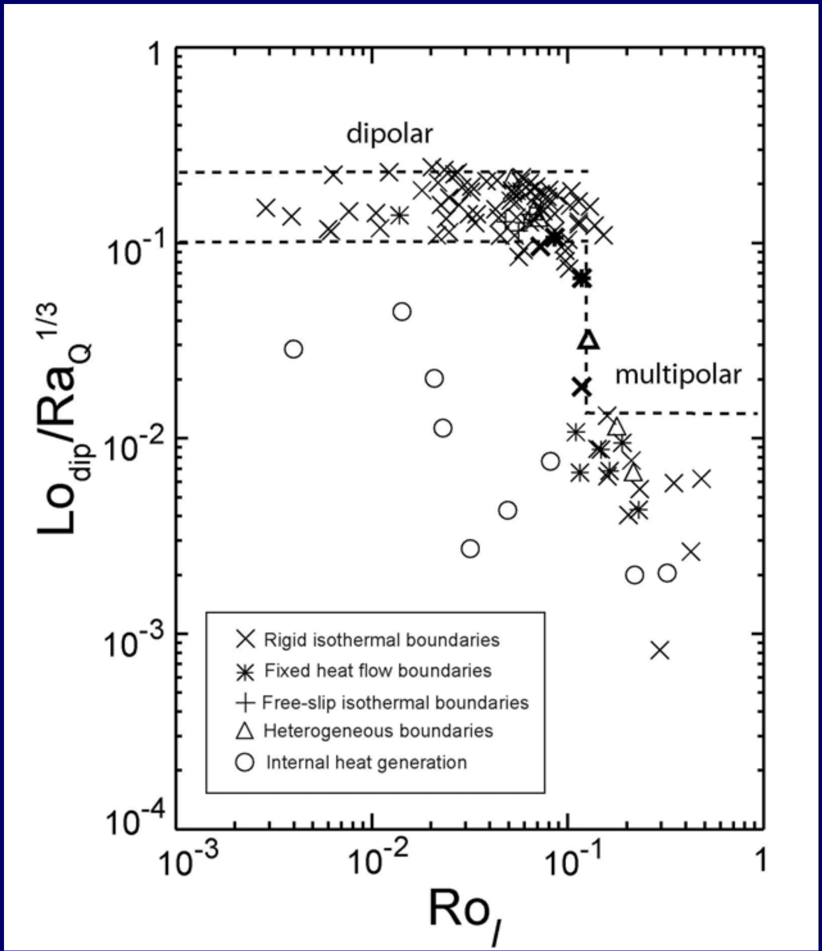


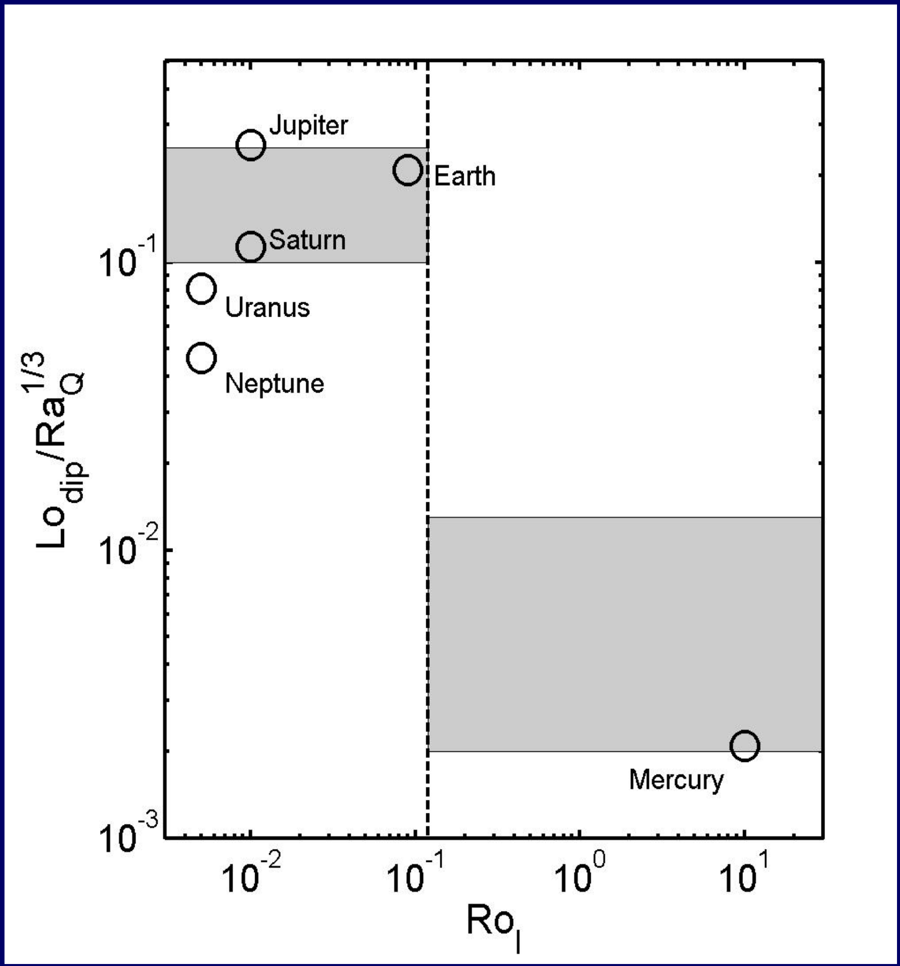
Fit involves all four control parameters

$$Ro_\ell \sim$$

$$Ra_Q^{*1/2}\, E^{-1/3}\, Pm^{-1/5}\, Pr^{1/5}$$

Dipole moment scaling





Earth predicted to lie close to transition dipolar - multipolar