CALCULATION OF THE COEFFICIENTS DEFINING THE MEAN ELECTROMOTIVE FORCE

$$\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

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MEAN–FIELD ELECTRODYNAMICS

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}\,, \quad \boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$$

$$\partial_t \overline{\boldsymbol{B}} - \boldsymbol{\nabla} \times \left(\eta \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} - \boldsymbol{\mathcal{E}} \right) = \boldsymbol{0}$$

$\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$ mean electromotive force due to fluctuations

$$\partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{G}) - \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})$$

$$\boldsymbol{G} = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}} \;\; (= (\boldsymbol{u} \times \boldsymbol{b})')$$

$\Rightarrow$ $\boldsymbol{b}$ is a functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$, which is linear in $\overline{\boldsymbol{B}}$

$\Rightarrow$ $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(\overline{\mathrm{B}})}$, with $\boldsymbol{\mathcal{E}}^{(0)}$ independent of $\overline{\boldsymbol{B}}$
and $\boldsymbol{\mathcal{E}}^{(\overline{\mathrm{B}})}$ linear and homogeneous in $\overline{\boldsymbol{B}}$

The mean electromotive force $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(\overline{\mathrm{B}})}$

Assume (for simplicity) that $\boldsymbol{b}$ decays to zero if $\overline{\boldsymbol{B}} = \mathbf{0}$
(purely hydrodynamic "background turbulence",
no small–scale dynamo)

$\Rightarrow$ $\boldsymbol{\mathcal{E}}^{(0)}$ decays to zero

$$\mathcal{E}_i(\boldsymbol{x},t) = \int_0^{\infty} \int_{\infty} K_{ij}(\boldsymbol{x},t;\boldsymbol{\xi},\tau)\overline{B}_j(\boldsymbol{x}+\boldsymbol{\xi},t-\tau)\mathrm{d}^3\xi\,\mathrm{d}\tau$$

K_{ij} depends on $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$ only (which may depend on $\overline{\boldsymbol{B}}$),
vanishes for large $|\boldsymbol{\xi}|$ and τ.

$\boldsymbol{\mathcal{E}}$ at $(\boldsymbol{x},t)$ depends on the behavior of $\overline{\boldsymbol{B}}$
in some surroundings of $(\boldsymbol{x},t)$ only.

The mean electromotive force

$$\mathcal{E}_i(\boldsymbol{x},t) = \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x},t;\boldsymbol{\xi},\tau)\overline{B}_j(\boldsymbol{x}+\boldsymbol{\xi},t-\tau)\mathrm{d}^3\xi\,\mathrm{d}\tau$$

$$\overline{B}_j(\boldsymbol{x}+\boldsymbol{\xi},t-\tau) = \overline{B}_j(\boldsymbol{x},t) + \frac{\partial \overline{B}_j(\boldsymbol{x},t)}{\partial x_k}\xi_k - \frac{\partial \overline{B}_j(\boldsymbol{x},t)}{\partial t}\tau - \cdots$$

$$\Rightarrow \quad \boxed{\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k} + b_{ij}\frac{\partial \overline{B}_j}{\partial t} + \cdots}$$

$$a_{ij} = \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x},t;\boldsymbol{\xi},\tau)\mathrm{d}^3\xi\mathrm{d}\tau\,, \quad b_{ijk} = \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x},t;\boldsymbol{\xi},\tau)\xi_k\mathrm{d}^3\xi\mathrm{d}\tau\,, \quad \cdots$$

The "ansatz" $$\boxed{\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k}}$$

is an approximation, which needs to be checked in any application !!!

It requires (length) scale separation !!!

Calculation of a_{ij}, b_{ijk}

Recall

$$\partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{G}) - \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}) \,, \quad \boldsymbol{G} = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

- Second-order correlation approximation (SOCA, FOSA)

$$\boldsymbol{G} = \boldsymbol{0}$$

$$a_{ij}(\boldsymbol{x}, t) = \int_0^\infty \int_\infty \Gamma_{ijmn}(\boldsymbol{x}, t, \boldsymbol{\xi}, \tau; \overline{\boldsymbol{U}}) Q_{mn}(\boldsymbol{x}, t; -\boldsymbol{\xi}, -\tau) \mathrm{d}^3\xi \mathrm{d}\tau \,, \cdots$$

$$Q_{mn}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = \overline{u_m(\boldsymbol{x}, t)\, u_n(\boldsymbol{x} + \boldsymbol{\xi}, t + \tau)}$$

If $\boldsymbol{u}$ homogeneous and isotropic and $\overline{\boldsymbol{U}} = \boldsymbol{0}$

$$\alpha = -\frac{1}{3} \int_0^\infty \int_\infty G(\boldsymbol{\xi}, \tau) \overline{\boldsymbol{u}(\boldsymbol{x}, t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{\xi}, t - \tau))} \, \mathrm{d}^3\xi \mathrm{d}\tau$$

$$G(\boldsymbol{\xi}, \tau) = (4\pi\eta t)^{-3/2} \exp(-\boldsymbol{\xi}^2/4\eta t)$$

- Higher-order correlation approximations ...

• TESTFIELD METHOD

(Schrinner et al. *Geophys. Astrophys. Fluid Dyn. 101 (2007) 81-116)*

Assume (for example) $\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$

Choose (suitable) test fields $\overline{\boldsymbol{B}}^{(n)}$

Calculate the corresponding $\boldsymbol{b}^{(n)}$ from

$$\partial_t \boldsymbol{b}^{(n)} - \boldsymbol{\nabla}\times(\overline{\boldsymbol{U}}\times\boldsymbol{b}^{(n)} + \boldsymbol{G}^{(n)}) - \eta\boldsymbol{\nabla}^2\boldsymbol{b}^{(n)} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\overline{\boldsymbol{B}}^{(n)})$$

Calculate the $\boldsymbol{\mathcal{E}}^{(n)} = \overline{\boldsymbol{u}\times\boldsymbol{b}^{(n)}}$

Solve $a_{ij}\overline{B}_j^{(n)} + b_{ijk}\partial\overline{B}_j^{(n)}/\partial x_k = \mathcal{E}_i^{(n)}$

to obtain the a_{ij} and b_{ijk}

- The testfields should be linearly independent
  and all higher than first-order spatial derivatives should be small

- The testfields need not to satisfy any boundary conditions,
  and they need not to be solenoidal

- The testfield method works independent on whether $\boldsymbol{u}$ or $\overline{\boldsymbol{U}}$
  depend on $\overline{\boldsymbol{B}}$,
  is therefore suitable for investigating magnetic quenching

A simple case

Assume that $\overline{\boldsymbol{B}}$ does not depend on x and y.
Then all first–order spatial derivatives of $\overline{\boldsymbol{B}}$ can be expressed by $\overline{\boldsymbol{J}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}$.

$$\Rightarrow \ \mathcal{E}_i = \alpha_{ij}\overline{B}_j - \eta_{ij}\boldsymbol{J}_j \quad (1 \leq i, j \leq 2)$$

Choose testfields

$$\overline{\boldsymbol{B}}^{(1c)} = B(\cos kz, 0, 0)\,, \quad \overline{\boldsymbol{B}}^{(2c)} = B(0, \cos kz, 0)$$

$$\overline{\boldsymbol{B}}^{(1s)} = B(\sin kz, 0, 0)\,, \quad \overline{\boldsymbol{B}}^{(2s)} = B(0, \sin kz, 0)$$

Then

$$\alpha_{ij} = B^{-1}(\mathcal{E}_i^{(jc)} \cos kz + \mathcal{E}_i^{(js)} \sin kz)$$

$$\eta_{ij} = \cdots$$

Results apply exactly in the limit $k \to 0$

Papers using testfield method

Sur et al. *MNRAS* 385 (2008) L15
α and η_t in isotropic turbulence

Brandenburg et al. *ApJ* 676 (2008) 740
effects of shear and rotation, shear-current dynamo

Brandenburg et al. *A&A* 482 (2008) 789
scale dependence of α and η_t

Brandenburg et al. *ApJ L* submitted
magnetic quenching of α and η_t

Rädler et al. *MNRAS(?)* under preparation
mean-field effects in Galloway-Proctor flow