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Circuit Complexity for Gaussian States in QFT

Novel Approaches to Quantum Dynamics

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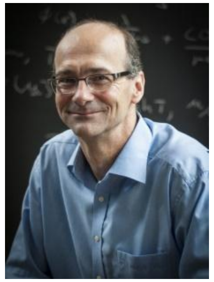

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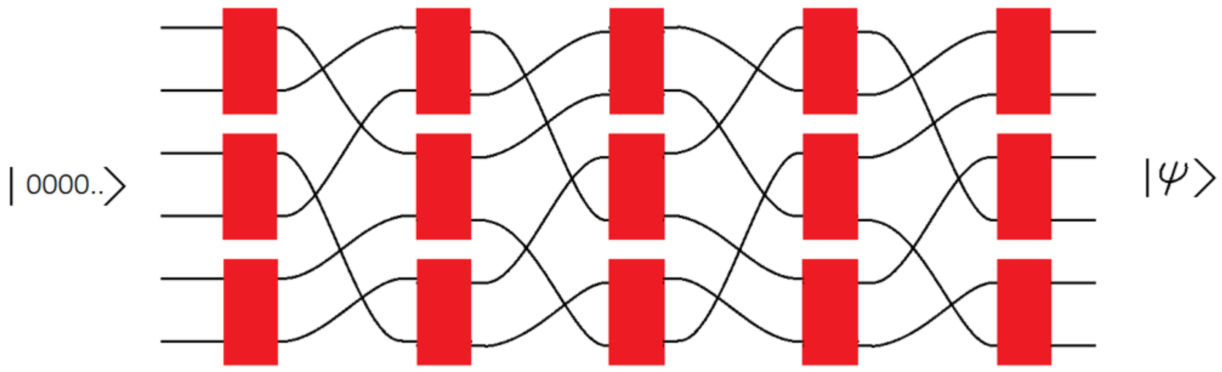
Quantum Computational Complexity

Complexity of a quantum state is defined as the minimal number of elementary unitary operations applied to a simple (unentangled) reference state in order to obtain the state of interest.



Circuit complexity of $|\psi\rangle$ is the minimum number of gates needed to go from $|0000..\rangle$ to $|\psi\rangle$.

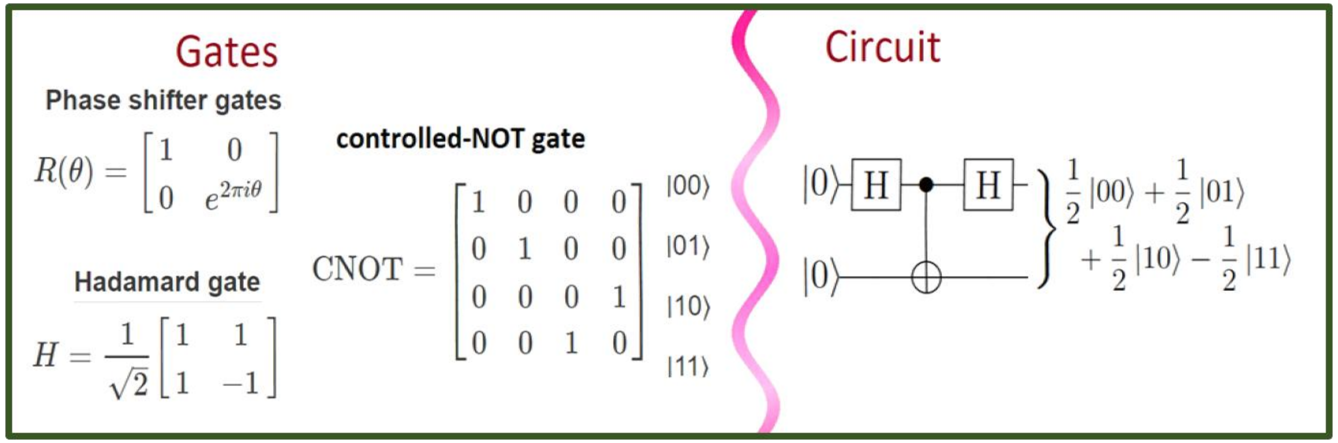
Example - Spin/Qubit Chain

- Quantum circuit will start with simple unentangled reference state, e.g., $|R_i\rangle = |\uparrow\uparrow \cdots \uparrow\rangle$.

- Apply simple gates taken from a universal set, acting on a small number of qubits, e.g., phase shift/Hadamard/CNOT.

- Approximate the state of interest - target state - with unitary operations built from those gates

$$|T\rangle = U|R\rangle = g_1 g_2 \dots g_n |R\rangle.$$



- With tolerance $|T\rangle \approx U|R\rangle$.

- Complexity is the minimal number of gates needed.

- Depends on the various choices!

Complexity Geometry

- The problem of identifying the optimal circuit was first addressed by Nielsen and collaborators by employing a geometric approach in the context of n-qubit chains.

  M. A. Nielsen, M. R. Dowling, M. Gu, and A. M. Doherty, Science 311 (2006) 1133-1135.
  M. A. Nielsen, quant-ph/0502070. M. A. Nielsen and M. R. Dowling, quant-ph/0701004.

- The basic idea is that the elementary gates form a representation of the Lie group $SU(2^N)$, and one can then define a natural geometry on the associated Lie manifold.
- This allows to translate the question of finding the optimal circuit to that of finding the minimal geodesic in the space of unitaries U.

- Neilsen et al. approach this question as the Hamiltonian control problem of finding a time-dependent Hamiltonian $H(t)$ that constructs the desired unitary

$$U = \overleftarrow{\mathcal{P}} \exp\left(-i \int_0^1 dt\, H(t)\right) \qquad H(t) = \sum_I Y^I(t) M_I$$

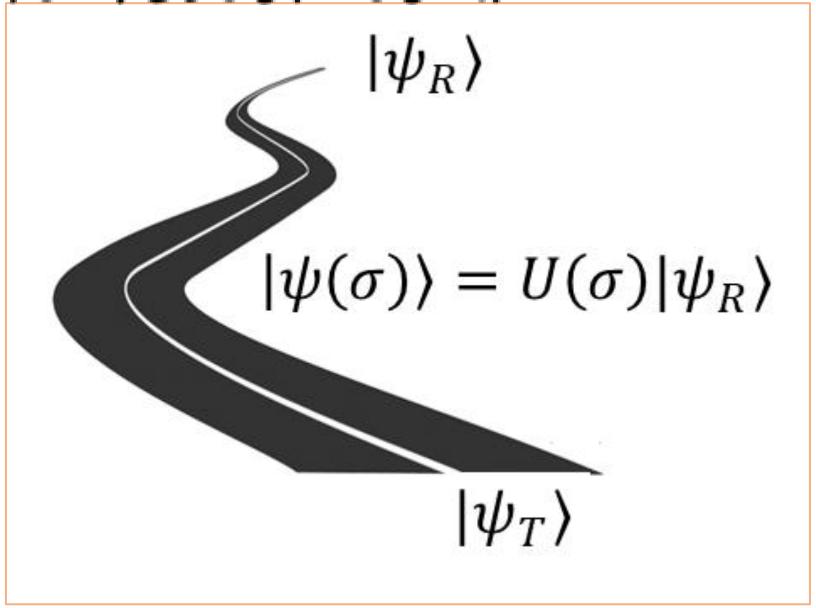
M_I - Generalized Pauli matrices

- The control functions $Y^I(t)$ specify a tangent vector to a trajectory in the space of unitaries

$$U(\sigma) = \overleftarrow{\mathcal{P}} \exp\left(-i \int_0^\sigma dt\, Y^I(t) M_I\right) \qquad \sigma \in [0,1]$$

- We would like to find the shortest path such that

$$U(0) = 1 \qquad U(1)|\psi_R\rangle = |\psi_T\rangle$$



Circuit Complexity from Nielsen Geometry

- Cost function should satisfy certain properties: continuity, positivity, triangle inequality.
- Still a lot of freedom

$$d_1 = \int dt \sum_I |Y^I(t)| \qquad d_k = \int dt \sum_I \left|Y^I(t)\right|^k$$

$$d_P = \int dt \sum_I p_I |Y^I(t)|$$

- Penalty factors can be chosen to favor certain directions in the circuit space, e.g., one or two-qubit operations.
- Alternative: use the Fubini-Study metric as distance function for Gaussian states.

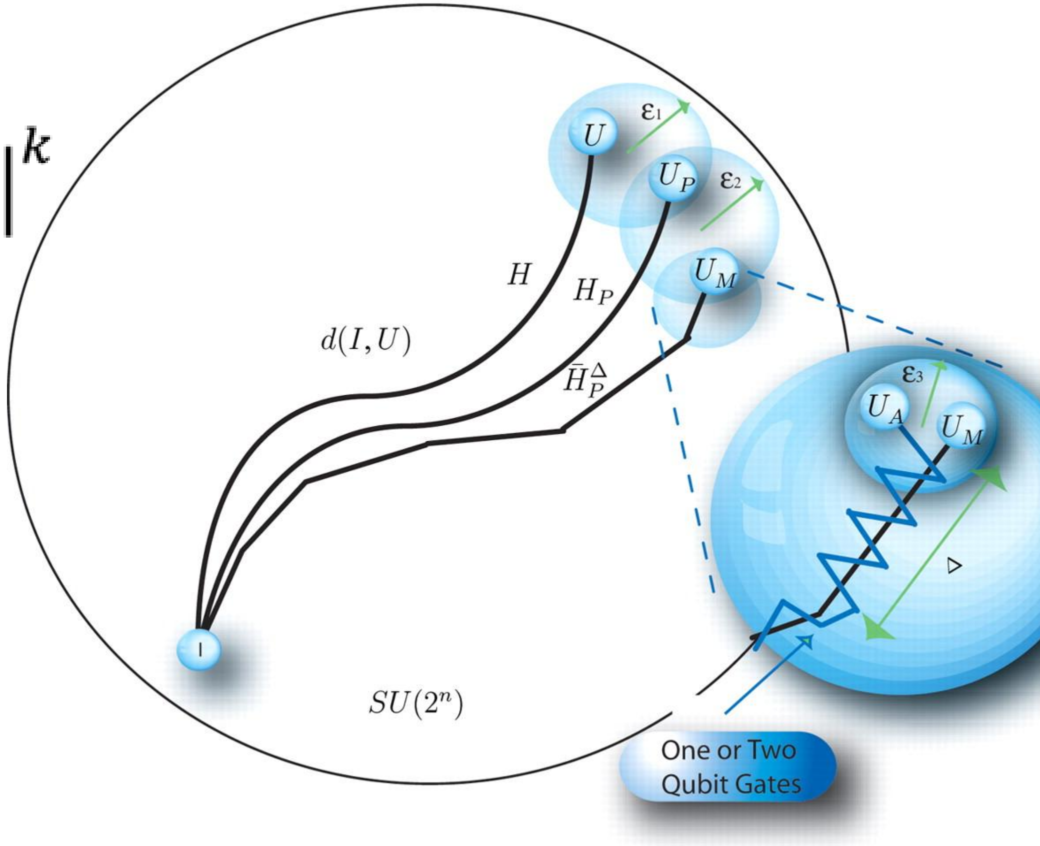


Illustration from “Quantum Computation as Geometry” M. A. Nielsen, M. R. Dowling, M. Gu, and A. M. Doherty, Science 311 (2006) 1133–1135.

Complexity evolution

For fast scramblers (systems that spread the effects of localized perturbations over all the degrees of freedom in a time logarithmic in the entropy), the complexity of the state evolved by the Hamiltonian grows linearly for a very long time.

K - number of d.o.f

$C_{max} = 4^K$ - maximal complexity.

This is based in counting arguments in random unitary circuits on <u>spin chains</u>.

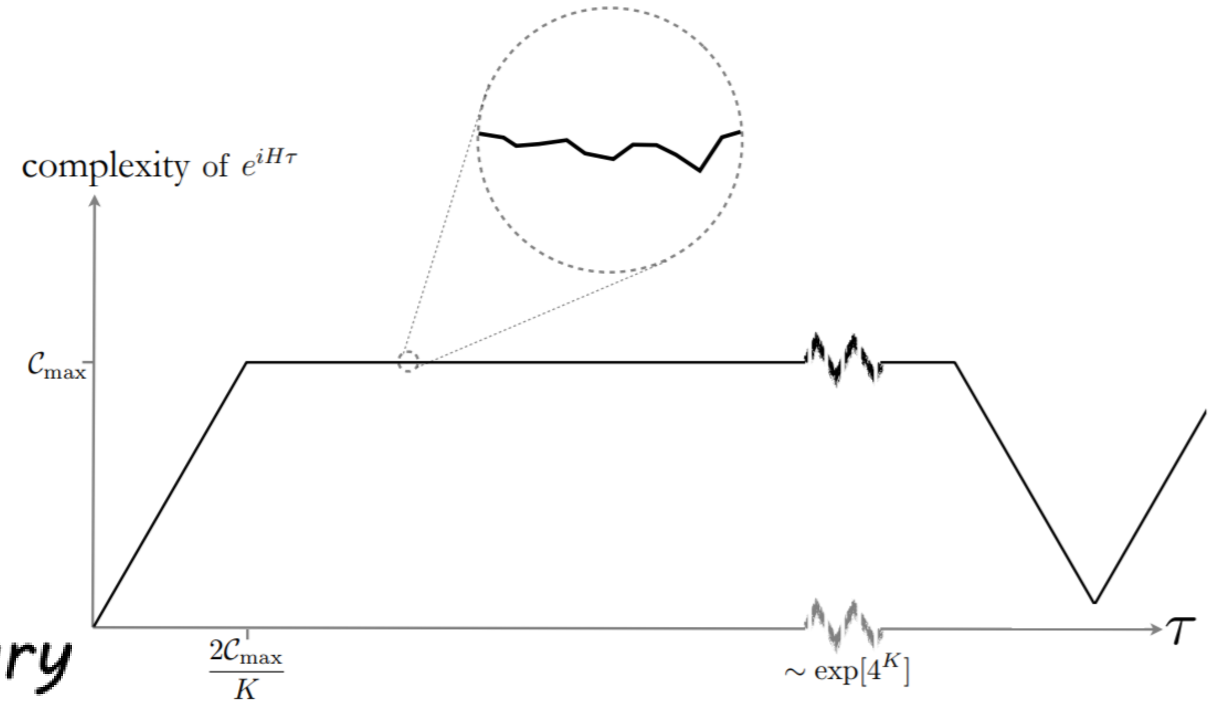
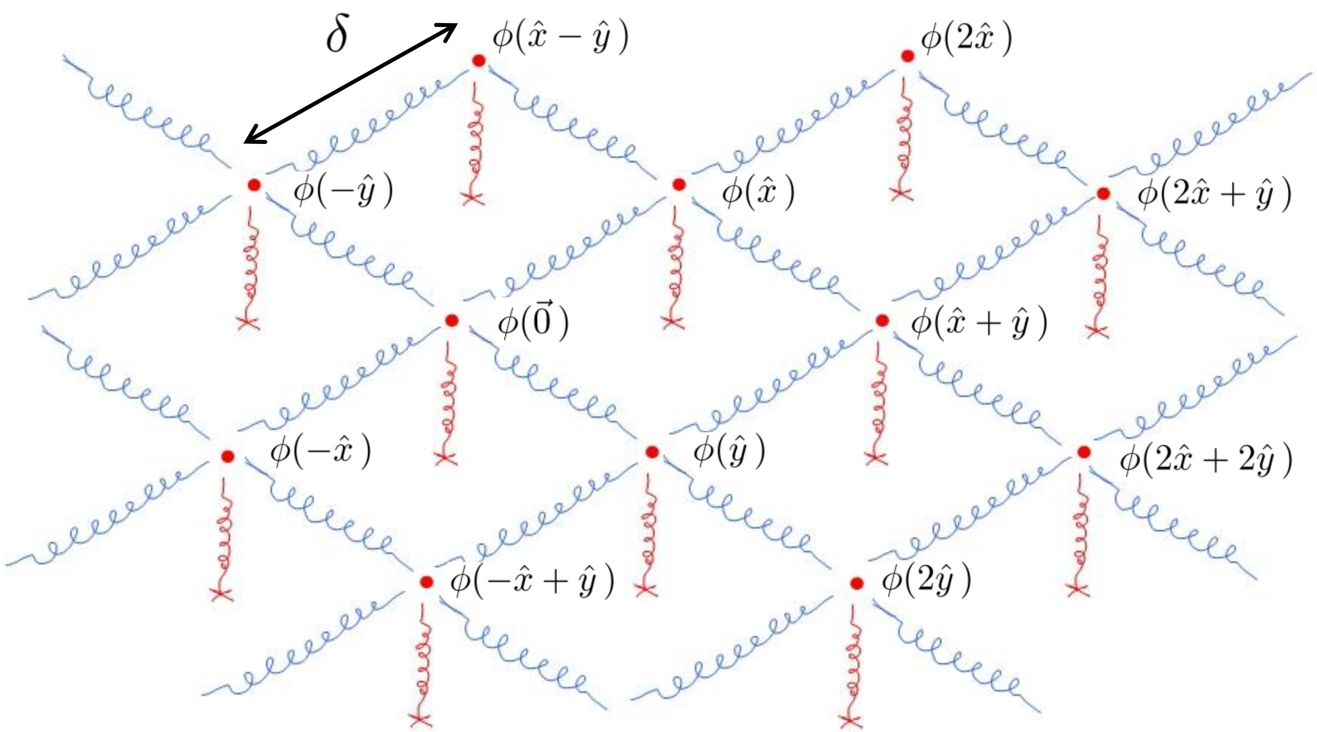


Illustration from "quantum complexity and negative curvature" A. Brown, L. Susskind, and Y. Zhao

Complexity in QFT

- Nielsen construction can be generalized to study the complexity of Gaussian states in QFT.
- For example - the ground state of a free scalar QFT.
- Discretize the field theory on the lattice and obtain a system of coupled harmonic oscillators



$H = \frac{1}{2}\int d^{d-1}x \left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + m^2\phi(x)^2\right]$ - (d spacetime dimension)

Complexity in QFT

- Can be discretized on the lattice and diagonalized by discrete Fourier transform for periodic lattice

$$H = \frac{1}{2} \sum_{\{k_i\}=0}^{N-1} \delta^{d-1} \left[|\pi_k|^2 + \omega_{\vec{k}}^2 |\phi_k|^2 \right]$$

with

$$\omega_k = m^2 + \frac{4}{\delta^2} \Sigma_{i=1}^{d-1} \sin^2\left(\frac{\pi k_i}{N}\right), \quad x_{\vec{k}} = \frac{1}{\sqrt{N}} \Sigma_{\{a_i\}=0}^{N-1} \exp\left(-\frac{2\pi i \vec{k}\vec{a}}{N}\right) x_{\vec{a}}$$

- The wavefunction of the ground state is Gaussian.

$$\langle \phi | 0 \rangle = e^{-\Sigma_k \frac{\delta^{d-1} \omega_k \phi_k^2}{2}}$$

- Take the reference state to be the ground state of the ultralocal Hamiltonian

$$H = \frac{1}{2} \int d^{d-1}x \left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + \mu^2 \phi(x)^2 \right]$$

(the term $\vec{\nabla}\phi(x)^2$ is crossed out)

- Use bilinear gates in field and momentum operators to move between Gaussian states, e.g.,

$$Q_{ab} = e^{i\epsilon \phi_a \pi_b}$$

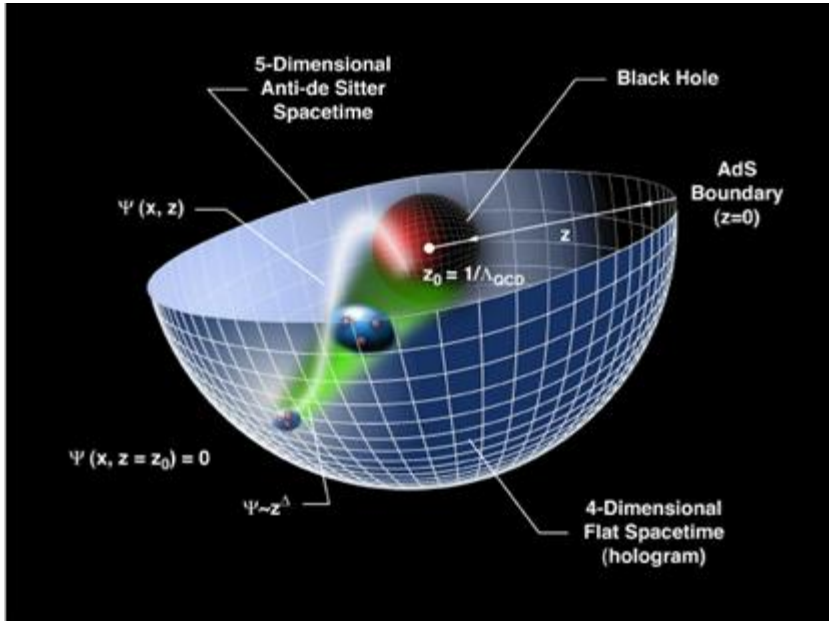
- *structure of UV divergences*

$$C_k = \sum_{\{k_i\}=0}^{N-1} \left| \log \frac{\omega_{\vec{k}}}{\mu} \right|^k \approx \frac{V}{\delta^{d-1}} \left| \log \left(\frac{1}{\mu\delta} \right) \right|^k + \cdots$$

- *Matches expectations from holography where the complexity is conjectured to be related to the gravitational action/volume of a certain region in AdS.*

- *Similarly for fermions, e.g., 3+1d, continuum, infinite line*

$$C_{k=1} = V \int^{1/\delta} \frac{d^3p}{(2\pi)^3} \tan^{-1} \frac{|p|}{m} \approx V/\delta^3 + \cdots$$



Vacuum of free scalar: R. A. Jefferson and R. C. Myers, JHEP 10 (2017) 107,
SC, M. P. Heller, H. Marrochio, and F. Pastawski, Phys. Rev. Lett. 120 no. 12, (2018) 121602.
Vacuum of free fermions: R. Khan, C. Krishnan, and S. Sharma, arXiv:1801.07620,
L. Hackl and R. C. Myers, JHEP 1807 (2018) 139

Thermofield Double State

To appear: Shira Chapman, Jens Eisert, Lucas Hackl, Michal P. Heller, Ro Jefferson, Hugo Marrochio, and Robert C. Myers.

$$|TFD(t_L, t_R)\rangle = Z_\beta^{-1/2} \sum_n e^{-\frac{\beta E_n}{2} - iE_n(t_L+t_R)} |E_n\rangle_L |E_n\rangle_R$$

Tracing over the right Hilbert space leaves a thermal density matrix for the Left system.

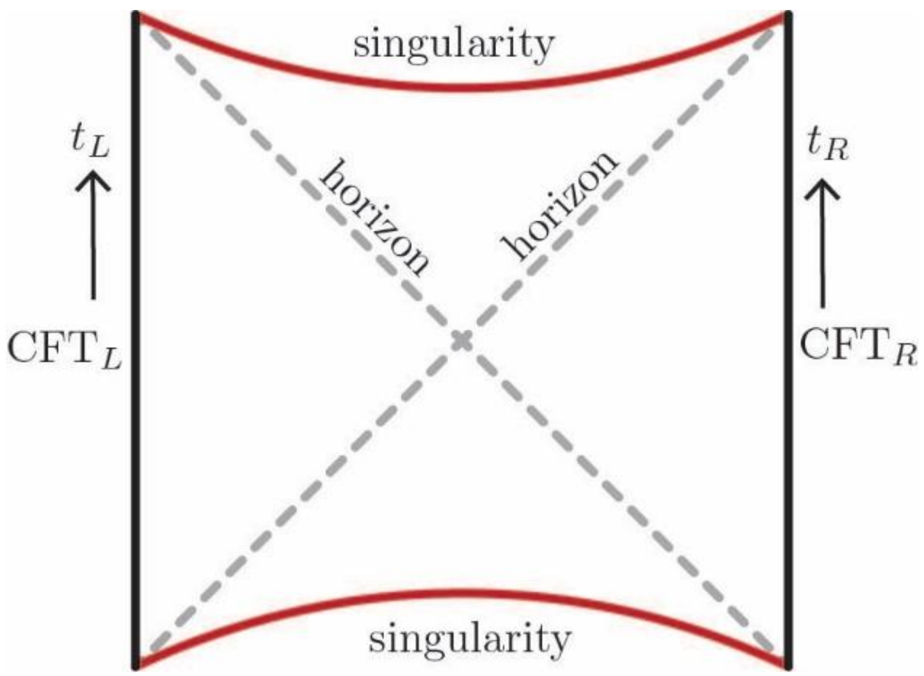
$$\rho_L = \frac{1}{Z_\beta} \sum_n e^{-\beta E_n} |E_n\rangle\langle E_n|$$

Dual to a two sided AdS black hole

Pernrose diagram –

light rays travel at 45 degrees,

distances are not faithfully represented.



TFD for Free QFT

We will focus on the thermofield double state of a free scalar QFT with Hamiltonian

$$H = H_L + H_R = \frac{1}{2}\int d^d x \left[\pi_L^2 + \left(\vec{\nabla}\phi_L\right)^2 + m^2\phi_L^2\right] + (L \leftrightarrow R)$$

Again discretizing on the lattice and diagonalizing we end up with a sum of independent momentum modes.

Every mode consists of two simple harmonic oscillators

$$H = \sum_n H_n \qquad H_n = \frac{1}{2M}\left[P_L^2 + P_R^2 + M^2\omega_n^2(Q_L^2 + Q_R^2)\right]$$

$$P = \pi\delta^{\frac{d}{2}-1}$$
$$Q = \phi\delta^{\frac{d}{2}}$$

$M = \delta^{-1}$ $\quad \omega_n = \left(m^2 + \frac{4}{\delta^2}\sum_i \sin^2\left(\frac{\pi n_i}{N}\right)\right)^{1/2}$ $\quad \delta$ – lattice spacing

The thermofield double for two harmonic oscillators

$$H = \frac{1}{2M}\left[P_L^2 + P_R^2 + M^2\omega^2(Q_L^2 + Q_R^2)\right]$$

is given by

$$|TFD\rangle = \mathcal{N}\sum_{n=0}^{\infty} e^{-\frac{\beta n\,\omega}{2} - i\left(n+\frac{1}{2}\right)\omega t}\,|n\rangle_L|n\rangle_R$$

$$E_n = \left(n + \tfrac{1}{2}\right)\omega$$
$$t_R = t_L = t/2$$

It is time dependent! The reduced density matrix of one side is time independent.

Can be also expressed as a unitary acting on the vacuum

$$= \mathcal{N} e^{(z\,a_L^\dagger a_R^\dagger - z^* a_L a_R)}|0\rangle_L|0\rangle_R$$

$$z = \frac{1}{2}\log\left(\frac{1+e^{-\frac{\beta\omega}{2}}}{1-e^{-\frac{\beta\omega}{2}}}\right)e^{-i\,\omega\,t}$$

Use diagonal modes to decouple the left and right sides

$$\boldsymbol{Q}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{Q}_L \pm \boldsymbol{Q}_R), \qquad \boldsymbol{P}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{P}_L \pm \boldsymbol{P}_R)$$

then TFD takes the form

$$\alpha = \frac{1}{2}\log\left(\frac{1+e^{-\frac{\beta\omega}{2}}}{1-e^{-\frac{\beta\omega}{2}}}\right)$$

$$|TFD(t)\rangle = e^{-i\,\alpha\,\hat{O}_+(t)}|0\rangle_+ \times e^{i\,\alpha\,\hat{O}_-(t)}\,|0\rangle_-$$

$$\hat{O}_\pm(t) = \frac{1}{2}\cos(\omega t)\left(Q_\pm P_\pm + P_\pm Q_\pm\right) + \frac{1}{2}\sin(\omega t)\left(M\omega Q_\pm^2 - \frac{1}{M\omega}P_\pm^2\right)$$

Focus on one of the modes.

We will need to consider more general bilinear generators.

Develop better framework for Gaussian states.

Gaussian States

Focus on pure Gaussian states with vanishing one point function.

In wave-function representation

$$\psi(q) = \langle q|\psi\rangle = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{2}(a + i\, b)\, q^2\right]$$

All data is given by the two point function (Wick's theorem).

Use the covariance matrix

$$G = \langle\psi|\xi^a\xi^b + \xi^b\xi^a|\psi\rangle; \qquad \xi^a = (q, p)$$

Explicitly

$$G = 2\begin{pmatrix} \langle\psi|q^2|\psi\rangle & \langle\psi|\frac{pq+qp}{2}|\psi\rangle \\ \langle\psi|\frac{pq+qp}{2}|\psi\rangle & \langle\psi|p^2|\psi\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ -\frac{b}{a} & \frac{a^2+b^2}{a} \end{pmatrix}$$

Acting with quadratic operators leaves us in this class of Gaussian states.

Quadratic operators

$$\hat{K}_1 = \frac{1}{2}(qp + pq) \qquad \hat{K}_2 = \frac{q^2}{\sqrt{2}} \qquad \hat{K}_3 = \frac{p^2}{\sqrt{2}}$$

These gates go beyond those used for the vacuum, but are needed to construct the TFD.

The generators above form an $\mathfrak{sp}(2,R)$ algebra and the U-s form the associated symplectic group $SP(2,R)$

We want to act with them to build our circuits

$$|\psi_I\rangle = e^{-i\,\epsilon \hat{K}_I}|\psi\rangle$$

Don't act linearly on the parameters of the wave-function, however they do act linearly on the covariance matrix:

$$G_1 = U_1 G U_1^T \qquad U_1 = e^{\epsilon K_1} \qquad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$G_2 = U_2 G U_2^T \qquad U_2 = e^{\epsilon K_2} \qquad K_2 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}$$

$$G_3 = U_3 G U_3^T \qquad U_3 = e^{\epsilon K_3} \qquad K_3 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$$

Group Structure

Rephrase the circuit in terms of trajectories between covariance matrices

Circuits will take the form $G(\sigma) = U(\sigma) G U^T(\sigma)$.

A general group element can be parameterized as

$$U = \begin{pmatrix} \cos\tau\cosh\rho - \sin\theta\sinh\rho & -\sin\tau\cosh\rho + \cos\theta\sinh\rho \\ \sin\tau\cosh\rho + \cos\theta\sinh\rho & \cos\tau\cosh\rho + \sin\theta\sinh\rho \end{pmatrix}$$

With $\tau \in [-\pi, \pi)$ and (ρ, θ) can be viewed as polar coordinates in the plane.

For the d_2 cost function the distance will be given according to the following metric

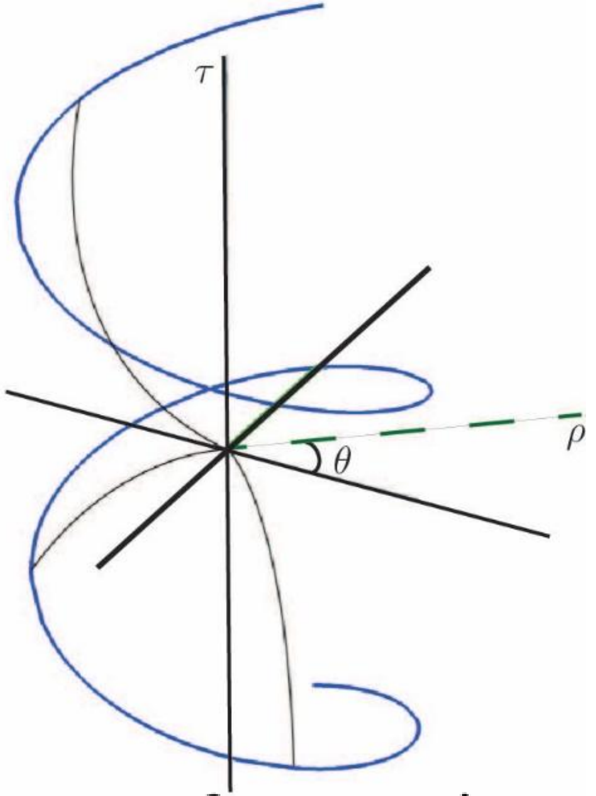
$$ds^2 = d\rho^2 + \cosh(2\rho)\cosh^2(\rho)\, d\tau^2 + \cosh(2\rho)\sinh^2(\rho)\, d\theta^2 - \sinh^2(2\rho) d\tau d\theta$$

Finding the minimal geodesics

- Look for geodesic in the space of circuits - symplectic transformations $U(\sigma) = U\big(\rho(\sigma), \theta(\sigma), \tau(\sigma)\big)$ that satisfy

$$U(0) = 1 \qquad U(1) G_R U(1)^T = G_{TFD}(t)$$

- The boundary condition is not unique - there is a one parameter family of transformations that achieve the task.
- This fixes a spiral in the (ρ, θ, τ) space.



- One needs to find the shortest geodesic which reaches to the spiral under the metric from the previous slide.

⟶ Its length will give the complexity (for the d_2 cost function).

-Use the same trajectory to bound the d_1 complexity.

Results - Complexity of Formation

For a massive theory
The ratio increases exponentially
with the mass (and the dimension).

nta up to a cutoff).

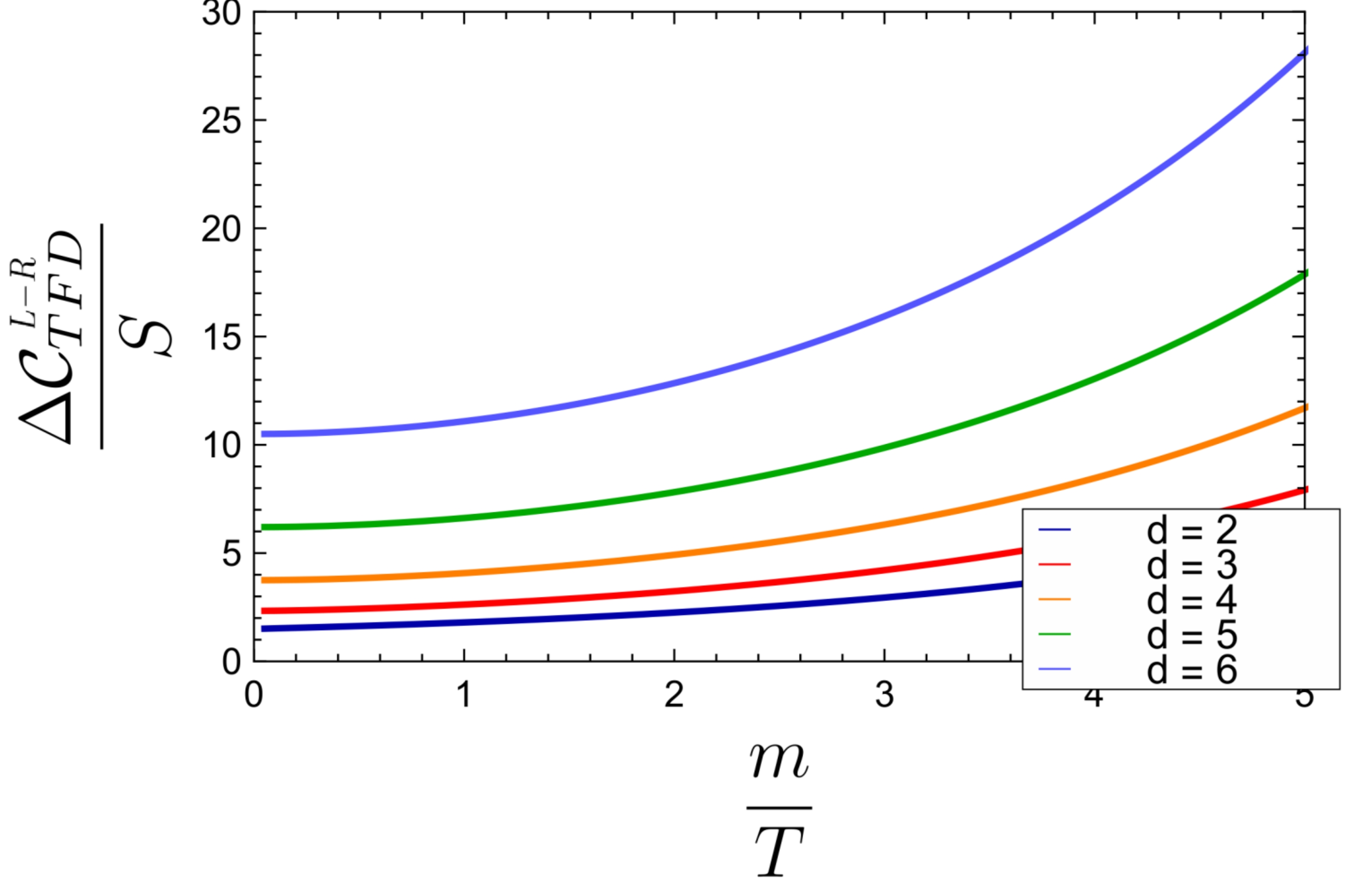
For the d_1 norm

massless

$\Delta C = $

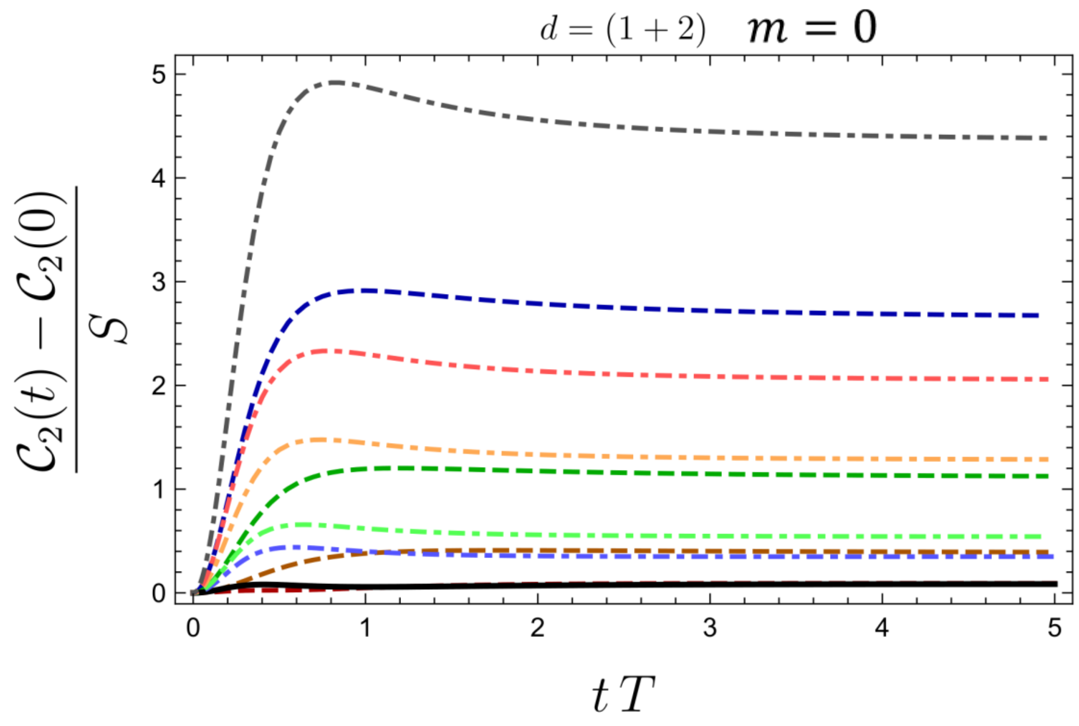
$\Delta C_{m=0} = VT^{d-1}$

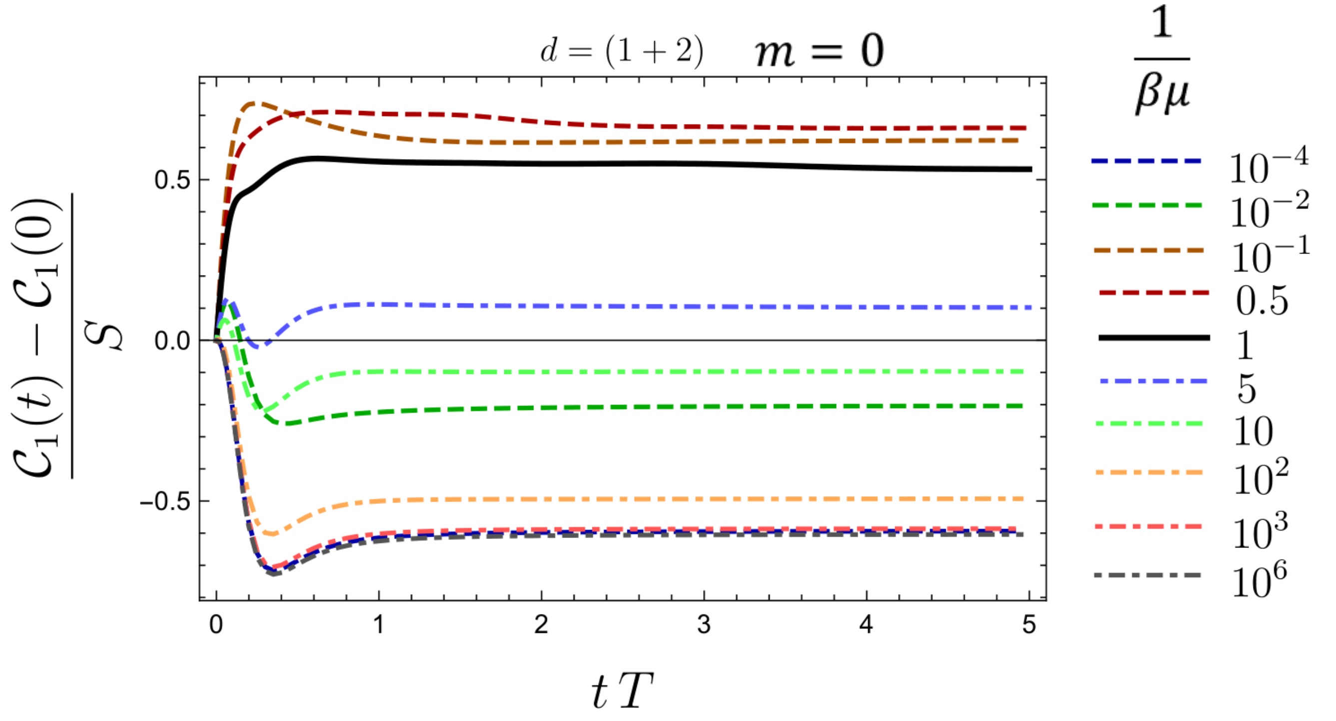
properties: Δ



Time dependence – infinite line

time-dependent TFD (with $t_L = t_R = t/2$): $|TFD(t)\rangle \simeq \bigotimes_{\vec{k}} \sum_{n_{\vec{k}}} e^{-n_{\vec{k}}\omega_{\vec{k}}\beta/2}\, e^{-i(n_{\vec{k}}+\frac{1}{2})\omega_{\vec{k}} t}\, |n_{\vec{k}}\rangle_L |n_{\vec{k}}\rangle_R$





C decreases/increases; saturates with $C \sim S$ at $t \sim \beta$

⟶ *free vs fast scrambler,*

⟶ *state remains Gaussian vs explores full Hilbert space.*

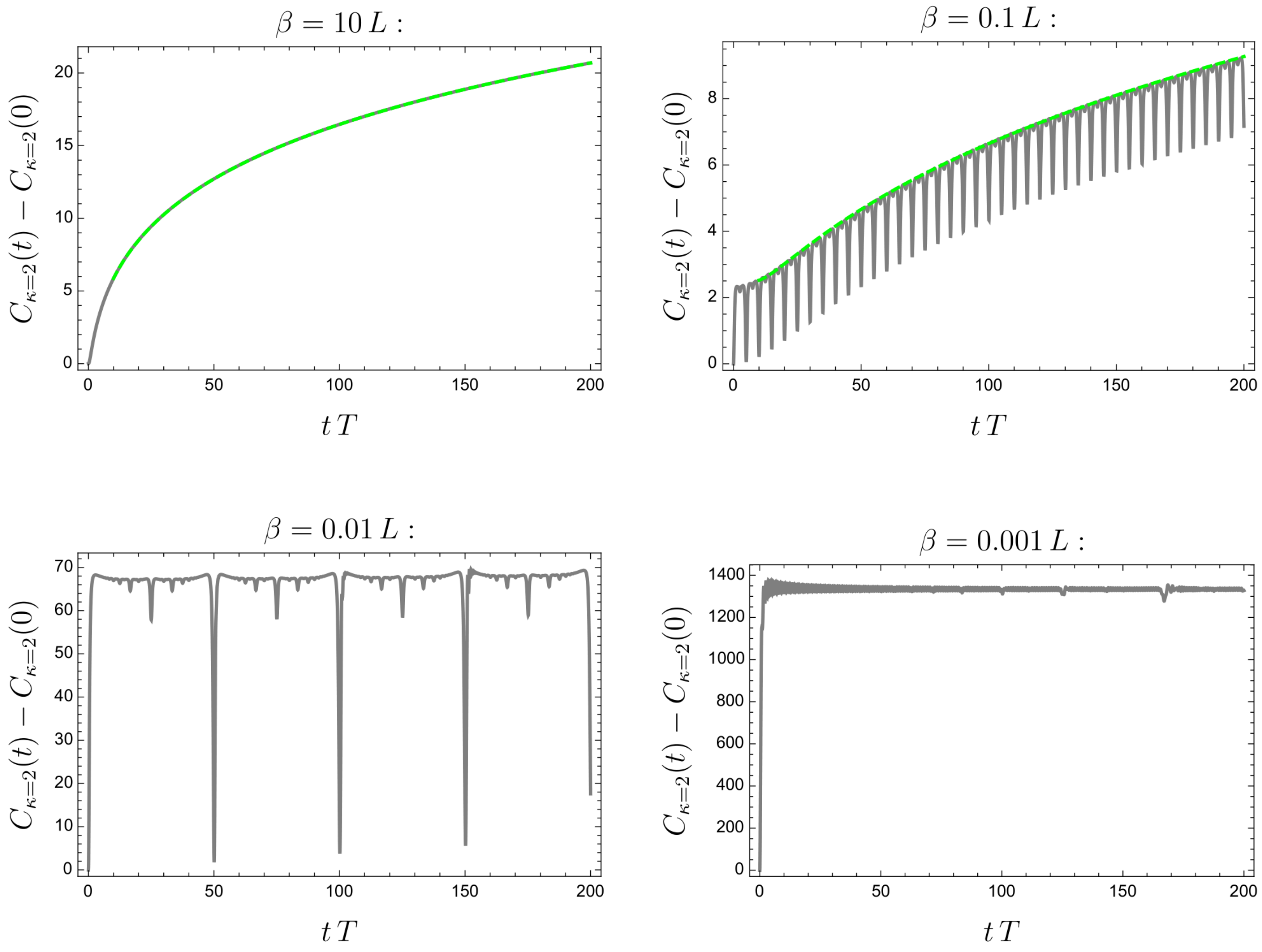
(early time transient depends on μ)

Time dependence on the circle

- On the circle have a zero mode (mode of lowest frequency).
- When m=0 complexity not well defined.
- Introduce small mass as IR regulator.
- Zero mode causes logarithmic growth which terminates at times which are inversely proportional to the mass.

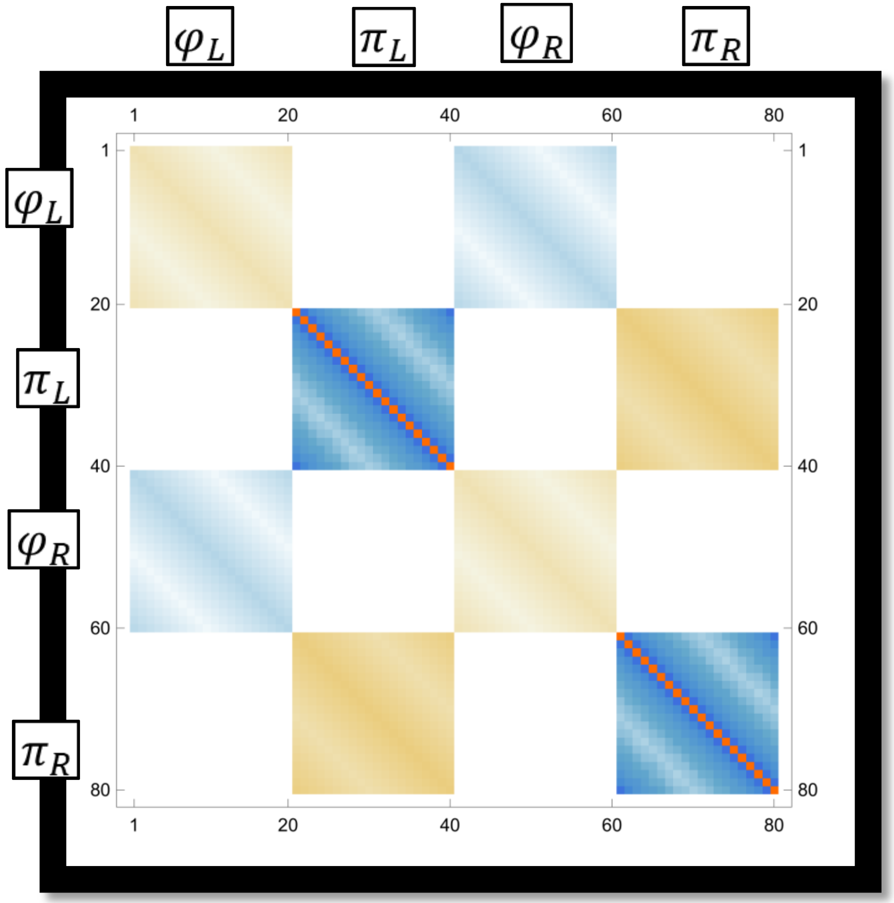
- Behaves similar to the line for high temperatures where the zero mode becomes less dominant.

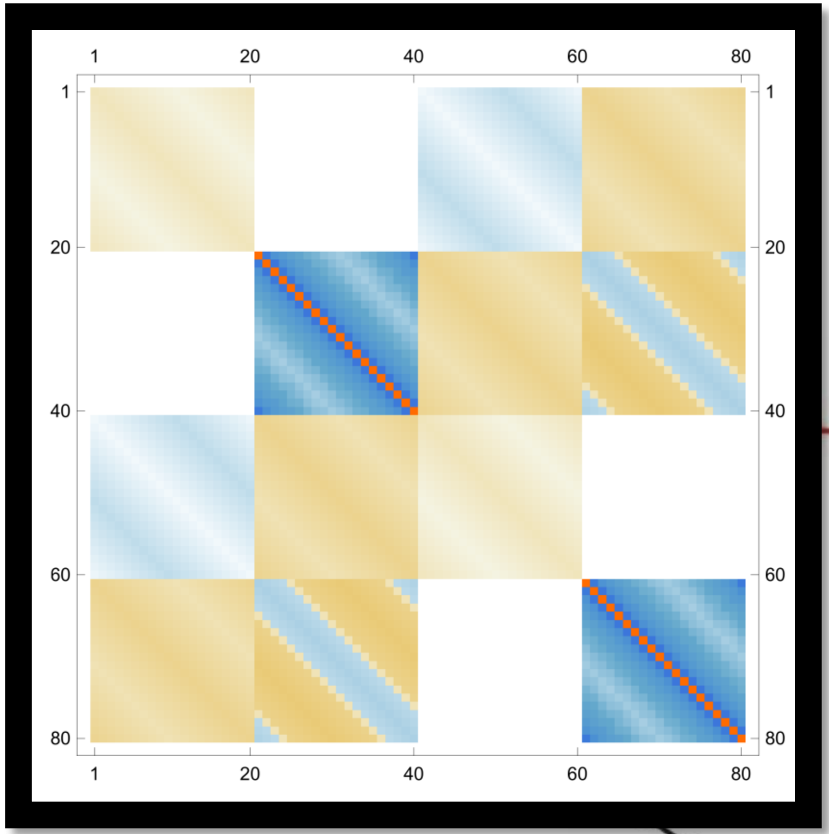
- Oscillations with frequency which is inversely proportional to the circle length as if two wave packets were propagating on a circle with the speed of light in opposite directions.

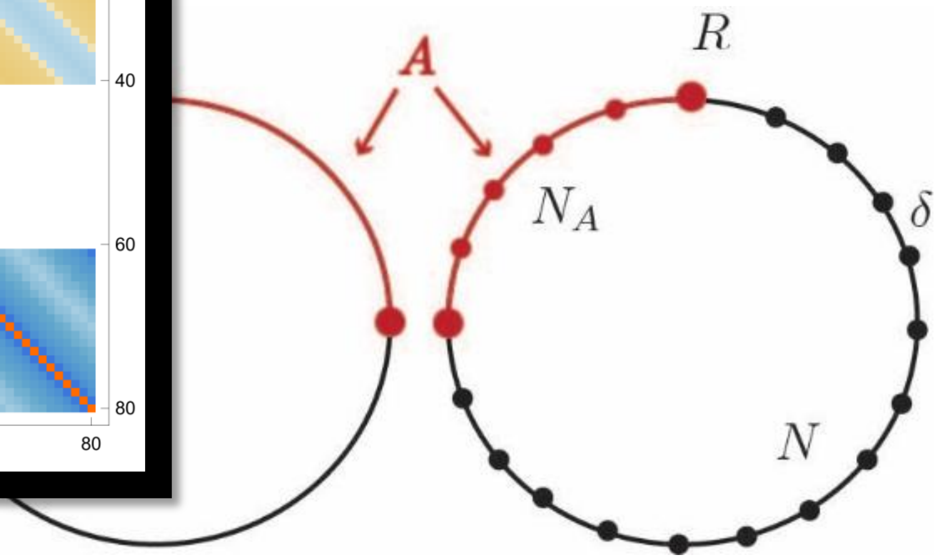


$m = 10^{-6}/L$, 1601 lattice sites on each side.

Entanglement entropy





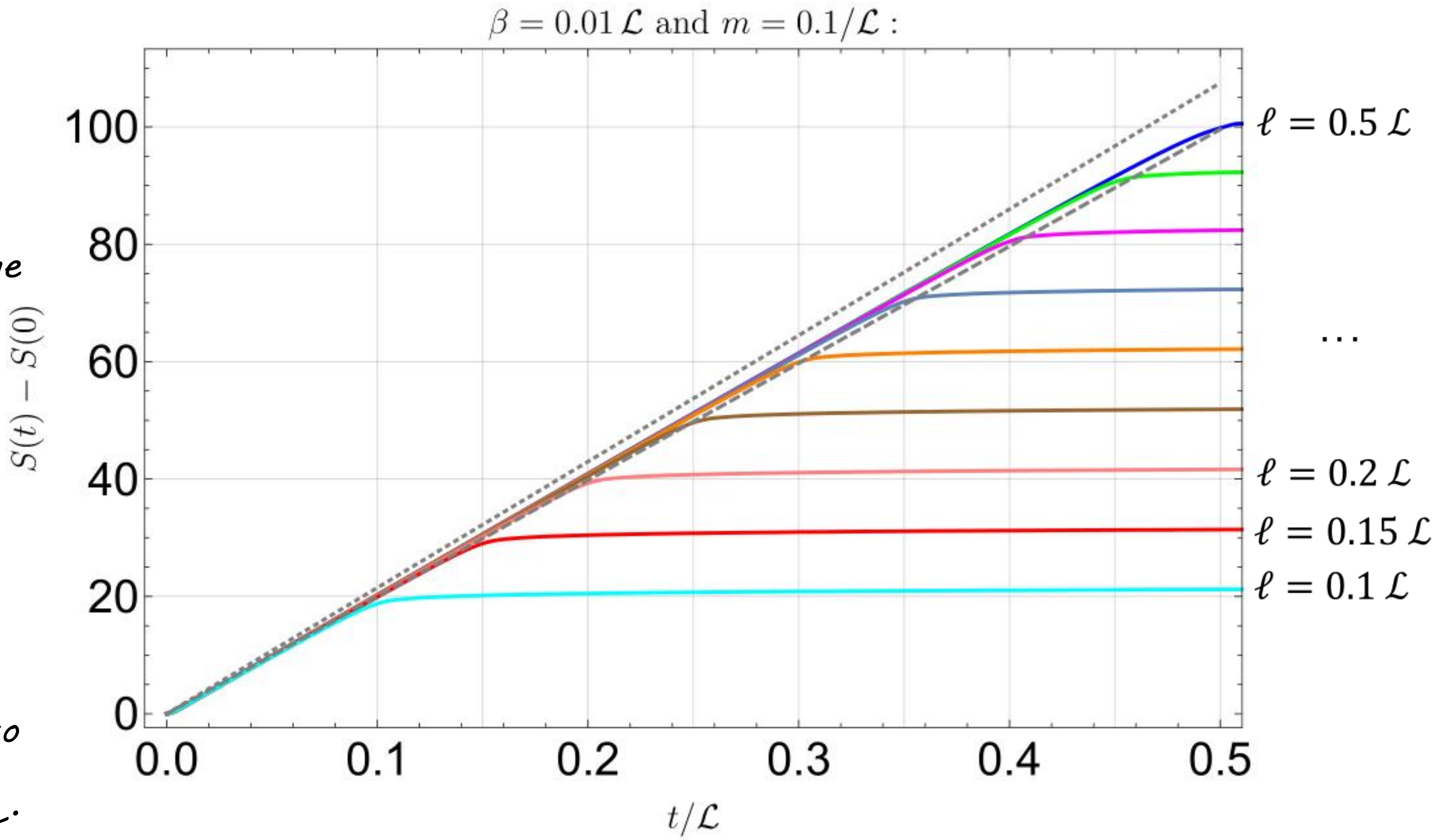


Block structure of the covariance matrix.
Time evolution does not influence the thermal blocks.
But the mixed Right-Left blocks change, and so we see a change of the EE with time.

- When $m\beta \ll 1$ there is an initial regime with linear growth at slope proportional to the thermodynamic entropy.

- The growth terminates at times of the order of half the size of the interval.

- For longer times there are oscillations. These would disappear on the infinite line.



- T. Hartman, J. Maldacena, JHEP 05 (2013) 014 pointed out that this system is similar to a quench.
- See also: P. Calabrese and J. L. Cardy, J. Stat. Mech. 0504, P04010 (2005).

Future direction

- Possible extensions of QFT model - Go beyond Gaussian states:
  - Complexity for excited states?
  - Ground states of interacting QFTs?
    [see: Bhattacharyya, Shekar, Sinha, hep-th/1808.03105]
  - TFD with random phases?
  - Gauge theories?
  - Complexity
    [see: Agon, I n, quant-ph/9806029]

Lots to explore!

- QFT/path in … ary CFT?

  build $\rho_A \to S_{EE} = -\Sigma\lambda_n \log\lambda_n$ ⟶ Replica trick

  build optimal $U \to C = \#gates$ ⟶ ?????

[see: Caputa et al hep-th/1703.00456, hep-th/1706.07056, Czech hep-th/1706.00965]

- Strongly interacting systems and concrete connection to "holographic complexity"?

Thank you!