Exactly Solvable Minimal Model of Maximal Quantum Chaos

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B Bertini, P Kos, TP, arXiv:1805.00931

Outline

1. Spectral correlations in quantum systems and
   The Quantum Chaos Conjecture
2. Self-dual Kicked Ising model:
   a minimal solvable model for maximal many body quantum chaos
   (*No small parameter, such as $\hbar$ or inverse local Hilbert space dimension!*)
3. Sketch of the derivation/proof

Consider hamiltonian H of a quantum system with finite volume L (length, in 1D) and let $\{E_n\}_{n=1,\ldots,\mathcal{N}=2^L}$ be its **spectrum**.

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Analogous object in periodically driven systems

$$H(t) = H(t+T)$$

is the set of **quasi-energies** $\{\varphi_n \in [0, 2\pi]\}_{n=1,\ldots,\mathcal{N}}$ such that $\{e^{-i\varphi_n}\}$ is the spectrum of the Floquet operator

$$U = T\exp\left(-i\int_0^T \mathrm{d}s\, H(s)\right).$$

The **spectrum** as a **gas** in one dimension

Spectral density (1-point function):

$$\rho(\varphi) = \frac{2\pi}{\mathcal{N}} \sum_n \delta(\varphi - \varphi_n).$$

Spectral pair correlation function (2-point function):

$$r(\vartheta) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi \rho(\varphi + \tfrac{1}{2}\vartheta)\rho(\varphi - \tfrac{1}{2}\vartheta) - 1.$$

Spectral Form Factor (SFF) (Fourier transform of 2-point function):

$$\begin{aligned} K(t) &= \frac{\mathcal{N}^2}{2\pi} \int_0^{2\pi} \mathrm{d}\vartheta r(\vartheta) e^{it\vartheta} = \sum_{m,n} e^{i(\varphi_m - \varphi_n)} - \mathcal{N}^2 \delta_{t,0} \\ &= \left|\mathrm{tr}\, U^t\right|^2 - \mathcal{N}^2 \delta_{t,0}. \end{aligned}$$

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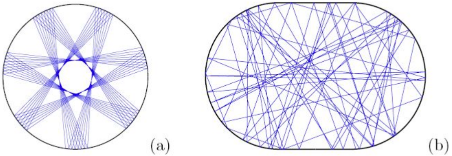
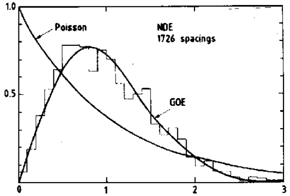
Caveat: SFF is not self-averaging! Consider instead $\bar{K}(t) = \mathbb{E}[K(t)]$.

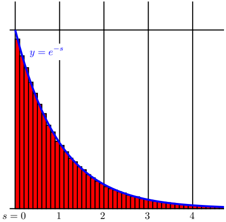
Casati, Guarnerri, Valz-Gris 1980, Berry 1981, Bohigas, Giannoni, Schmidt 1984

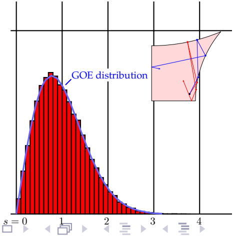
The spectral fluctuations of quantum systems with chaotic and ergodic classical limit are *universal* and described by Random Matrix Theory (RMT).

The same holds for periodically-driven systems if one considers the statisrtics of quasi-energies instead.









RMT (No time reversal symmetry):

$$K_{\mathrm{CUE}}(t) = t, \quad t < \mathcal{N}.$$

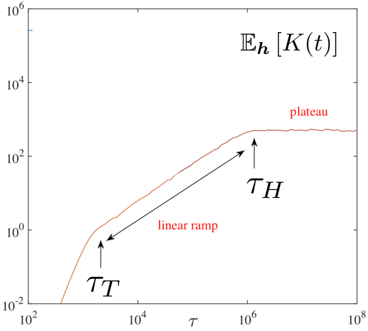
RMT (With time teversal symmetry):

$$K_{\mathrm{COE}}(t) = 2t - \log(1 + 2t/\mathcal{N}), \quad t < \mathrm{N}.$$

Random (uncorrelated, Poissonian) spectrum $\{\varphi_n\}$:

$$K_{\mathrm{Poisson}} \equiv \mathcal{N}.$$

Real System:



Review: Chen and Ludwig 2017

$$\mathbb{E}[K(t)] = \mathbb{E}\left[\sum_{m,n} e^{i(\varphi_m - \varphi_n)}\right].$$

Saturation $\bar{K}(t) \sim \mathcal{N}$ beyond Heisenberg time $t > t_{\mathrm{H}} = \mathcal{N} = 1/\Delta\varphi$.

Non-universal (system-specific) behaviour below Ehrenfest/Thouless time $t < t_{\mathrm{T}}$.

To first order, this is captured by the *diagonal approximation* (Berry 1985)

$$K(\tau) \sim \sum_{p}^{\tau} \sum_{p'}^{\tau} A_p e^{\mathrm{i} S_p/\hbar} A_{p'}^* e^{-\mathrm{i} S_{p'}/\hbar} \simeq (2) \sum_{p}^{\tau} |A_p|^2 = (2) t$$

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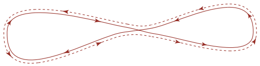
To second order, the RMT term is reproduced by considering so-called Sieber-Richter (2001) pairs of orbits



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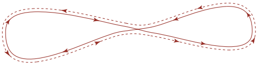
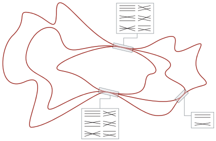


To all orders, RMT terms is reproduced by considering full combinatorics of self-encountering orbits (Müller et al, 2004)



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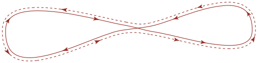

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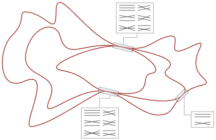

Rigorous proof only possible for very specific class of models: Fully connected incommensurate quantum graphs [Pluhař and Weidenmüller, PRL 2014]

What about QCC for many-body systems at '$\hbar \sim 1$'? (say for interacting spin 1/2 or fermionic systems)

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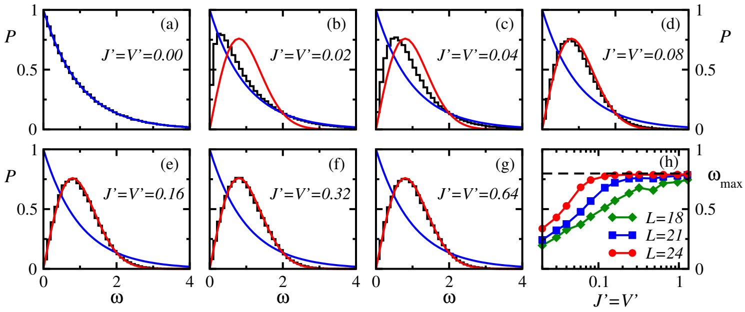
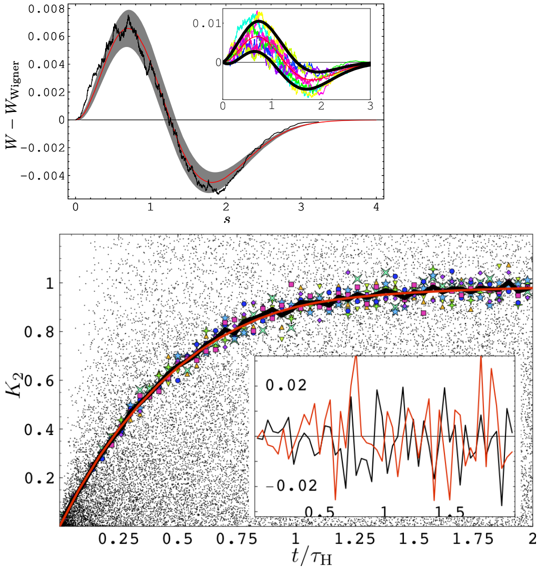
$$H = \sum_{j=0}^{L-1} (-J c_j^\dagger c_{j+1} - J' c_j^\dagger c_{j+2} + \mathrm{h.c.} + V n_j n_{j+1} + V' n_j n_{j+2}), \quad n_j = c_j^\dagger c_j.$$



From [Rigol and Santos, 2010]



From [Pineda and TP, PRE 2007]

Only very recently first analytic results arrived..

Floquet long-ranged (non-integrable/non-mean field) spin 1/2 chains
[arXiv:1712.02665]

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Many-Body Quantum Chaos: Analytic Connection to Random Matrix Theory

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(Received 5 February 2018; revised manuscript received 12 April 2018; published 8 June 2018)

Floqeut local quantum circuits with random unitary gates in the limit of large local Hilbert space dimension $q \to \infty$
[arXiv:1712.06836,arXiv:1803.03841]

Solution of a minimal model for many-body quantum chaos

Amos Chan, Andrea De Luca and J. T. Chalker
Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom
(Dated: December 20, 2017)

Spectral statistics in spatially extended chaotic quantum many-body systems

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Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom
(Dated: April 4, 2018)

What about fermionic or spin 1/2 systems with strictly local interactions?

Kicked Ising model [TP, JPA 1998; PTPS 2000; PRE 2002]

$$H_{\rm KI}[\boldsymbol{h};t] = H_{\rm I}[\boldsymbol{h}] + \delta_p(t) H_{\rm K}, \quad H_{\rm I}[\boldsymbol{h}] \equiv \sum_{j=1}^{L} \left\{ J\sigma^z_j \sigma^z_{j+1} + h_j \sigma^z_j \right\}, \quad H_{\rm K} \equiv b \sum_{j=1}^{L} \sigma^x_j,$$

with Floquet propagator

$$U_{\rm KI} = e^{-iH_{\rm K}} e^{-iH_{\rm I}}.$$

J, b: homogeneous spin-coupling and transverse field
h_j position dependent longitudinal field

Kicked Ising model [TP, JPA 1998; PTPS 2000; PRE 2002]

$$H_{\mathrm{KI}}[\boldsymbol{h};t] = H_{\mathrm{I}}[\boldsymbol{h}] + \delta_p(t) H_{\mathrm{K}}, \quad H_{\mathrm{I}}[\boldsymbol{h}] \equiv \sum_{j=1}^{L} \left\{ J\sigma_j^z \sigma_{j+1}^z + h_j \sigma_j^z \right\}, \quad H_{\mathrm{K}} \equiv b \sum_{j=1}^{L} \sigma_j^x,$$

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$$U_{\mathrm{KI}} = e^{-iH_{\mathrm{K}}} e^{-iH_{\mathrm{I}}}.$$

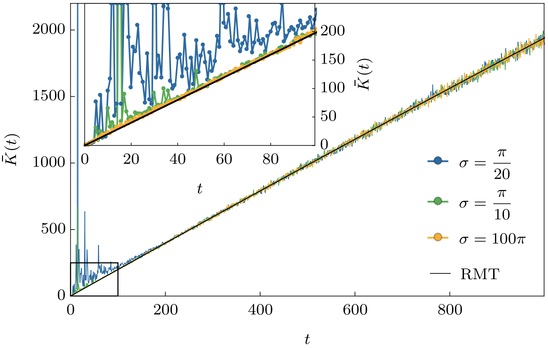
J, b: homogeneous spin-coupling and transverse field
h_j position dependent longitudinal field

Remarks:

- KI model is integrable if $b = 0$ or $h_j \equiv 0$.
- For generic h_j and $b \neq 0$, the model has no symmetries.
- The clean model $h_j \equiv h$, for $J \sim b \sim h \sim 1$ appears to be ergodic and its spectral statistics well described by RMT
- The clean model appears to display non-trivial non-ergodicity – ergodicity transition when h is varied [TP PRE 2002, TP JPA 2007, see also Vajna, Klobas, TP, Polkovnikov, PRL 120, 200607 (2018)]

Consider longitudinal magnetic field h_j to be i.i.d. (Gaussian) variable with mean $\bar{h}$ and standard deviation σ

$$\bar{K}(t) = \mathbb{E}_{\boldsymbol{h}}[K(t)] = \int_{-\infty}^{\infty} \left(\prod_{j=1}^{L} \frac{\mathrm{d}h_j}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(h_j - \bar{h})^2}{2\sigma^2} \right) \right) K(t).$$



For $|J| = |b| = \pi/4$ and σ large enough the behaviour seems immediately RMT-like ($t_{\mathrm{T}} \sim 1$)

Interpreting $\bar{K}(t)$ in terms of a partition function of $2d$ classical statistical model, we can study SFF analytically in thermodynamic limit!

Key result: Exact SFF in thermodynamic limit

Theorem: For *odd* t:

$$\lim_{L\to\infty} \bar{K}(t) = \begin{cases} 2t-1\,, & t \leq 5 \\ 2t\,, & t \geq 7 \end{cases}.$$

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Conjecture: For *even* t:

$$\bar{K}(2) = 2,\ \bar{K}(4) = 7,\ \bar{K}(6) = 13,\ \bar{K}(8) = 18,\ \bar{K}(10) = 22,$$
$$\bar{K}(t) = 2t + 1, \quad t \ge 12.$$

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Remarks:

- Results independent of $\sigma > 0$: The model is ergodic for any disorder strength (**no Floquet-MBL!**). In particular, we can take the limit of a clean system at the end $\sigma \searrow 0$.
- Results independent of $\bar{h}$: We can set $\bar{h} = 0$ which corresponds to a limiting integrable system.

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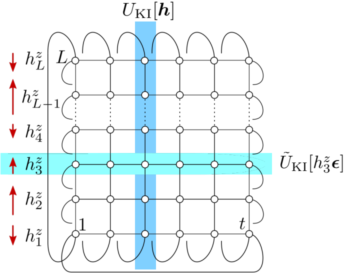
- Results independent of $\bar{h}$: We can set $\bar{h} = 0$ which corresponds to a limiting integrable system.

We found a simple locally interacting model with finite dimensional local Hilbert space with proven RMT spectral correlations at all time-scales!

Sketch of the proof. Space-time duality

The trace of U_{KI}^t is equivalent to a partition sum of a classical 2d Ising model with **row-homogeneous field** h_j:

$$\mathrm{tr}\,(U_{\mathrm{KI}}^t[\boldsymbol{h}]) \sim$$



Duality relation

$$tr\,(U_{\mathrm{KI}}[\boldsymbol{h}])^t = tr\left(\prod_{j=1}^{L} \tilde{U}_{\mathrm{KI}}[h_j \boldsymbol{\epsilon}]\right)$$

where $\boldsymbol{\epsilon} = (1, 1 \ldots, 1)$ and $\tilde{U}_{\mathrm{KI}}$ is a KI model on a ring of size t with twisted parameters $\tilde{J}(J, b)$, $\tilde{b}(J, b)$.

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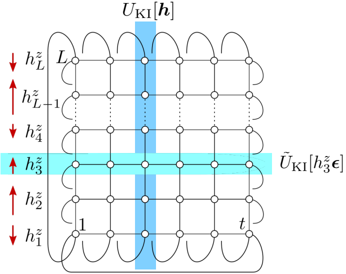

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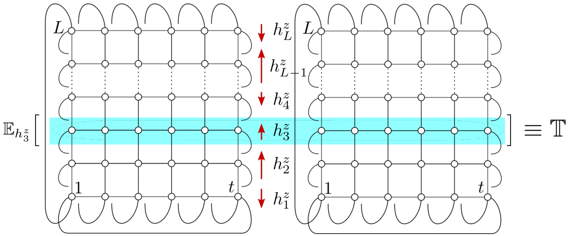
Remarkably: $\tilde{U}_{\mathrm{KI}}$ is **unitary** for $|J| = |b| = \pi/4$ (Self-dual, $J = \pm\tilde{J}$, $b = \pm\tilde{b}$)

Space-time duality allows to simply express the disorder averaging:

$$\mathbb{E}_{\boldsymbol{h}}\left[K(t)\right]$$

$$\equiv$$



$$=$$

$$\mathrm{tr}\left(\mathbb{T}^{L}\right)$$

Unitary

$$\mathbb{T} \equiv \mathbb{E}_h\left[\tilde{U}_{\mathrm{KI}}[h\boldsymbol{\epsilon}] \otimes \tilde{U}_{\mathrm{KI}}[h\boldsymbol{\epsilon}]^*\right] = (\tilde{U}_{\mathrm{KI}} \otimes \tilde{U}^*_{\mathrm{KI}}) \cdot \mathbb{O}_\sigma$$

Non-Unitary

Contraction

$$\mathbb{O}_\sigma = \exp\left[-\tfrac{1}{2}\sigma^2\left(M_z \otimes I - I \otimes M_z\right)^2\right]$$

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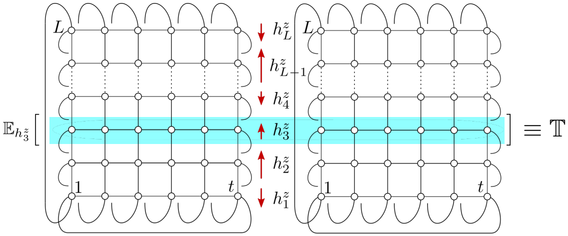

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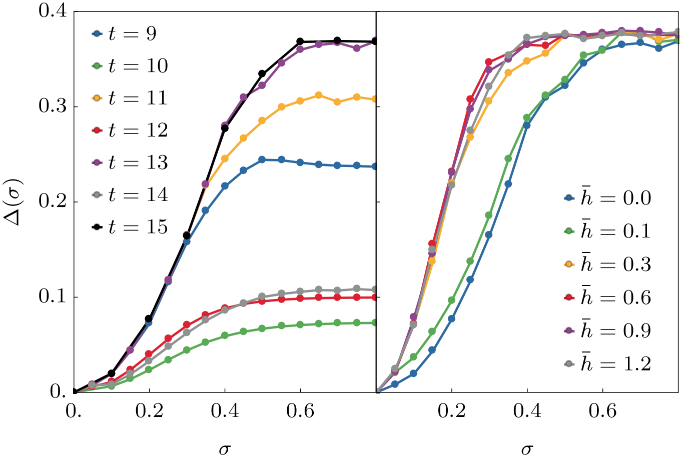
Computation of thermodynamic SFF $\lim_{L\to\infty} \mathbb{T}^L$ thus amounts to determining the multiplicity of eigenvalue 1 of $\mathbb{T}$ and proving positive spectral gap.

Empirical convergence of the spectral gap



The following is straightforward to show:

Property 1

1. The eigenvalues of $\mathbb{T}$ of maximal (unit) magnitude are either $+1$ or -1.
2. Each eigenvector associated to the eigenvalue ± 1 is uniquely parametrized by an operator $A \in \mathrm{End}((\mathbb{C}^2)^{\otimes t})$ satisfying

$$UAU^\dagger = \pm A, \quad [A, M_\alpha] = 0, \quad \alpha \in \{x, y, z\}. \tag{1}$$

where we have defined $M_\alpha = \sum_{\tau=1}^{t} \sigma_\tau^\alpha$, $U = \exp\left[i\frac{\pi}{4}\sum_{\tau=1}^{t}(\sigma_\tau^z \sigma_{\tau+1}^z - 1)\right]$.

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Observation: The operators U, M_α are translationally invariant and reflection symmetric $\Rightarrow$ All elements of $\mathcal{D}_t = \{\Pi^n R^m, n \in \{0, 1, \ldots, t-1\}, m \in \{0, 1\}\}$ fullfil (1) with $+1$, where

$$\Pi = \prod_{\tau=1}^{t-1} P_{\tau,\tau+1}, \quad R = \prod_{\tau=1}^{\lfloor t/2 \rfloor} P_{\tau,t+1-\tau}$$

are *translation* and *reflection* on a spin ring of length t.

Property 2

The number of linearly independent elements of $\mathcal{D}_t$ is $2t$ for $t \geq 6$, $2t-1$ for $t \in \{1,3,4,5\}$, and 2 for $t=2$.

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For odd t, Eq. (1) can be fulfilled only for eigenvalue $+1$.

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For odd t, all A satisfying (1) are given by linear combination of elements of $\mathcal{D}_t$.

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Observation: For even t, we find generically exactly one additional operator A satisfying Eq. (1). For special values of $t \leq 10$ we find an extra additional operator, and also solutions of Eq. (1) for eigenvalue -1.

t	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\#_{+1}$	2	5	7	9	13	14	18	18	22	22	25	26	29	30	33	34
$\#_{-1}$	0	0	0	0	2	0	0	0	2	0	0	0	0	0	0	0

Conclusions

- The first exact result on ergodicity in terms of spectral correlations for an interacting quantum many-body problem
- Self-dual instances of Kicked Ising chain provide a minimal model of quantum many-body chaos with no intrinsic time scales (Thouless time $= 1$)

Pending open problems and promising future directions:

1. Complete the picture by rigorous analysis of the even t case.
2. Structural stability of the self-dual point: Perturbation theory may have a finite radius of convergence?
3. Potentially accessible ergodicity – MBL transition from the ergodic side?
4. Computing dynamics of entanglement entropy via space-time duality.
5. Path to a rigorous approach to ETH?

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