Cellular automata as a “classical limit” of Floquet dynamics

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arXiv:1802.07729 [with Bahti Zakirov (CUNY)]

arXiv:1806.04156

+ unpublished work with Romain Vasseur

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Floquet models can be "nondispersing"

- Simple unitary matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Moves all particles one step left

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- Moves all particles one step left
- Eigenstates w/ eigenvalues $\lambda_k = e^{i\pi k/6}$

$$|\psi\rangle = \frac{1}{\sqrt{6}}(|1\rangle + e^{i\theta}|2\rangle + \dots e^{5i\theta}|6\rangle)$$

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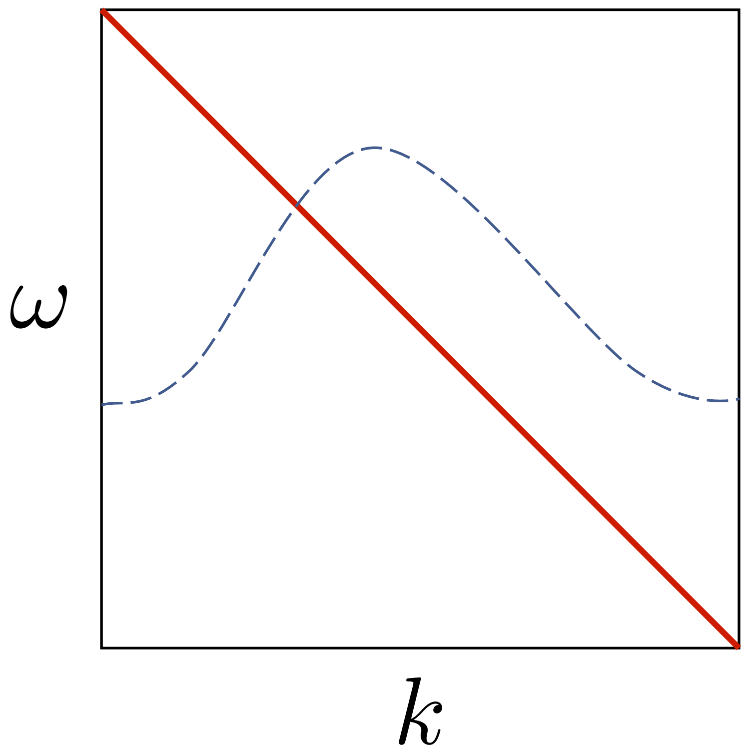
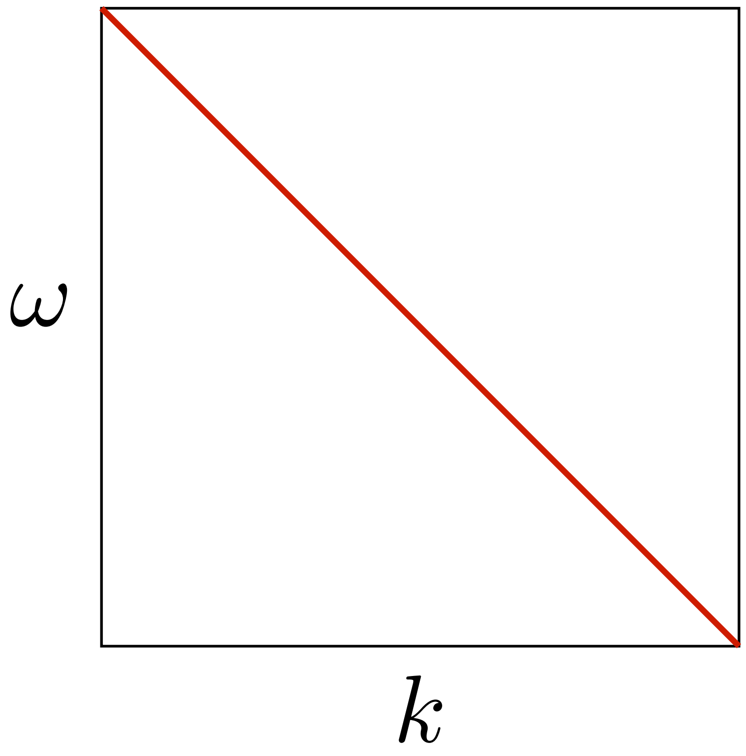
Dispersion relation:



Dispersionless systems are "classical"

- Particle has sharply defined velocity; wavepackets don't spread
- Dynamics in "special" basis can be understood classically
  - Building blocks for more interesting models
  - Eigenstates are still delocalized and entangled
- Key feature: product states map to product states



Interacting "classical" Floquet systems

- Maps from z-basis product states to other product states:

$$|\ldots \uparrow\uparrow\downarrow\uparrow \ldots\rangle \mapsto |\ldots \uparrow\downarrow\downarrow\uparrow \ldots\rangle$$

- Constraints:
  - Must be unitary/reversible (we're interested in quantum dynamics)
  - Update rule must be local

- General class of models that work: reversible cellular automata

Why study these models?

- Could be building blocks of more interesting models (cf. time crystals)
- Useful in their own right
  - Many aspects of thermalization are under-explored
- For example:
  - Are all nonthermal systems either conventionally integrable or MBL?
  - Does ETH for small [o(1)] subsystems always imply ETH for large [o(L)] subsystems?
  - How do operators spread in nonthermal systems?

This talk

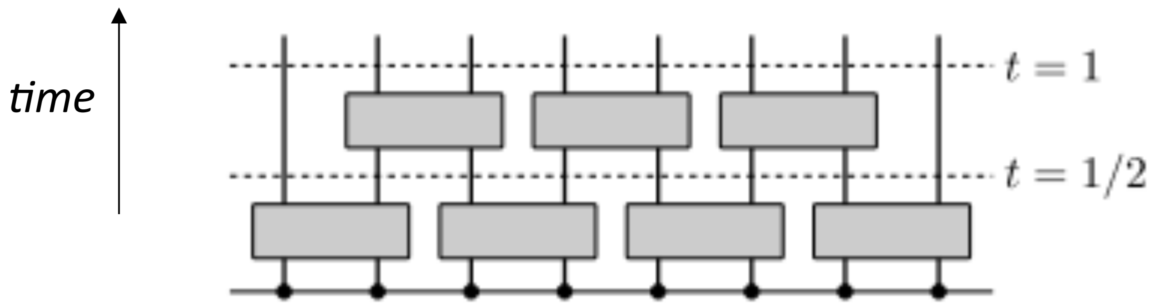
- Some generalities

- Model 1 (Clifford-East)
  - Small subsystems are thermal, large subsystems may violate ETH depending on system size
  - Operators spread as a fractal inside the light-cone

- Model 2 (“Floquet-Fredrickson-Andersen”)
  - Simple model of an interacting integrable system
  - Operator spreading “chaotic” in some respects (although model is integrable)

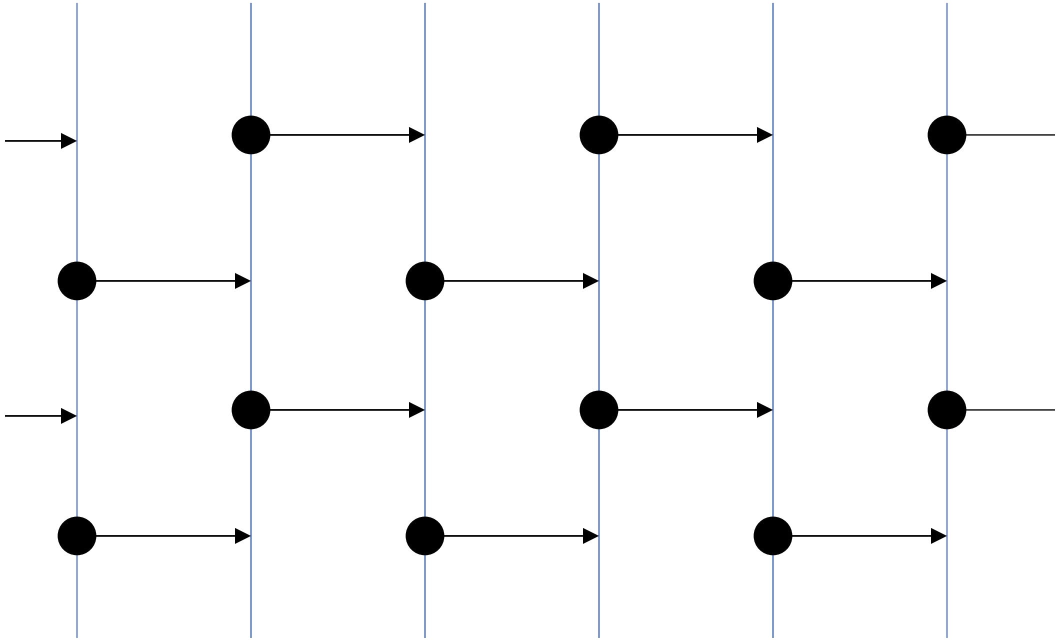
Some generalities

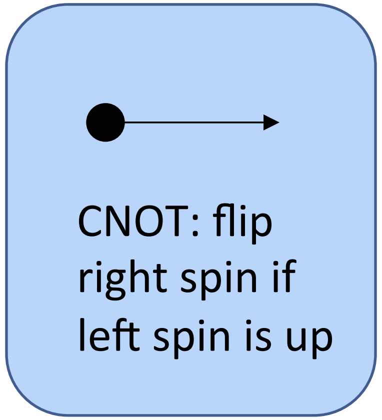
Quantum circuit models



- Apply local quantum gates
  - Randomly in both space and time (Nahum et al. 2016, von Keyserlingck et al. 2016)
  - Randomly in space but periodically in time
    (Chandran, Laumann 2015; Chan, De Luca, Chalker 2018)
  - Periodically in both space and time (SG+Zakirov 2018; also Schumacher, Werner, Carr…)
- In the periodic case, a single cycle is a "Floquet unitary" and has eigenstates
  - But no energy conservation, unlike a Hamiltonian

An extremely simple circuit





- No entanglement growth for z-basis initial product states
- General principle: flip A conditioned on neighboring B's, then vice versa

Constructing eigenstates

- Under time evolution

$$|C_1\rangle \mapsto |C_2\rangle \mapsto |C_3\rangle \ldots \mapsto |C_1\rangle \mapsto |C_2\rangle \ldots$$

- Evidently, this is an eigenstate with eigenvalue unity:

$$|\psi\rangle = \mathcal{N}(|C_1\rangle + |C_2\rangle + |C_3\rangle \ldots + |C_N\rangle)$$

- Eigenstates are classically constructible!
  - Start with a random configuration
  - Evolve until recurrence
  - Sum up over the “orbit” with appropriate phases

Operator spreading

- Standard method: squared commutator / "OTOC" for initial product state

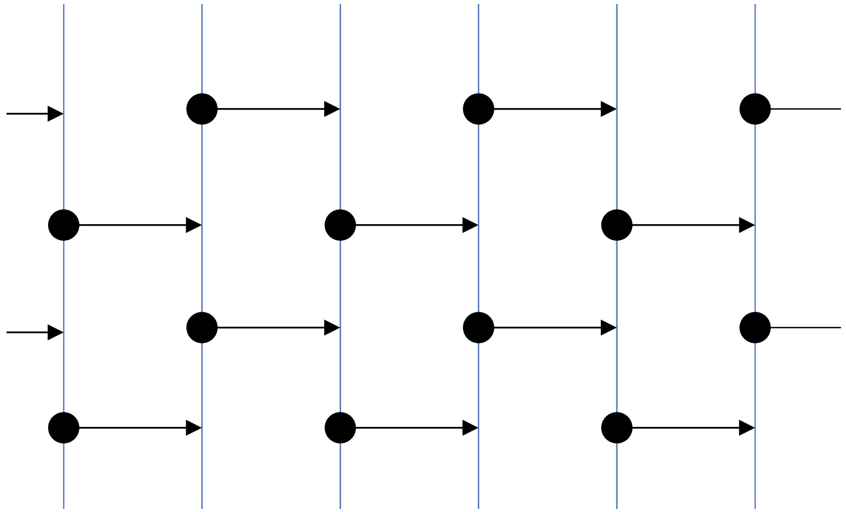
$$\langle\psi|\sigma_i^z(t)\sigma_j^x(0)\sigma_i^z(t)\sigma_j^x(0)|\psi\rangle = \{\langle\psi(t)|\sigma_i^z|\psi(t)\rangle\}\left\{(\langle\psi|\sigma_j^x U^\dagger)\sigma_i^z(U\sigma_j^x|\psi\rangle)\right\}$$

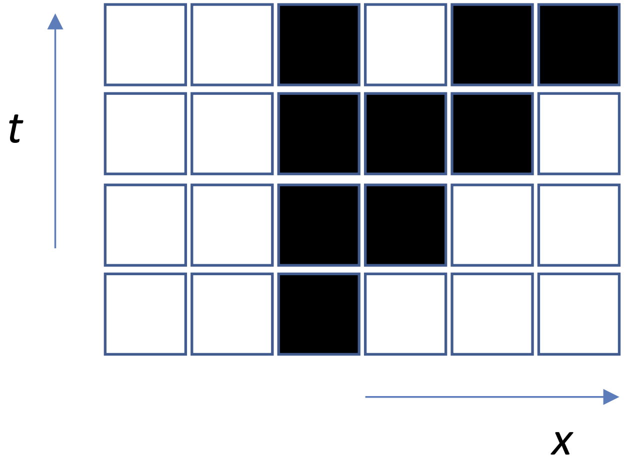
- Corresponds to overlap between trajectories with and without a perturbation at time t = 0

- I.e., flipped bits between the unperturbed and perturbed configurations

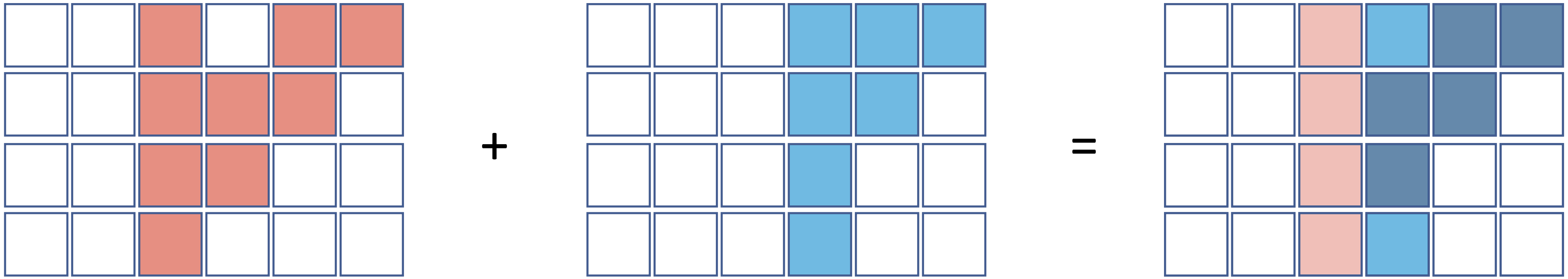
CNOT circuits

Time evolution of product states





Additive dynamics



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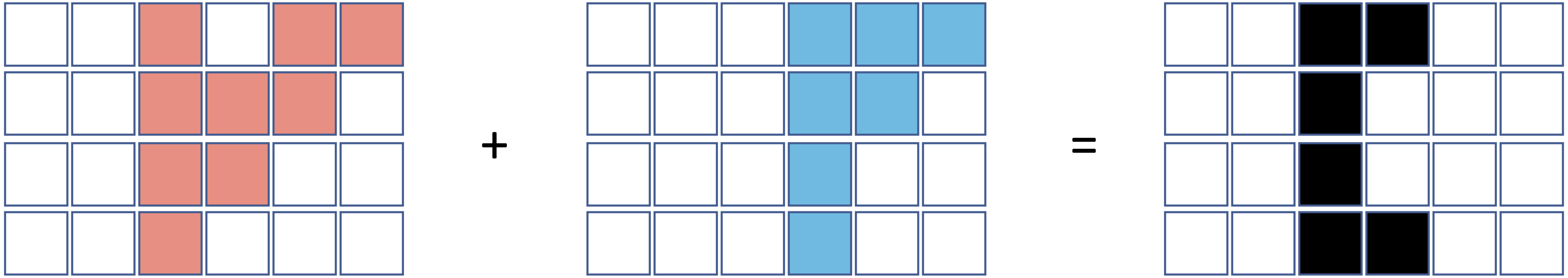
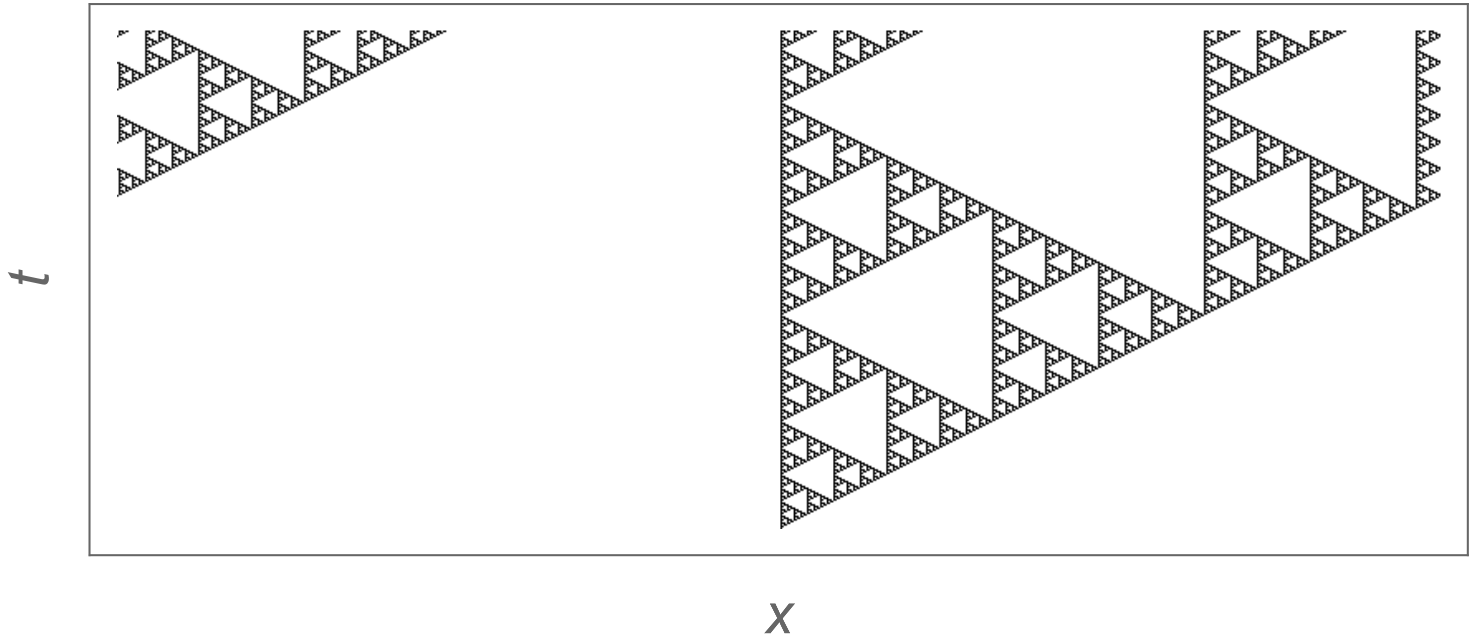


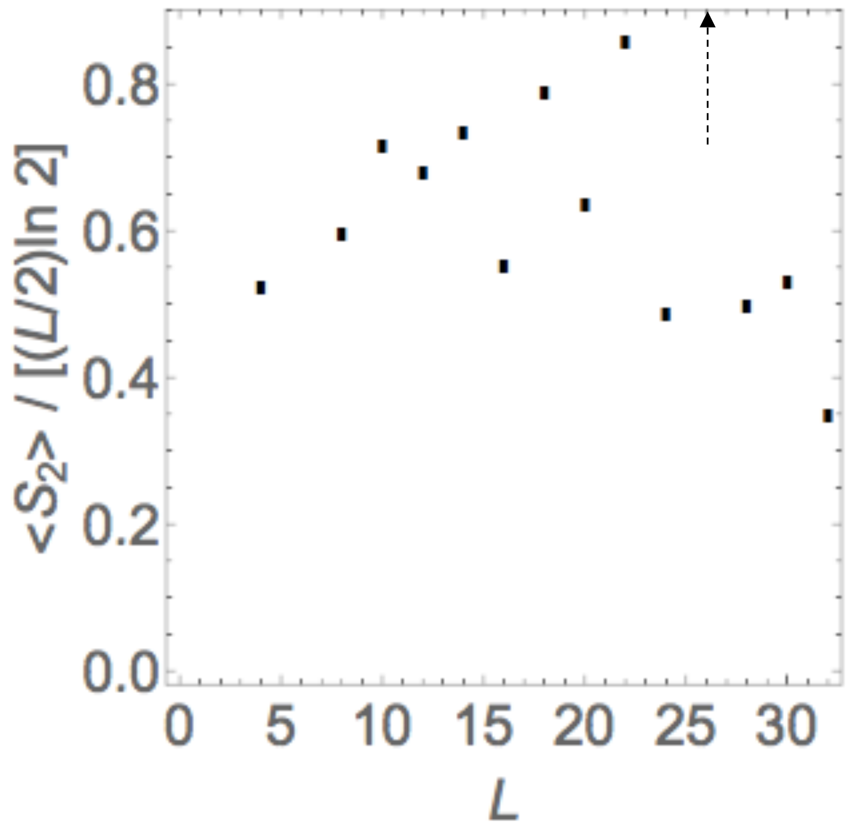
- CNOT circuit dynamics is noninteracting histories of single spins
- Consequence of “Clifford” nature of CNOT gates
- Implication: OTOC is the history of a single perturbed spin
  - What does that look like in this model?

Operators spread as spacetime fractals

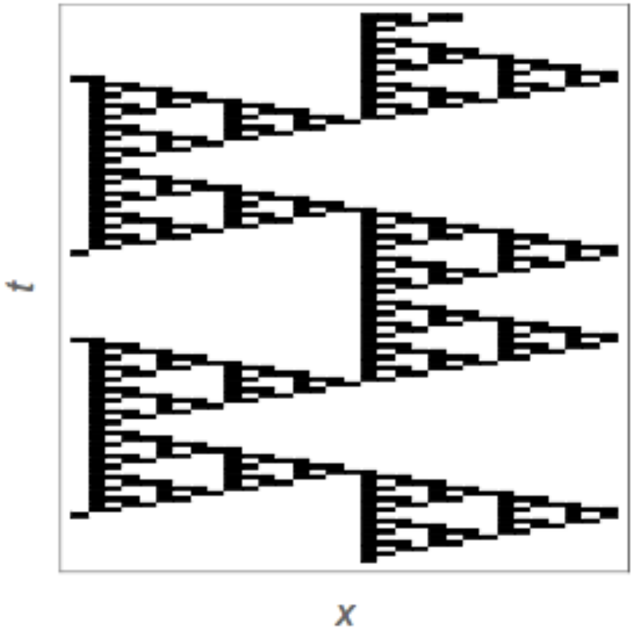


Eigenstate entanglement

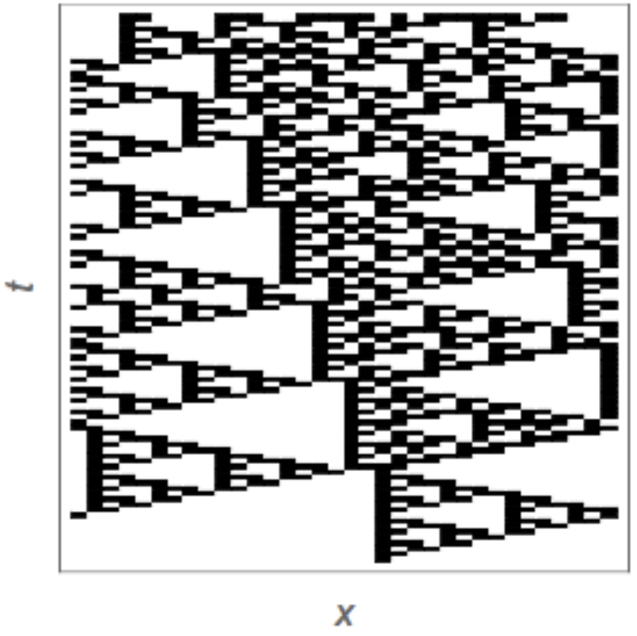
half-chain



L=32



L=34



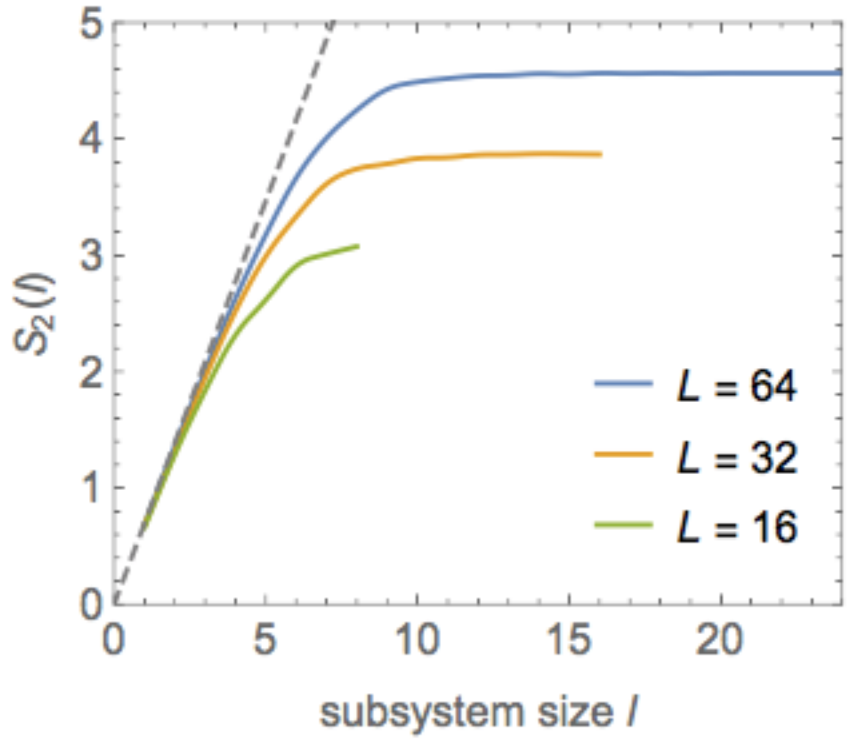
Scaling of eigenstate entanglement

- Origin of anomalously low entanglement:
  - Recurrence time ~ system size L
  - "Multiplicity" of L, so entropy ~ log(L)
  - However, small subsystems are completely thermal

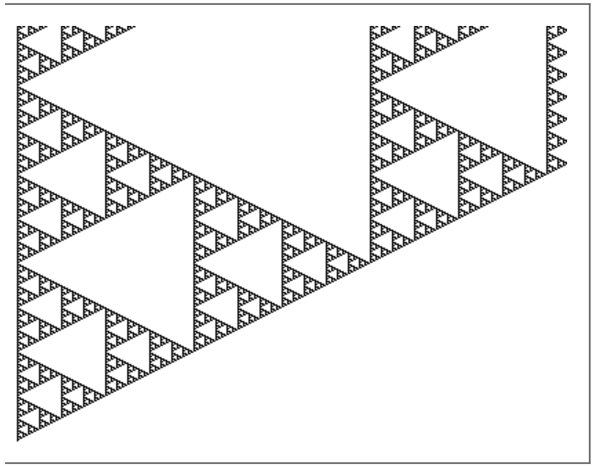
- All local observables are also close to thermal

- This system obeys "conservative" ETH but violates strong ETH…

- depending on prime factorization of system size!



Is this system integrable or chaotic?



- Neither (in a conventional sense)
- Chaotic behavior:
  - Heisenberg operator "fills in" the light-cone
- Conventional integrable behavior:
  - Some operators have the property

$$\left[\sum_i \hat{O}_i, \hat{U}\right] = 0$$

  - No such operators can exist in this model: every Pauli string grows with time
  - This appears to be a different type of integrable dynamics
  - Self-similar behavior of some autocorrelation functions

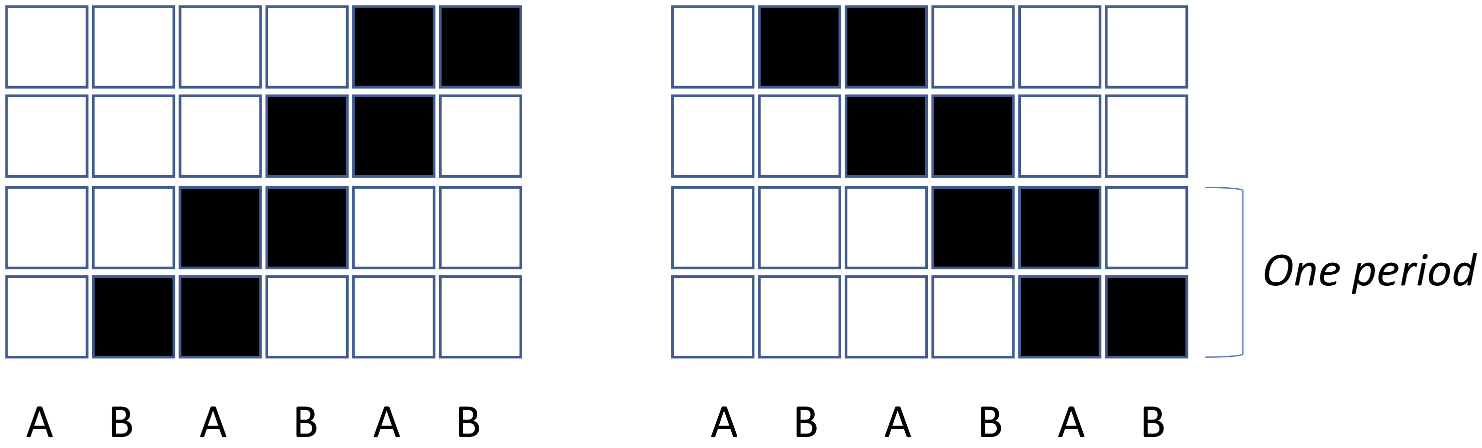
Model 2: soliton gas

Dynamical rule and quasiparticles

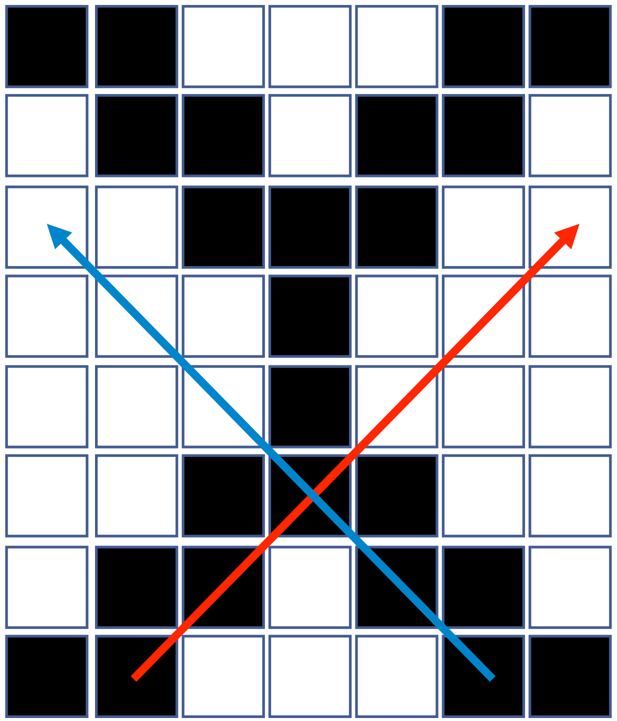
- Flip each A site if one or both neighbors is up
  - Cannot make up out of Clifford gates; need a “Toffoli” gate
  - Inherently interacting (though still “classical”) model [related to Bobenko Rule 54 CA]

Dynamical rule and quasiparticles

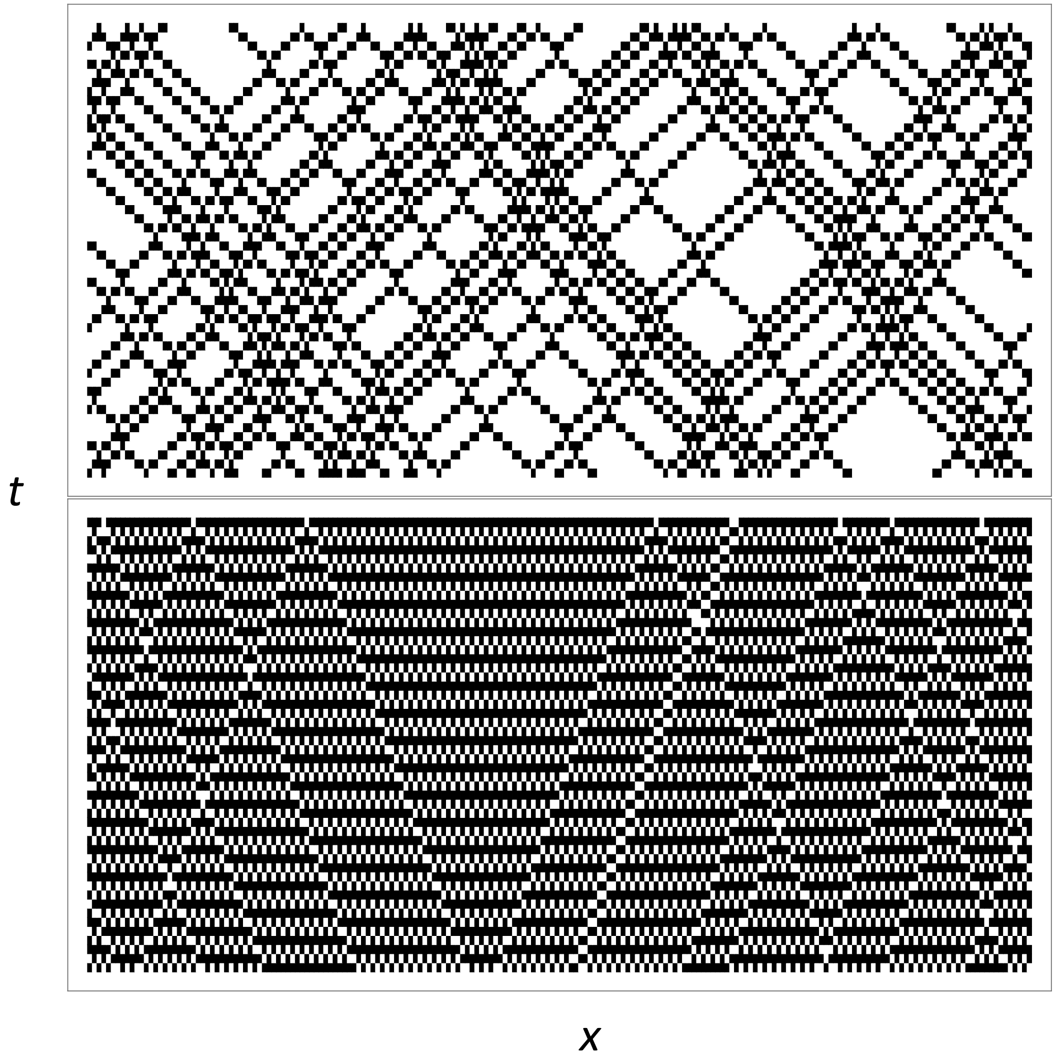
- Flip each A site if one or both neighbors is up
  - Cannot make up out of Clifford gates; need a “Toffoli” gate
  - Inherently interacting (though still “classical”) model [related to Bobenko Rule 54 CA]
- Quasiparticles are left and right movers:



How quasiparticles interact



Analogous to hard rods
with length –1 and only one velocity



The quantization condition

- Time for R movers to wrap around depends on total number of L movers

$$t_r \simeq L/2 + 2N_l, \quad E_k^{(r)} = 2\pi k/t_r$$

- Higher density of L's -> more states for R's -> more R's -> more states for L's
  - Number fluctuations of L and R movers are not independent

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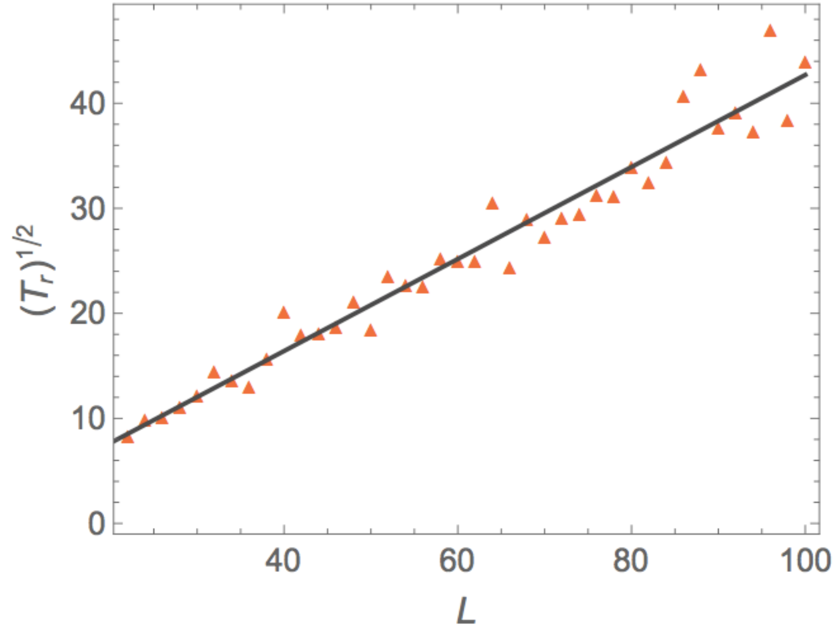
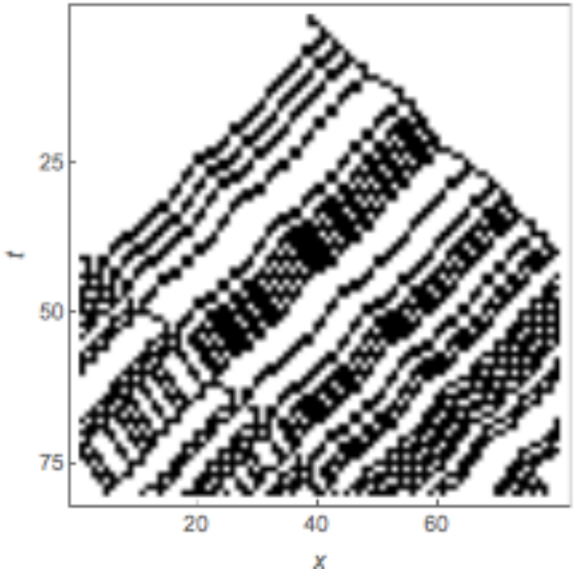


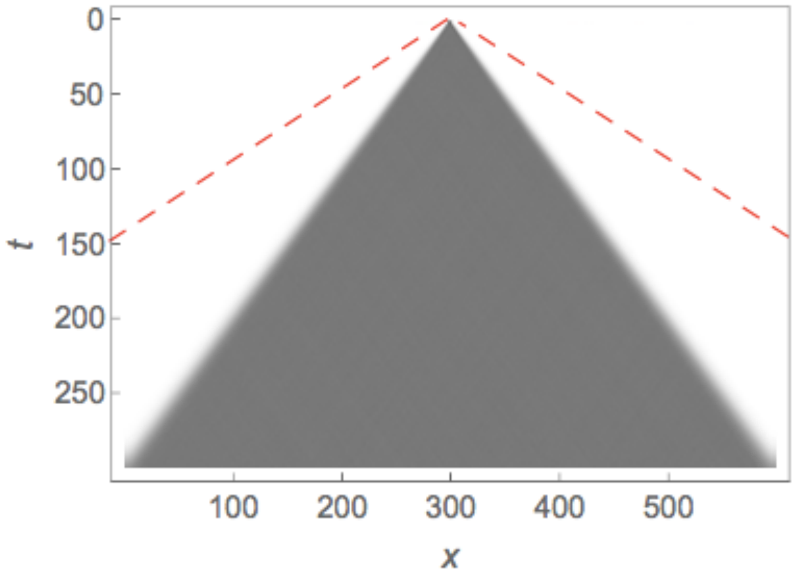
- Recurrence time is LCM of orbits, hence quadratic in system size (half-chain entanglement entropy is ~2 log L)

Operator growth

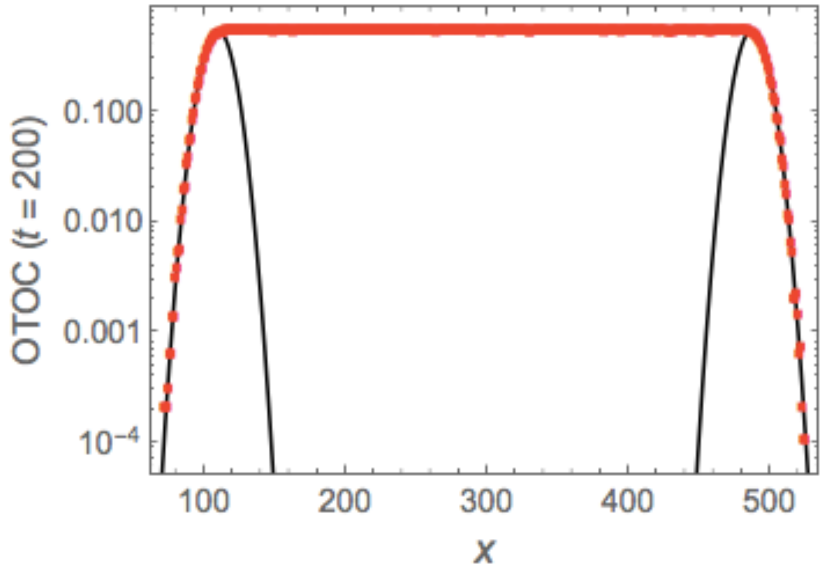
Operator "front" in this model broadens diffusively (when averaged) because of random time-delays due to collisions [expected to be generic for integrable systems]



Single product state



Averaged over states



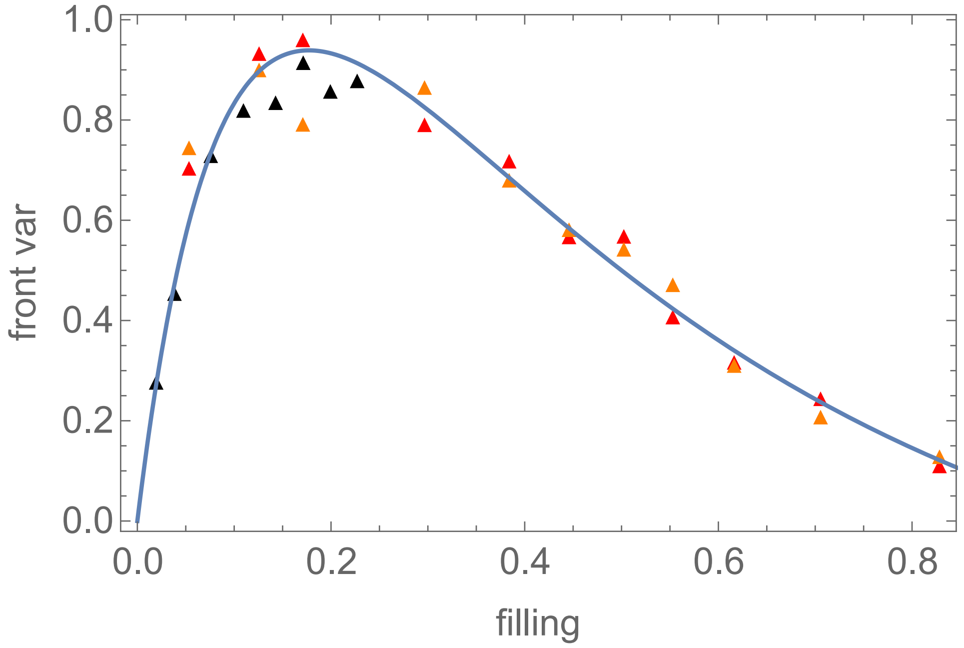
At a fixed late time

with R. Vasseur

Details of operator growth

- Velocity of a right-mover:

$$v_r = 2 - \frac{4n_l}{1 + n_l + n_r}$$

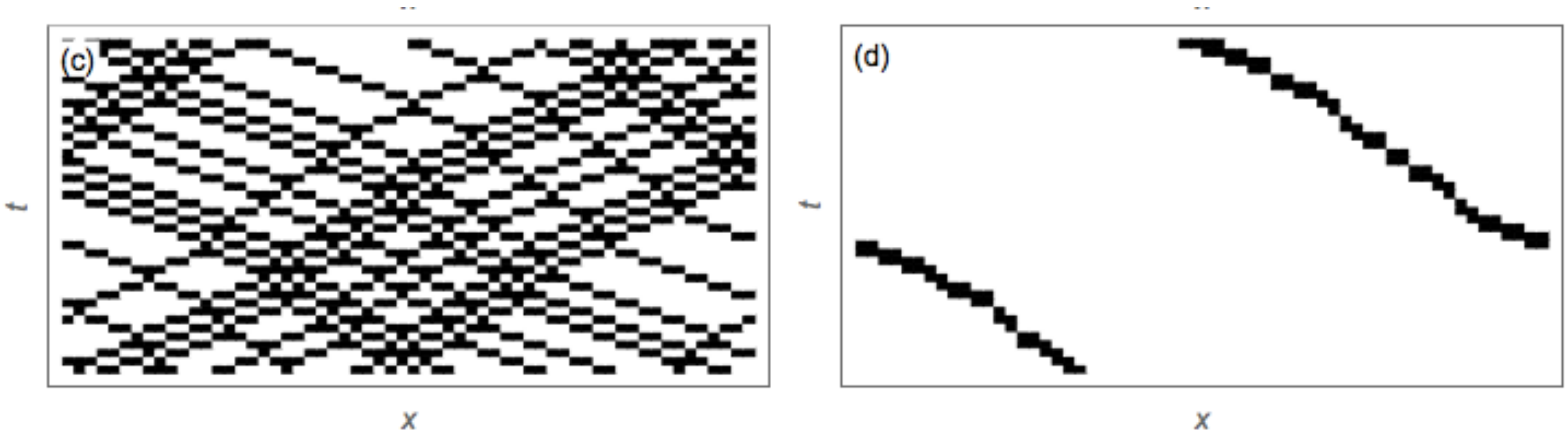


- Qualitative "hydrodynamic" picture: density fluctuations cause velocity fluctuations, which cause spreading

- Works very well up to factor of 2

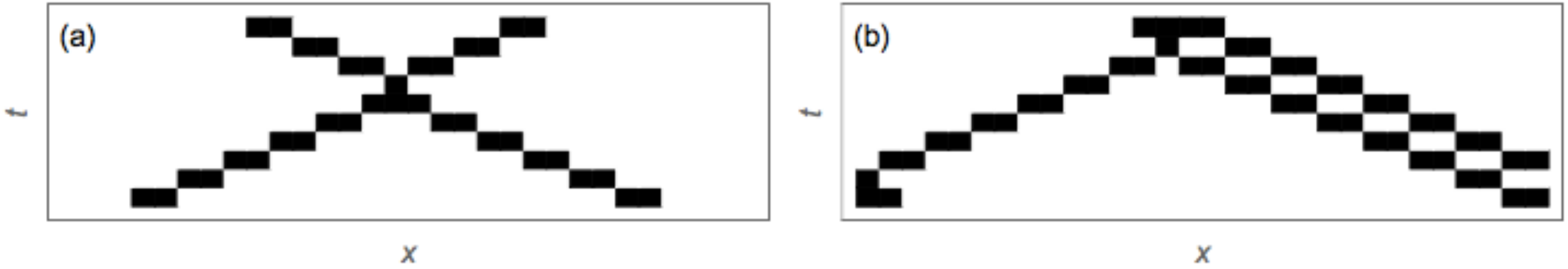
Why does the front fill in?

- In the dilute limit, we can address individual quasiparticles
- Moving a single quasiparticle does not create a butterfly effect, just gives you the trace of the moved quasiparticle:



Why does the front fill in?

- Processes where the operator creates/destroys quasiparticles
- All very local operators do this because you can create quasiparticles even without changing the total number of up spins



- Adding a qp is disruptive because it changes the phase shifts of all the opposite-moving qps

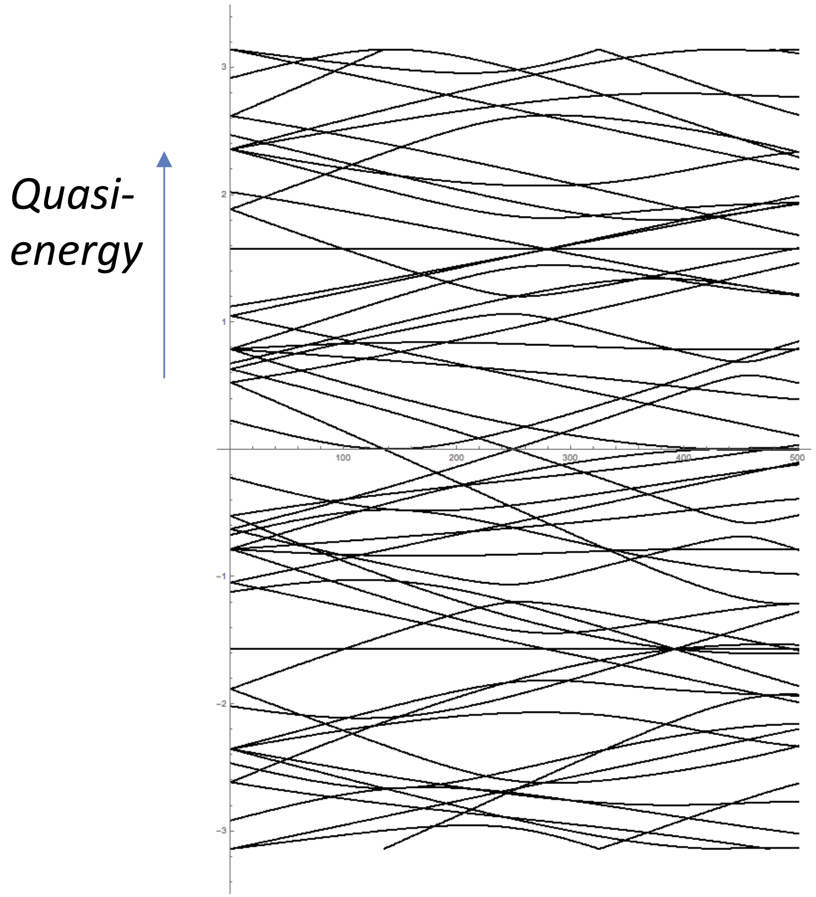
Adding quantum fluctuations

- Would like to add non-commuting piece $U = U_0 U_1, U_1 = \exp(iHt)$
- What to choose for H?
- Terms adding dispersion for quasiparticles should preserve integrability as long as no qps are being created or destroyed
- Simplest operators that do this:

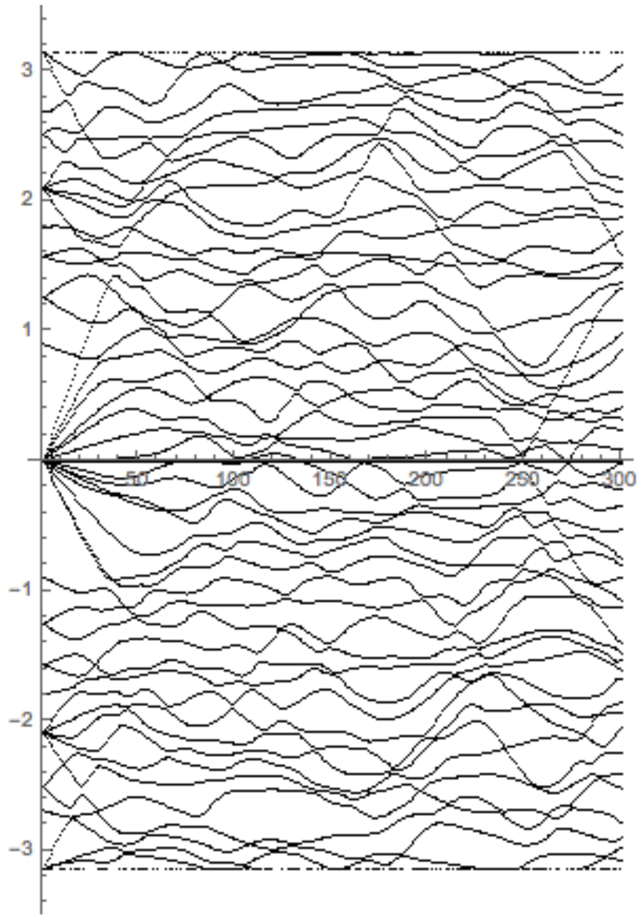
$$(1-\sigma_1^z)\sigma_2^+\sigma_3^+\sigma_4^-\sigma_5^-(1-\sigma_6^z) + \text{h.c.}, \quad (1-\sigma_1^z)\sigma_2^+\sigma_3^-(1-\sigma_4^z) + \text{h.c.}$$

Preliminary results: level statistics

- Integrability-preserving:
$$(1-\sigma_0^z)\sigma_1^+\sigma_2^-(1-\sigma_3^z)$$



- Integrability-breaking: $\sigma_1^+\sigma_2^-$



Perturbation strength

Summary/outlook

- Reversible cellular automata are useful starting points for interesting many-body dynamics
  - Models that are neither chaotic nor integrable
  - Dispersionless hard-rod gas: an extremely simple interacting integrable model

- Can explicitly compute eigenstate entanglement, OTOC, spatio-temporal correlations,…

- How robust are these phenomena?
  - Integrability appears to survive adding a dispersion
  - What about noise and disorder that couple to specific quasiparticles?
  - What about integrability-breaking perturbations?
  - Are there interacting Floquet systems with fractal light-cones?
  - Generalized "prethermal time crystals" based on arbitrary cellular automata?