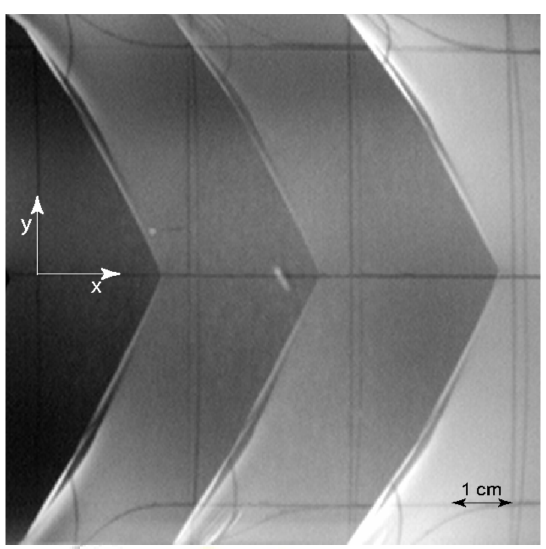
Rupture of Rubber

Experiment: Paul Petersan, Robert Deegan, Harry Swinney

Theory: Michael Marder

Center for Nonlinear Dynamics
and Department of Physics
The University of Texas at Austin



PRL **93** 015505 (2004)

PRL **88** 014304 (2002)

PRL **94** 048001 (2005)

cond-mat/0504613

Supported by the National Science Foundation

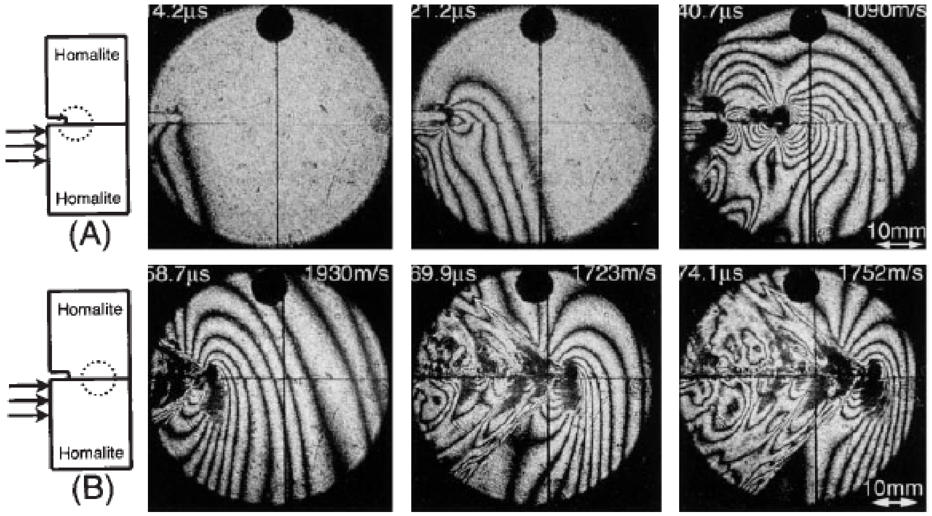
Outline

- Background: Limiting Speed of Cracks
- Experimental Observations
  - Oscillations
  - Sound and Rupture Speeds
- Numerical Investigations
  - Multi-Particle Modeling
  - Explorations
- Theory
  - Elementary Shock Theory
  - Continuum Theory
  - Discrete Theory and Scaling

Background

Limiting Speed of Cracks

- Rayleigh wave speed limits cracks in tension (Yoffe, 1951; Stroh, 1957; Freund, 1972)
- Shear waves can also travel at $\sqrt{2}c_s$ (Andrews, 1976; Burridge, Conn and Freund, 1979; Rosakis, Samudrala, Coker, 1999; Gao, Huang, Abraham, 2001)
- In a discrete medium, there is no limit to the speed of tensile cracks (Slepyan, 1982; MM, 1995; Buehler, Gao, and Abraham, 2003)
- Integrity of crack surface behind tip is key to whether supersonic solutions survive (Ravi-Chandar and Knauss, 1984; Fineberg et al, 1991; MM)

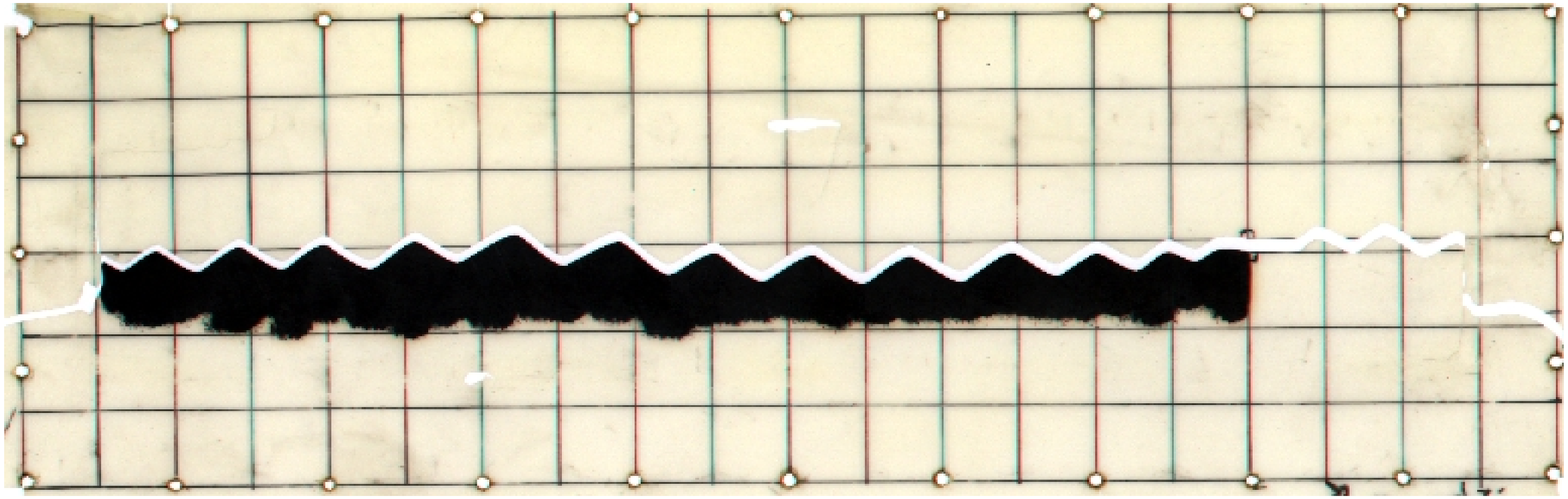


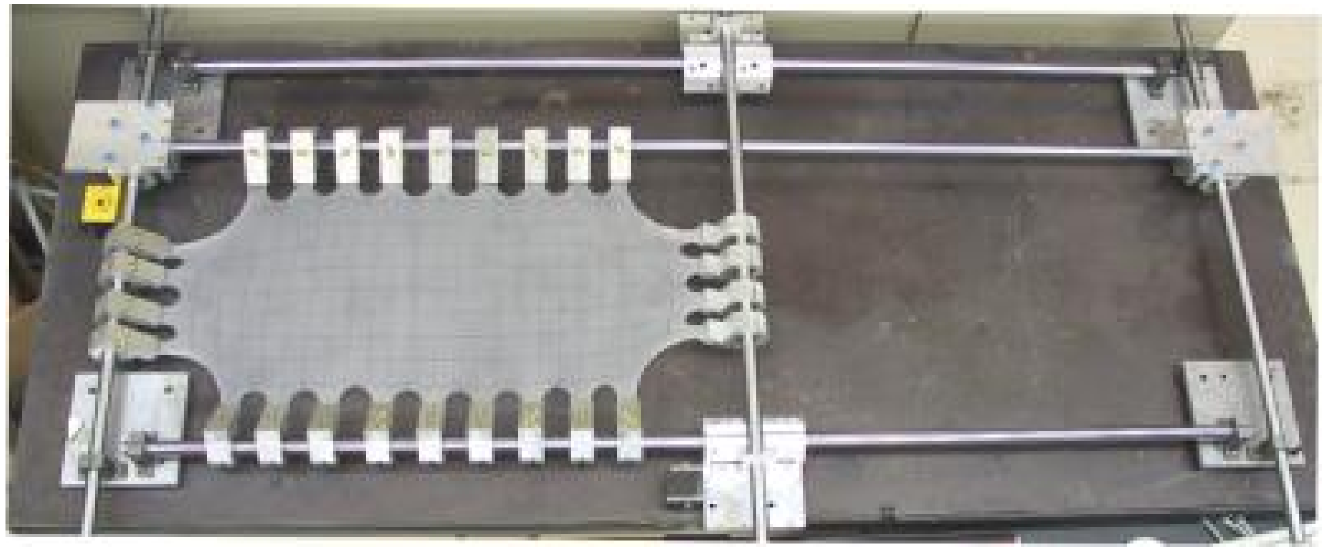
Experimental Observations

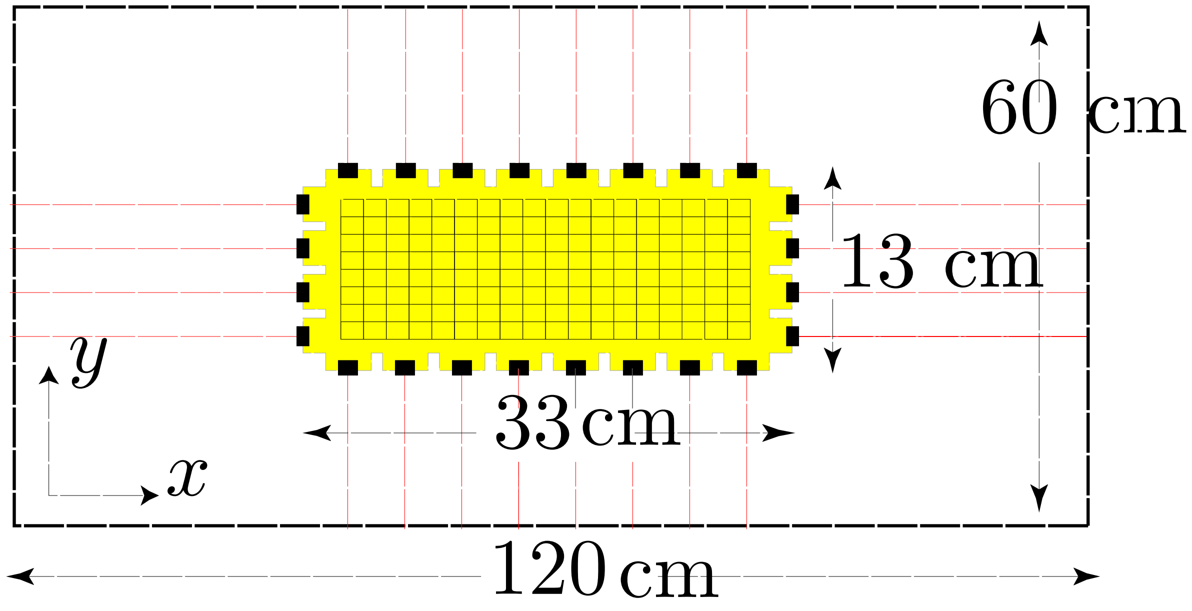
Balloons and First Experiments

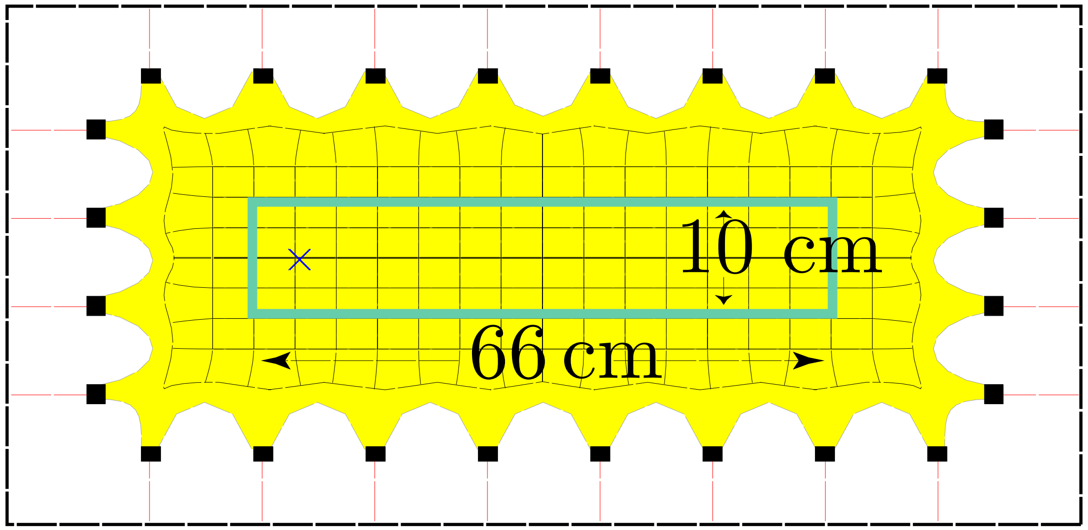
Balloons

Controlled Experiments



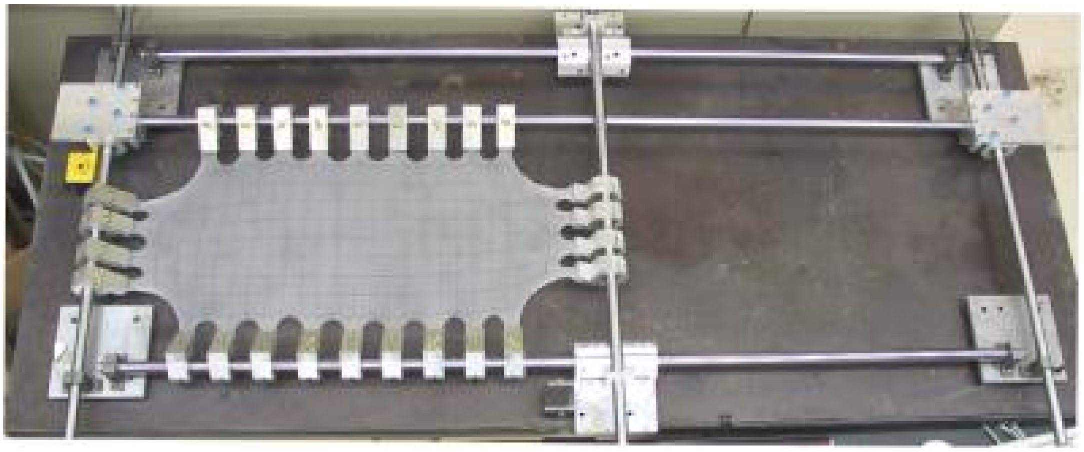


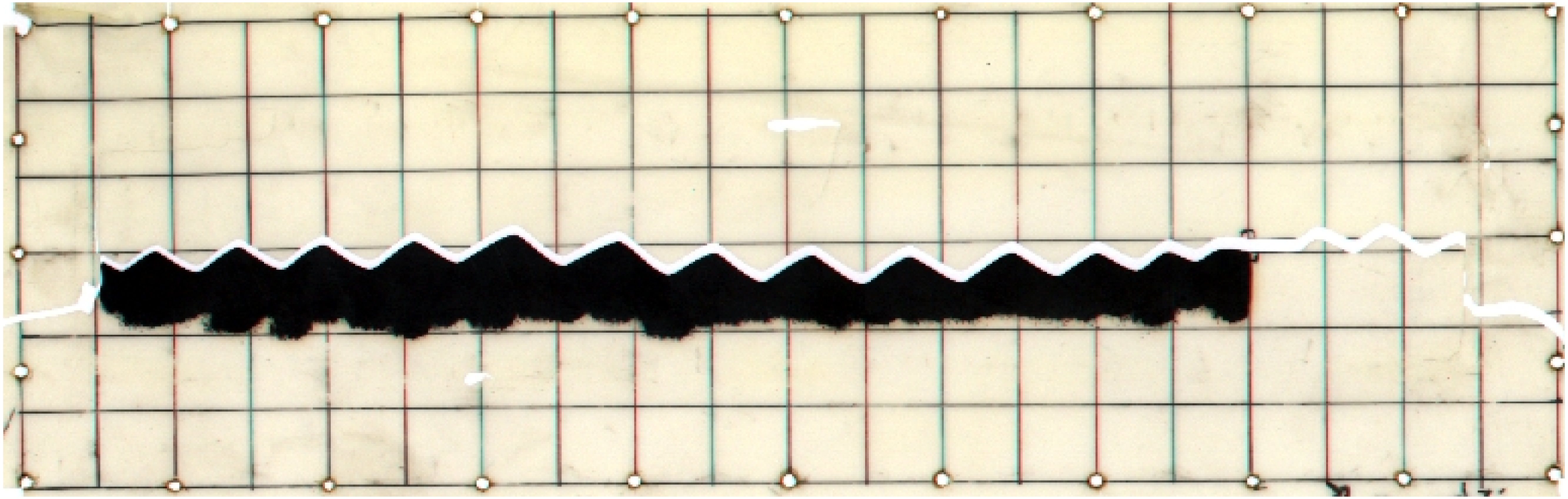

60 cm
13 cm
y
33 cm
x
120 cm

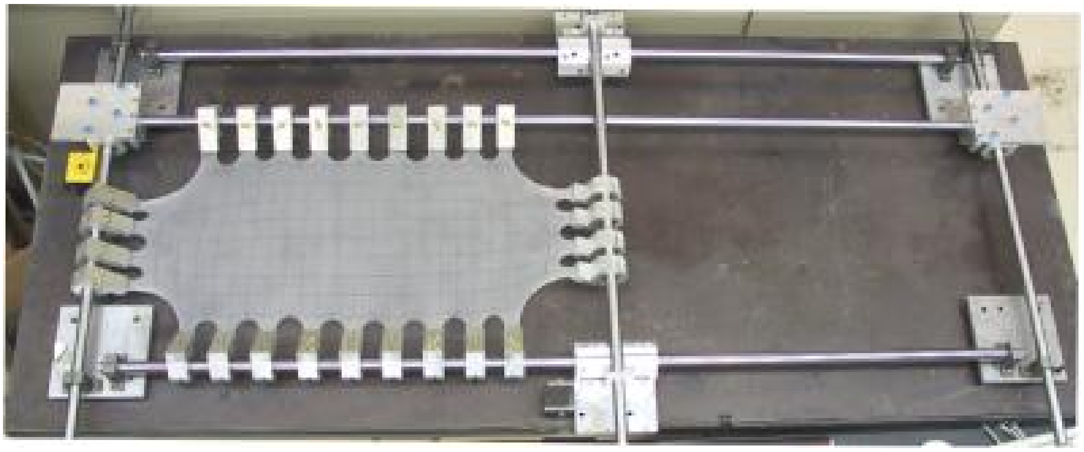

10 cm
66 cm

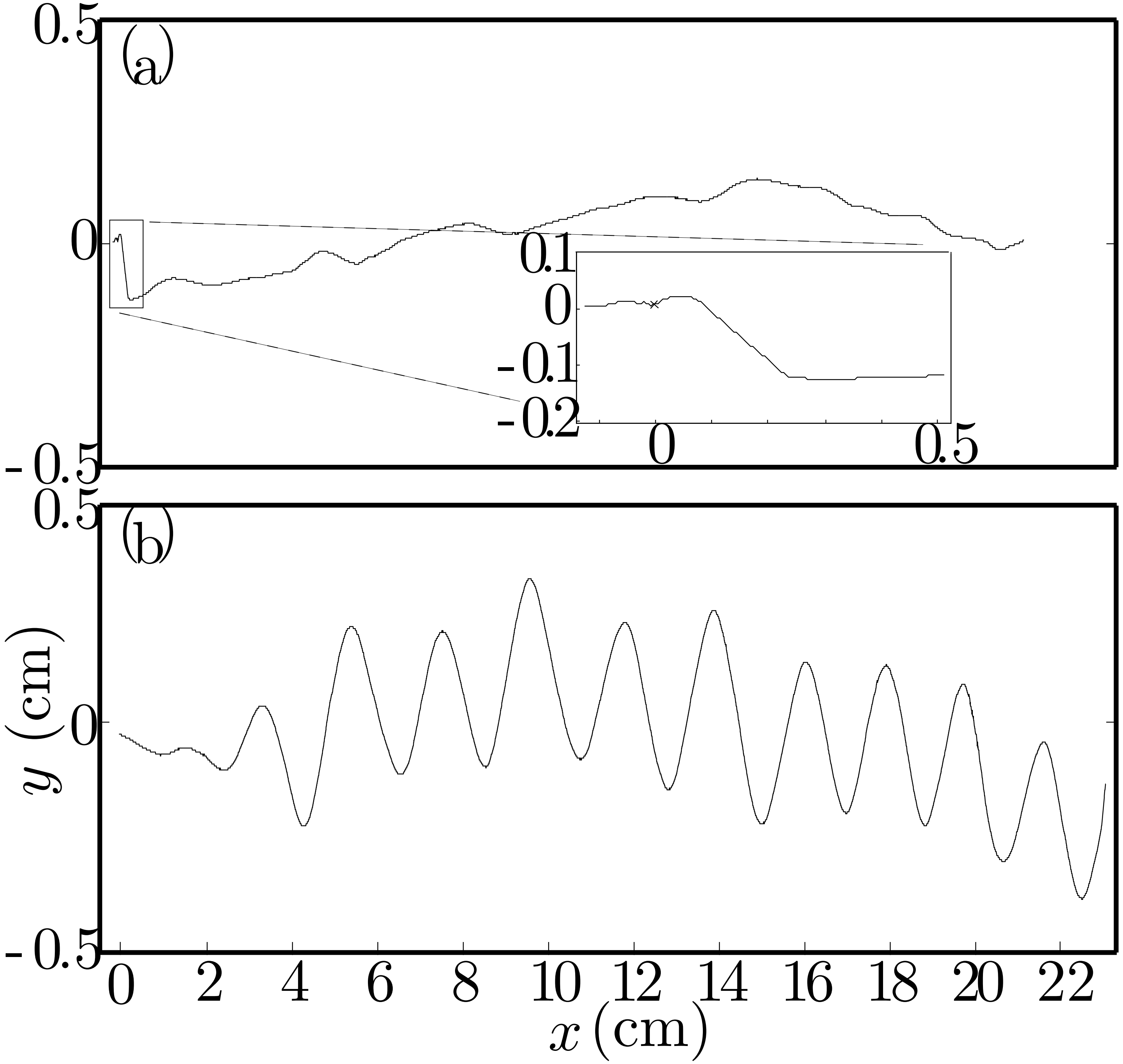
Experimental Observations

Oscillations



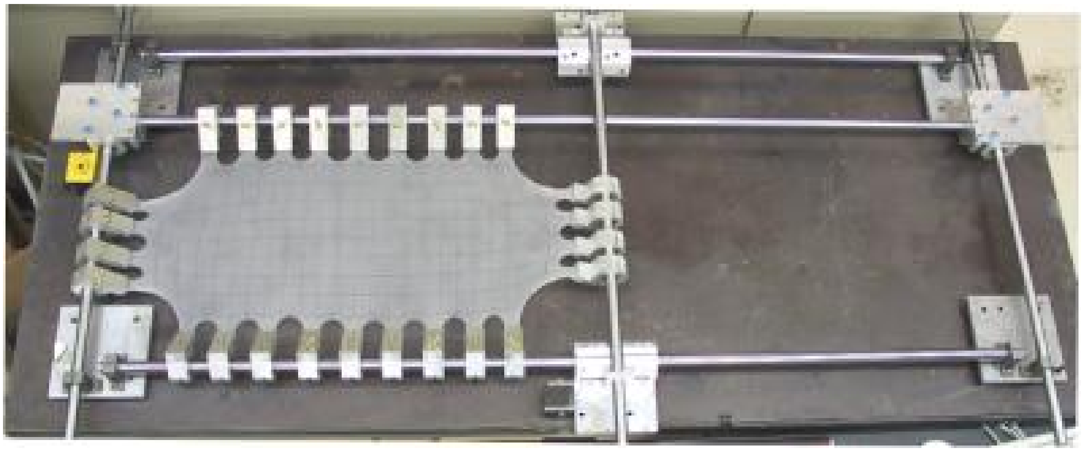


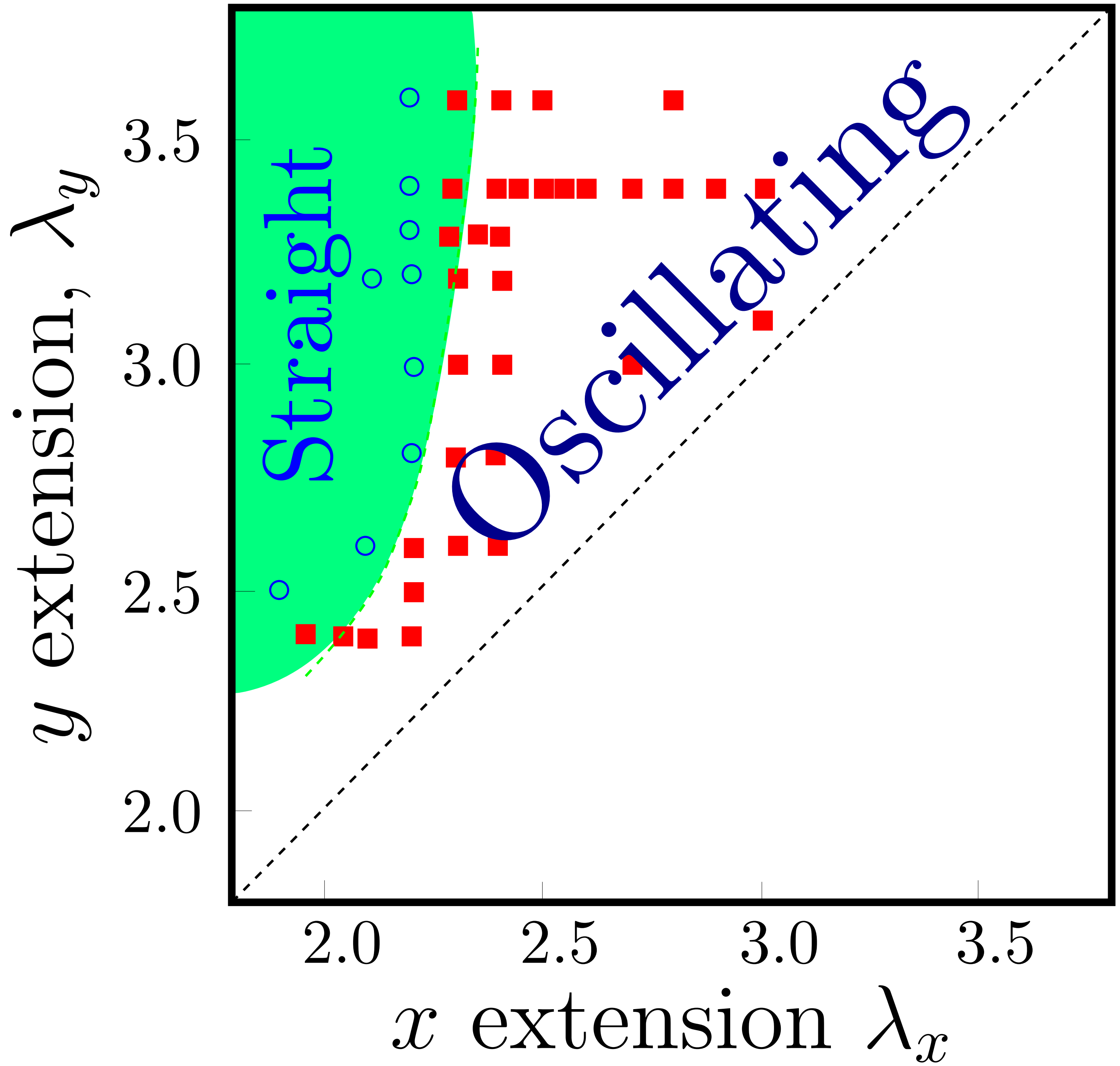



(a)
(b)
0.5
0
-0.5
0.1
-0.1
-0.2
y (cm)
0 2 4 6 8 10 12 14 16 18 20 22
x (cm)

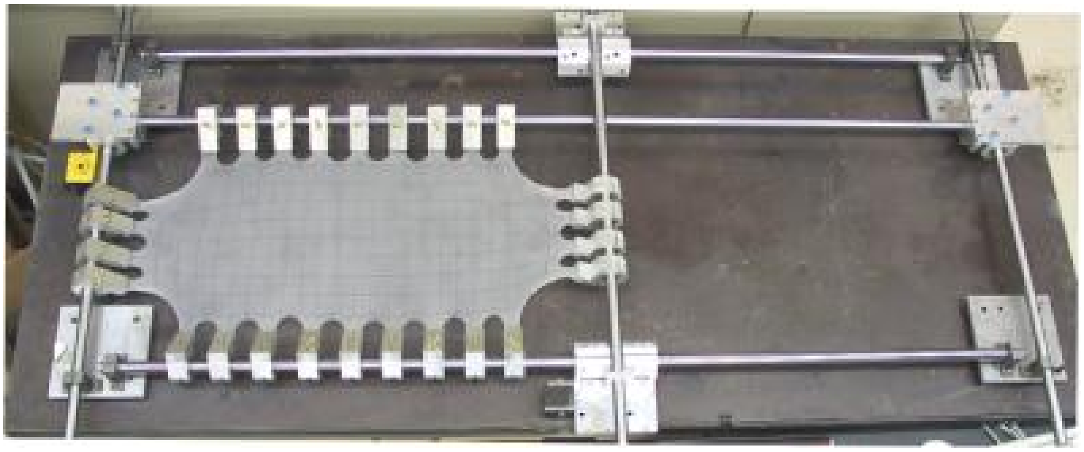
Experimental Observations

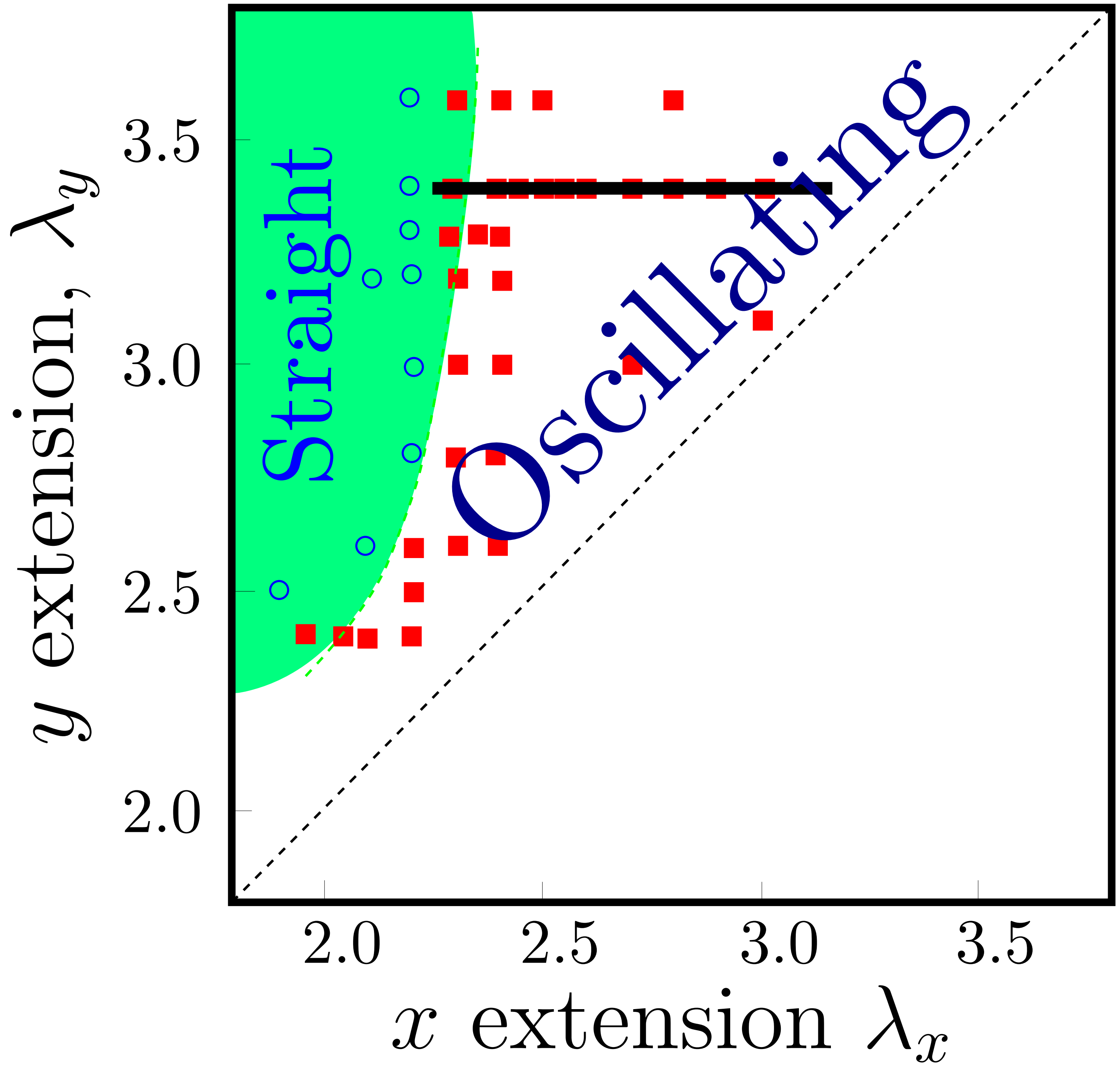
Oscillations





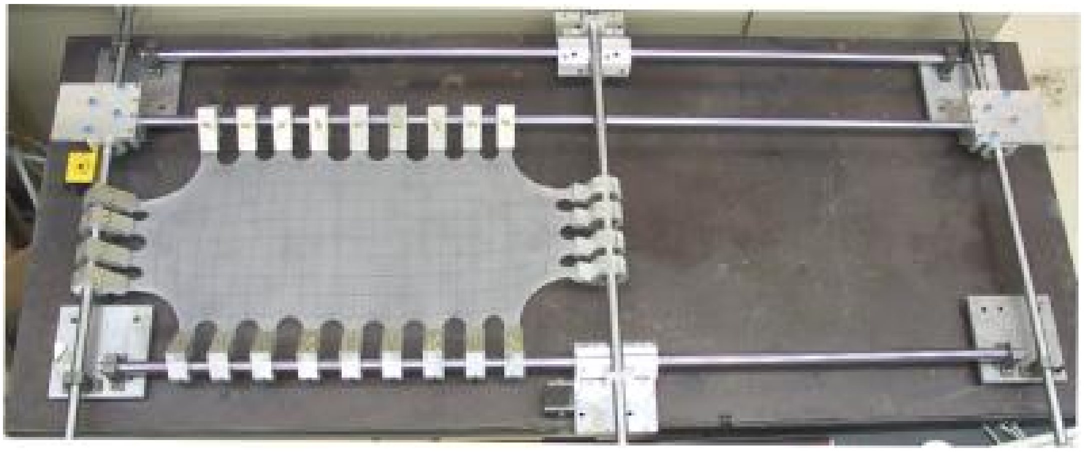
Experimental Observations



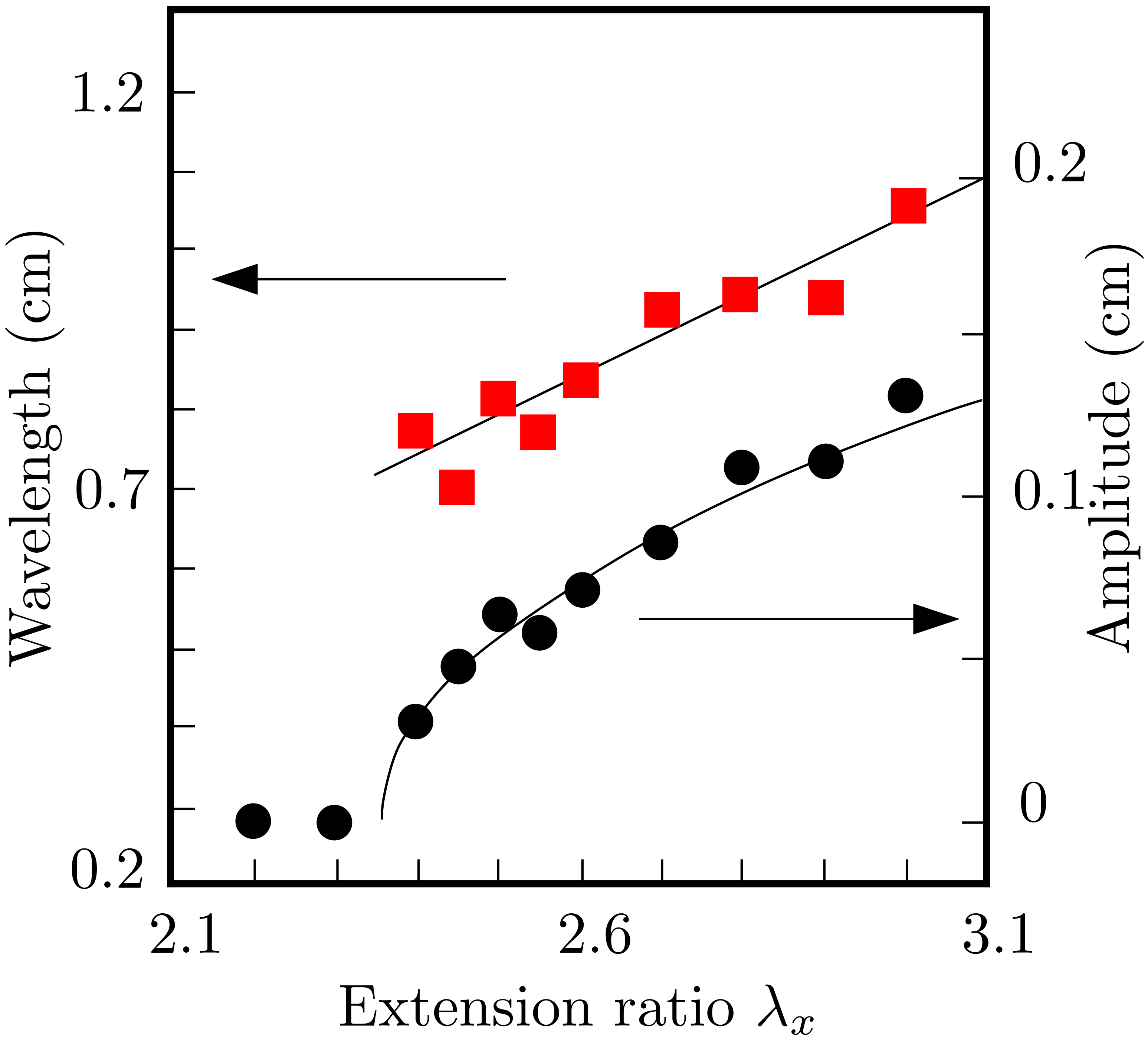


Experimental Observations

Oscillations

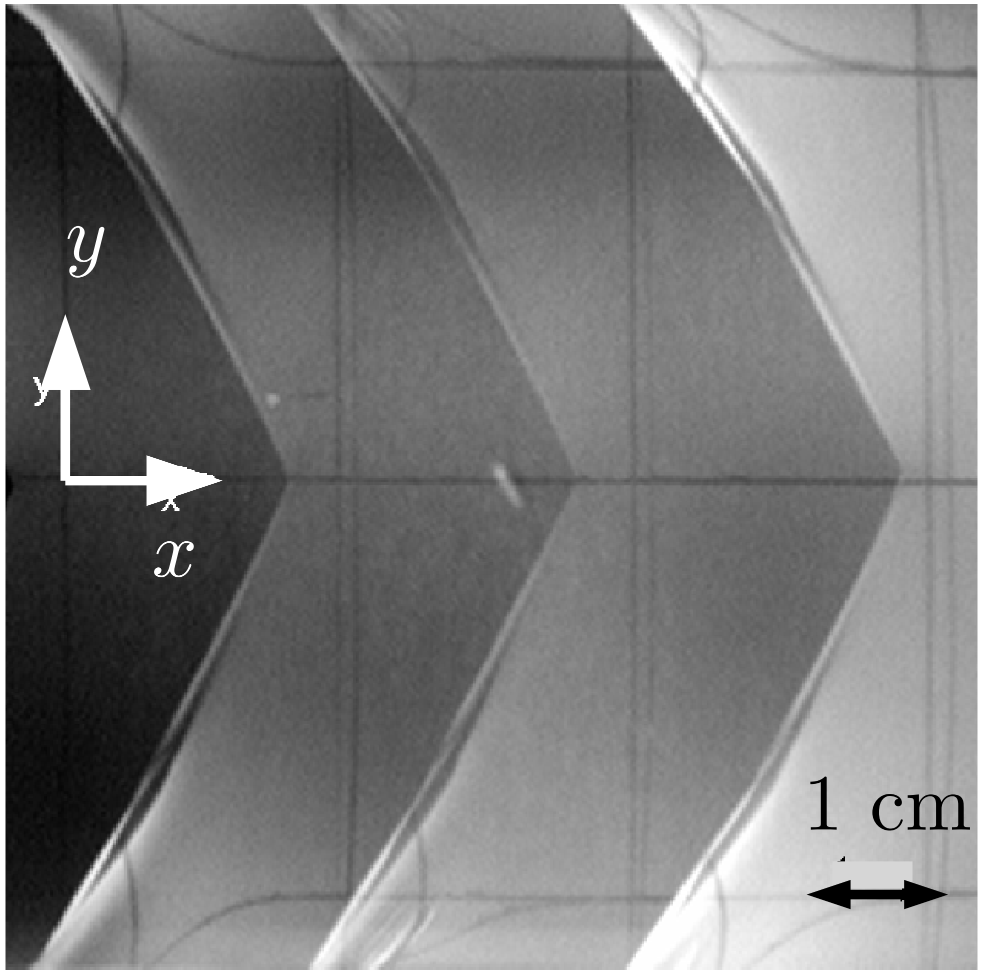


No detailed theory yet for oscillations... but see Yang and Chen, *Phys. Rev. Lett.* **95**, 144301 (2005)



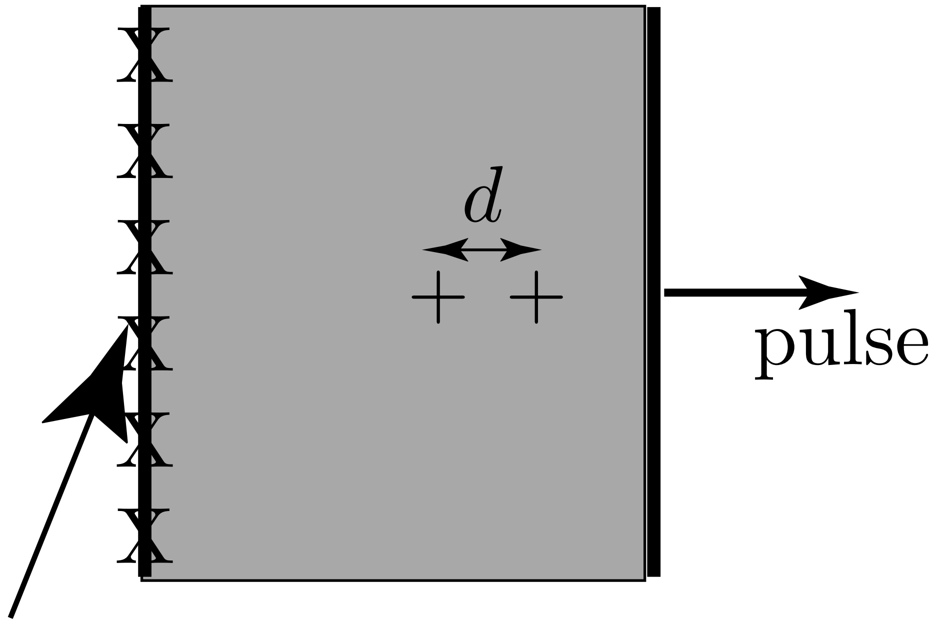
Experimental Observations

Sound Speeds and Rupture Velocities



Tip has wedge-like shape like Mach cone

Longitudinal speed measured by time of flight



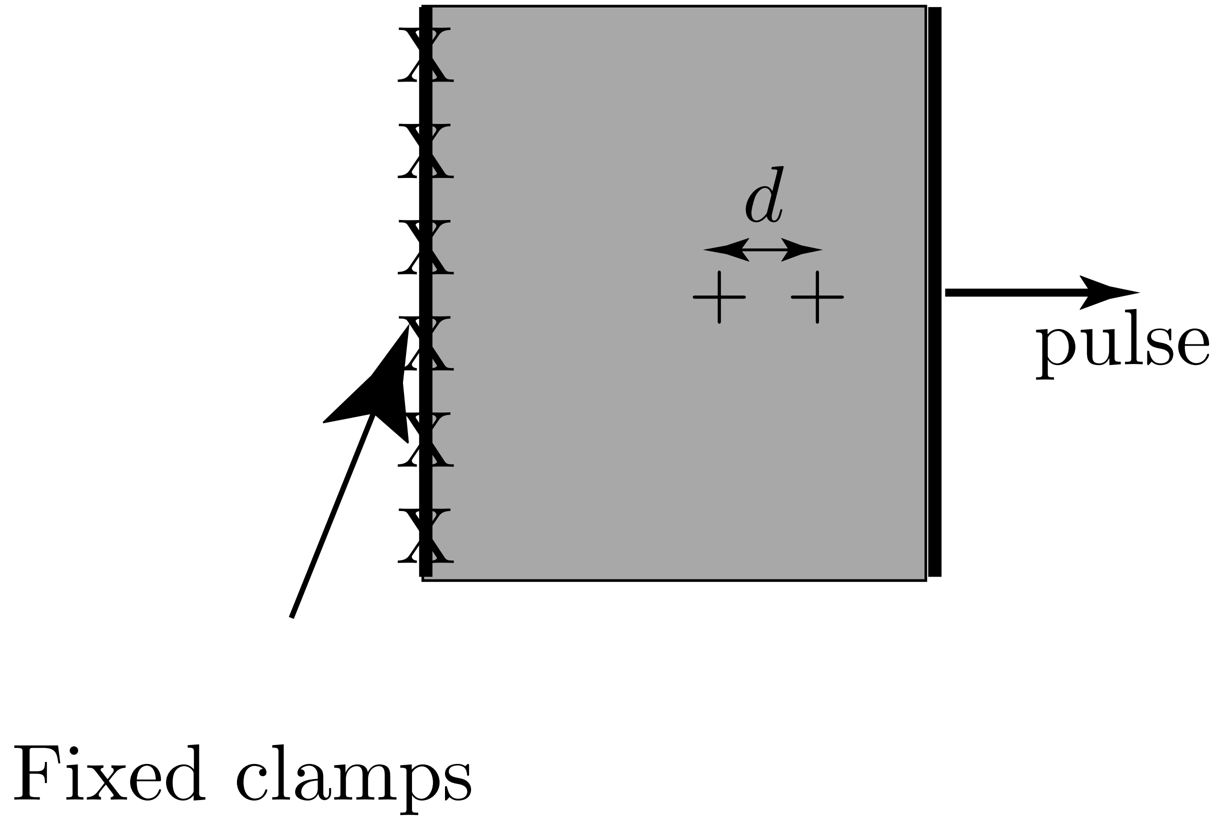
Fixed clamps

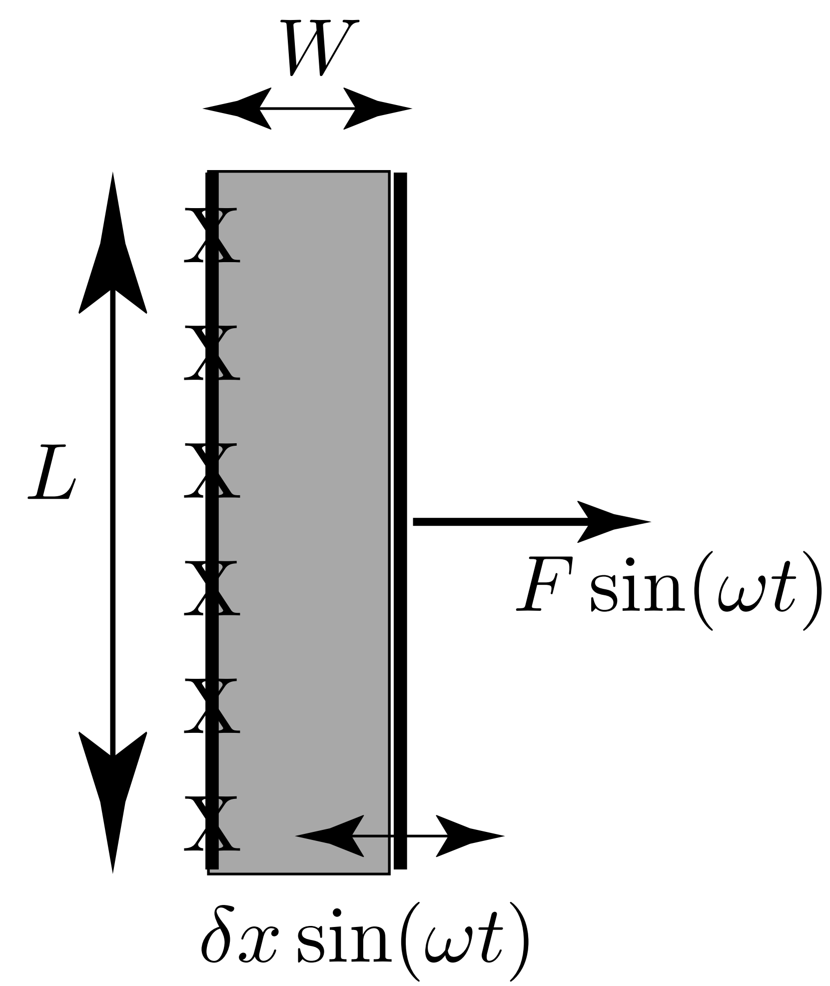
Experimental Observations

Sound Speeds and Rupture Velocities

Longitudinal speed measured by time of flight

Longitudinal speed measured from force extension curves.

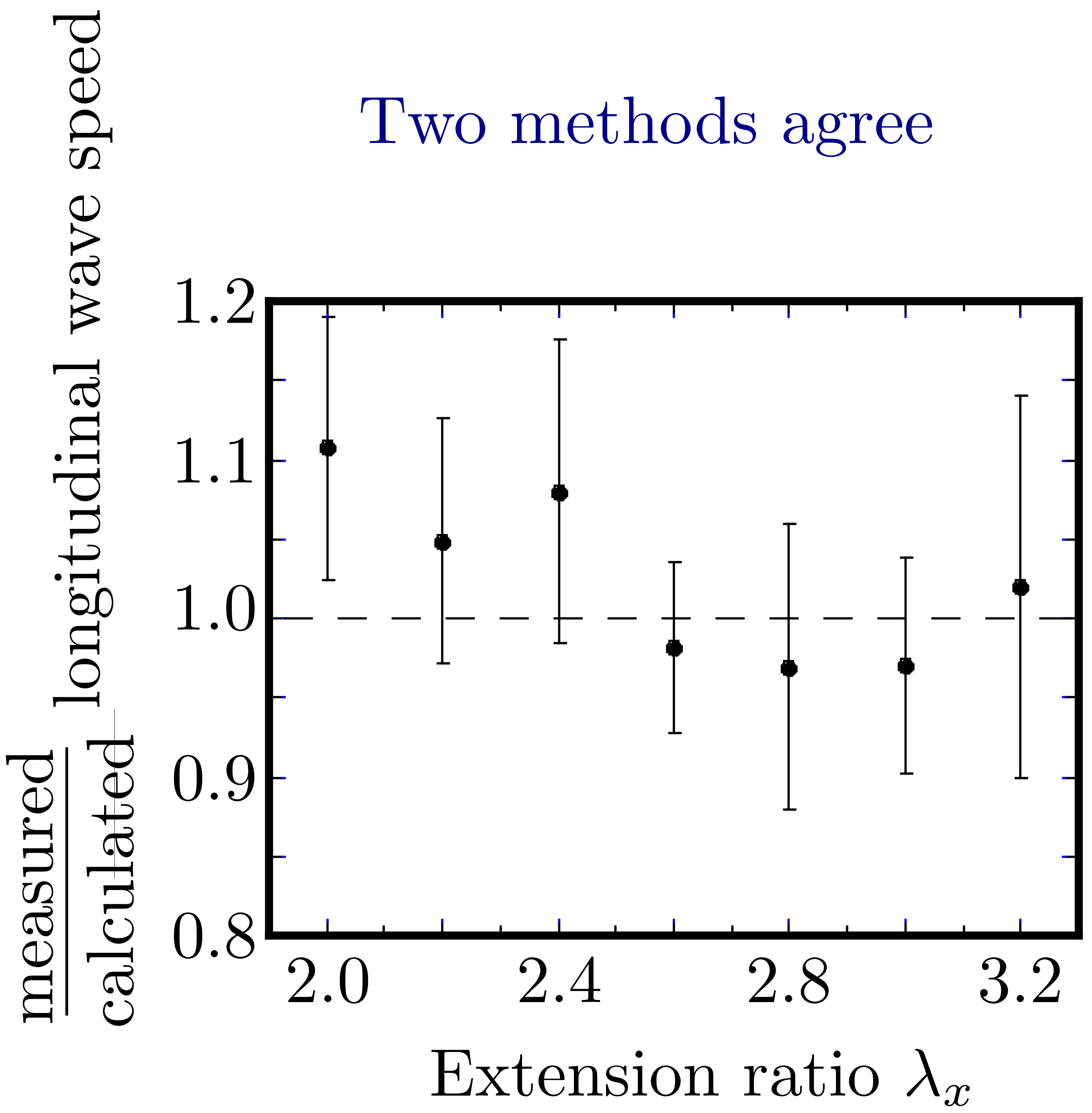




Experimental Observations

Sound Speeds and Rupture Velocities

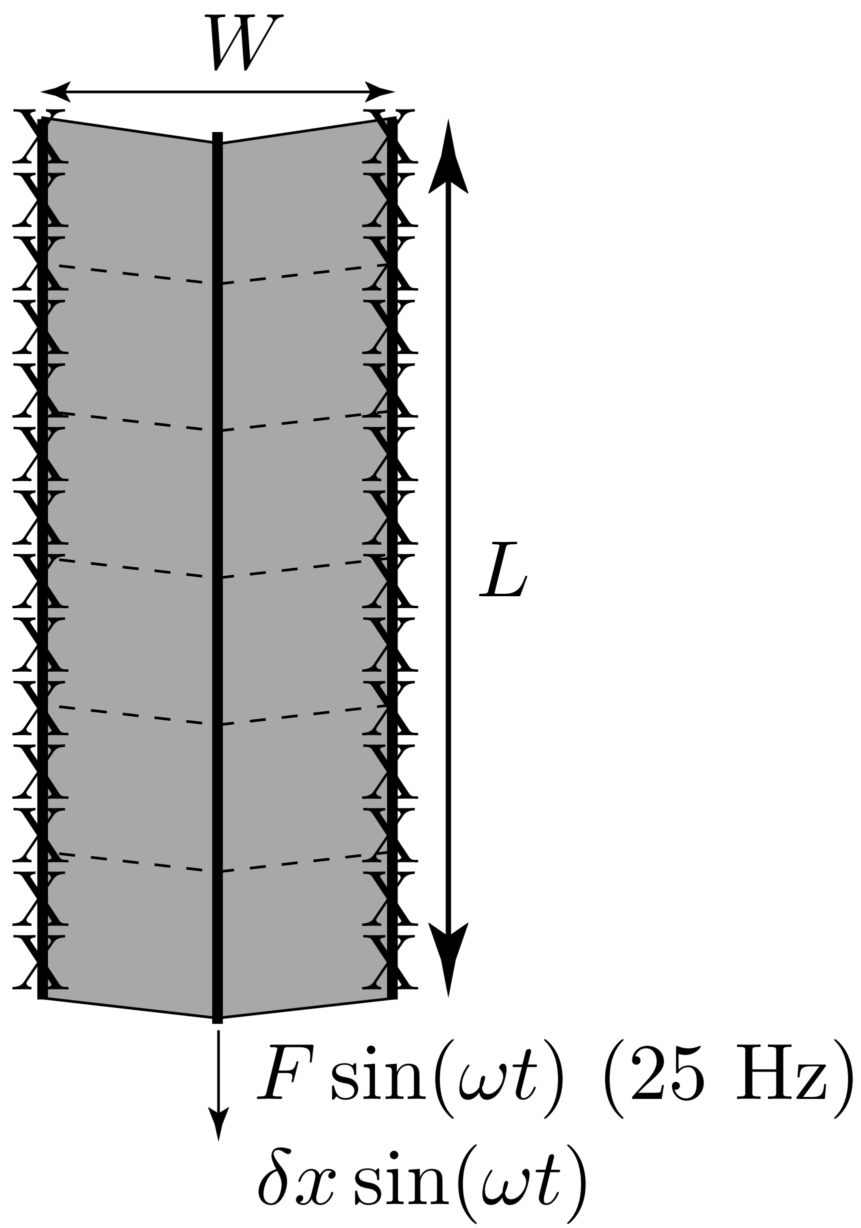
Two methods agree



Experimental Observations

Sound Speeds and Rupture Velocities

Shear wave speed measured from force extension curves.



Experimental Observations

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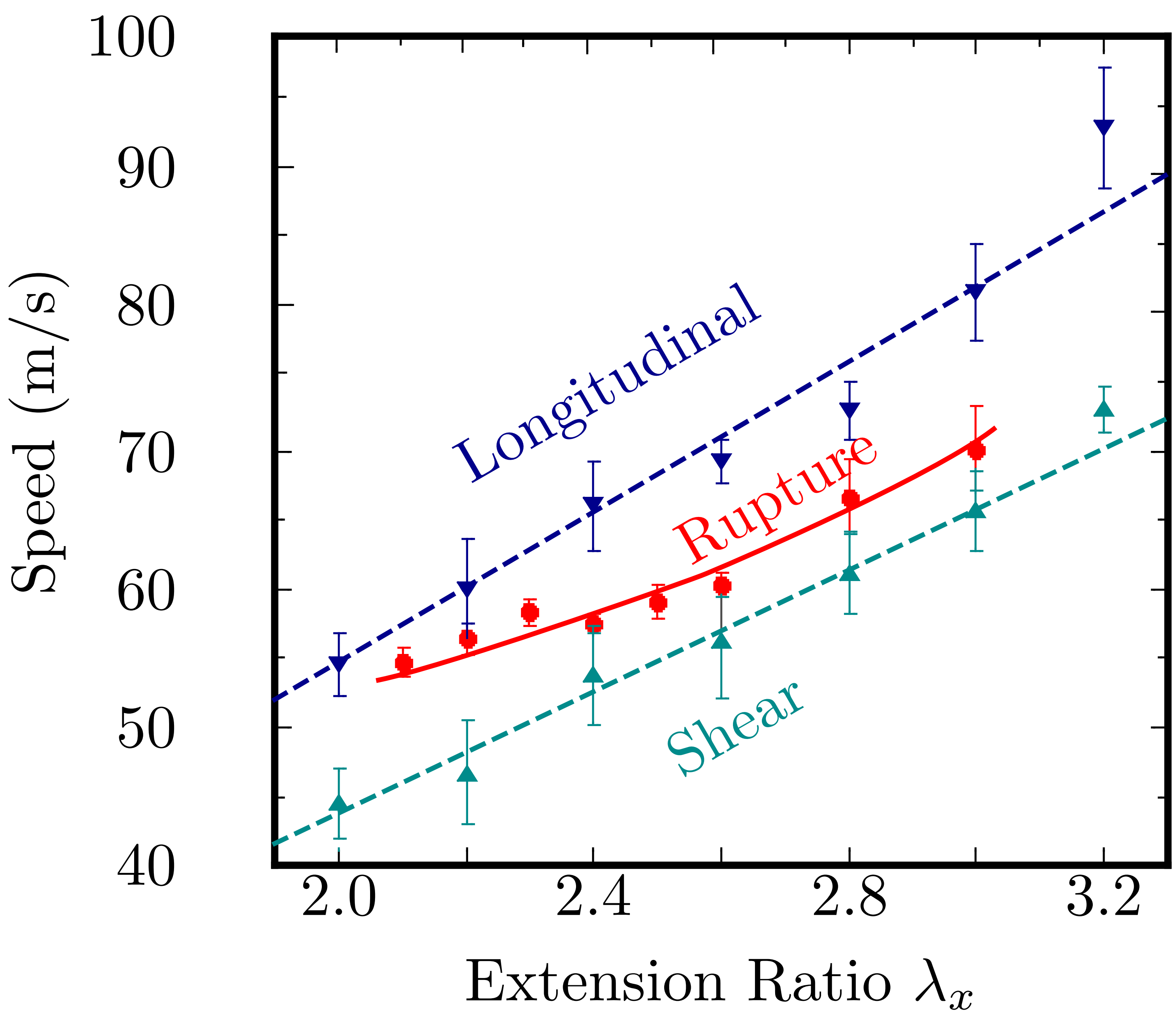
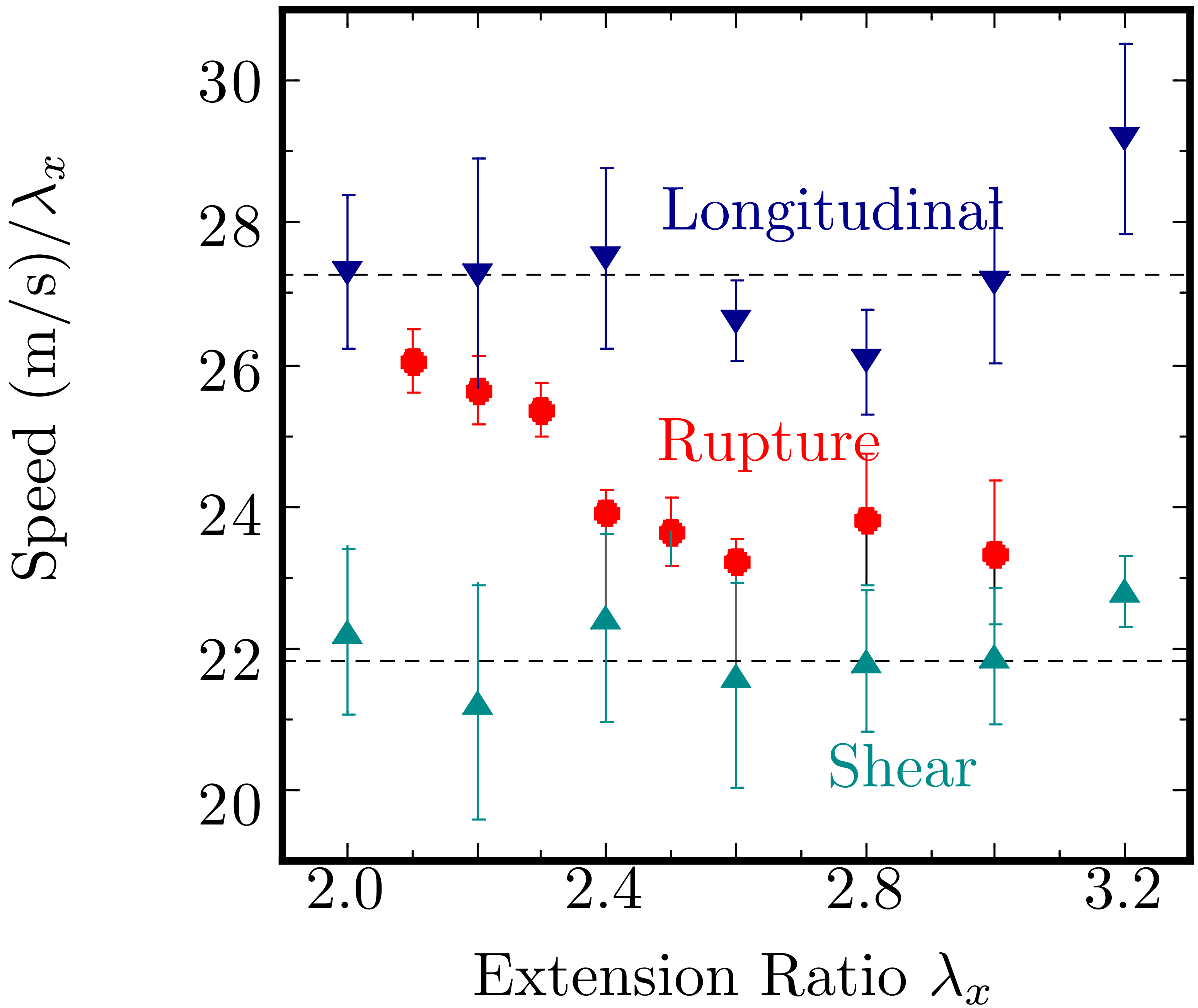


Experimental Observations

Sound Speeds and Rupture Velocities

Transform to reference frame



Experimental Observations

Sound Speeds and Rupture Velocities

Over range of extensions covered by our experiments ($\lambda_x, \lambda_y \sim$ 200%–350%), the speed of sound waves is well described by the Mooney-Rivlin free energy:

$$
\begin{aligned}
e(I_1, I_2) &= A(I_1 + 2BI_2) \\
&= A\left[(E_{xx} + E_{yy} + E_{zz}) + 2B\left(E_{xx}E_{yy} - E_{xy}^2\right) + 2BE_{zz}I_2\right]
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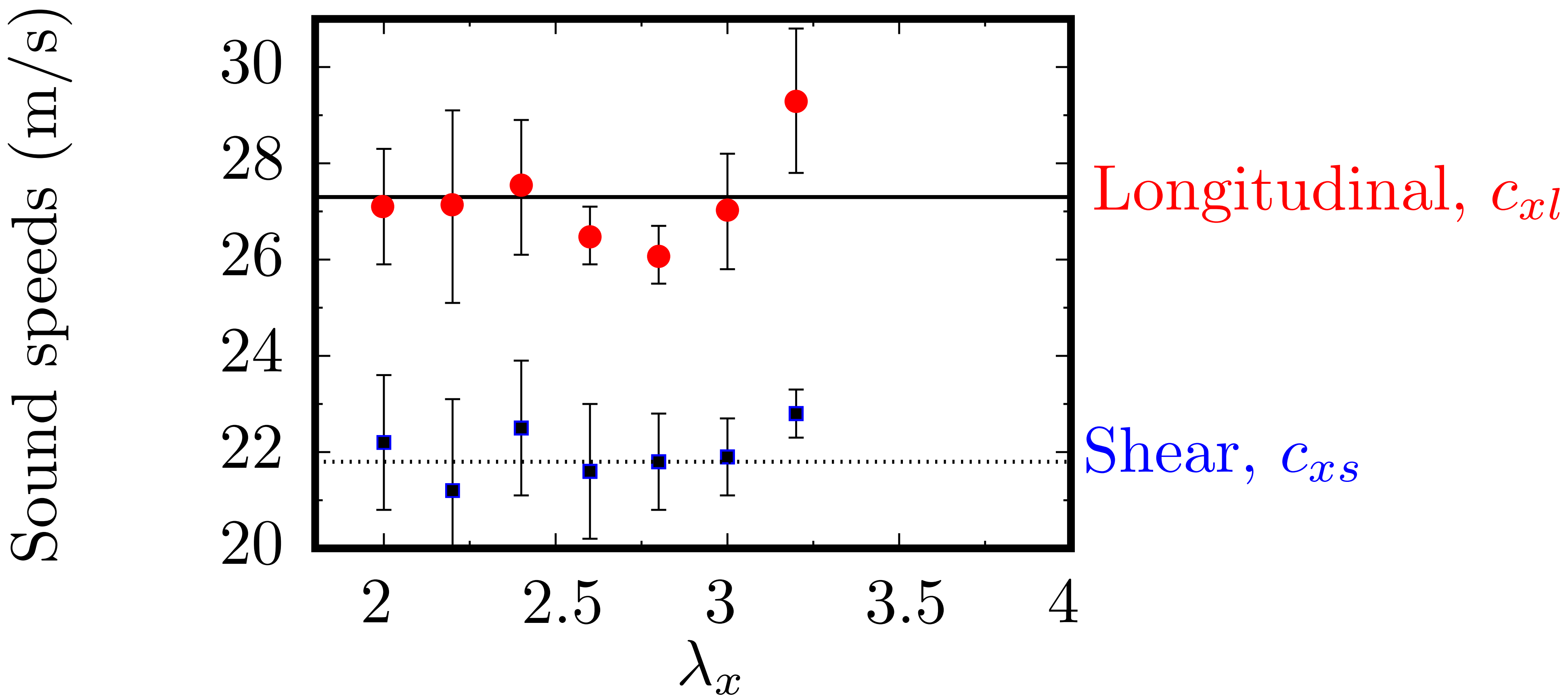

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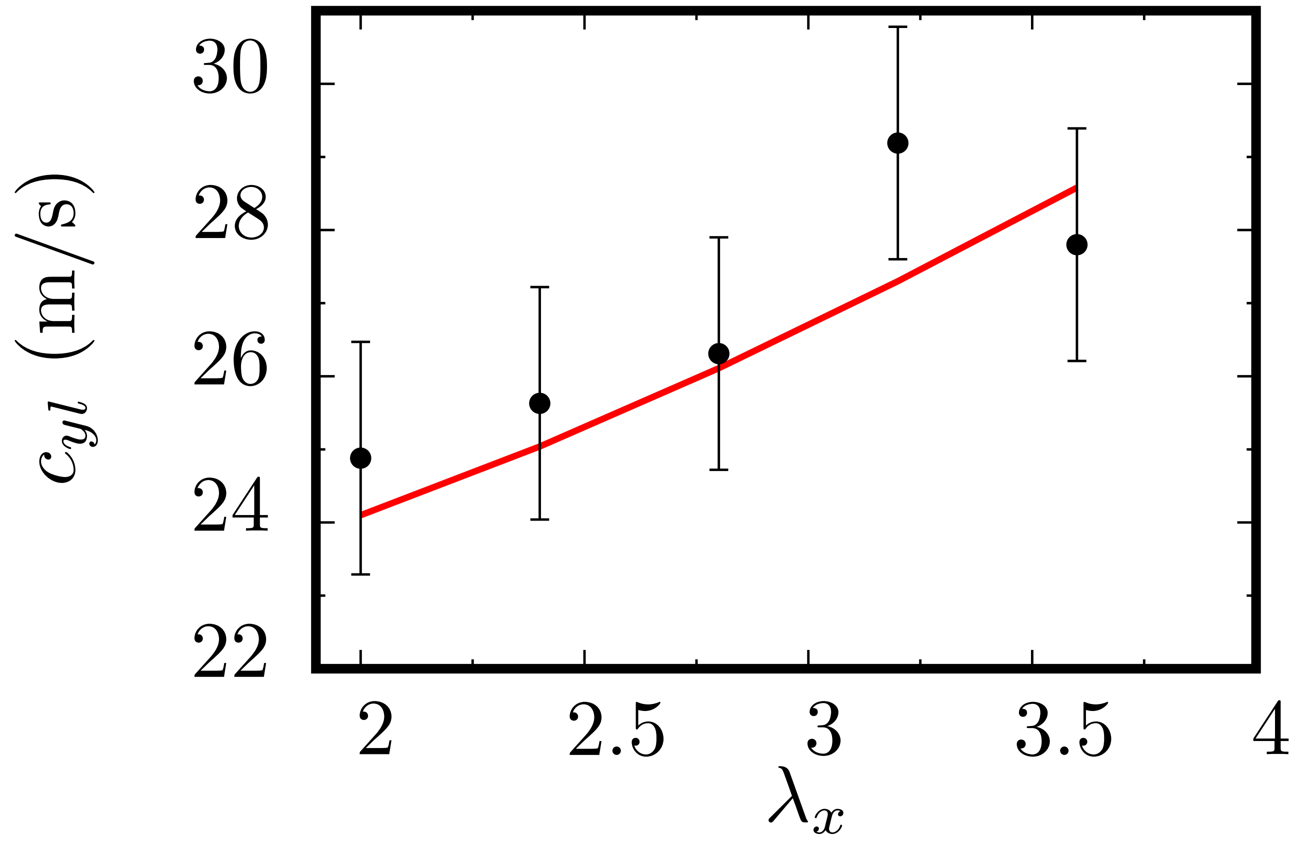


Numerical Studies

Multi-Particle Modeling

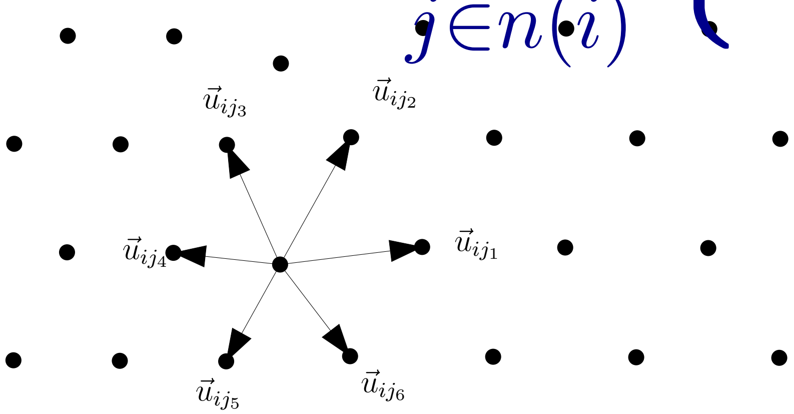
- Tethered Membranes (Nelson, *et al.*)
- Virtual Bond Method (Gao and Klein)
- Peridynamics (Silling and Bobaru)
- Mesodynamics (Holian)

Distance between near neighbors

Lattice spacing

$$I_{1i} = \frac{1}{6a^2} \sum_{j \in n(i)} \begin{cases} (\vec{u}_{ij} \cdot \vec{u}_{ij} - a^2) & \text{if } u_{ij} < \lambda_f \\ \lambda_f^2 - a^2 & \text{else} \end{cases}$$

Failure extension



$$F_i = \frac{1}{6} \sum_{j \in n(i)} \begin{cases} \left(\vec{u}_{ij} \cdot \vec{u}_{ij} - a^2\right) & \text{if } u_{ij} < \lambda_f \\ \lambda_f^2 - a^2 & \text{else} \end{cases}$$

$$G_i = \frac{1}{9} \sum_{j \in n(i)} \begin{cases} \left(\vec{u}_{ij} \cdot \vec{u}_{ij} - a^2\right)^2 & \text{if } u_{ij} < \lambda_f \\ \left(\lambda_f^2 - a^2\right)^2 & \text{else} \end{cases}$$

$$H_i = \frac{1}{27} \sum_{j \neq k \in n(i)} h(u_{ij}) h(u_{ik}) \left(\vec{u}_{ij} \cdot \vec{u}_{ik} + 2a^2\right)^2 ,$$

$$\text{and} \quad h(u) = 1/(1 + e^{(u-\lambda_f)/u_s}).$$

$$I_1^i = \frac{F_i}{a^2}$$

$$I_2^i = \frac{3}{4} \frac{1}{a^4} \left(F_i^2 - G_i\right) ,$$

or alternatively,

$$I_2^i = \frac{9}{8} \frac{1}{a^4} \left(G_i - H_i + 4\right) .$$

Numerical Studies

Investigations

- Obtained ruptures resembling experiment. What was needed, what was not?

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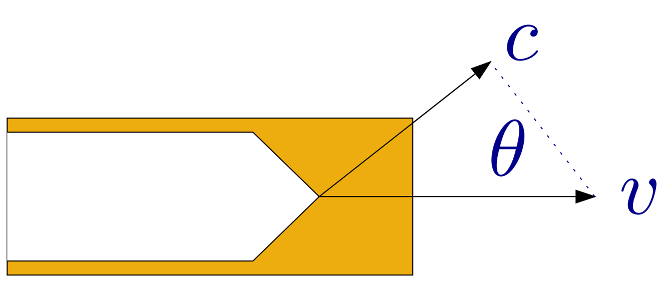
Theoretical Views

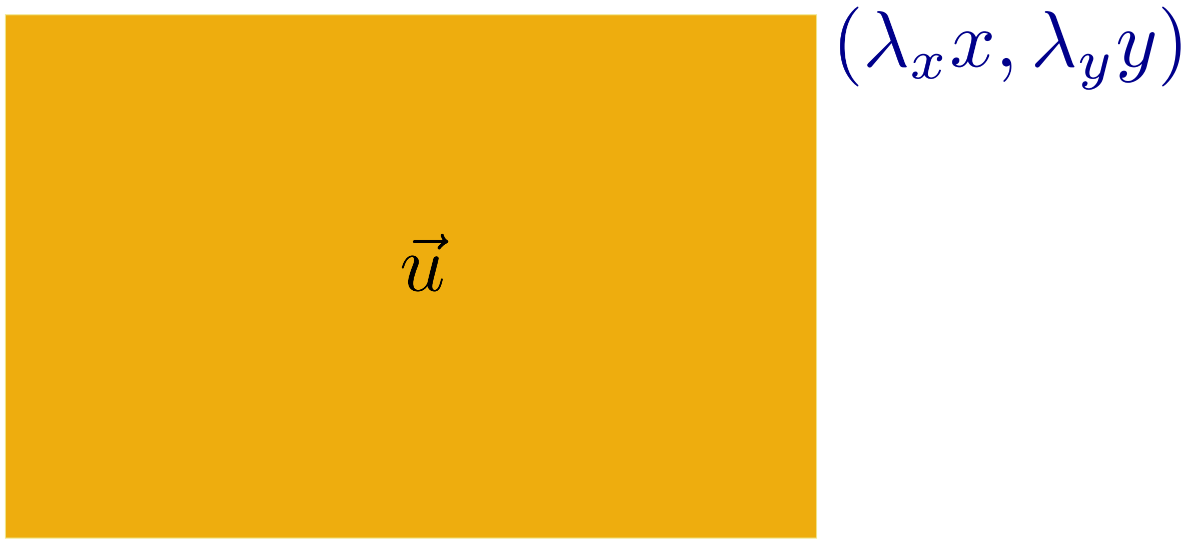
Elementary Considerations

Reference Frame



$$\frac{c}{v} = \sin\theta$$

Laboratory Frame



$$\text{laboratory slope} = \frac{\lambda_y}{\lambda_x\sqrt{v^2/c^2 - 1}}$$

Theoretical Views

Continuum Theory

Following suggesion of Rice (following Shield) adopt Neo–Hookean theory. Horizontal displacements are static, so obtain theory for vertical displacement u.

$$\ddot{u} = c^2\nabla^2 u + c^2\beta\nabla^2\dot{u}$$

- u is not small.
- β describes Kelvin dissipation ($E_\infty \to \infty$)

Boundary conditions:

$$\frac{\partial u}{\partial y} = -\beta\frac{\partial^2 u}{\partial t\partial y} \quad \text{for} \quad x < 0; \quad u = 0 \quad \text{for} \quad x > 0.$$

Theoretical Views

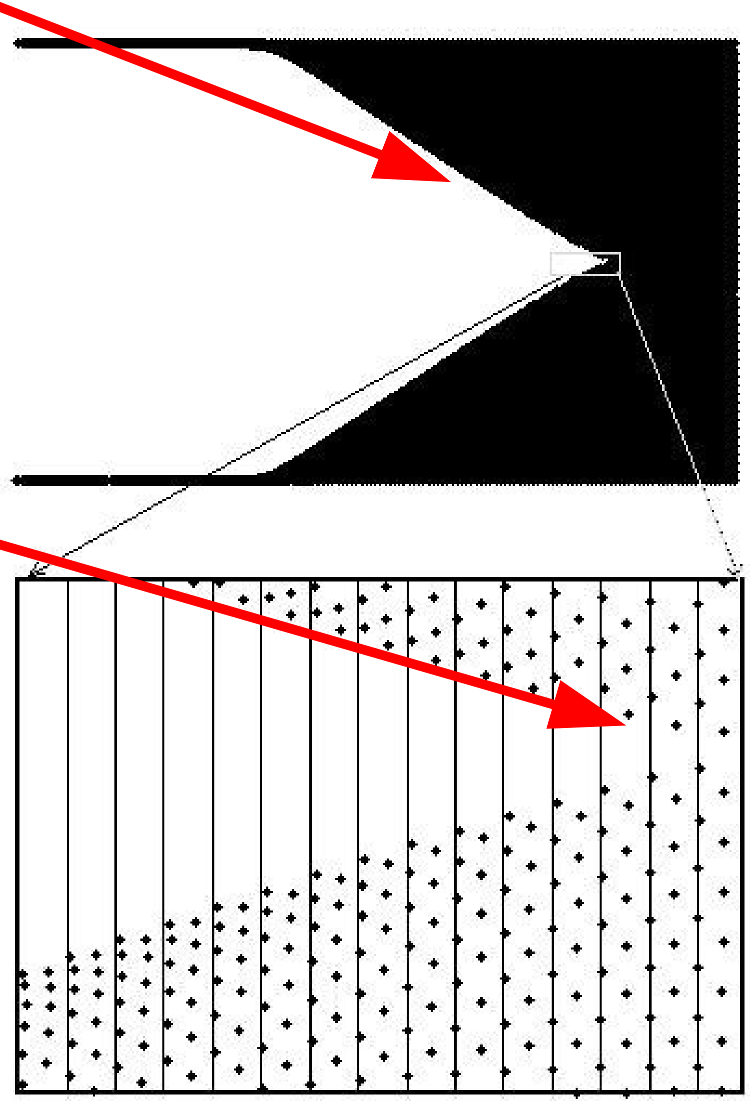
Continuum Theory

This theory has supersonic solutions ($v > c$).

Recover laboratory slope of back edge $= -\dfrac{\lambda_y}{\lambda_x\sqrt{v^2/c^2-1}}$

At the origin, the slope of the rupture is

$$\left.\frac{\partial u}{\partial y}\right|_{x=0,y=0} = \frac{\lambda_y}{\sqrt{1-c^2/v^2}}.$$



The theory can be closed with rupture criterion

$$\frac{\lambda_y}{\sqrt{(4\lambda_f^2-\lambda_x^2)/3}} = \sqrt{1-c^2/v^2}$$

Theoretical Views

Continuum Theory

Kelvin dissipation Sound speed

Lattice spacing

- Dimensionless measure of dissipation is $\beta c/\Delta$: this theory applies when $\beta c/\Delta \geq 1$.
- Displacements and strains are finite near tip.
- Stress diverges as $\exp[-x/(\beta c)]/\sqrt{x}$ near tip
- There is no energy release.

Theoretical Views

Discrete Theory

Discrete theory can be solved using Wiener-Hopf techniques (Slepyan, MPM); find rupture speeds of 800-million-particle systems in five minutes

$$\ddot{u}_i^y = \frac{2c^2}{3a^2} \sum_{j \in n(i)} \left(u_{ij}^y + \beta \dot{u}_{ij}^y \right) \theta(\lambda_f - u_{ij}).$$

Theoretical Views

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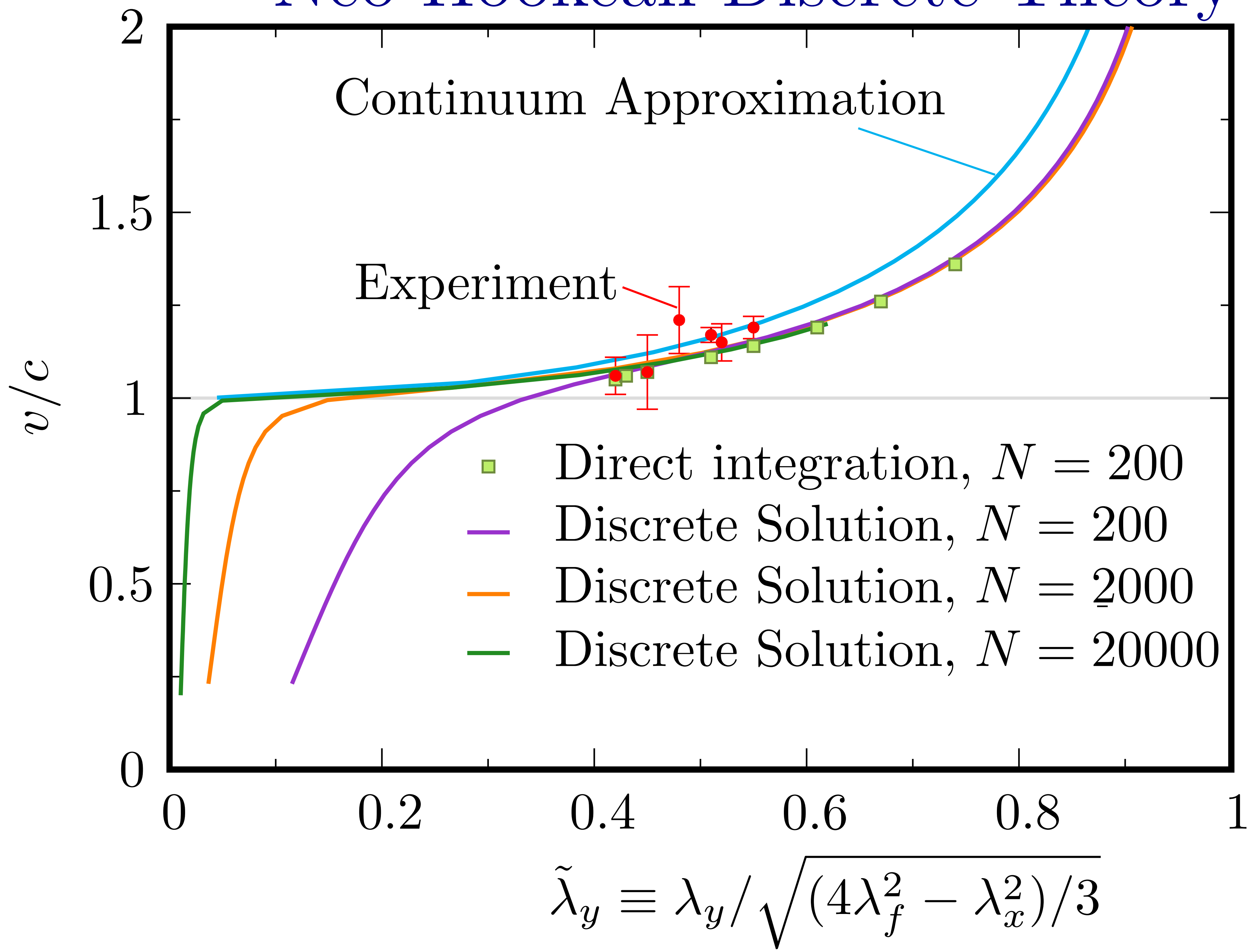
$$\tilde{v} = v/c, \;\; \tilde{\beta} = \beta c/a; \, z = \frac{3 - \cos(\omega/\tilde{v}) - 3\omega^2/[4(1 - i\tilde{\beta}\omega)]}{2\cos(\omega/2\tilde{v})}$$

$$y = z + \sqrt{z^2 - 1}\text{with} \quad \text{abs}(y) > 1, ; F(\omega) = \left\{ \frac{y^{[N-1]} - y^{-[N-1]}}{y^N - y^{-N}} - 2z \right\} \cos(\omega/2\tilde{v}) + 1$$

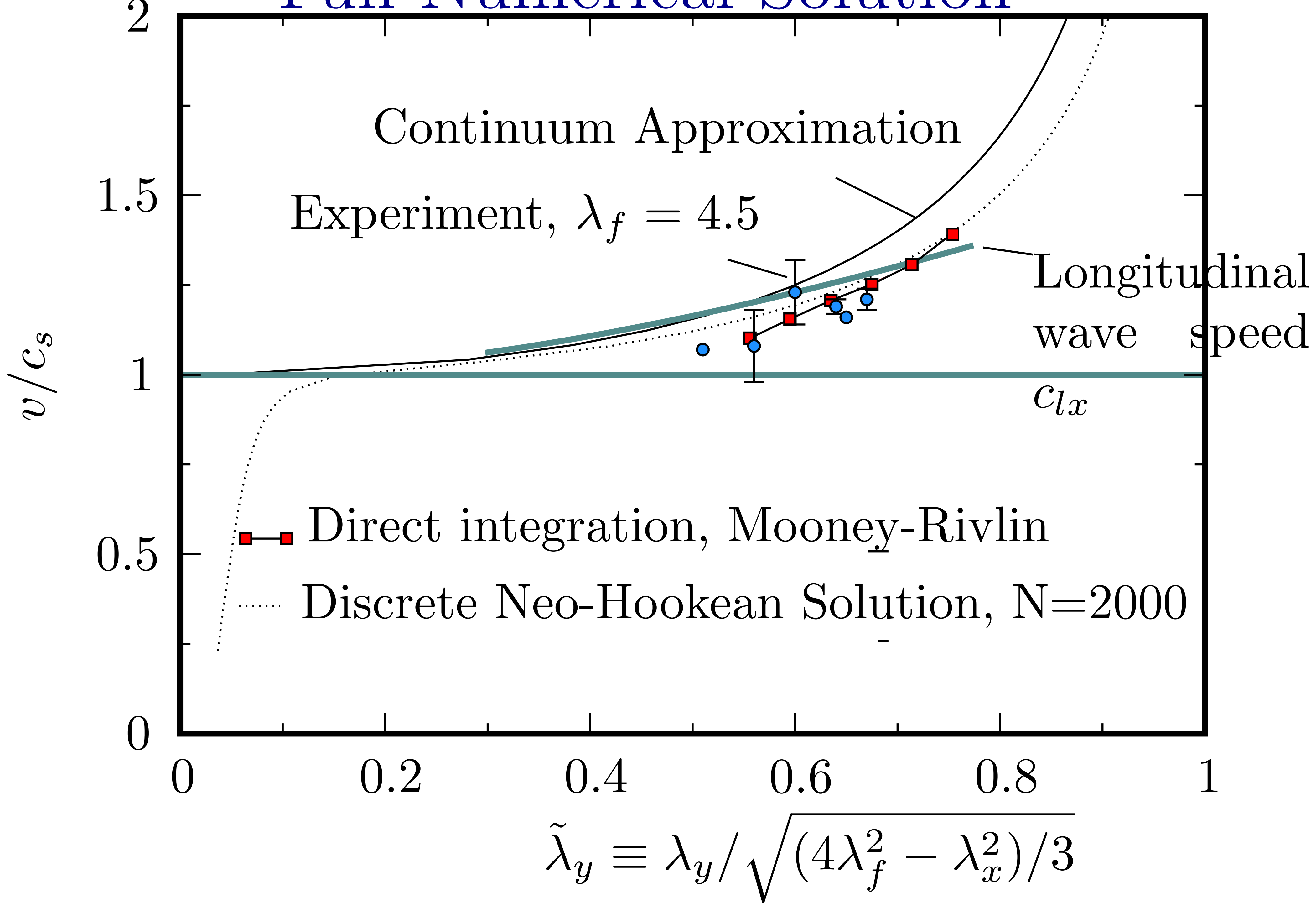
$$Q(\omega) = \frac{F}{F - 1 - \cos(\omega/2\tilde{v})}; \tilde{\lambda}_y = \lambda_y / \sqrt{(4\lambda_f^2 - \lambda_x^2)/3}$$

$$\tilde{\lambda}_y \quad = \quad \frac{1}{\sqrt{2N+1}} \exp\left[- \int \frac{d\omega'}{4\pi} \left\{ \frac{1}{i\omega'(1 + \tilde{\beta}^2 \omega'^2)} \left[\ln Q(\omega') - \overline{\ln Q(\omega')} \right] + \frac{\tilde{\beta} \ln |Q(\omega')|^2}{1 + \tilde{\beta}^2 \omega'^2} \right\} \right].$$

Neo-Hookean Discrete Theory

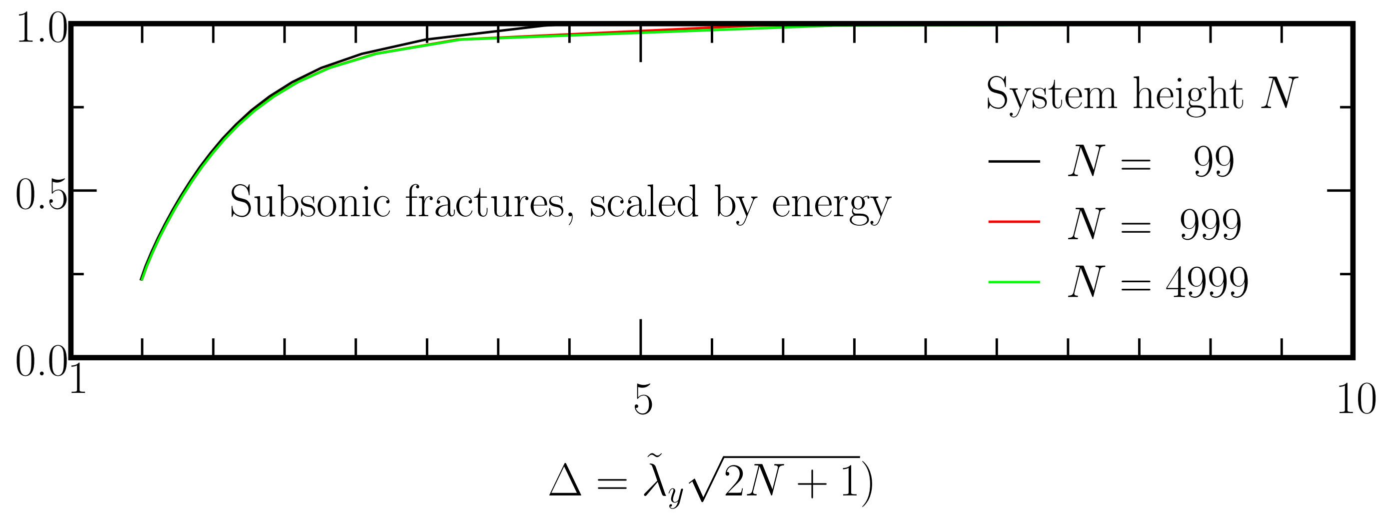


Full Numerical Solution



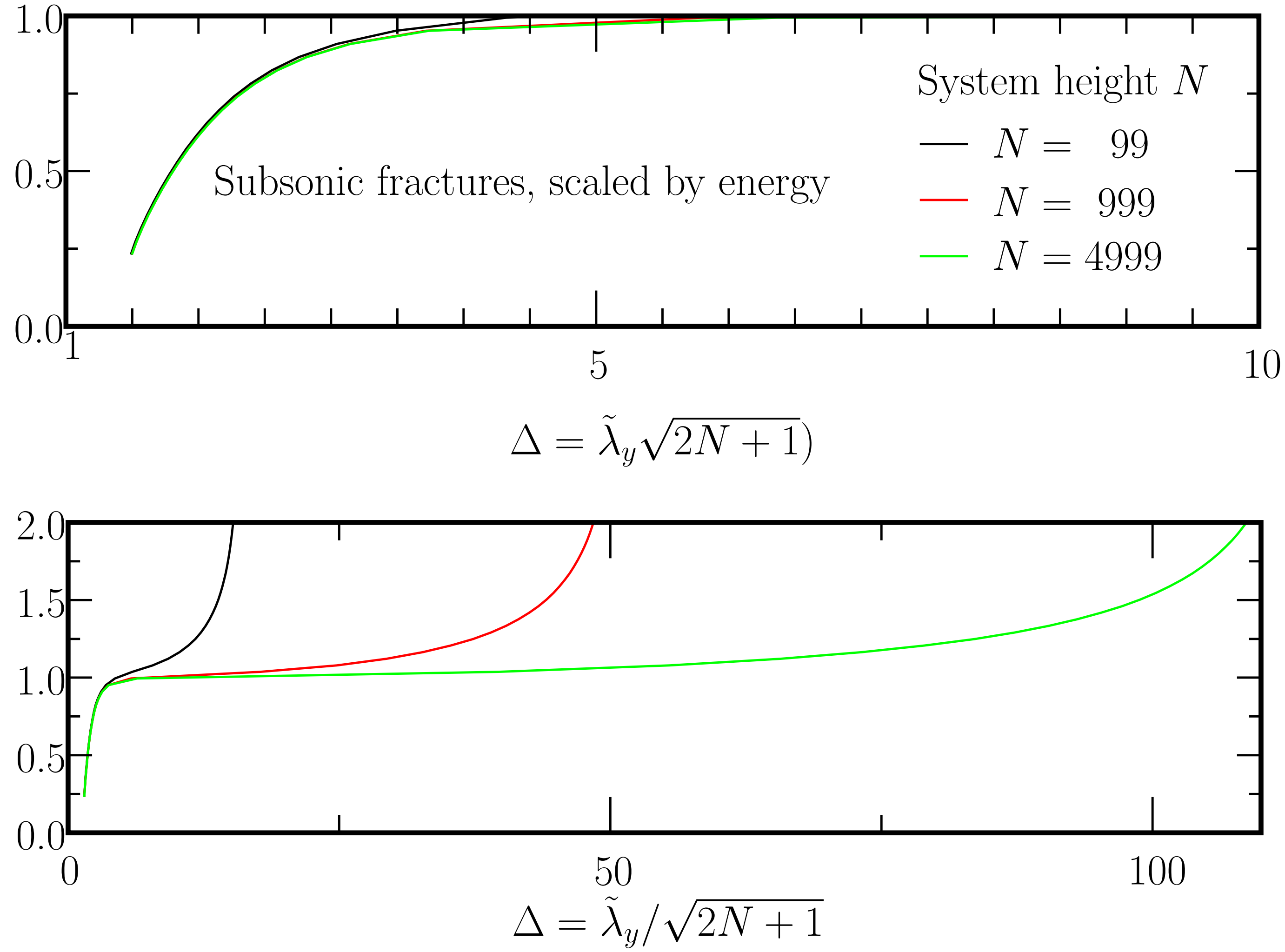
Theoretical Views

Scaling Theory



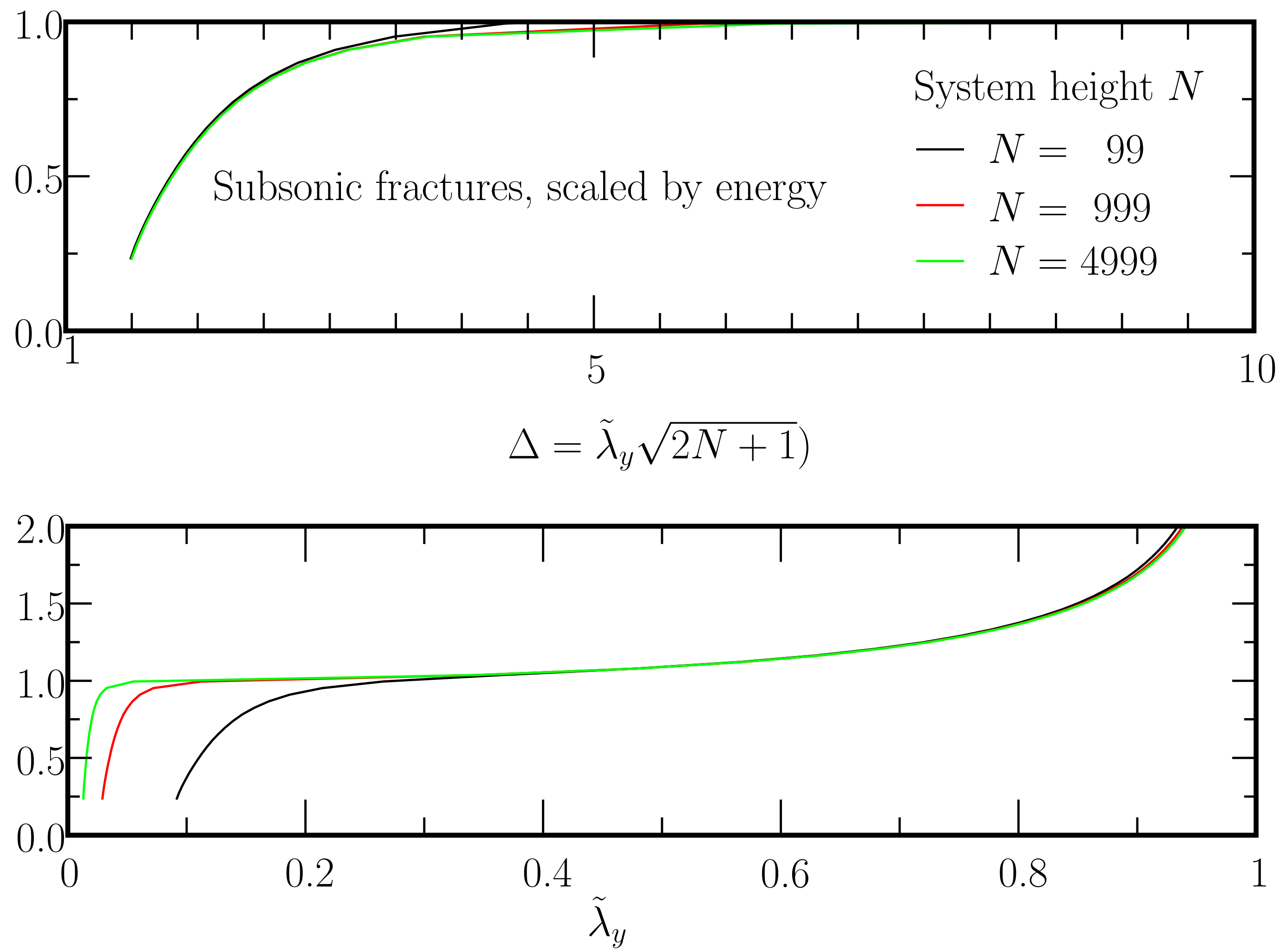
Theoretical Views

Scaling Theory



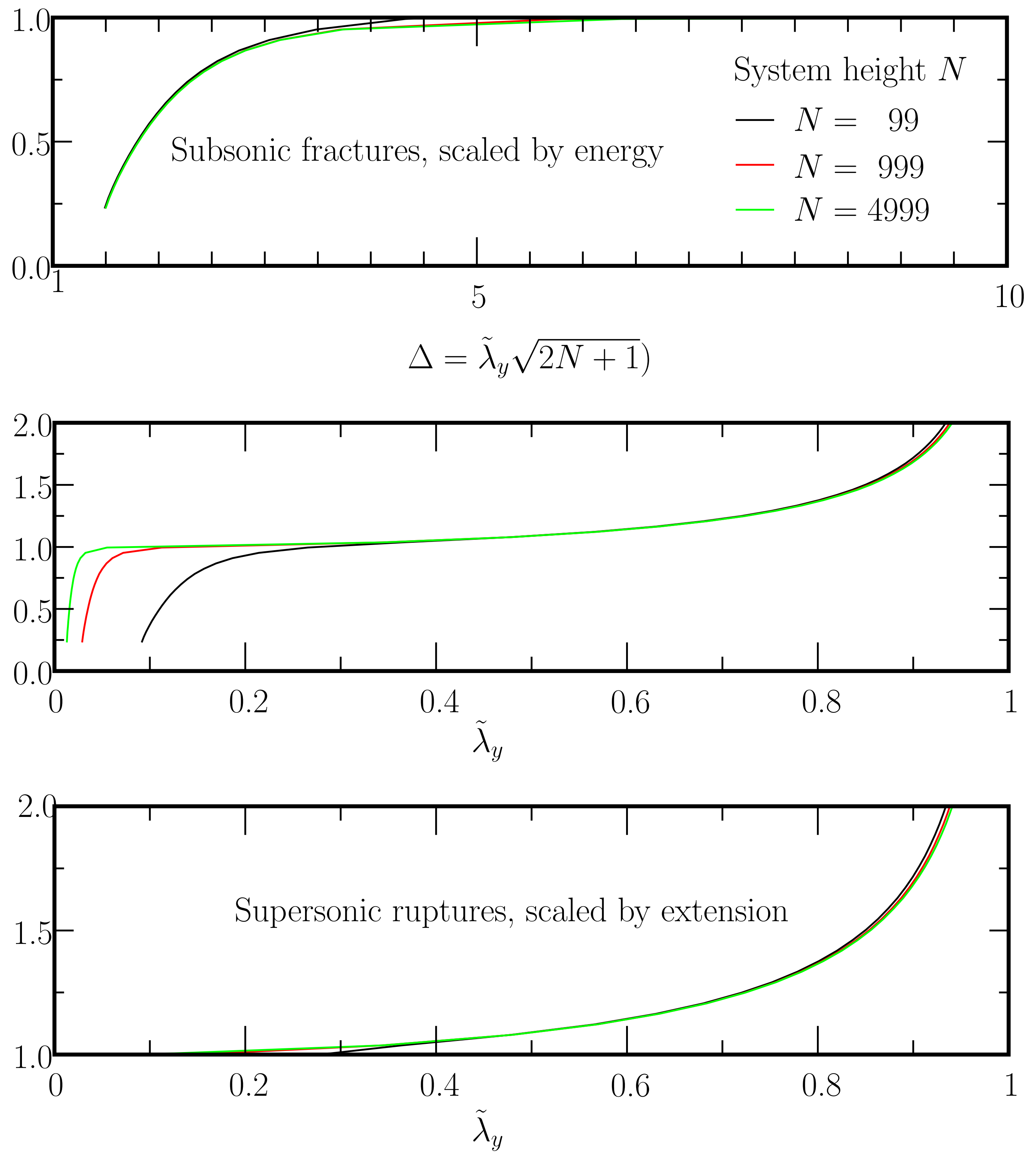
Theoretical Views

Scaling Theory



Theoretical Views

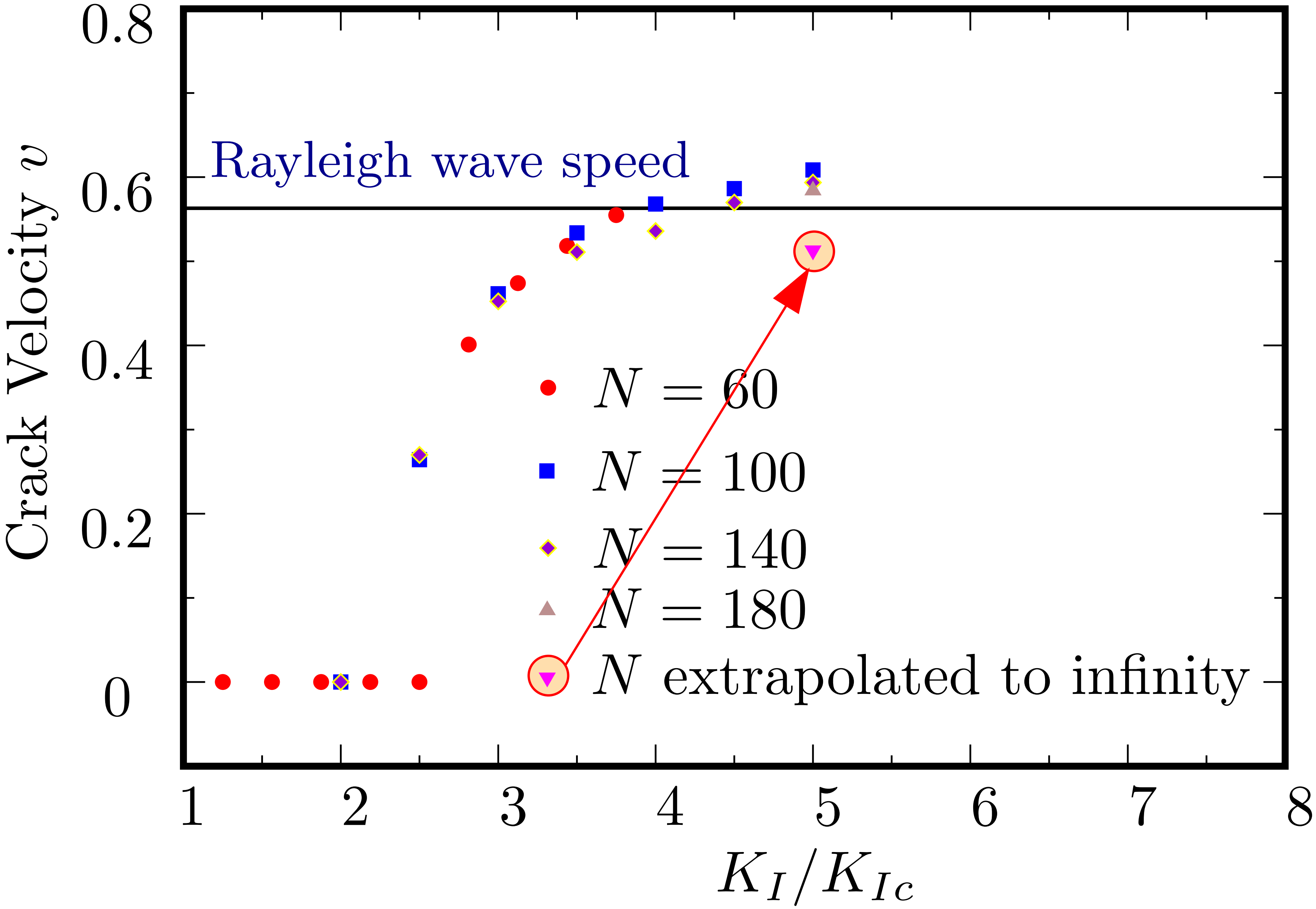
Scaling Theory



Theoretical Views

Scaling Theory

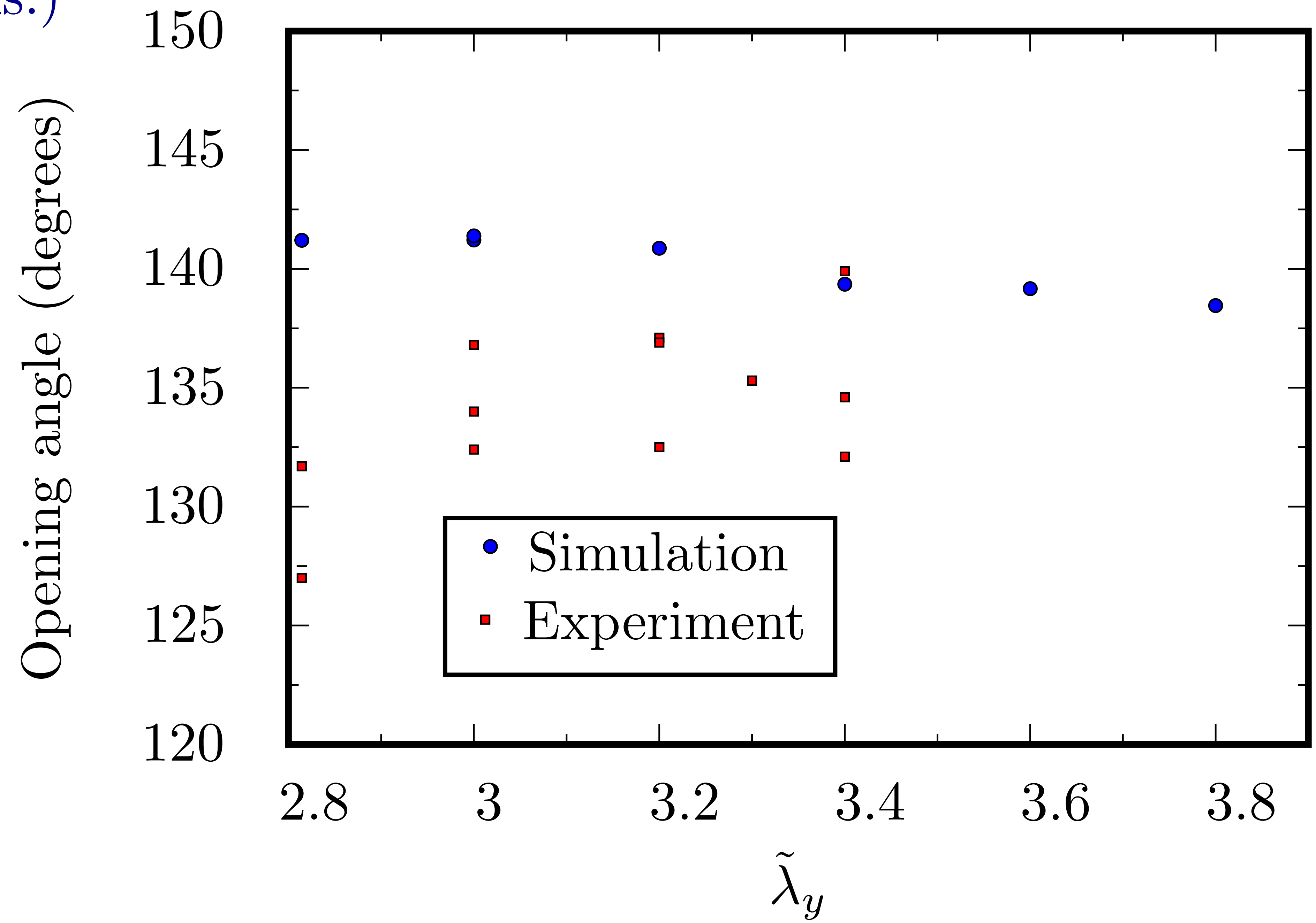
Lattice dynamics, constraint of crack along line, explains work of Buehler, Abraham, and Gao (2003), not hyperelasticity



Theoretical Views

Opening Angles

Opening angle comparisons are less successful (simulations involve all measured features of experiment, including all sound speed variations.)



Conclusions

New Sort of Failure

- The rupture is supersonic
- There is no energy release; energy arrives from very near tip.
- Dissipation and toughening of front behind tip are key physical ingredients; variations in sounds speed are present in real system but play no role in theory.
- Problem can be solved exactly in continuum and discrete formulations
- Comparison of rupture speeds with experiment is good.
- Comparison of rupture angles with experiment less satisfactory.
- Oscillatory instability and application to other systems remain to be studied.