Chern-Weil Symmetries
and how gravity avoids them

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1. **Generalized Global Symmetries and the Weak Gravity Conjecture**

2. Chern-Weil Global Symmetries and the Necessity of Axions

Review: differential forms for currents

Conserved current: $\partial_\mu j^\mu = 0$

Rewrite in terms of $(d-1)$-form $J = \star j$:

$$J_{\mu_1 \cdots \mu_{d-1}} = \varepsilon_{\mu_1 \cdots \mu_d} j^{\mu_d} \quad \text{and} \quad \partial_\mu j^\mu = 0 \quad \Rightarrow \quad \mathrm{d}J = 0.$$

Conserved currents $\Longleftrightarrow$ Closed forms (related by $\star$)

Total charge:

$$Q = \int \mathrm{d}^{d-1}x\, j^0 \quad \Leftrightarrow \quad Q = \int_{M_{d-1}} J$$

Gauging a conserved current:

$$A_\mu j^\mu \quad \Leftrightarrow \quad A \wedge J_{d-1}$$

Equation of motion:

$$\partial^\mu F_{\mu\nu} = j_\nu \quad \Leftrightarrow \quad \mathrm{d}(\star F) = J$$

A current is gauged when it is *exact*, not just *closed*.

Gauging removes currents from the cohomology.

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Disclaimer:

I'm being sloppy by not writing the $\sqrt{|\det g|}$ factors, but they all work out so the equations on the right are exactly correct.

A current is gauged when it is *exact*, not just *closed*.
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Ordinary Global Symmetries

For an ordinary U(1) global symmetry in Euclidean *d*-dimensional spacetime, we can associated a charge with ***any* (*d*–1)*-dimensional submanifold***,

$$Q = \int_{M_{d-1}} J \in \mathbb{Z}$$

In the quantum theory, this means that we have a family of ***operators***,

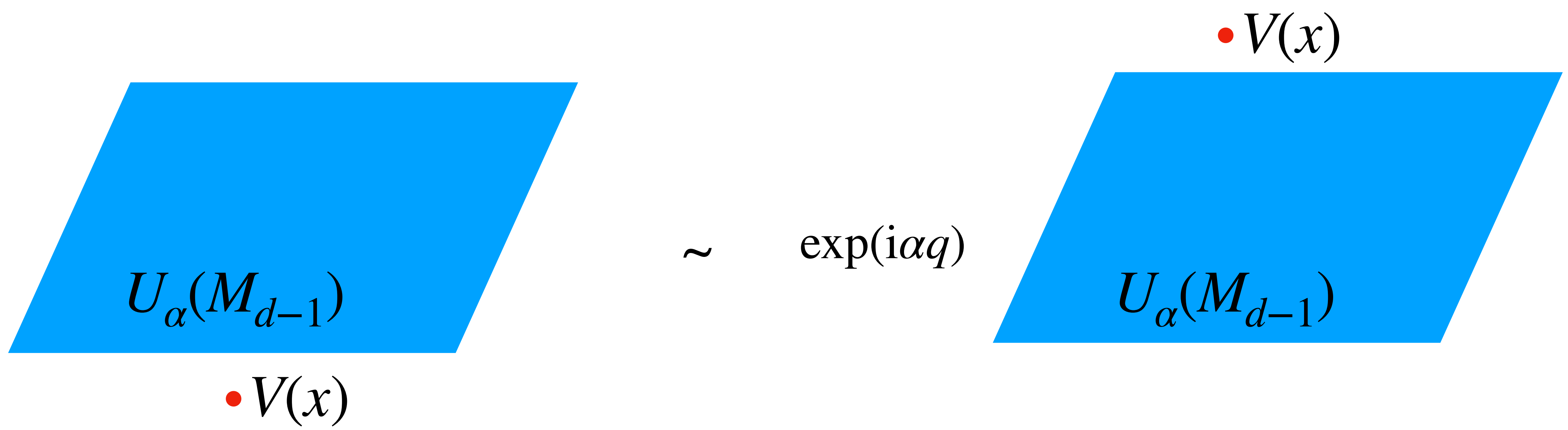
$$U_\alpha(M_{d-1}) = \exp\left(\mathrm{i}\alpha \int_{M_{d-1}} J \right).$$

associated to **codimension-1 surfaces**.

These operators are ***topological***: if a charged local operator is inserted in the theory, then the state picks up a phase when this operator **crosses through** the surface M_{d-1}.

Ordinary Global Symmetries

When a local operator $V(x)$ of charge q crosses the surface operator associated with the element $\exp(\mathrm{i}\alpha)$ of the global group U(1), it gains a phase $\exp(\mathrm{i}q\alpha)$

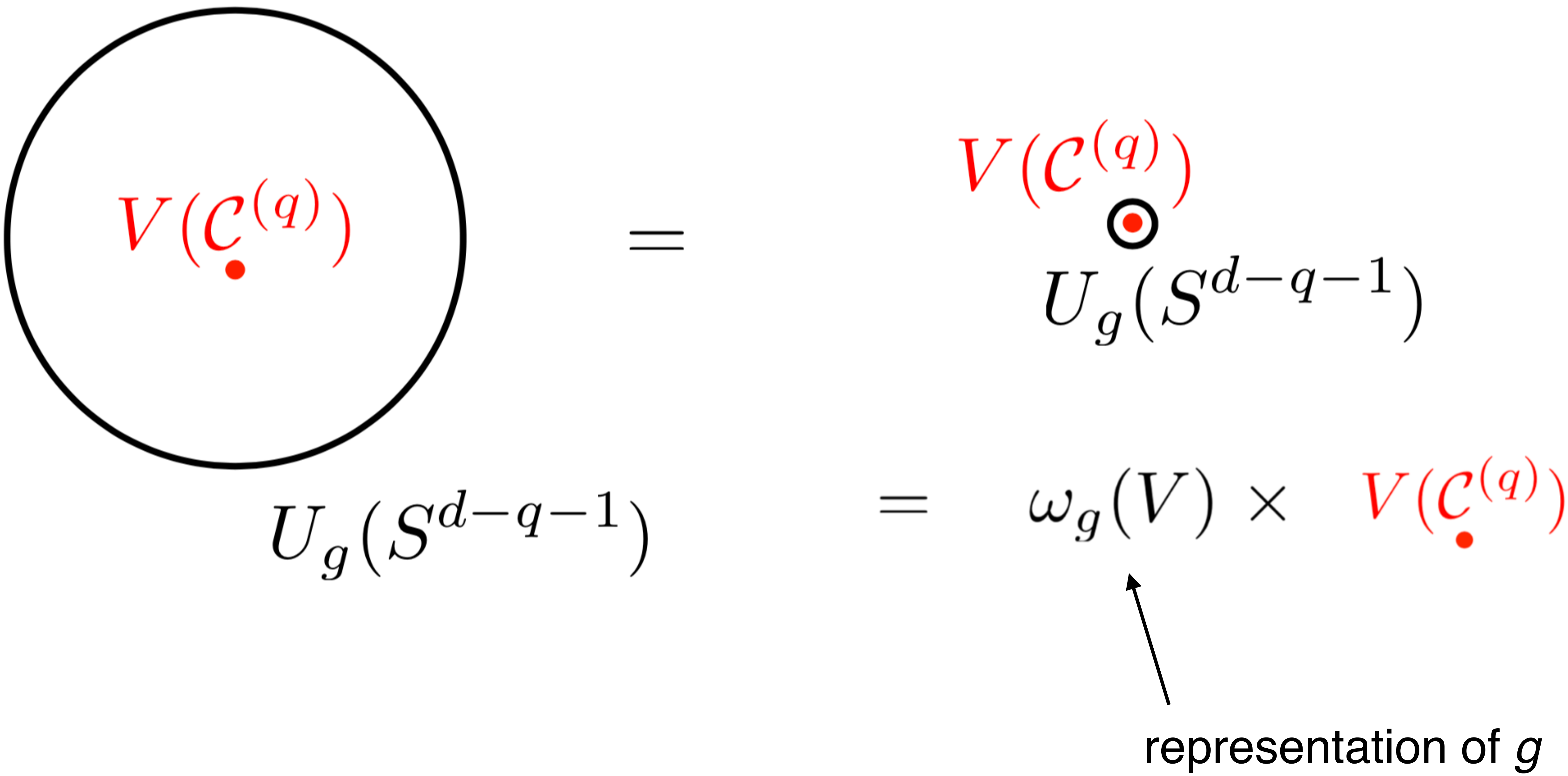


When the operator U is constructed out of the conserved current j^μ, this is familiar.

This formulation also works nicely for discrete symmetries, which have no local conserved current.

Generalized Global Symmetries

(Figure from a nice talk by Tom Rudelius at the 2019 Madrid workshop “Navigating the Swampland”)



Generalized Global Symmetries

arXiv:1412.5148 by Gaiotto, Kapustin, Seiberg, and Willett

A ***p*-form** G **global symmetry** has:

- Charge/symmetry operators $U_g(M_{(d-p-1)})$ which are **topological**
- *Charged* operators $V(M_p)$ associated with p-dimensional manifolds, which can be **"linked"** with the charge operators on $(d-p-1)$-manifolds.
- Dynamical charged *objects* with $(p+1)$-dimensional worldvolumes.
- Continuous G: local conserved $(d-p-1)$-form currents J
- Group law $U_g(M_{d-p-1})U_{g'}(M_{d-p-1}) = U_{gg'}(M_{d-p-1})$
- If $p > 0$, the only symmetries acting nontrivially are **abelian**

1-form Symmetries of U(1) Gauge Theory

In free Maxwell theory, we have no electric or magnetic sources, so

$$\mathrm{d}(\star F) = 0$$

Closed (d–2)-form current
$\Longrightarrow$ **Global (d–3)-form symmetry**

$$\mathrm{d}F = 0$$

Closed 2-form current
$\Longrightarrow$ **Global 1-form symmetry**

The quantization of fluxes means that these are both U(1) symmetries.
In 4d, they are both *1-form* global symmetries.

- **Electric symmetry, current $\star$ *F*, charged objects are *Wilson loops*.**
- **Magnetic symmetry, current *F*, charged objects are *'t Hooft loops*.**

The symmetries basically *count* Wilson or 't Hooft loops.

Existence of charged particles vs. presence of global symmetries

$$\mathrm{d}(\star F) = J$$

Charged particles break the *1-form symmetry*'s conservation law
(while gauging a *0-form symmetry* with current *J*)

The symmetry operators *exist*, but are no longer topological. Wilson operators can end on local operators that create charged particles.

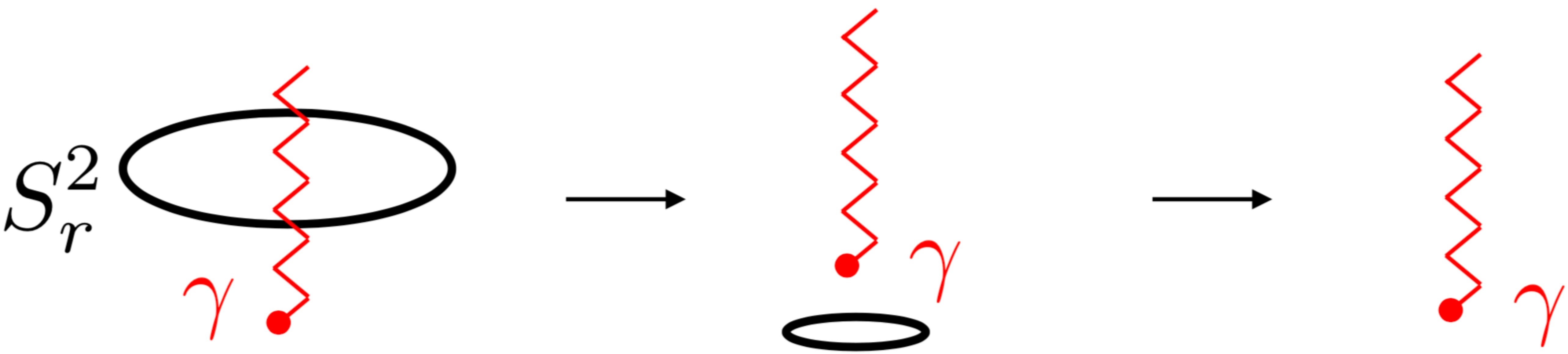


figure from Tom Rudelius

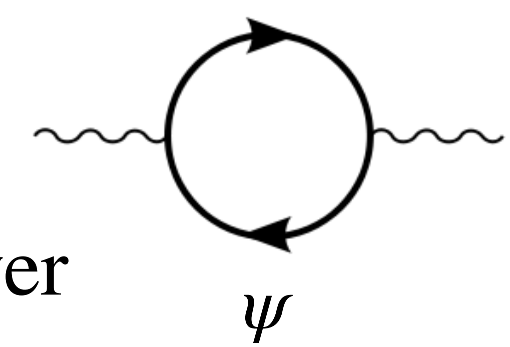
Wilson lines can *end* $\Longleftrightarrow$ 1-form electric symmetry is *explicitly broken*.

The WGC from no global symmetries?

For a U(1) gauge theory: absence of the 1-form generalized global symmetry requires electrically charged particles to exist.

Clay **Córdova**, Kantaro **Ohmori**, and Tom **Rudelius** (forthcoming work):

Asking that the 1-form symmetry be **badly** ***broken at the QG cutoff energy requires a tower of charged particles that parametrically obey the WGC.***

$\sum_{\psi \in \text{tower}}$ 

$\Rightarrow$ $V(r)$ deviating strongly from $1/r$ for $r \sim \Lambda_{\text{QG}}^{-1}$

$\Rightarrow$ $U_\alpha(M) \sim \exp(\mathrm{i}\alpha \int_M \star F)$ is far from topological in the UV

(Effectively, reproduce the strong coupling argument for Tower WGC [Heidenreich, MR, Rudelius '17].)

1. Generalized Global Symmetries and the Weak Gravity Conjecture

2. **Chern-Weil Global Symmetries and the Necessity of Axions**

work in preparation with Ben Heidenreich, Jake McNamara, Miguel Montero, Tom Rudelius, and Irene Valenzuela

Conservation of Chern-Weil currents

In an abelian gauge theory, if $\mathrm{d}F = 0$ (no magnetic monopoles), then

$$\mathrm{d}(F \wedge F) = \mathrm{d}F \wedge F + F \wedge \mathrm{d}F = 0,$$

so $F \wedge F$ is a conserved 4-form current, and generates a $(d-5)$-form symmetry. It is broken if magnetic monopoles exist (but the story is not so simple—stay tuned).

A generalization is true in nonabelian gauge theories:

$$\begin{aligned}\mathrm{d}\,\mathrm{tr}(F \wedge F) &= \mathrm{tr}(\mathrm{d}F \wedge F + F \wedge \mathrm{d}F) \\ &= \mathrm{tr}\big((\mathrm{d}F + [A, F]) \wedge F + F \wedge (\mathrm{d}F + [A, F])\big) \\ &= \mathrm{tr}(\mathrm{d}_A F \wedge F + F \wedge \mathrm{d}_A F) = 0\end{aligned}$$

This is a lemma in the construction of the Chern-Weil homomorphism, an important step in the theory of characteristic classes.

Conservation of Chern-Weil currents

More generally, we have a family of conservation laws,

$$\mathrm{d}\,\mathrm{tr}\left(\bigwedge^k F\right) = 0$$

Here $\bigwedge^k F$ denotes $F \wedge F \wedge \ldots \wedge F$, with k copies of F.

These conservation laws all follow from the nonabelian Bianchi identity,

$$\mathrm{d}_A F \equiv \mathrm{d}F + [A, F] = 0$$

Each $(2k)$-form conserved current means there is a generalized $(d - 2k - 1)$-form global symmetry, which we call a ***Chern-Weil global symmetry***.

Chern-Weil global symmetries vs. quantum gravity?

Chern-Weil global symmetries are ubiquitous in gauge theories. **They are not easy to break**, as they follow from the Bianchi identity.

In 4 dimensions, the current $\mathrm{tr}(F \wedge F)$ is a 4-form, so it is trivially conserved. Nonetheless, there is a sense in which it generates a U(1) global "(–1)-form symmetry," because it has quantized (integer) integrals (periods). The charge is **instanton number**.

In 5 dimensions, this becomes an honest 0-form global symmetry and instantons are particles that carry a conserved charge.

Quantum gravity cannot have global symmetries. How does it remove these apparent Chern-Weil global symmetries?

Chern-Weil meets ’t Hooft-Polyakov

Consider d-dimensional SU(2) gauge theory higgsed to U(1) with an adjoint VEV. This theory contains the semiclassical, ’t Hooft-Polyakov magnetic monopole, whose worldvolume has codimension 3.
(We consider $d \geq 4$; the case $d = 4$ is somewhat degenerate, but I think it does make sense.)

UV: $\mathrm{d}\,\mathrm{tr}(F \wedge F) = 0$ Conserved 4-form current

IR: $\mathrm{d}\,(F \wedge F) = 2\,J_{\mathrm{mag}} \wedge F$

Broken 4-form current, due to monopoles

So, it appears that the Higgsing process has eliminated the symmetry from our IR theory.

Dyons and ’t Hooft-Polyakov

However, the story is more interesting. The classical ’t Hooft-Polyakov monopole solution has **collective coordinates** or **zero modes**.

The obvious zero modes are translations. However, there is a less obvious one, corresponding to a global U(1) rotation. This is realized as a ***compact scalar boson*** σ living on the monopole worldvolume.

In the 4d case, σ is described by the QM of a particle on a circle, which has a spectrum labeled by integers. Exciting this particle above its ground state **transforms the monopole into a dyon**, and the integer is the electric charge. σ shifts under U(1) gauge transformations.

For $d > 4$, σ is still a compact scalar, described by a *QFT* on the monopole worldvolume.

[Julia, Zee ’75; Jackiw, ’76; Tomboulis, Woo ’76; Christ, Guth, Weinberg ’76]

Chern-Weil, Dyons, and ’t Hooft-Polyakov

We can *gauge* the SU(2) Chern-Weil current by adding a $(d-4)$-form gauge field C with a (Chern-Simons) coupling,

$$\frac{1}{8\pi^2} C \wedge \mathrm{tr}(\mathrm{F} \wedge \mathrm{F})\,.$$

After Higgsing, this coupling is inherited not only by the U(1) gauge field but by the theory on the monopole worldline:

$$C \wedge F \wedge F - C \wedge \mathrm{d}_A \sigma \wedge J_{\mathrm{mag}}$$

(I am not being careful about normalization of the terms here and subsequently)

You can think of J_{mag} as the delta functions that localize the latter coupling on the worldline. Thus, the existence of the monopole breaks the conservation law of $F \wedge F$, but it *preserves* another closed 4-form current,

$$\mathrm{d}\left[F \wedge F - \mathrm{d}_A \sigma \wedge J_{\mathrm{mag}}\right] = 0.$$

This current had to exist, or our gauging with *C* would have been inconsistent!

Chern-Weil and the Witten effect

In the 4d case, *C* is a **“0-form gauge field”**, which is to say, a periodic scalar boson—**an axion!**

$$\frac{1}{8\pi^2}\theta\,\mathrm{tr}(\mathrm{F}\wedge\mathrm{F})\,.$$

The localized coupling on the monopole worldline, that is, the familiar theta term of a particle on a circle in QM,

$$\theta\,\mathrm{d}_A\sigma$$

serves to implement the **Witten effect**: magnetic monopoles acquire an electric charge when a theta angle is turned on,

$$q_{\mathrm{el}} = q_{\mathrm{mag}}\frac{\theta}{2\pi}\,.$$

We see that this whole story fits together nicely: the Witten effect is essential in order to allow us to consistently gauge the Chern-Weil symmetry of the nonabelian theory.

Chern-Weil gauging on D-branes

In string theory, gauge fields can live on a stack of D*p*-branes, which have a (*p*+1)-dimensional worldvolume. In these cases, we always find that the Chern-Weil current $\mathrm{tr}(F \wedge F)$ is gauged by a closed string $(p-3)$-form field:

$$C_{p-3} \wedge \mathrm{tr}(F \wedge F)$$

So far, so good. But this field actually propagates into the bulk, where it couples to lower-dimensional membranes, so a more complete story is:

$$C_{p-3} \wedge \left[\mathrm{tr}(F \wedge F) \wedge J_{Dp} + J_{D(p-4)}\right]$$

Where J_{Dq} is a $(9-q)$-form (the number of delta functions needed to localize on the brane).

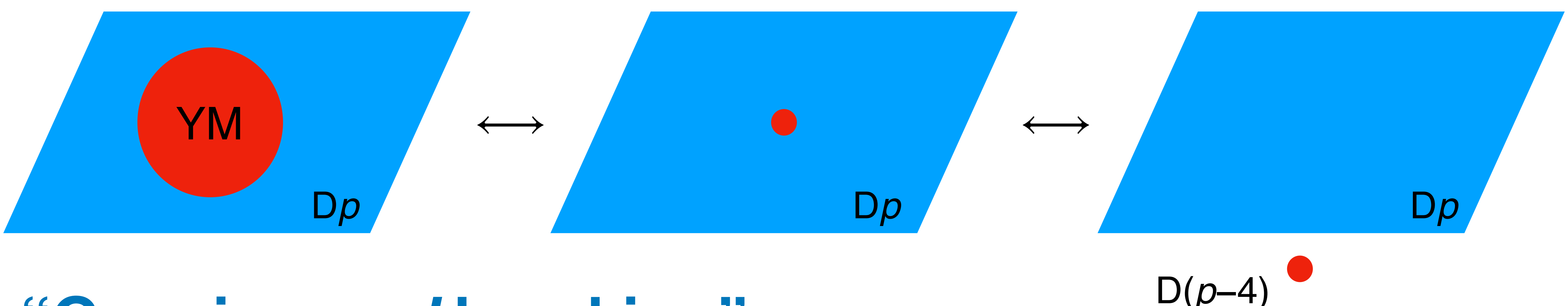
Chern-Weil gauging on D-branes

If the closed string gauge field C_{p-3} is gauging the current in brackets,

$$C_{p-3} \wedge \left[\mathrm{tr}(F \wedge F) \wedge J_{Dp} + J_{D(p-4)} \right]$$

then **what happens to the other linear combination of these two conserved currents?**

The answer is a well-known effect in string theory: **zero-size Yang-Mills instantons on the D*p*-brane are *the same thing* as D(*p* – 4)-branes.**
(Witten '95; Douglas '95; Green, Harvey, Moore '96).



"Gauging *and* breaking"

Chern-Weil and GUTs

Consider a nonabelian gauge group that is higgsed to a product group, as in the SM embedding in a GUT, for instance:

$$\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$$

The IR theory has more Chern-Weil currents than the UV theory. Some of these are "accidental": selecting out SU(3) within SU(5) requires Higgs insertions, so the IR $\mathrm{tr}(F \wedge F)$ contains Higgses in the UV theory, and d(Higgs) is nonzero.

An IR theorist might overcount Chern-Weil symmetries and expect more gauge fields (or axions). However, there will always be at least one. This UV explicit breaking of IR Chern-Weil symmetries only happens for "unifiable" gauge groups.

Summary of examples

Once you start looking for Chern-Weil symmetries and mechanisms to remove them, you get a fresh perspective on many familiar phenomena.

Chern-Weil symmetries are ubiquitous in gauge theories. They are not easy to eliminate.

String theory removes many Chern-Weil symmetries by **gauging via Chern-Simons terms**. This might even be thought of as the reason why C-S terms are so generic in string theory.

Often, Chern-Weil symmetries are broken to the diagonal with another current through **intrinsically stringy UV effects**, e.g., turning YM instantons into branes.

[see also: “Chern-Simons pandemic”, Montero, Uranga, Valenzuela ’17]

Implications for axion physics

If SM gauge fields propagate in higher dimensions, the $\mathrm{tr}(F \wedge F)$ terms are symmetry currents. Expect at least one combination to be gauged. Reducing to 4d, this gives an axionic coupling,

$$\frac{1}{8\pi^2}\theta\,\mathrm{tr}(\mathrm{F} \wedge \mathrm{F})\,.$$

to a *fundamental* axion (compact scalar).

Even in 4d, the notion of a U(1) (–1)-form global symmetry may be well-defined and require such couplings, though this is subtle.

String theory *examples* with axions coupling to $\mathrm{tr}(F \wedge F)$ are common. Chern-Weil symmetry perspective sheds light on *why*—not just "looking under the lamp post."

Implications for axion physics

A common concern about axions for solving strong CP is the **axion quality problem**: misaligned contributions to the potential could lead to strong CP violation.

$$\Lambda_{\mathrm{UV}}^4 \left[\mathrm{e}^{-S_{\mathrm{QCD}}+\mathrm{i}\theta} + \mathrm{e}^{-S_{\mathrm{other}}+\mathrm{i}\theta+\mathrm{i}\phi} + \mathrm{h.c.}\right]$$

If $\phi \neq 0$, need $S_{\mathrm{other}} \gg S_{\mathrm{QCD}}$.

The Chern-Weil perspective ameliorates this worry. Given two kinds of instantons, **either** we expect two independent axions, **or** the different kinds of instantons can be transformed into each other.

Suggests we only worry about gauge sectors **"unifiable"** with QCD.

(Not a complete solution to the problem; e.g., what about $\theta \mathrm{d}C_3$ terms?)

Conclusions

Some messages to take away

The absence of charged particles often leads to generalized global symmetries (or related topological operators [Rudelius, Shao ’20]).

Towers of charged particles guarantee that **1-form symmetries are badly broken at the Planck scale**.

Chern-Weil global symmetries are ubiquitous in gauge theories. In gravitational theories, they must be gauged or broken.

Often they are gauged via Chern-Simons couplings. **Suggestive of why axions are necessary in QG.**

Future: what does it mean for those to be “badly broken”? What are implications for axion physics?