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Relaxation in mixed chaotic systems

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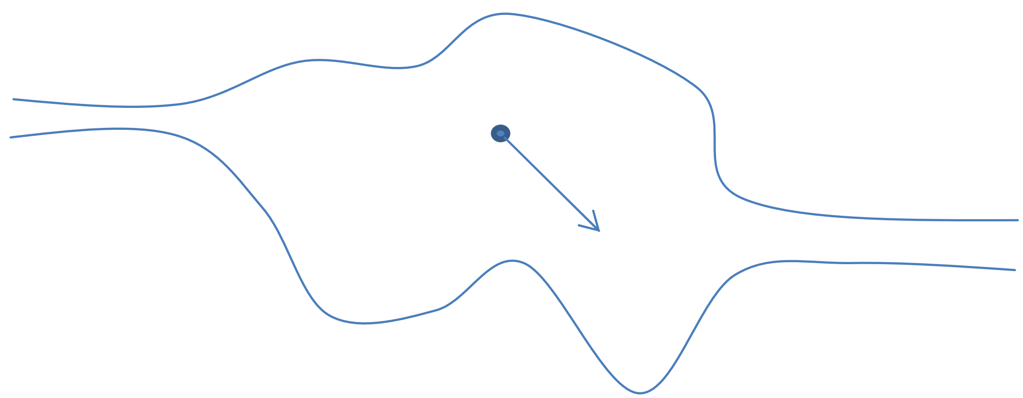
Survival Probability



$P(t)$ *The probability to find the particle in the system at time* t.

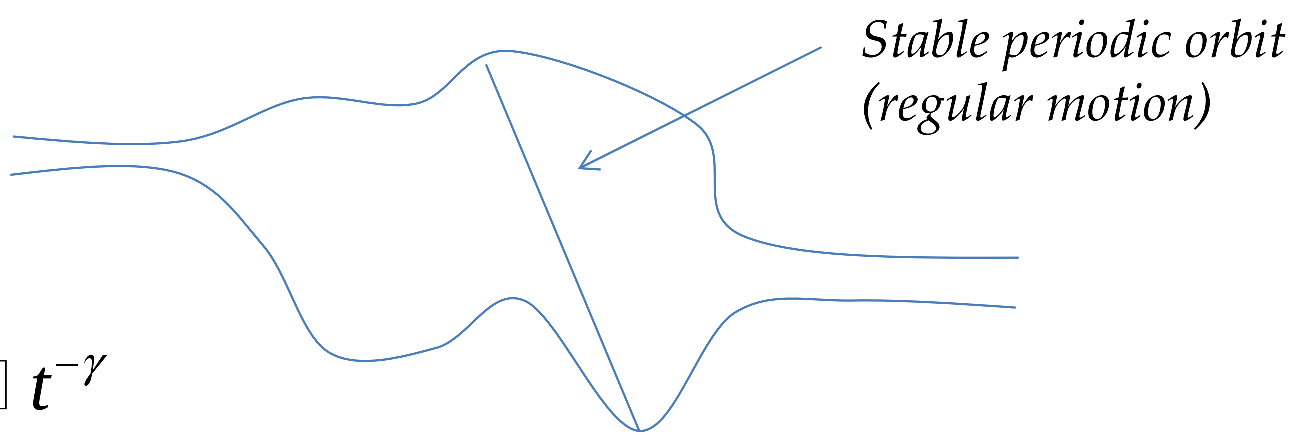
E.g. important for the characterization of transport properties (in the semiclassical limit) such as

Weak localization
Shot noise

In fully chaotic systems $P(t) \sim e^{-\frac{t}{\tau_c}}$.

(positive Lyapunov exponent- all trajectories are unstable)

However, generic systems are "mixed"



$P(t) \sim t^{-\gamma}$

"sticking"

The decay exponent : $\gamma \sim 1-3$

S.R. Channon and J.L. Lebowitz (1980)
B.V. Chirikov and D.L. Shepeliansky (1981)
B.V. Chirikov and D.L.Shepeliansky (1985)
P. Grassberger and H. Kantz (1985)
J.D. Meiss and E. Ott (1985).
Y.C. Lai, M. Ding, C. Grebogi and R.Blumel (1992)
T. Geisel A. Zacherl and G.Radons (1987)
R. Fleischmann, T.Geisel and R. Ketzmerick (1992)
G.M.Zaslavsky,M. Edelman and B.A.Niyazov, (1997).
M. Weiss, L. Hufnagel, and R. Ketzmerick, (2002)
G. Cristadoro and R. Ketzmeric (2008)
R. Venegeroles (2009)
A.V. Avetisov and S.K. Nechaev (2010).

Is it Universal?

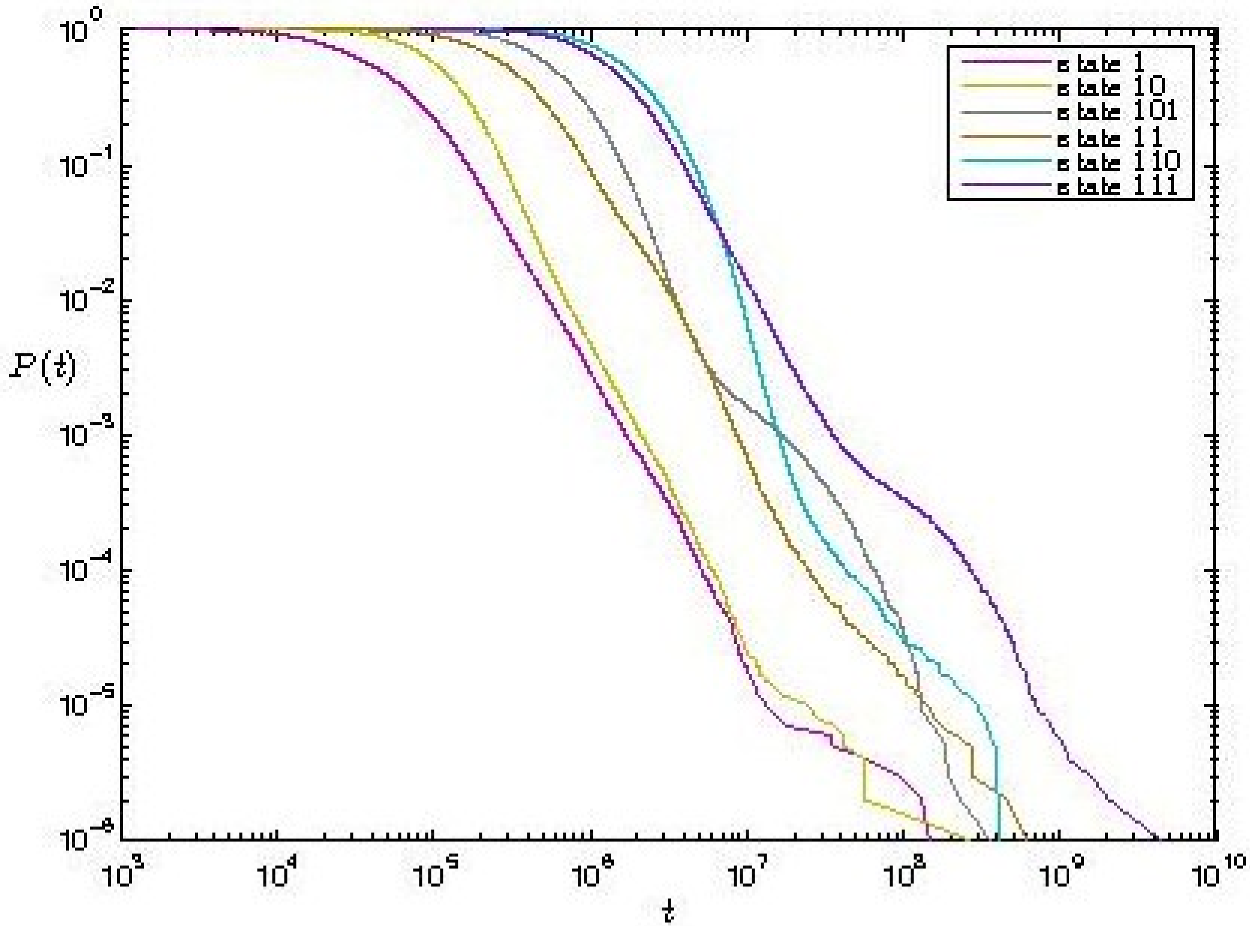
Theories involve uncontrolled approximations.

Numerical calculations are extremely difficult.

A refuge:
Numerical computations never get to the long time asymptotic limit.

We claim that even with ideal numerical accuracy it is impossible to extract a unique value for the decay exponent due to time fluctuations in the survival probability.

Example: fluctuations in the survival probability



A numerical study of the fluctuations suffers from the same difficulties mentioned above.

$$\Downarrow$$

Study the fluctuations by averaging over an ***ensemble*** *of mixed chaotic systems.*

$$\langle \cdots \rangle$$

Manifestations of fluctuations:

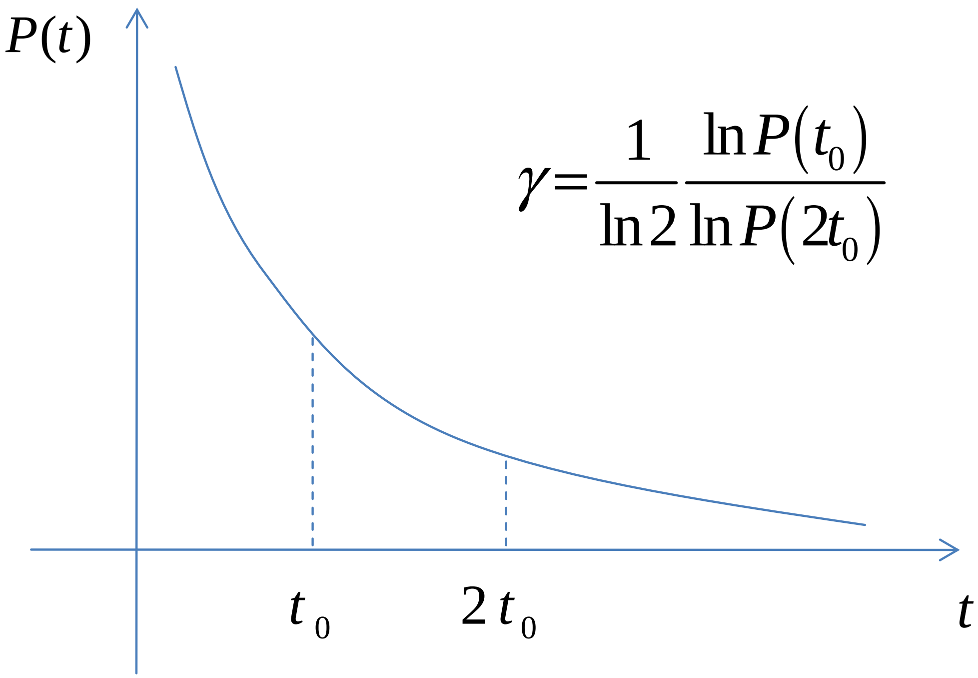
$$P(t) \sim t^{-\langle \gamma \rangle}\left(1+\delta p(t)\right)$$

Relative fluctuations (perturbative)

$$\left\langle \delta p\left(t+\frac{\Delta t}{2}\right)\delta p\left(t-\frac{\Delta t}{2}\right)\right\rangle \sim \left\langle \delta p^2(t)\right\rangle\left(1-\left(\frac{\Delta t}{2t}\right)^2\right)$$

Correlation decay over a time scale of order of the measurement time.

Manifestations of fluctuations (cont.):



$$\gamma = \frac{1}{\ln 2}\frac{\ln P(t_0)}{\ln P(2t_0)} \qquad t_0 \to \infty$$

$$\langle \delta\gamma^2 \rangle \approx \frac{1}{\ln^2 2}\left\langle \left(\delta p(t_0) - \delta p(2t_0)\right)^2 \right\rangle$$

$$\delta\gamma(t) \approx 0.68\sqrt{\langle \delta p^2(t) \rangle}$$

essentially a time independent constant of order unity.

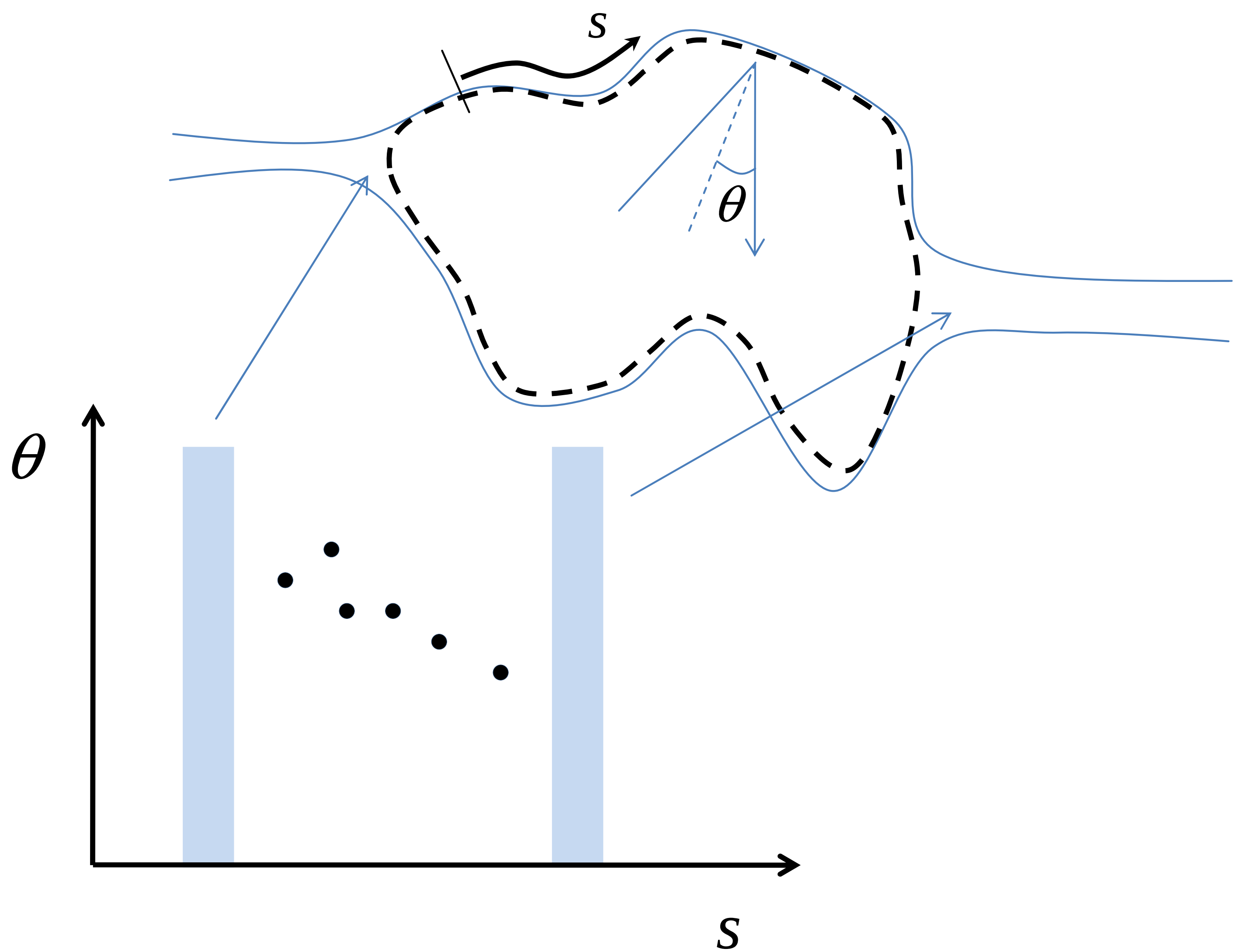
Our aim is to show that

1. $$\frac{\left\langle \delta p\left(t+\frac{\Delta t}{2}\right)\delta p\left(t-\frac{\Delta t}{2}\right)\right\rangle}{\left\langle \delta p^{2}(t)\right\rangle} \approx 1-\left(\frac{\Delta t}{2t}\right)^{2}$$

2. $$\left\langle \delta p^{2}(t)\right\rangle \approx const.$$

Ensemble of mixed chaotic systems?

Simplifying the description: Poincare sections

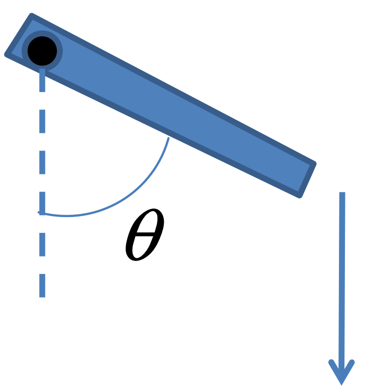


Simplifying the description: Maps

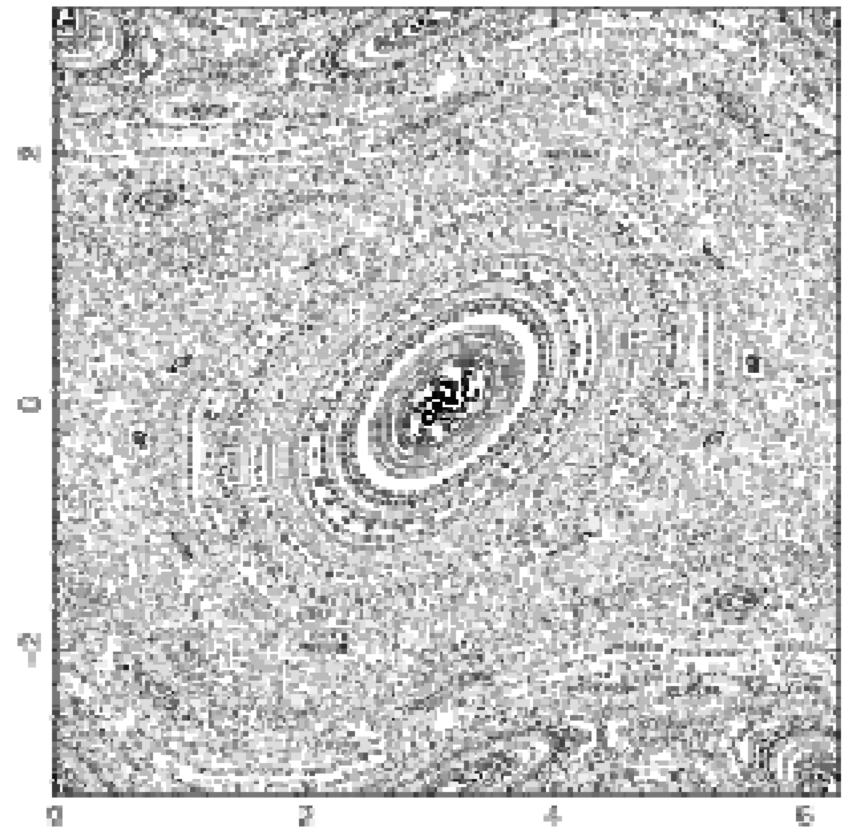
$$I_{n+1} = f\left(I_n, \theta_n\right)$$
$$\theta_{n+1} = g\left(I_n, \theta_n\right)$$

Example - the standard map:

$$I_{n+1} = I_n + K \sin \theta_n$$
$$\theta_{n+1} = \theta_n + I_{n+1} \left(\mathrm{mod}\, 2\pi\right)$$



$$g \sum_n \delta(t-n)$$

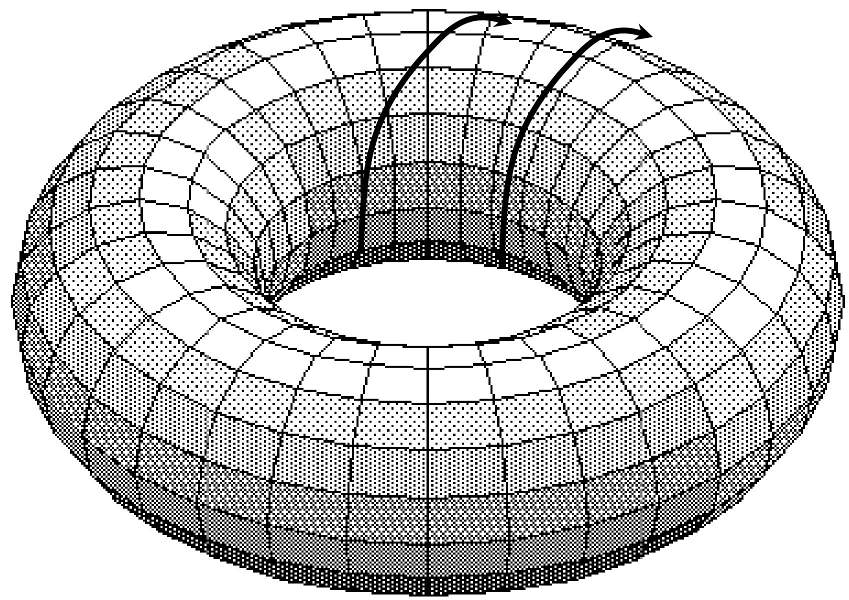


A comment on KAM theory & universality

$$H = H_0(I) + \varepsilon H_1(I, \theta)$$

$$p\omega_1 = q\omega_2$$

$$\frac{\omega_1}{\omega_2} = \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \ldots}}} \approx \frac{p}{q}$$

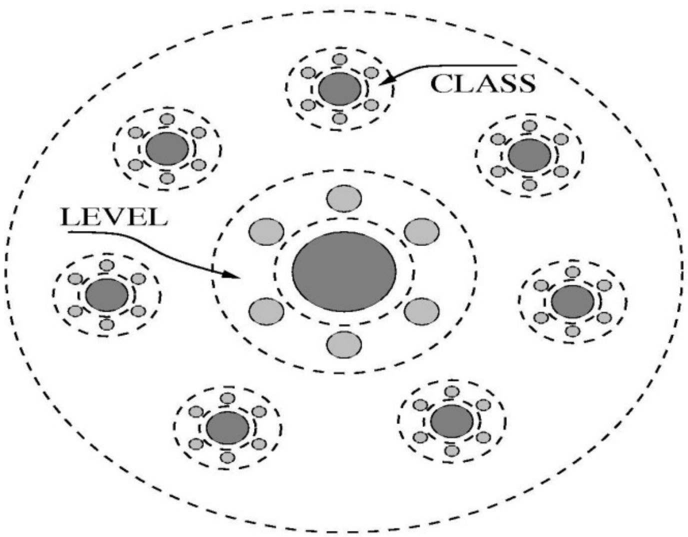


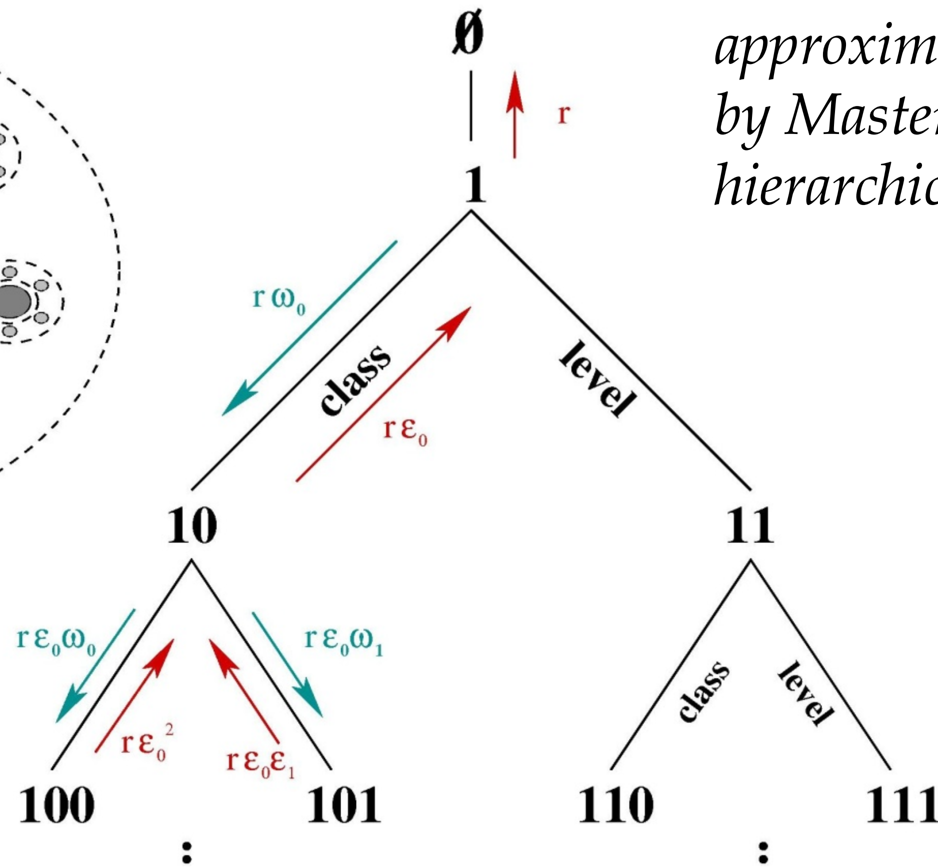
Preserved tori satisfy the condition

$$\left| \frac{\omega_1}{\omega_2} - \frac{p}{q} \right| > \frac{K(\varepsilon)}{q^{2.5}} \quad \textit{for all } q \textit{ and } p$$

Cantor set like structure of the phase space.

The Meiss-Ott model for mixed chaotic systems:

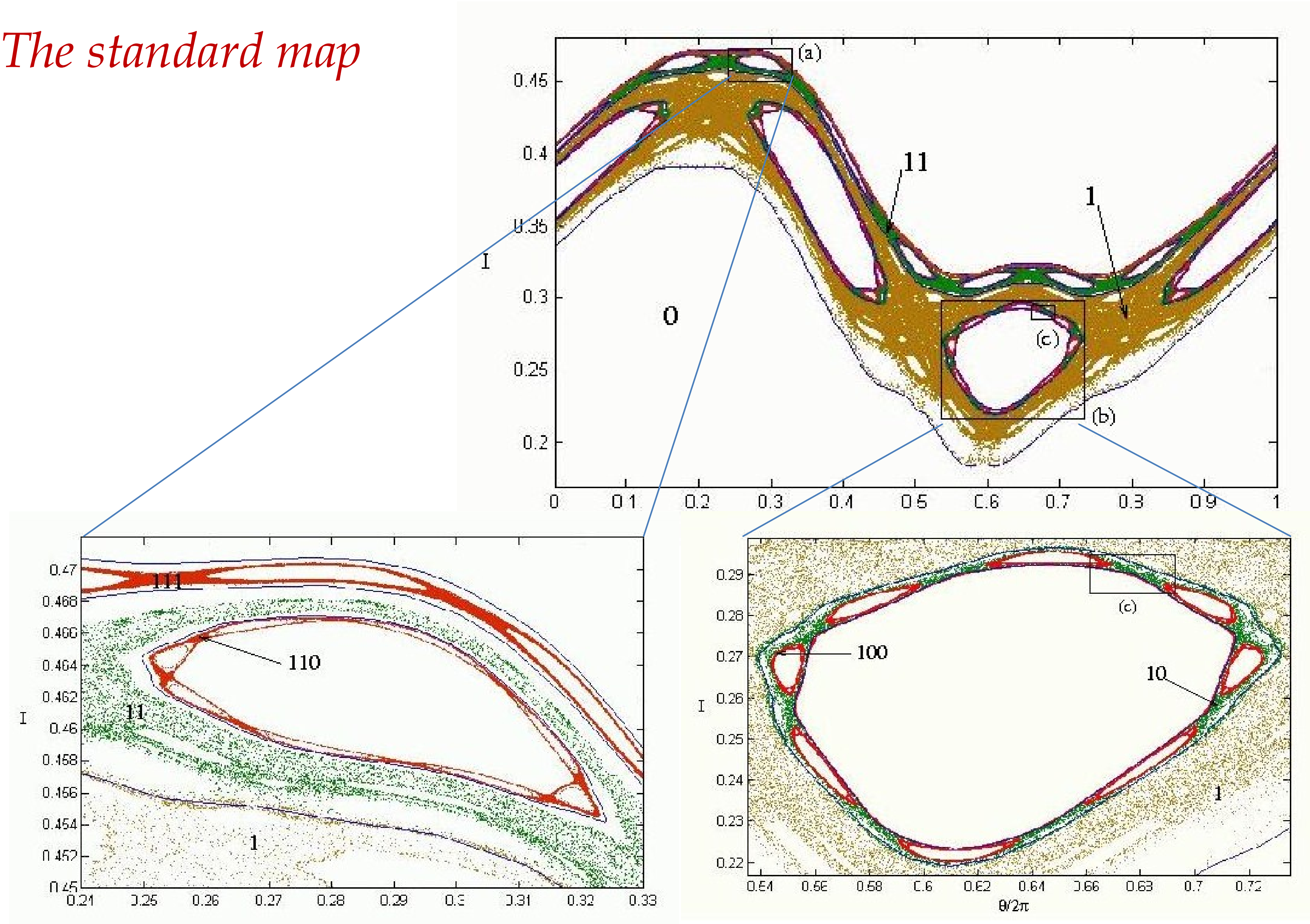




approximating the dynamics by Master's equations on hierarchical binary tree

The Meiss-Ott model: An example for the phase space division

The standard map



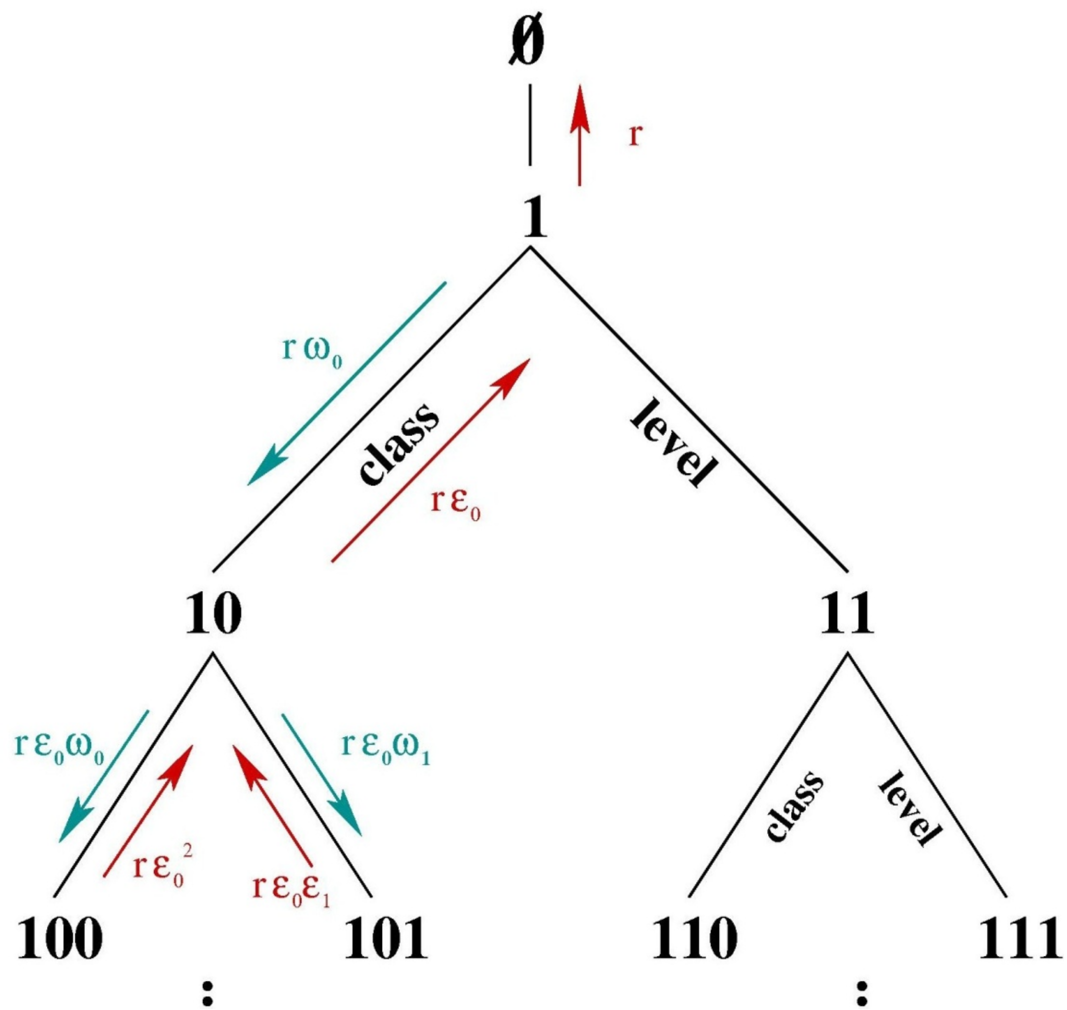
The Meiss-Ott model: Transition rates

$$\frac{W_{n0\to n}}{W_{n00\to n0}} = \varepsilon_0 \square\ 1$$

$$\frac{W_{n\to n0}}{W_{n0\to n}} = \frac{\omega_0}{\varepsilon_0} \square\ 1$$

$$\frac{W_{n1\to n}}{W_{n11\to n1}} = \varepsilon_1 \square\ 1$$

$$\frac{W_{n\to n1}}{W_{n1\to n}} = \frac{\omega_1}{\varepsilon_1} \square\ 1$$



$$\varepsilon_0 \square\ 0.143 \quad \varepsilon_1 \square\ 0.382 \quad \omega_0 \square\ 0.0142 \quad \omega_1 \square\ 0.0532$$

The Meiss-Ott solution:

$$\omega_0 \varepsilon_0^{-\gamma} + \omega_1 \varepsilon_1^{-\gamma} = 1$$

⇩

$$\gamma = 1.96$$

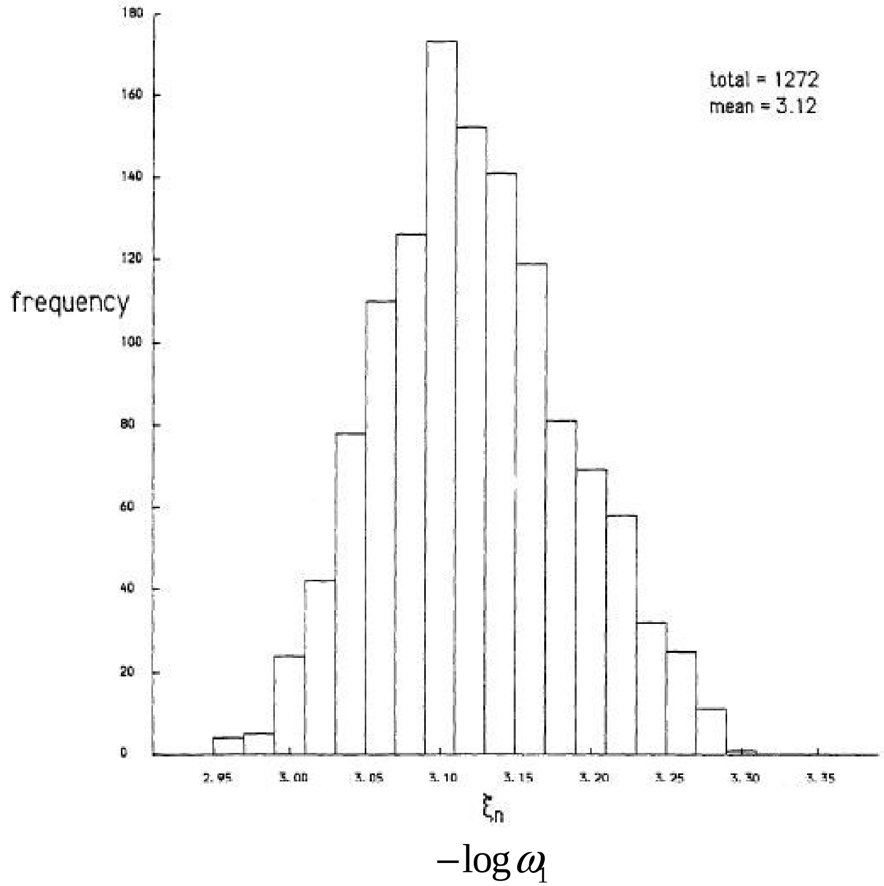
Chirikov and Shepeliansky ("level " scaling only) $\gamma = 3$

Zaslavsky, Edelman and Niyazov, ("class" scaling only) $\gamma = 2.25$

Fluctuations in the transition rates:

Greene, Mackay and Stark (1986)

(Level scaling only)



Ensemble of mixed chaotic systems

Assuming that fluctuations behave statistically in a self similarity manner (i.e. an invariant measure that respect the hierarchical structure of the phase space) :

$$W_{n\to m} \to W_{n\to m}\left(1+\xi_{nm}\right)$$

$$\left\langle \xi_{nm} \right\rangle = 0 \qquad \left\langle \xi_{nm}\xi_{n'm'} \right\rangle = \sigma^2\left(\delta_{nn'}\delta_{mm'} + \delta_{nm'}\delta_{mn'}\right)$$

"The random tree model"

Results of the model:

$$\frac{d\vec{P}}{dt} = \left(M_0 + M_{rand}\right)\vec{P}$$

Locator expansion

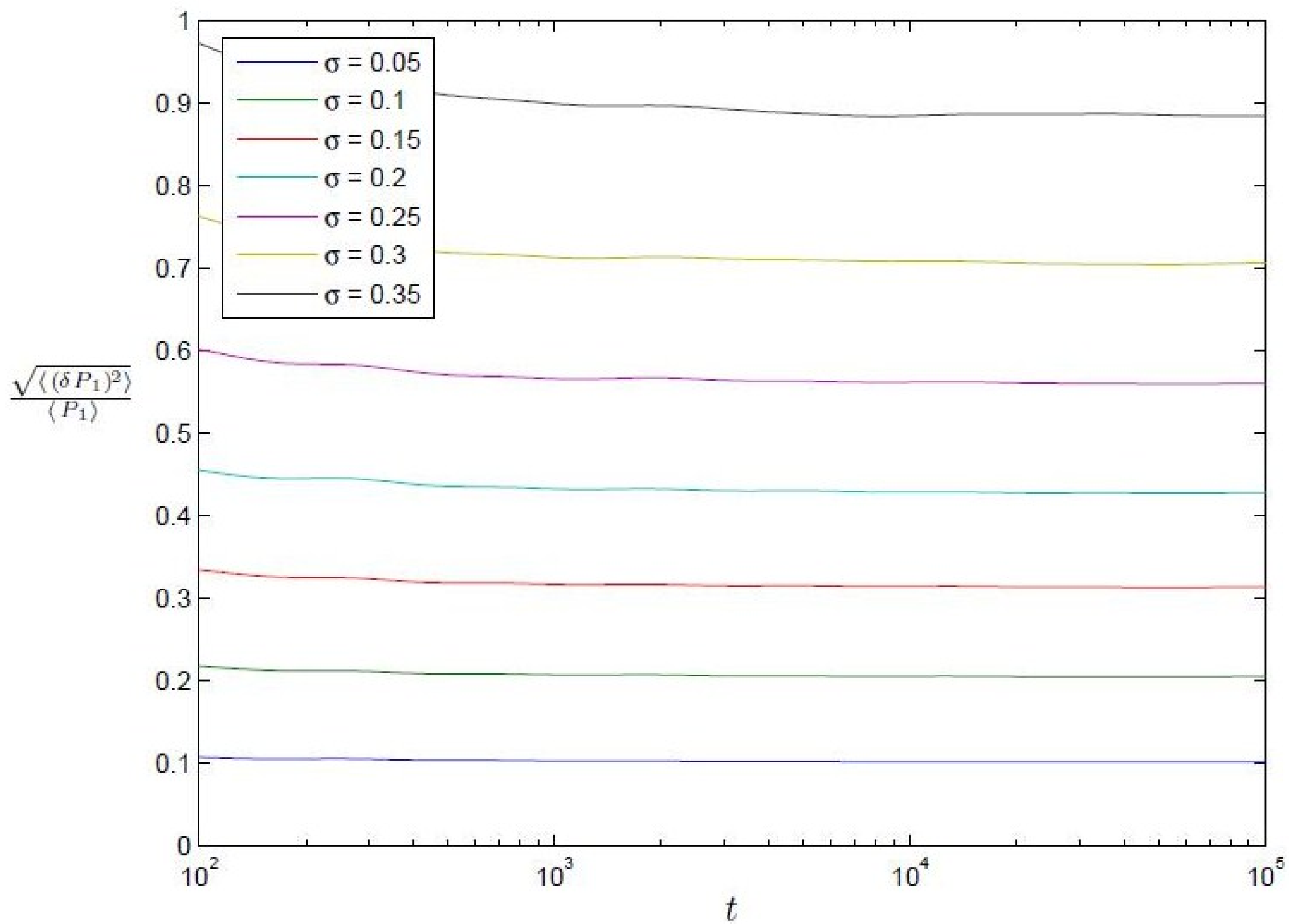
$$\omega_0 \varepsilon_0^{-\langle\gamma\rangle} + \omega_1 \varepsilon_1^{-\langle\gamma\rangle} = 1$$

$$\frac{\left\langle \delta p\left(t + \frac{\Delta t}{2}\right) \delta p\left(t - \frac{\Delta t}{2}\right) \right\rangle}{\left\langle \delta p^2(t) \right\rangle} \sim 1 - \left(\frac{\Delta t}{2t}\right)^2$$

$$\sqrt{\left\langle \delta p^2(t) \right\rangle} \sim \sigma \qquad \sqrt{\left\langle \delta p^2(t) \right\rangle} \sim \sigma t^{-0.174}$$

$$\left(\omega_0 \varepsilon_0^{-2\mu-\langle\gamma\rangle}\right)^2 + \left(\omega_1 \varepsilon_1^{-2\mu-\langle\gamma\rangle}\right)^2 = 1$$

Numerical results for the random tree model



$$\delta\gamma(t) \sim \sqrt{\left\langle \delta p^2(t) \right\rangle}$$

$$\delta\gamma(t) \sim \sigma$$

essentially independent of time

Numerical study:

The standard map with a small random (Gaussian) component:

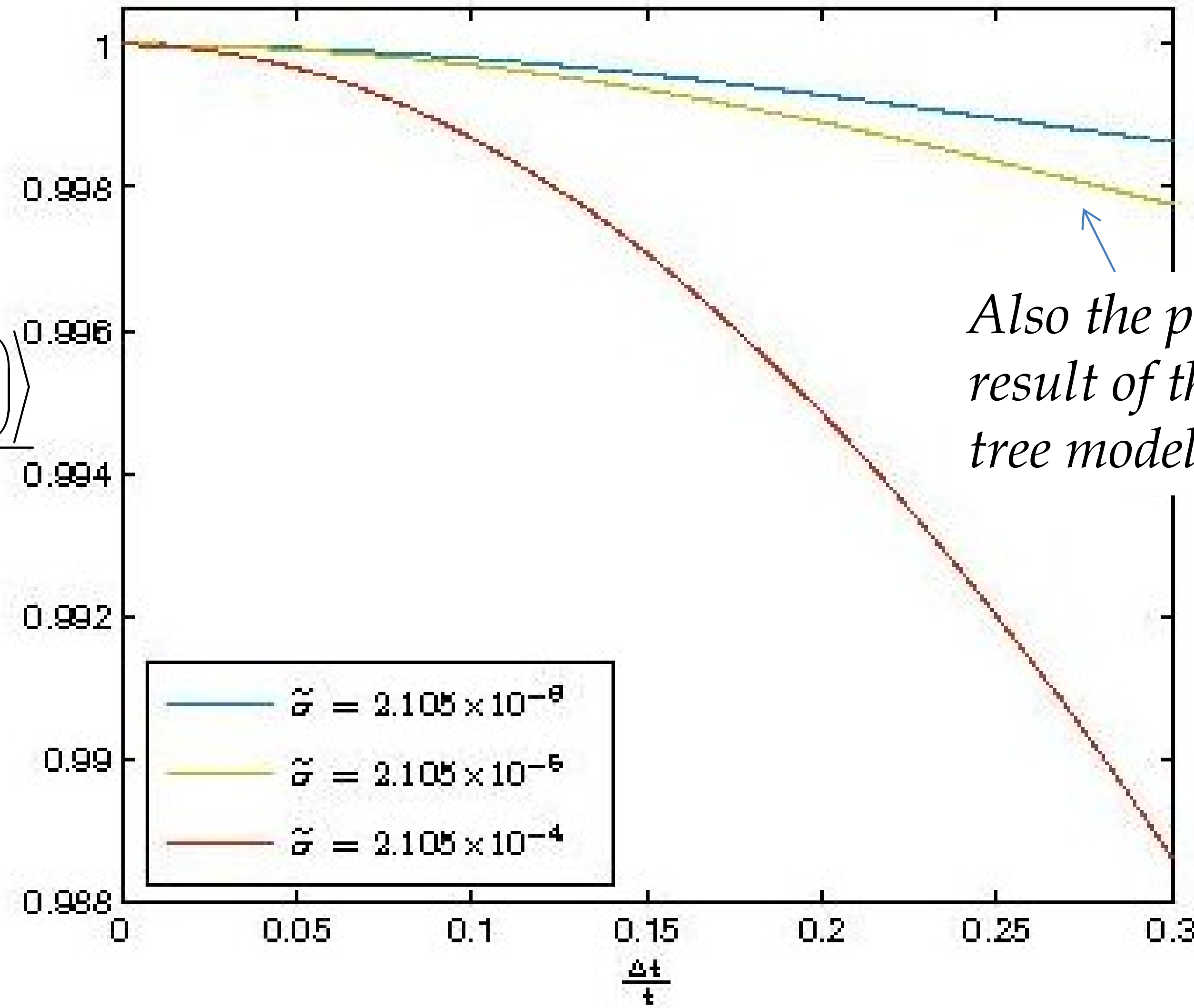
$$I_{n+1} = I_n + K \sin\theta_n + R(\theta_n)$$
$$\theta_{n+1} = \theta_n + I_{n+1}$$

$$\langle R(\theta) \rangle = 0$$

$$\langle R(\theta) R(\theta') \rangle = \tilde{\sigma}^2 \sum_m e^{-\left(\frac{\theta - \theta' - 2\pi m}{l}\right)^2}$$

Fluctuations correlations:

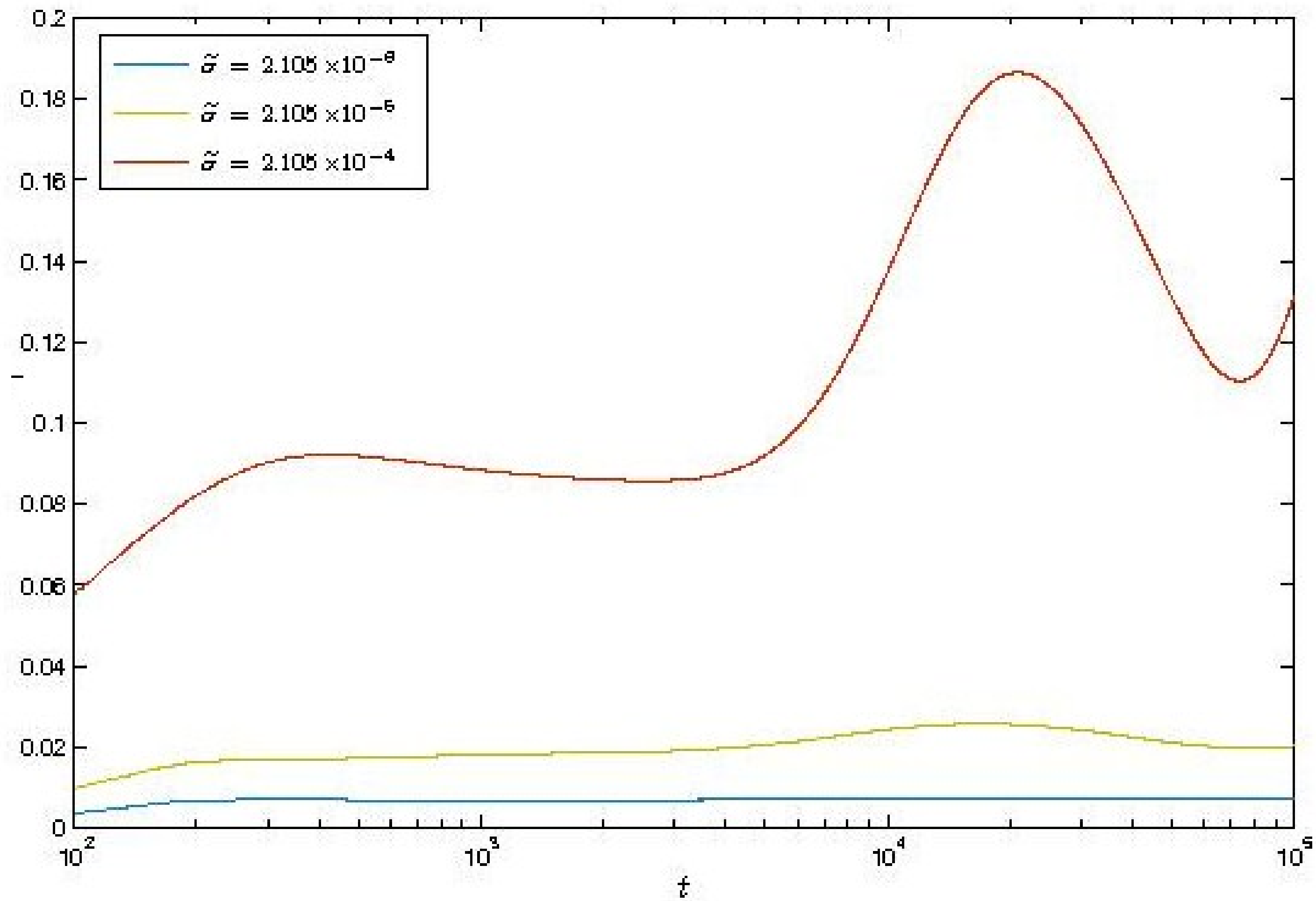
$$\frac{\left\langle \delta p\left(t+\frac{\Delta t}{2}\right)\delta p\left(t-\frac{\Delta t}{2}\right)\right\rangle}{\left\langle \delta p^2(t)\right\rangle}$$



Also the perturbative result of the random tree model: $1-\left(\frac{\Delta t}{2t}\right)^2$

Fluctuations magnitude

$$\sqrt{\left\langle \delta p^2(t) \right\rangle}$$



Summery & reservations

The model is Markovian, but this is only an approximation.

Correlations decay on the scale of the measurement time

Relative fluctuations essentially do not decay

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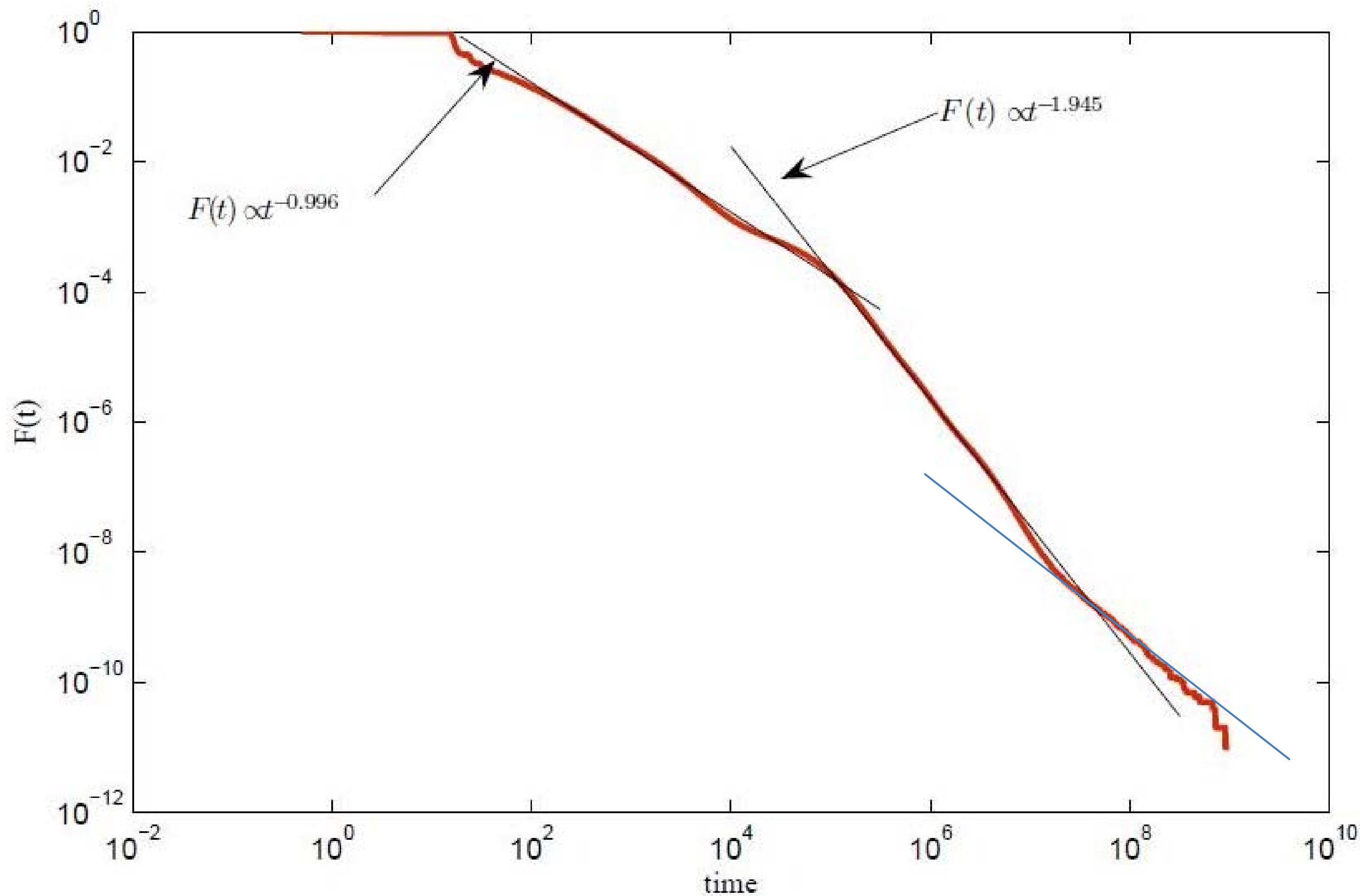
A unique value of γ may exists only in ensemble average sense.

Example: standard map



"Happy families are all alike, every unhappy family is unhappy in its own way." Tolstoy