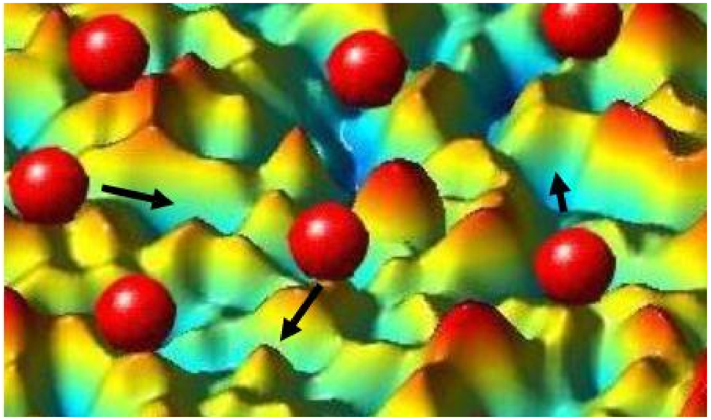
Localization, *anomalous diffusion* *and* *slow relaxations* in disordered systems

Ariel Amir , work with
Yuval Oreg and Yoseph Imry



מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Outline

- Electron glass model
    - slow relaxations and aging
    - statics (Coulomb gap)

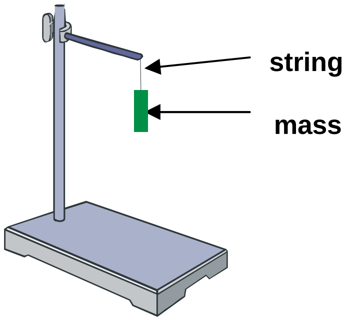
    Dynamics:
    - mapping to a new class of Random Matrix Theory (RMT)

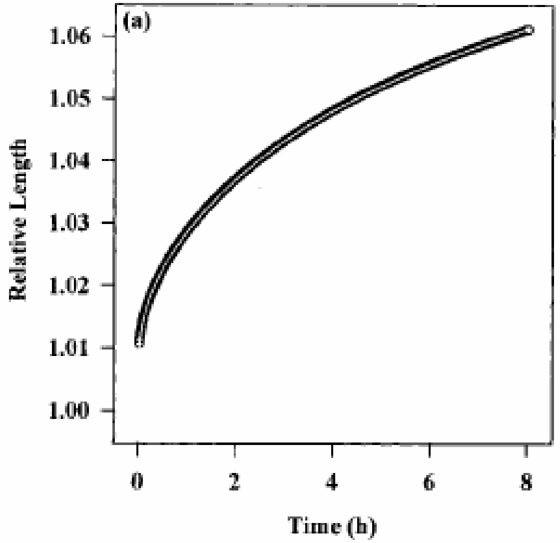
- Definition of Random Distance Matrices
- Solution of model: *eigenvalue distribution* through moment calculation
- Solution of model : RG approach → *localization properties*
- Implications: anomalous diffusion, slow relaxations, localized phonons
- Conclusions

Slow relaxations in nature

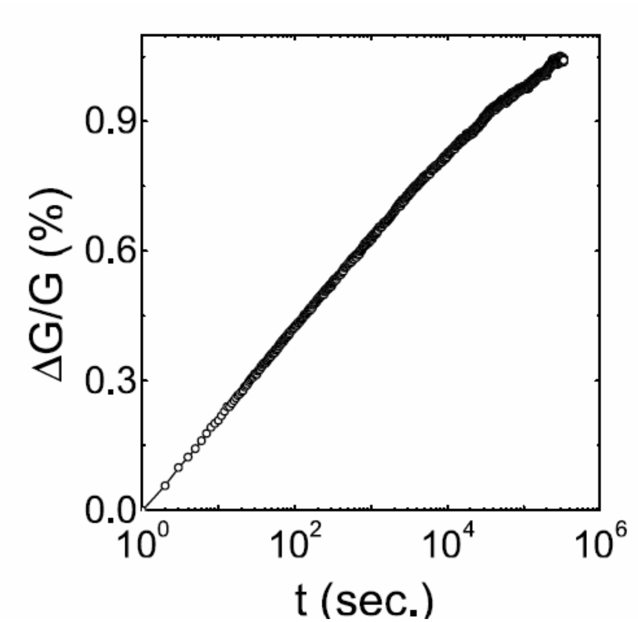
W. Weber, *Ann. Phys.* (1835)

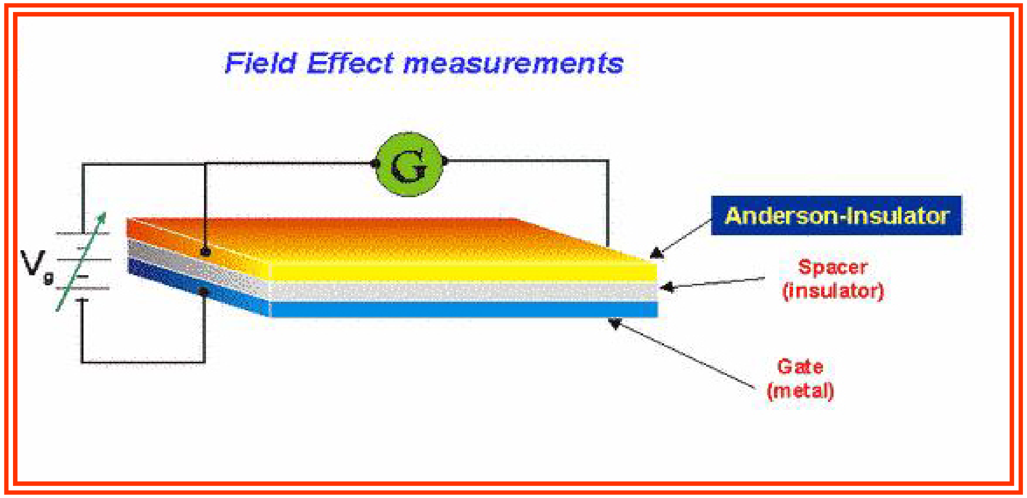
D. S. Thompson, *J. Exp. Bot.* (2001)





Electron glass- Experimental system





Logarithmic relaxations for 5 days!

What are the ingredients leading to slow relaxations?

Electron glass aging– experimental protocol

A. Vaknin and Z. Ovadyahu and M. Pollak, PRL 2000

Step I

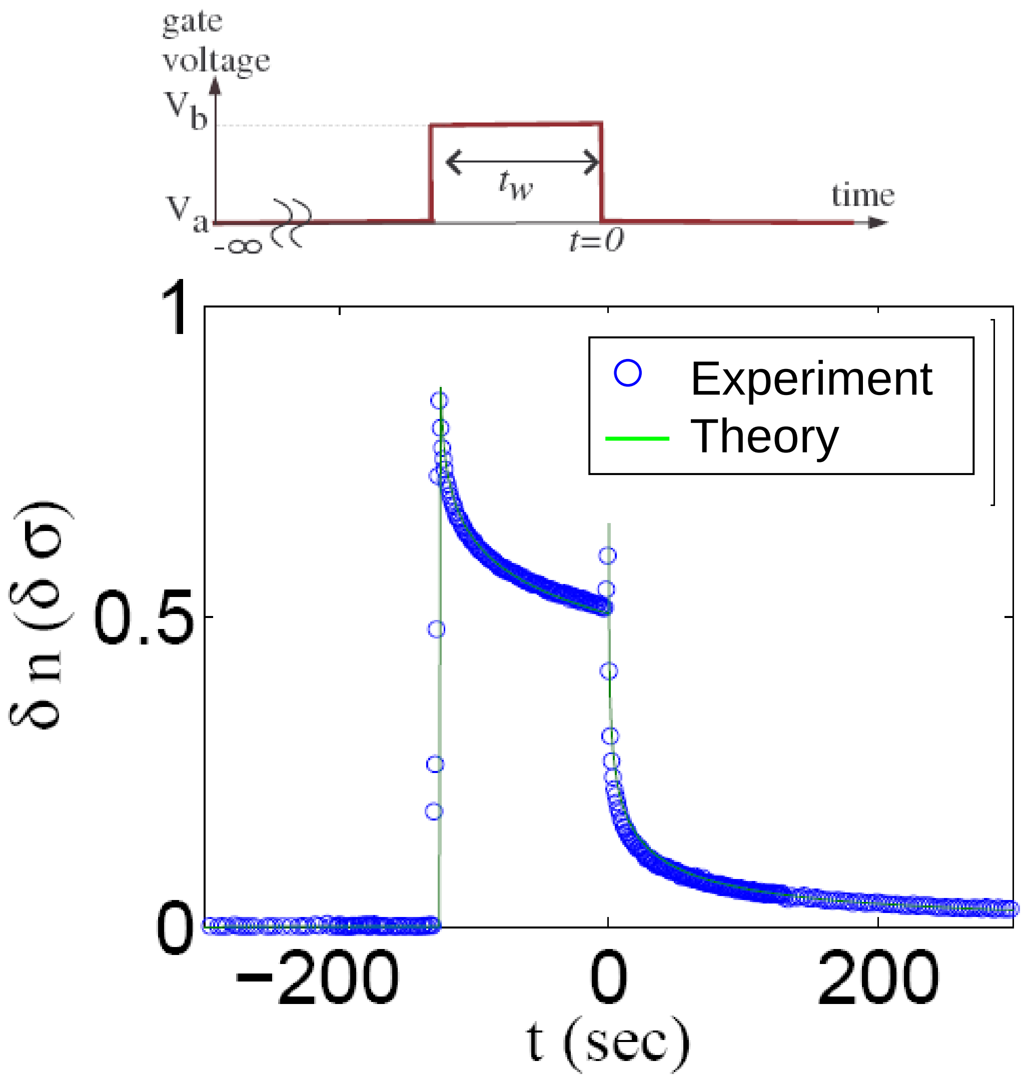
System equilibrates for long time

Step II

V_g is changed, for a time of t_w.

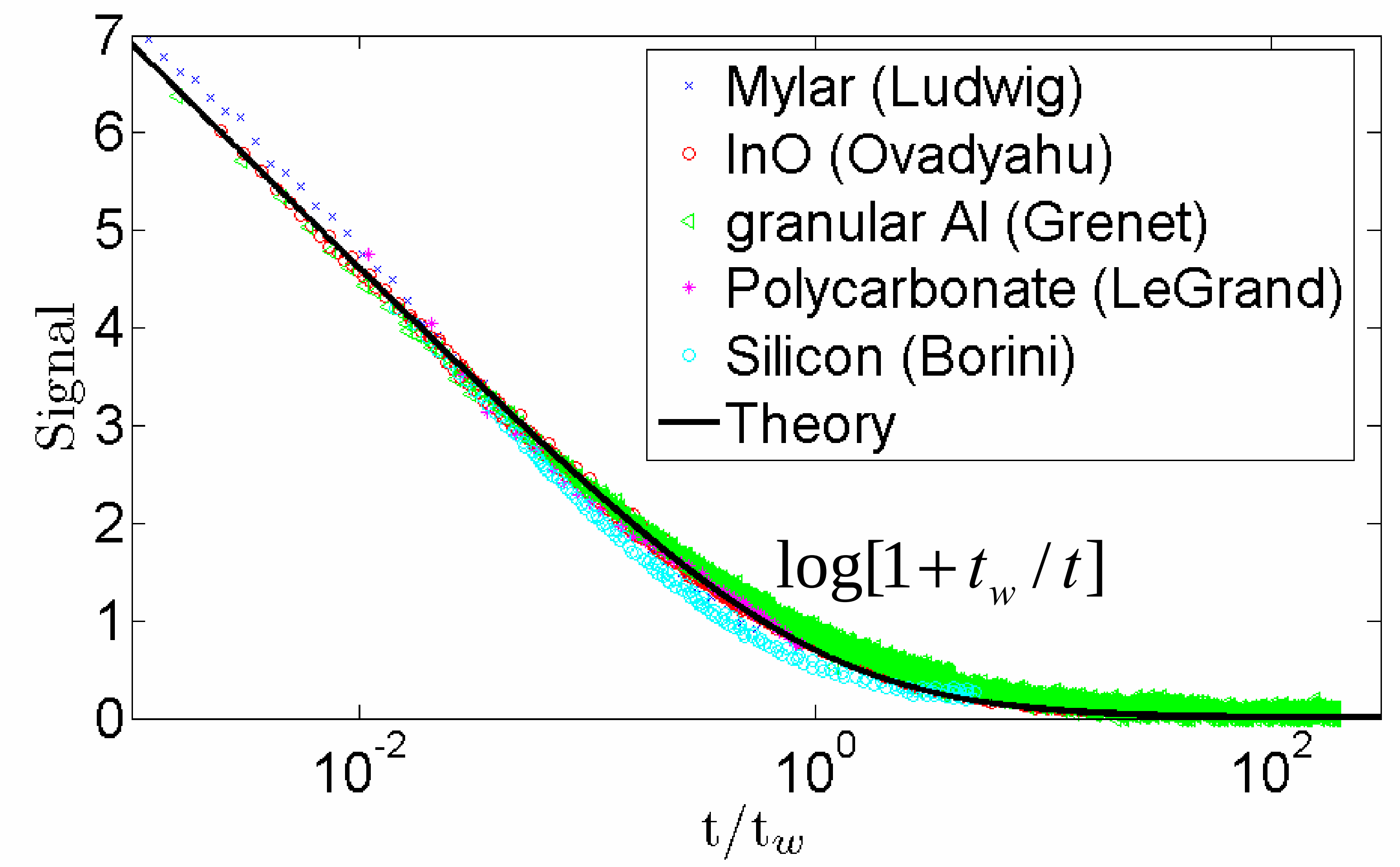
Throughout the experiment

Conductance is measured as a function of time.



Data: *Ovadyahu et al.*

Aging and universality



Amir, Oreg and Imry, to be published

The model

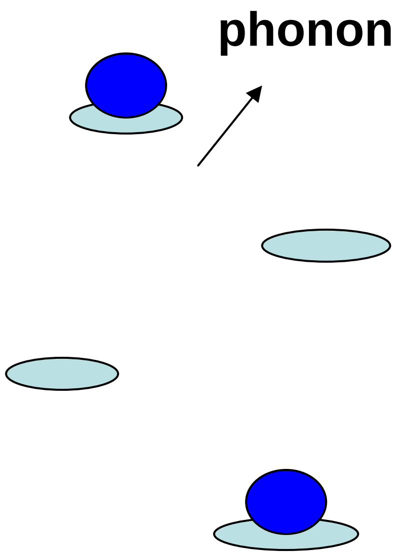
• Strong localization due to disorder

→ randomly positioned sites, on-site disorder.

• Coulomb interactions are included

• "Phonons" induce transitions between configurations.

• Interference (quantum) effects neglected.



e.g:

Pollak (1970)

Shklovskii and Efros (1975)

Ovadyahu and Pollak (2003)

Muller and Ioffe (2004)

"Local mean-field" approximation - Dynamics

AA, Oreg and Imry, PRB (2008)

$$n_i \rightarrow \langle n_i \rangle, \qquad \frac{dn_i}{dt} = \sum_j -\gamma_{i,j} + \gamma_{j,i}$$

$$\gamma_{i,j} = \exp(-2r_{ij}/\xi) n_i (1-n_j)[N(|\Delta E|) + \theta(\Delta E)]$$

- ΔE includes the interactions
- N is the Bose-Einstein distribution
- ξ - the localization length

$$\left(E_i = \varepsilon_i + \sum_j \frac{n_j}{r_{ij}} \right)$$

At long times:

- The system reaches a locally stable point (metastable state).
- Many metastable states

("Pseudo-ground-states", *Baranovski et al., J. Phys. C, 1979*)

“Local mean-field” approximation - Equilibrium

- Detailed balance leads to Fermi-Dirac statistics ($n_j \rightarrow f_j$).
- Self-consistent set of equations for the energies:

$$E_i = \varepsilon_i - \sum_j \frac{1}{2} \frac{e^2}{r_{ij}} \tanh(\frac{E_j}{2T})$$

(assuming half filling).

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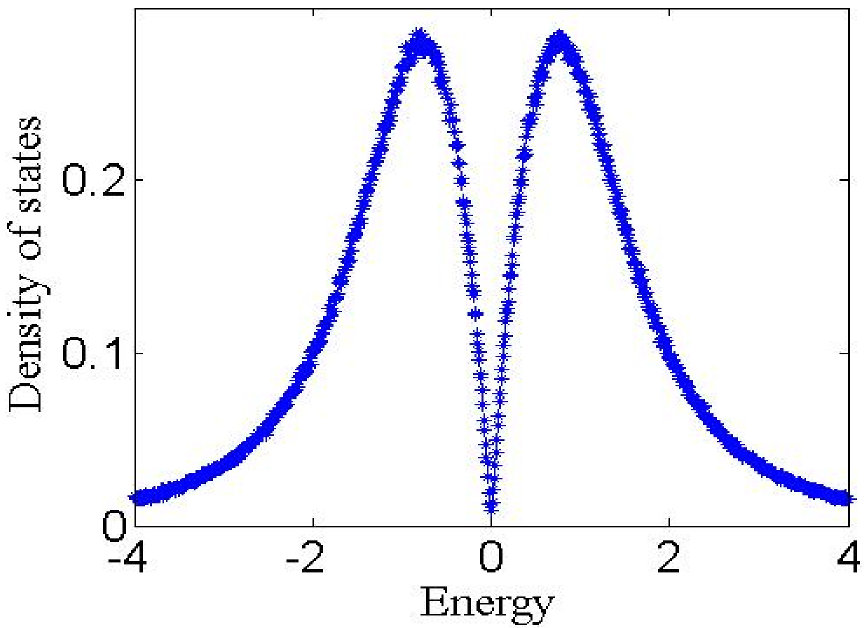

Yields linear **Coulomb gap**

(+ temperature dependence)

M. Pollak, Discuss. Faraday Soc (1970)

A.L. Efros and B.I. Shklovskii,

J. Phys. C (1975)

A. L. Efros, J. Phys. C (1976).

M. Grunewald et al., J. Phys. C. (1982)

A.A. Mogilyanskii and M.E. Raikh (1989)

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AA et al., PRB (2008)

Surer et al., PRL (2009)

Goethe et al., PRL (2009)

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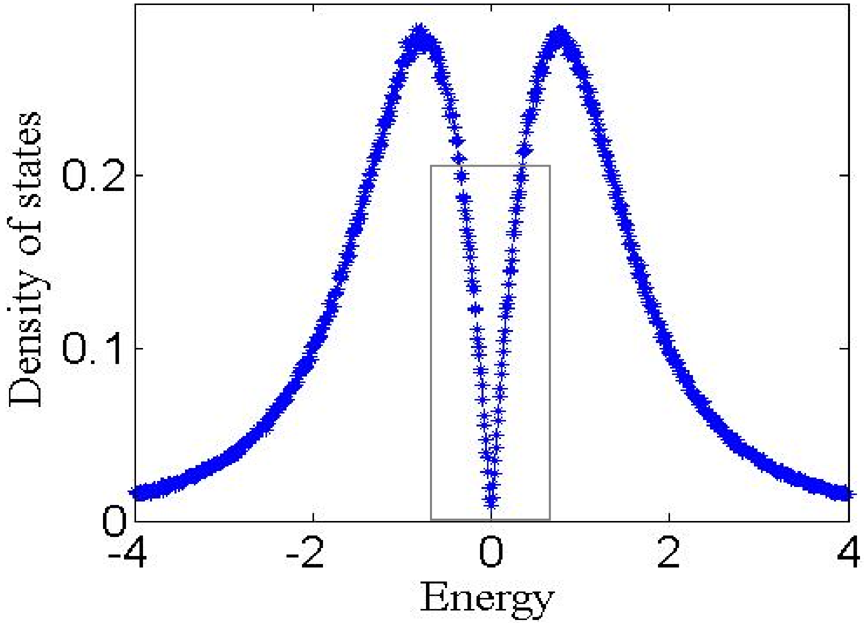

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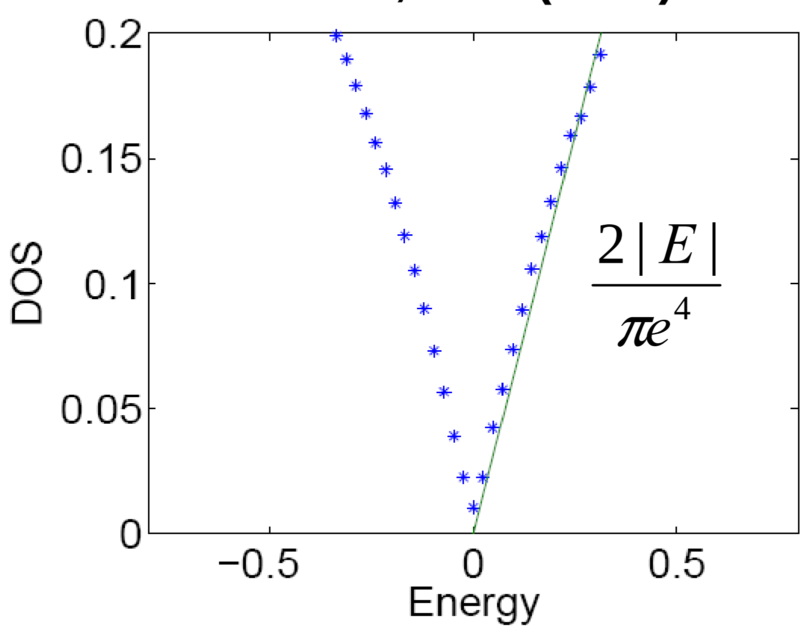

$$\frac{2|E|}{\pi e^4}$$

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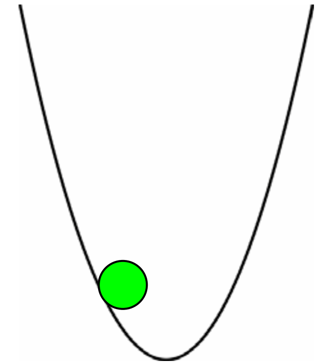
$$\gamma_{i,j} = \exp(-2r_{ij}/\xi) n_i (1-n_j)[N(|\Delta E|) + \theta(\Delta E)]$$

We saw: approach works well for statics

Moving on to dynamics…

Solution near locally stable point



Close enough to the equilibrium (locally) stable point,

one can linearize the equations, leading to the equation:

$$\frac{d\delta\vec{n}}{dt} = A \cdot \delta\vec{n}$$

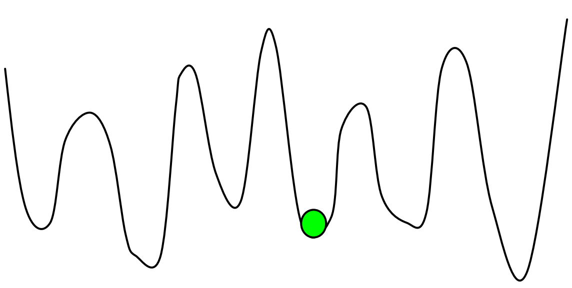
$$A_{i,j} = \frac{\gamma^0_{i,j}}{n^0_j(1-n^0_j)} - \frac{e^2}{T}\sum_{l \neq i,j} \gamma^0_{i,k}\left(\frac{1}{r_{i,j}} - \frac{1}{r_{i,k}}\right), \qquad (i \neq j)$$

$$\gamma^0_{i,j} \sim e^{-\frac{2r_{ij}}{\xi}}$$ (Anderson Localization)

Sum of columns vanishes (particle conservation number)

For low temperatures, near a local minimum,

second term is negligible →

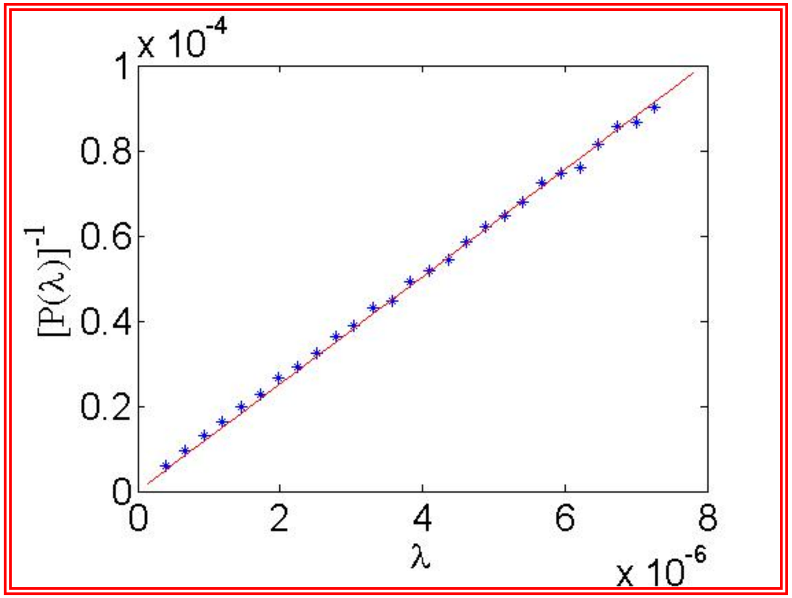


All eigenvalues are real and negative

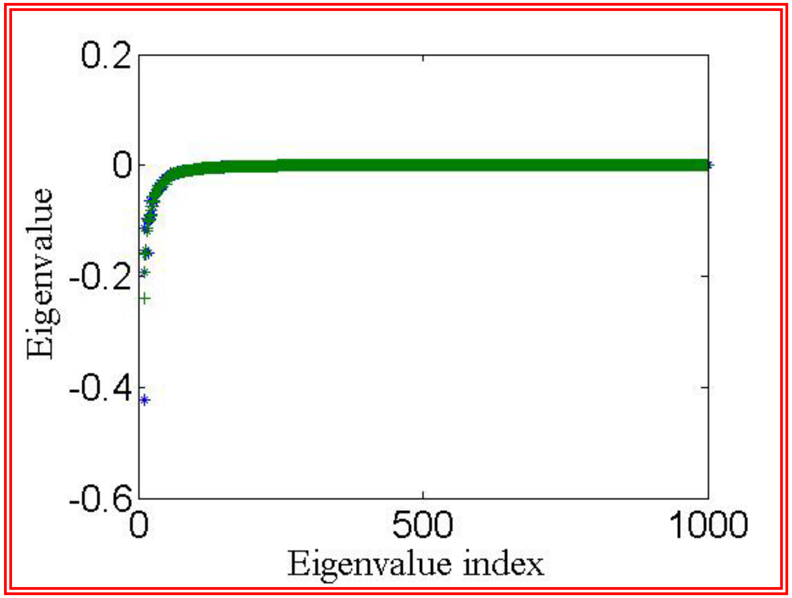
Eigenvalue Distribution

Solving numerically shows a distribution proportional to $1/\lambda$:

Eigenvalue distribution

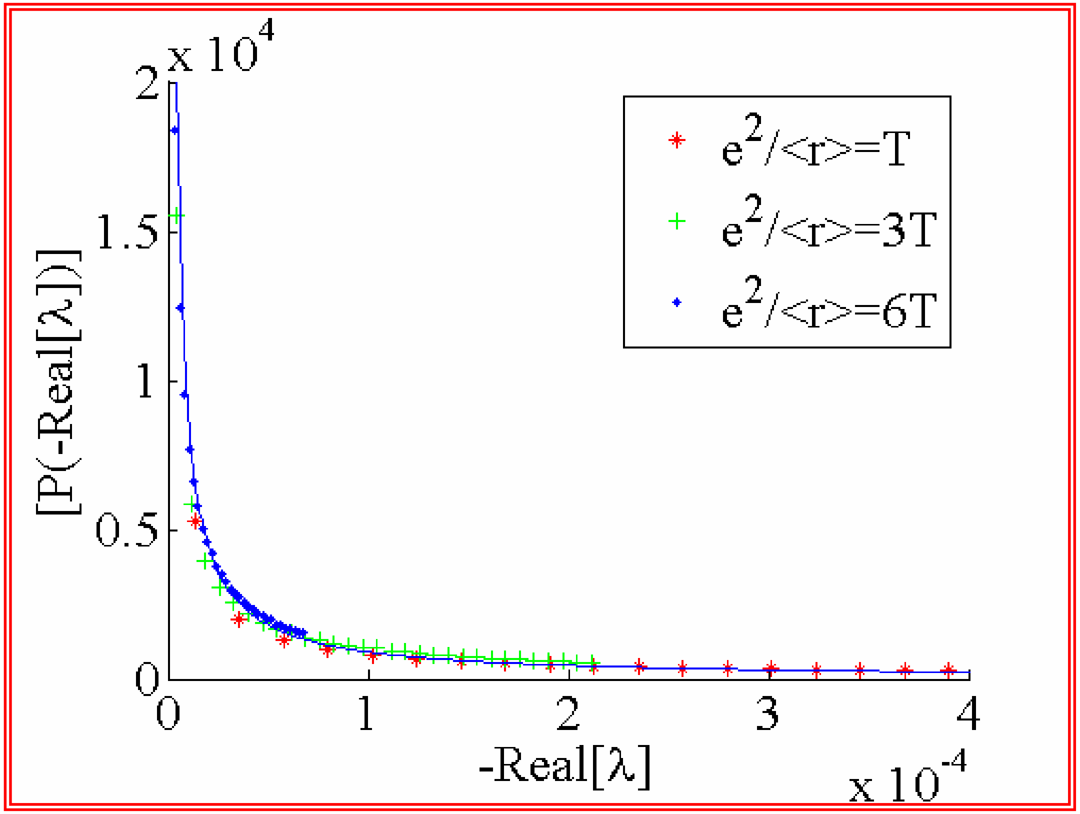


Eigenvalues



$$\sum_{\lambda} e^{-\lambda t} \longrightarrow \int P(\lambda) e^{-\lambda t} d\lambda \sim -\gamma_E - \log(\lambda_{\min} t)$$

Changing the interaction strength



- Slow relaxations occur also without interactions.
- Interactions push the distribution to (s)lower values.

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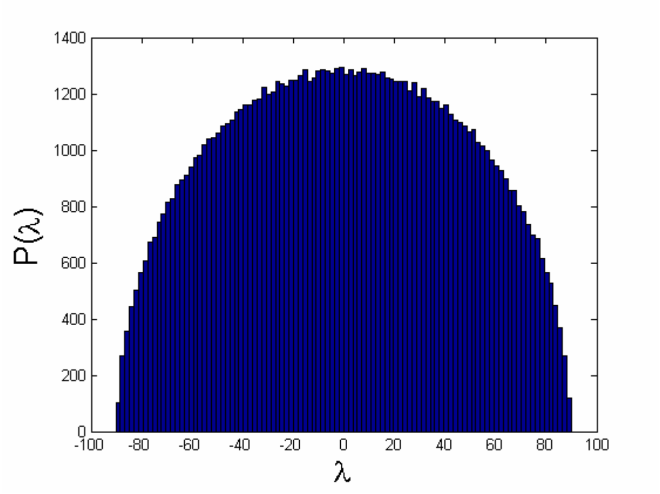
Digression: Random Matrix Theory in a nutshell

- Define a matrix with A_{ij} independent, Gaussian variables.
- Distribution of eigenvalues follows the “Wigner semicircle law”:



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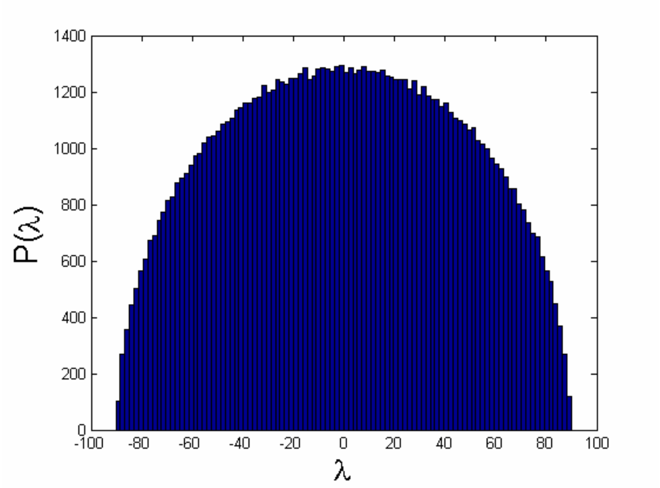

- Why is this interesting? Wide applications include:

Nuclear physics Wigner and Dirac (1951) Atomic physics Camarda and Georgopulos *PRL* (1983) Mesoscopics Imry, *EPL* (1986), Beenakker, *PRL* (1993) Chaotic systems Bohigas, Giannoni and Schmit, *PRL* (1984) …And many more.

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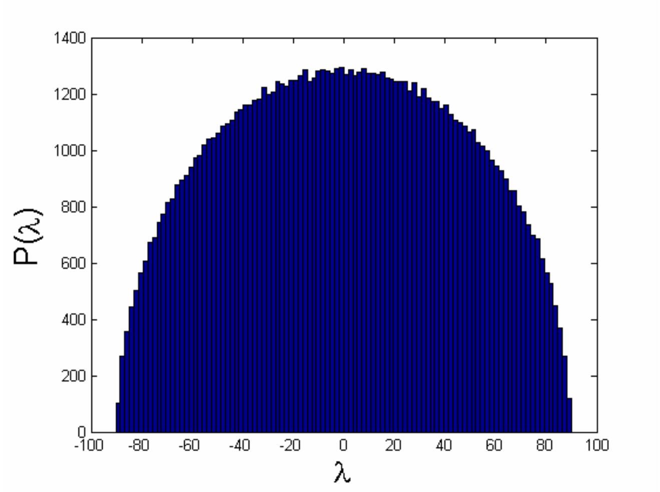

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In the following we present a different class of RMT, also broadly applicable

Solution near locally stable point

Close enough to the equilibrium (locally) stable point,

one can linearize the equations, leading to the equation:

$$\frac{d\delta\vec{n}}{dt} = A \cdot \delta\vec{n}$$

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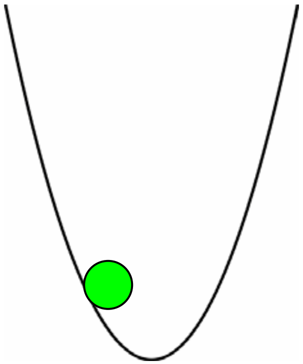
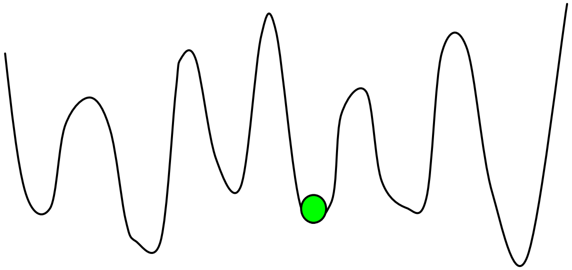
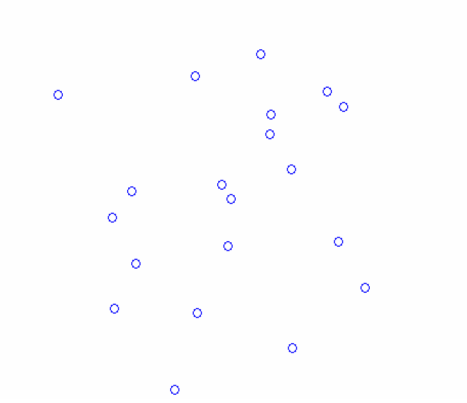




Digression: What are Random Distance Matrices?

1) Choose N points randomly and uniformly in a d-dimensional cube.



I. M. Lifshitz, *Adv. Phys* (1964).

Mezard, Parisi and Zee, *Nucl. Phys.* (1999)

Bogomolny, Bohigas, and Schmit, *J. Phys. A: Math. Gen.* (2003).

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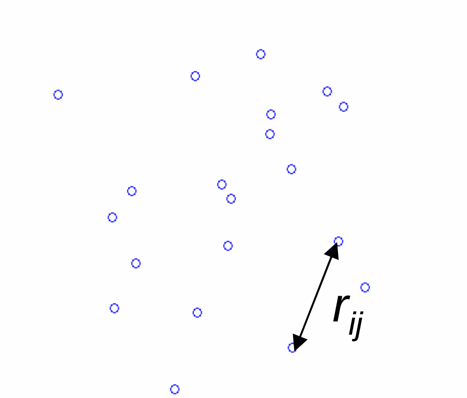
1) Choose N points randomly and uniformly in a d-dimensional cube.
2) Define the off-diagonals of our matrix as:

$$A_{i,j} = f(r_{ij}) \ , \ \ f(r) = e^{-r/\xi}$$

(Euclidean distance)

$$\varepsilon = \xi / \langle r \rangle$$



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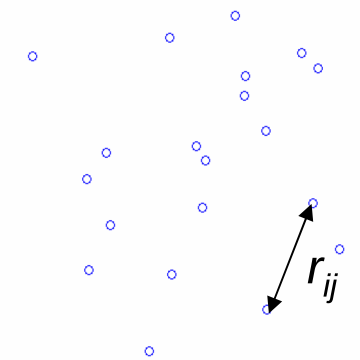

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sum of every column vanishes

(will come from a conservation law)

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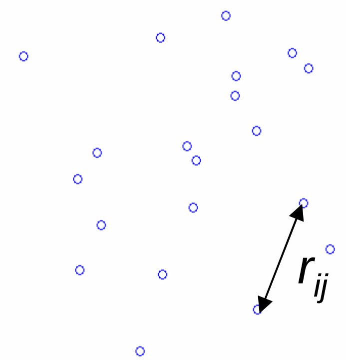

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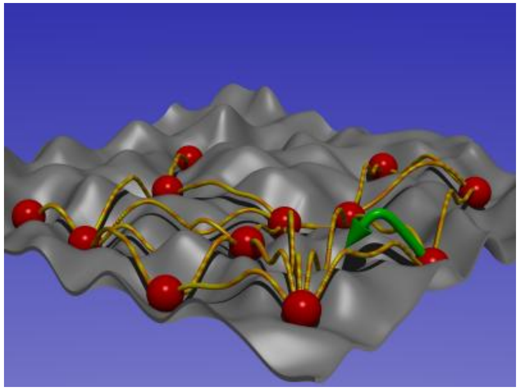
Q: What is the eigenvalue distribution?

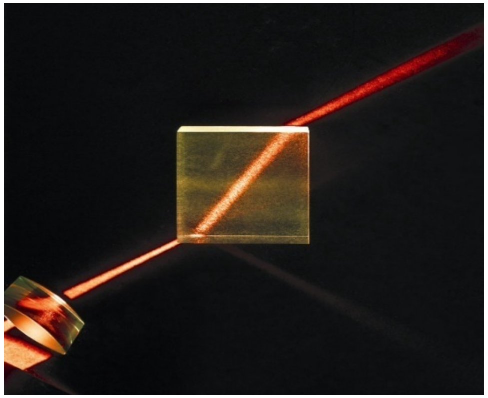
What are the eigenmodes?

Distance matrices – Motivation

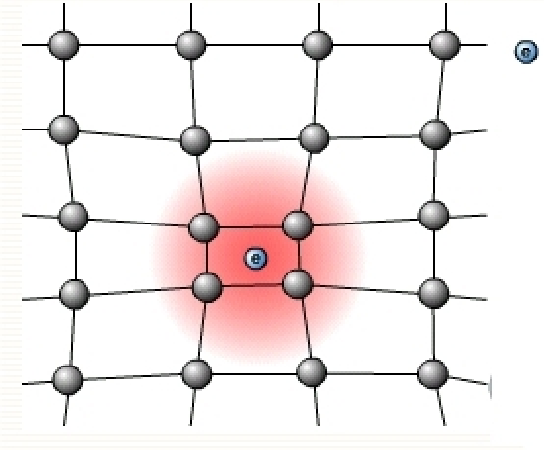
Relaxation in electron glasses
Amir, Oreg and Imry, PRB 2008



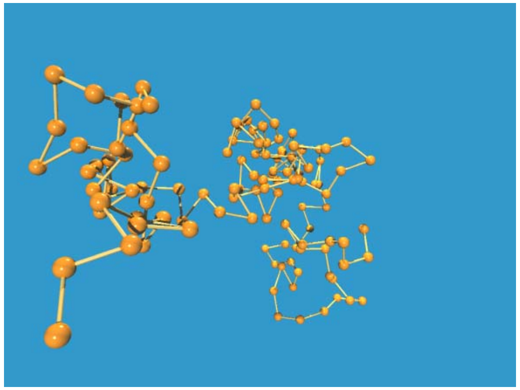
Photon propagation in a gas
Akkermans, Gero and Kaiser, PRL 2008



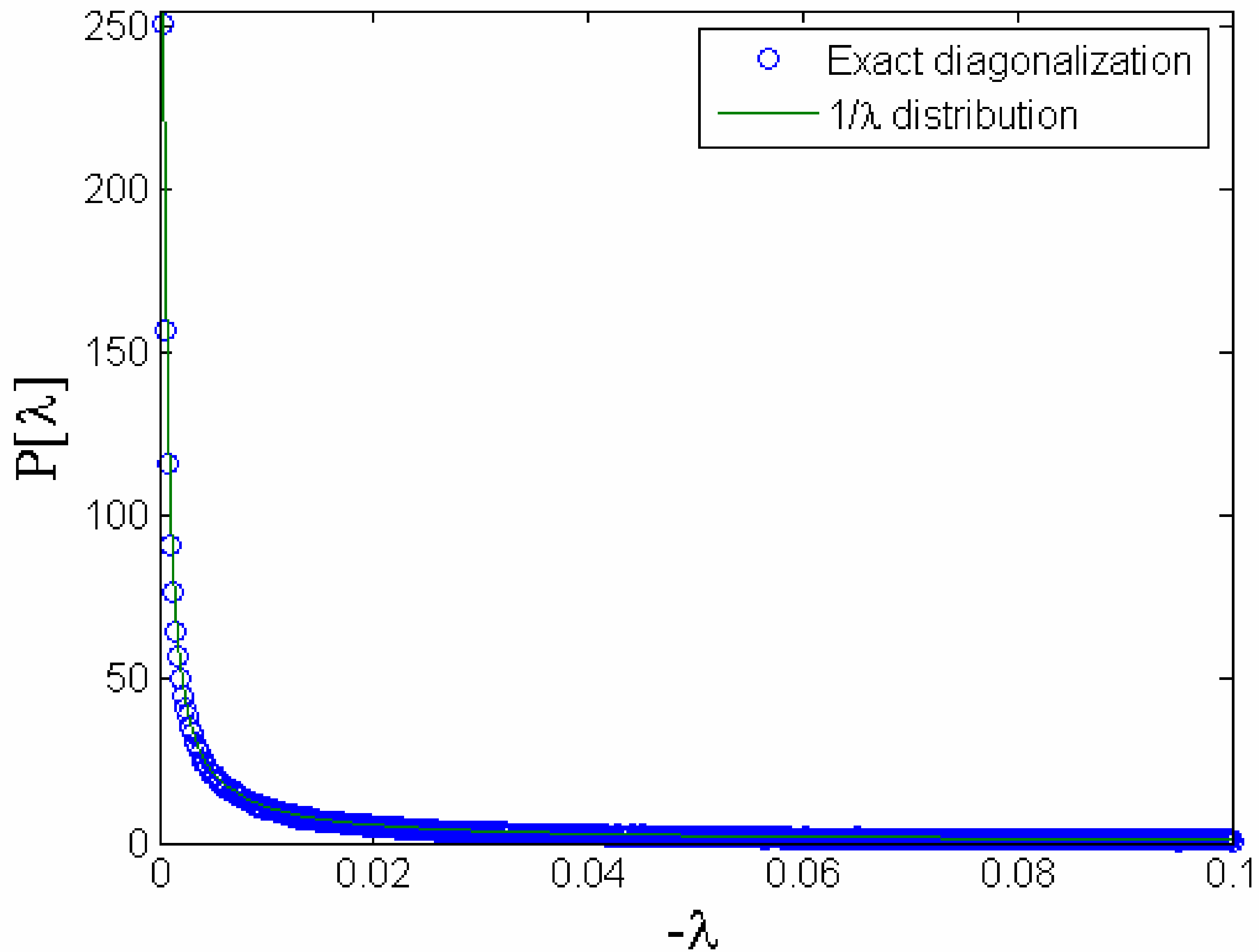
Localization of phonons
Ziman, PRL 1982
Vitelli et al., PRE 2010



Anomalous diffusion
Scher and Montroll, PRB 1975
Metzler, Barkai and Klafter, PRL 1999

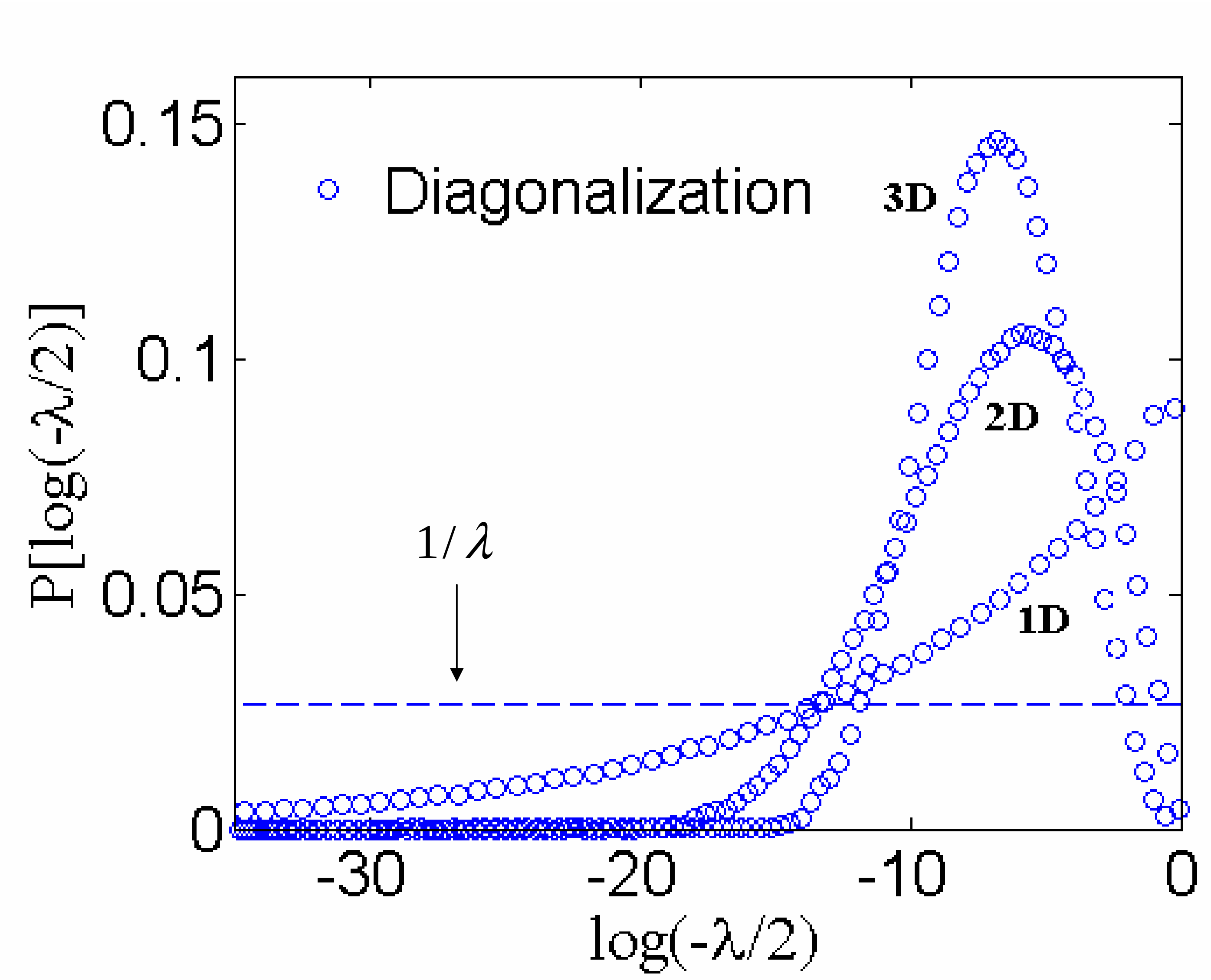


Results – 2D



250
200
150
100
50
0
0
0.02
0.04
0.06
0.08
0.1
P[λ]
-λ
Exact diagonalization
1/λ distribution

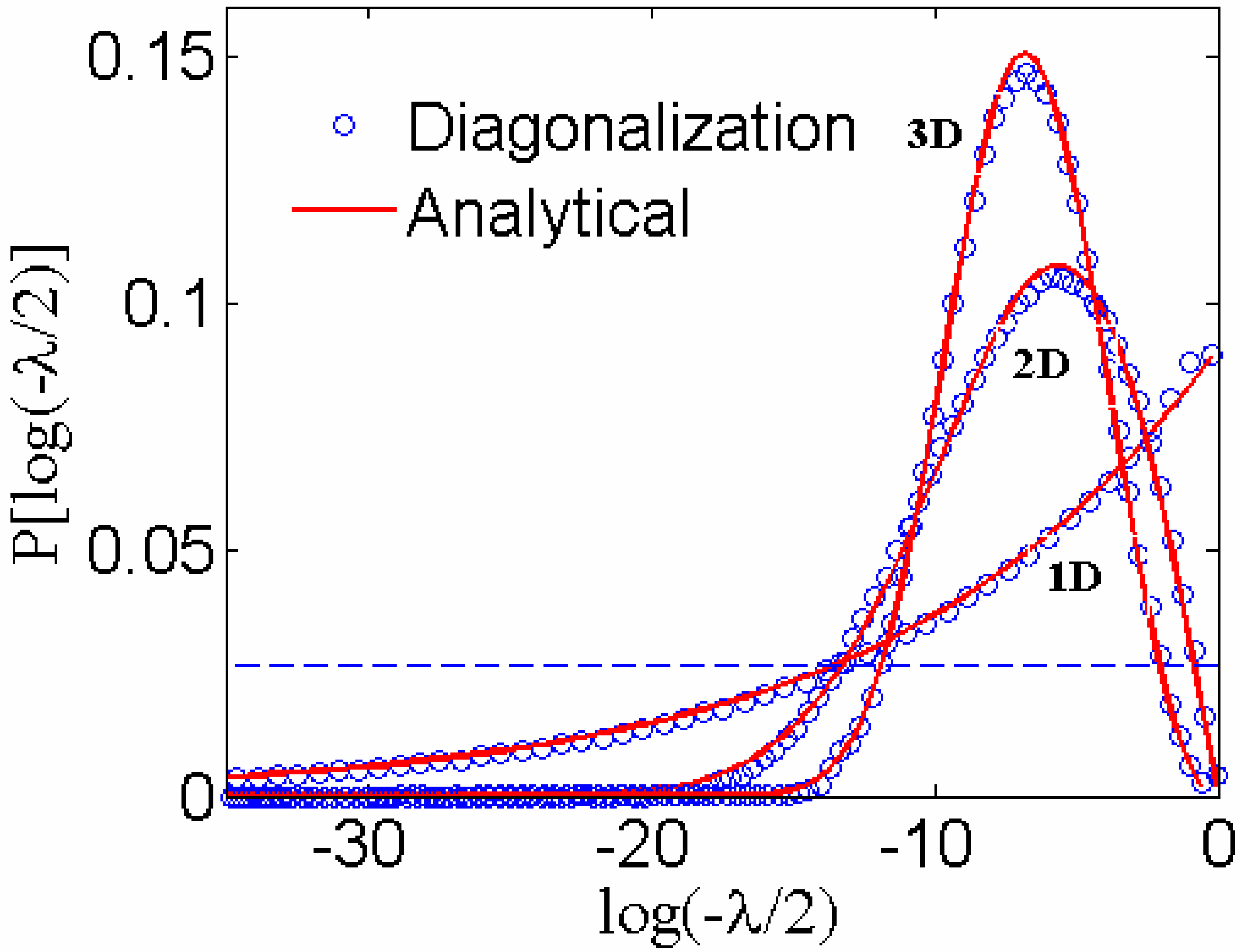
Results



0.15
0.1
0.05
0
Diagonalization
3D
2D
1D
1/ λ
P[log(-λ/2)]
-30
-20
-10
0
log(-λ/2)

Results

(no fitting parameters)



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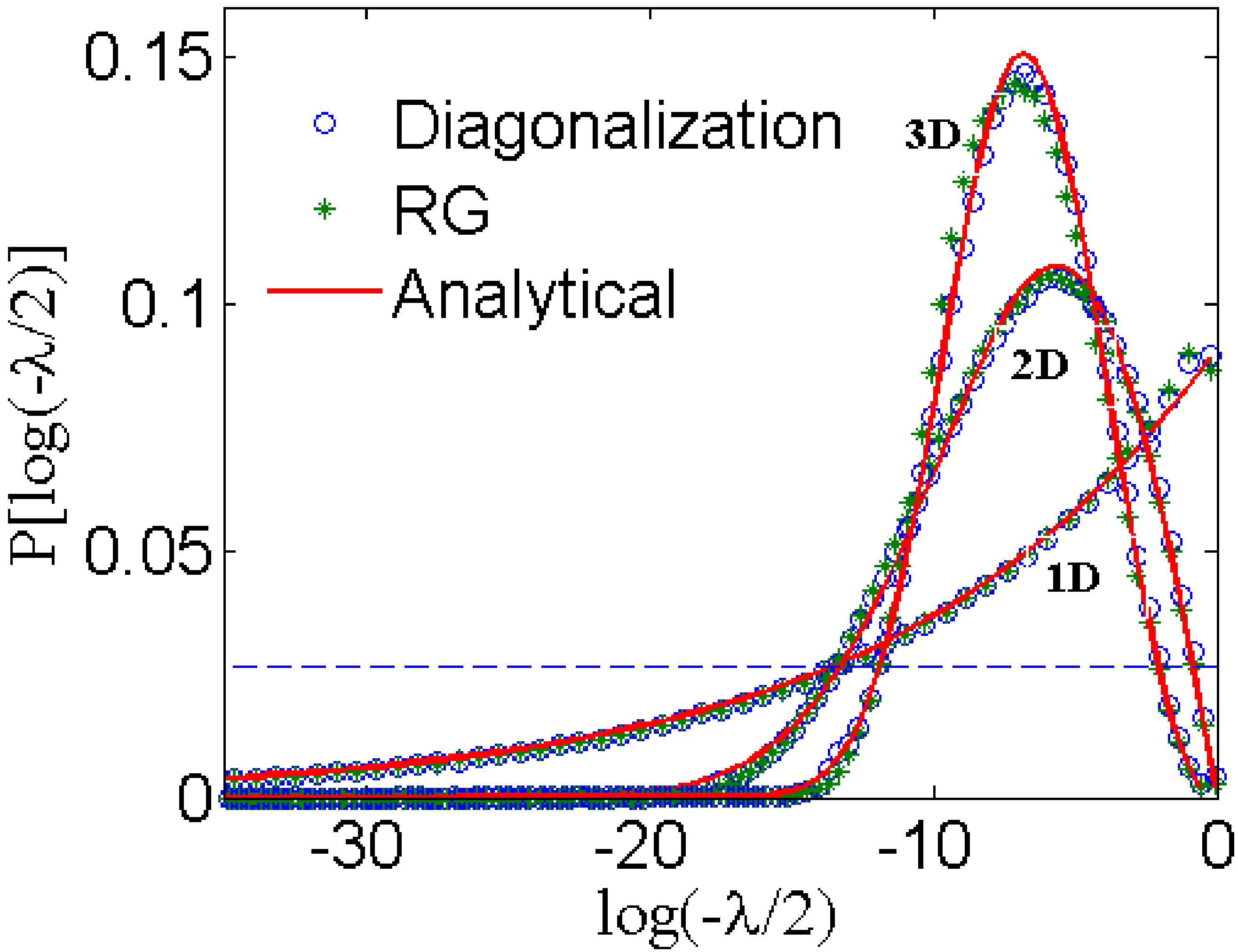


Exponential Distance Matrices- results

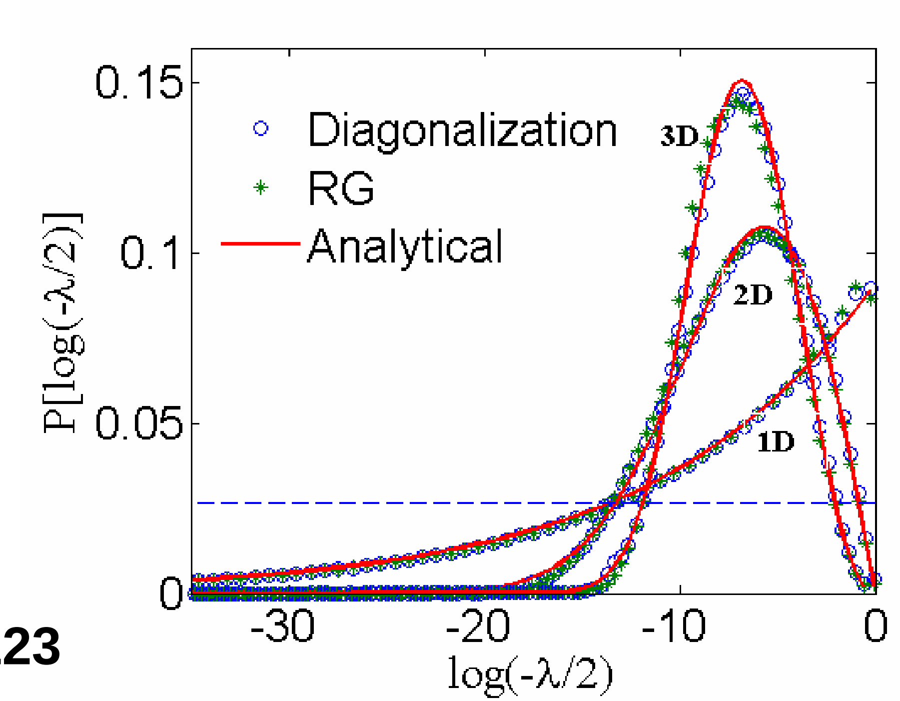
$$P(\lambda) = \frac{dC_d \varepsilon^d \log^{d-1}(\lambda/2) e^{-\frac{C_d}{2}\varepsilon^d \log^d(\lambda/2)}}{2\lambda}$$

(arbitrary dimension *d*)

$$\varepsilon = \xi / \langle r \rangle$$

C_d=volume of a d-dimensional sphere

- Logarithmic corrections to $1/\lambda$
- In dimensions > 1: cutoff at $e^{-C/\varepsilon^{d/(d-1)}}$

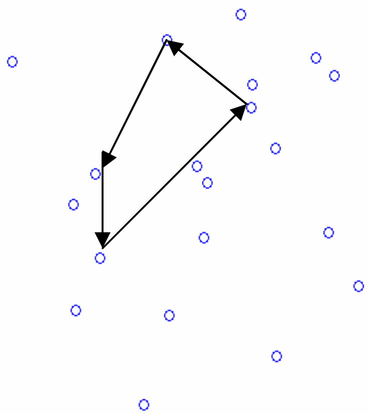


Amir, Oreg and Imry, arxiv: 1002.2123
To appear shortly in PRL

Analytical approach in a nutshell

Part I – Moment calculation

$$I_k = \int \lambda^k P(\lambda) d\lambda = \frac{1}{N} \left\langle A_{i_1,i_2} A_{i_2,i_3} \ldots A_{i_k,i_1} \right\rangle$$



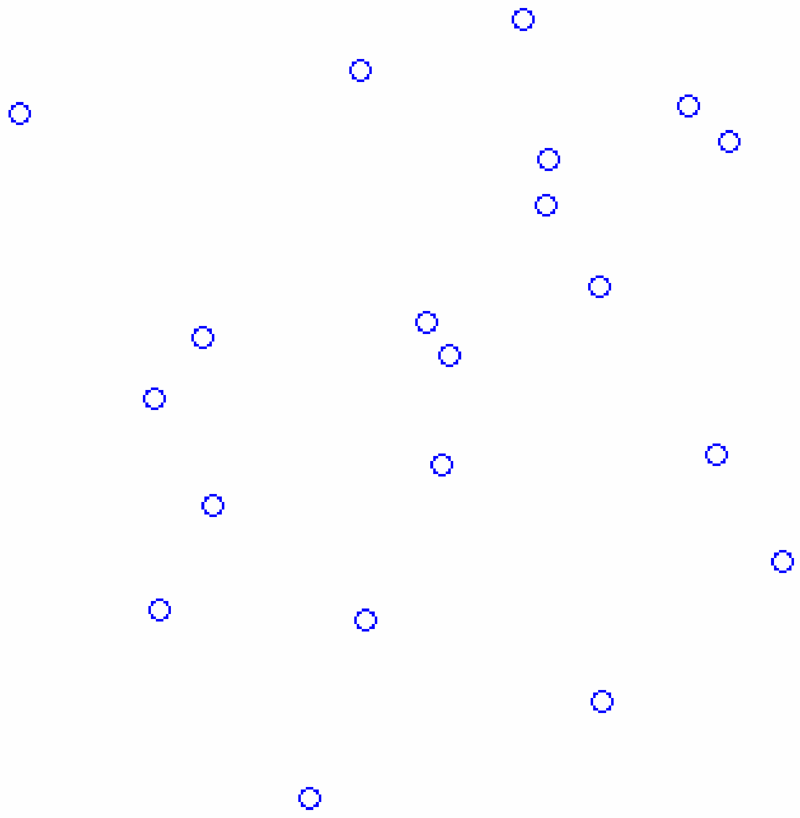
The *k*'th moment:

$$\mathrm{I_k} = 2^{\mathrm{k-1}} \mathrm{d!} \mathrm{C_d} (\varepsilon/\mathrm{k})^{\mathrm{d}}$$

leads to the distribution function.

$$P(\lambda) = \frac{d C_d \varepsilon^d \log^{d-1}(\lambda/2) e^{-\frac{C_d}{2}\varepsilon^d \log^d(\lambda/2)}}{2\lambda}$$

Renormalization group approach



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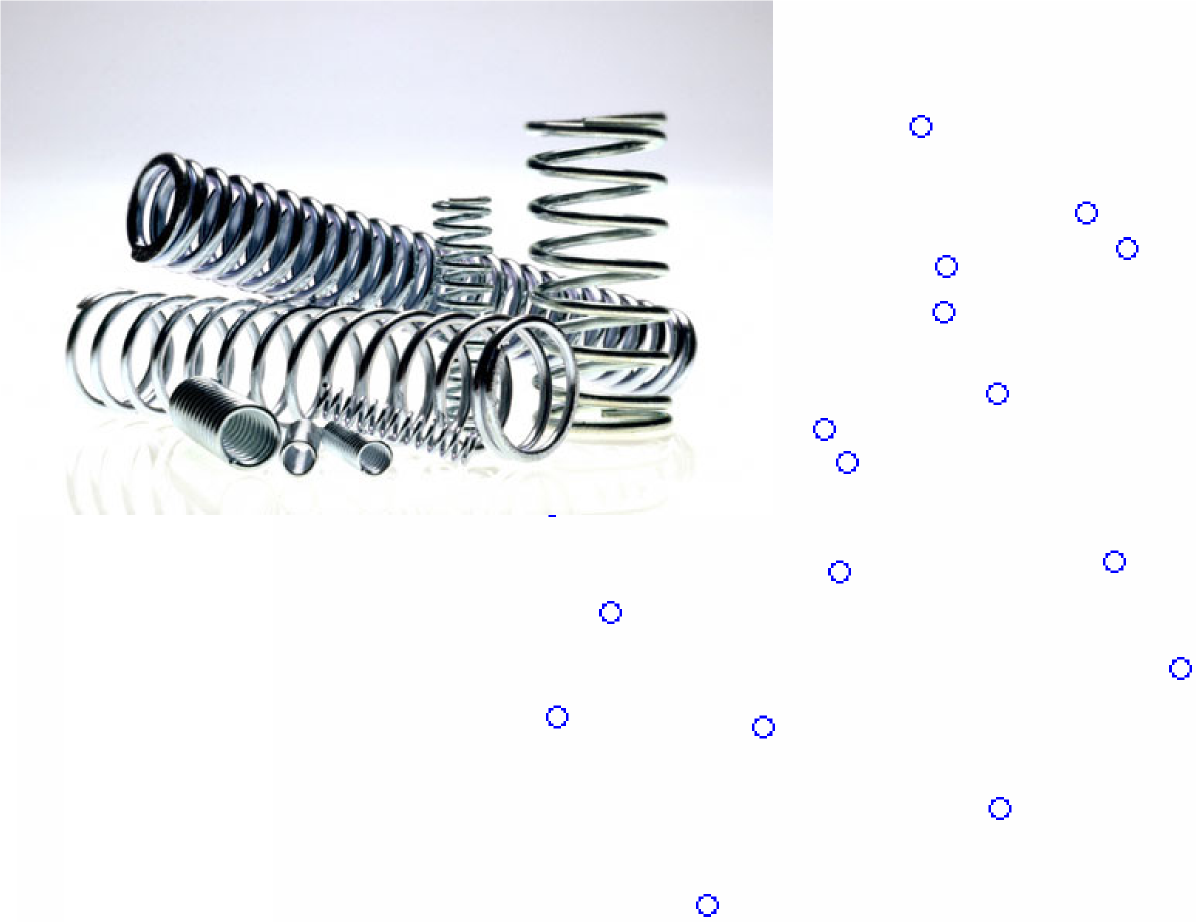
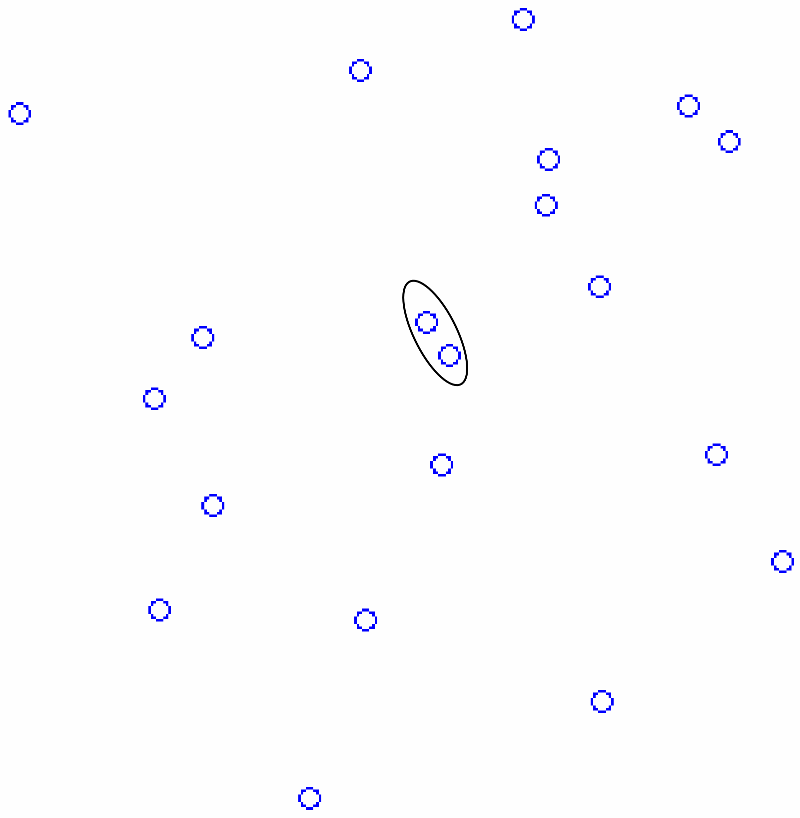


Mechanical intuition: *network of masses and springs*

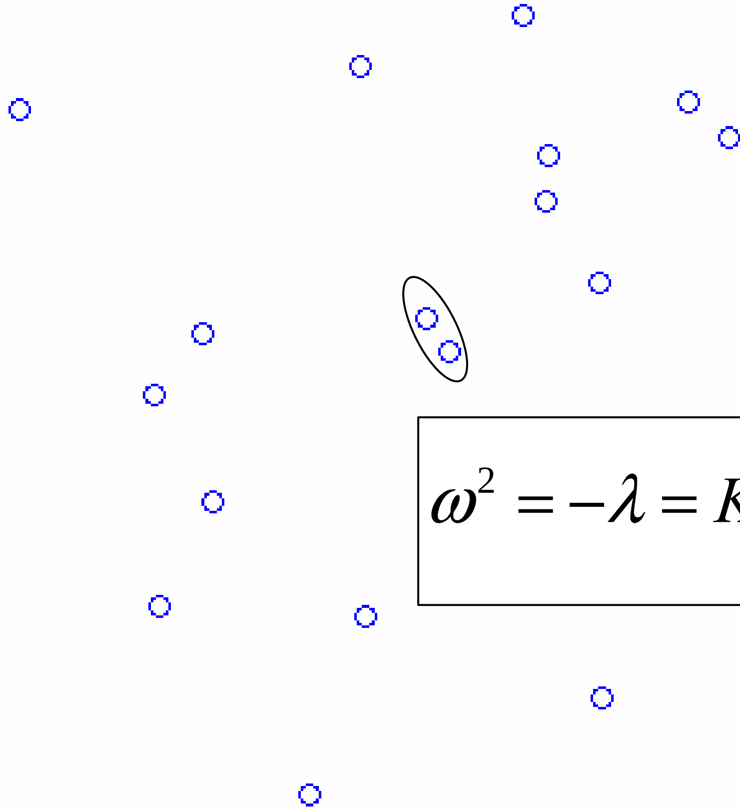
Renormalization group approach



Dasgupta and Ma, PRB 1980

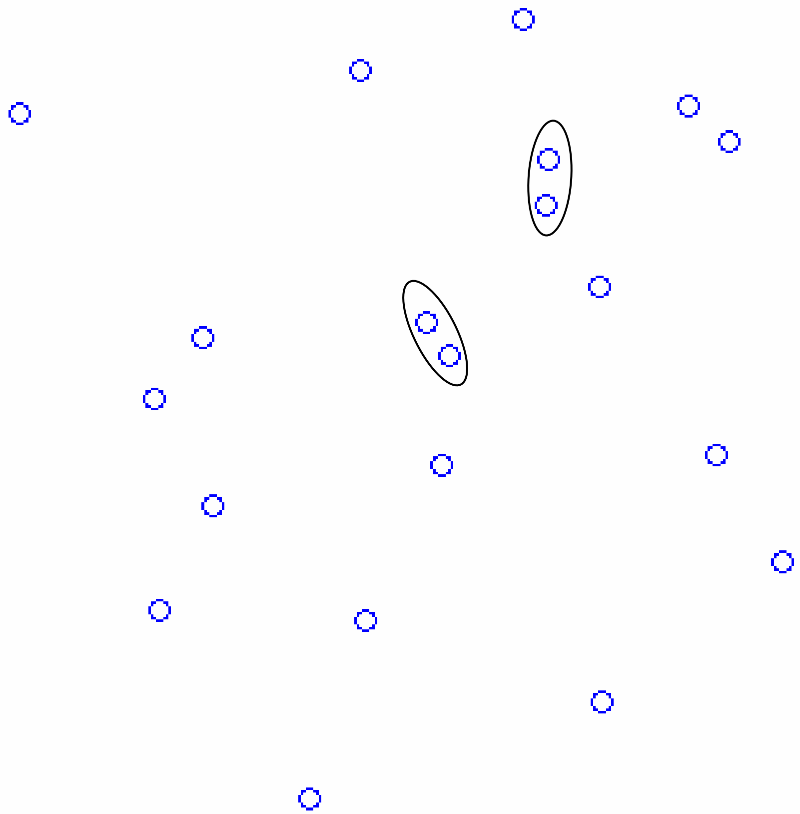
Fisher , PRL (1992) etc.

Renormalization group approach

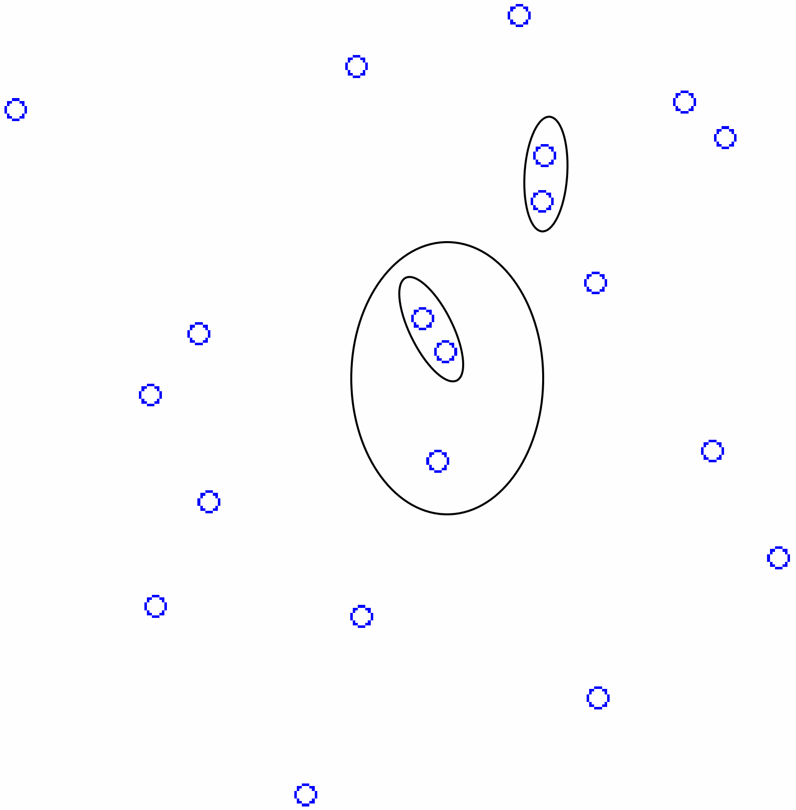


$$\omega^2 = -\lambda = K/\mu \ , \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

Renormalization group approach

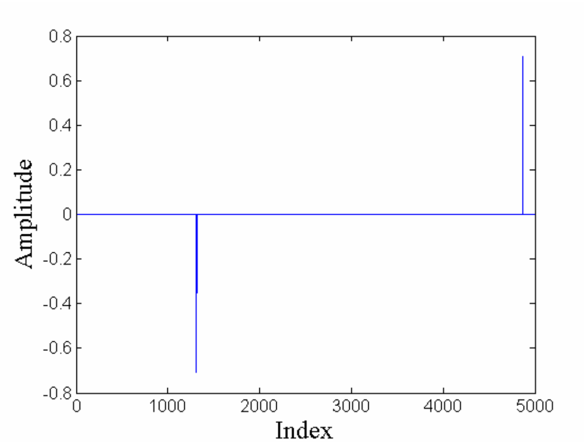


Renormalization group approach

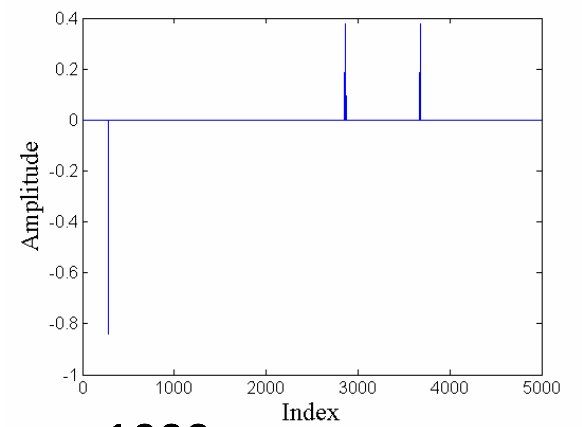


RG procedure yields growing clusters

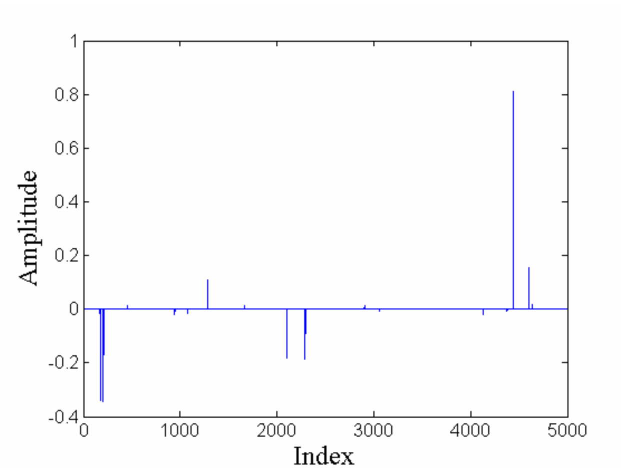
Renormalization group approach



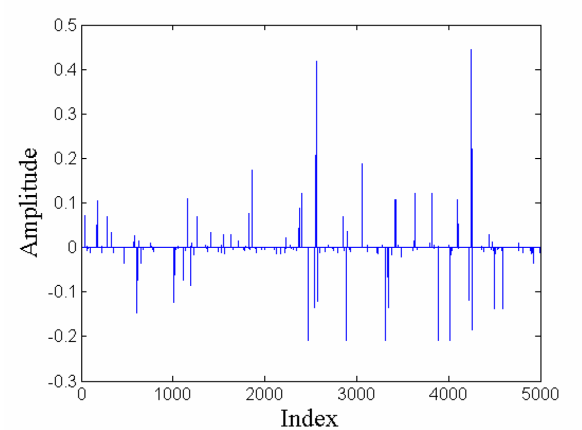
3'rd $\lambda \sim -1.86$



1000 $\lambda \sim -0.05$



3000 $\lambda \sim -9.6 \cdot 10^{-4}$



4000 $\lambda \sim -8.5 \cdot 10^{-5}$

Examples of eigenmodes of a 5000X5000 matrix

Renormalization group approach

Eigenvalue distribution + RG approach

$$\longrightarrow \quad n_c \sim e^{\frac{C_d}{2}\varepsilon^d |\log^d(-\lambda/2)|}$$

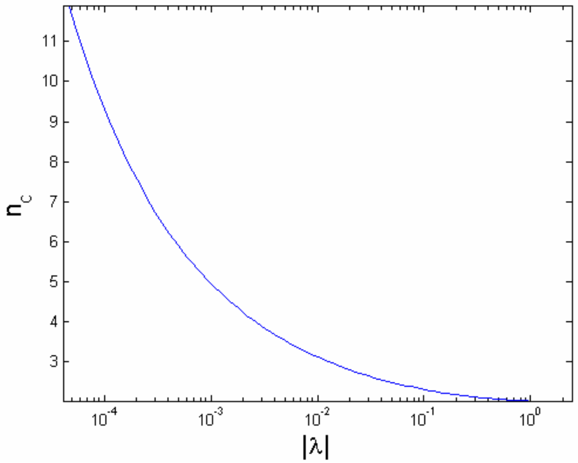
Number of points in a cluster of a given eigenvalue

Renormalization group approach

Eigenvalue distribution + RG approach

→

$$n_c \sim e^{\frac{C_d}{2}\varepsilon^d |\log^d(-\lambda/2)|}$$



Number of points in a cluster of a given eigenvalue

- Eigenmodes are localized clusters (“phonon localization”)
- Size of clusters diverges at low frequencies

Amir, Oreg and Imry**,** *Localization, anomalous diffusion and slow relaxations: a random distance matrix approach***, arxiv: 1002.2123**

Intermediate Summary

- Statics: Coulomb gap
- Dynamics: Microscopic model reduces to RMT problem of a different class.
- Eigenvalue distribution (relaxation times) follow approximately $1/\lambda$ distribution.
- Same RMT class relevant for various other problems.

Electron glass aging– experimental protocol

Step I

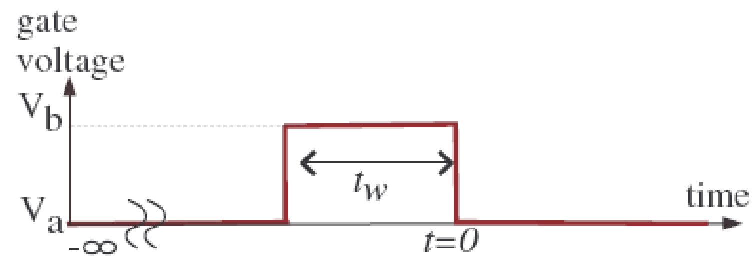
System equilibrates for long time

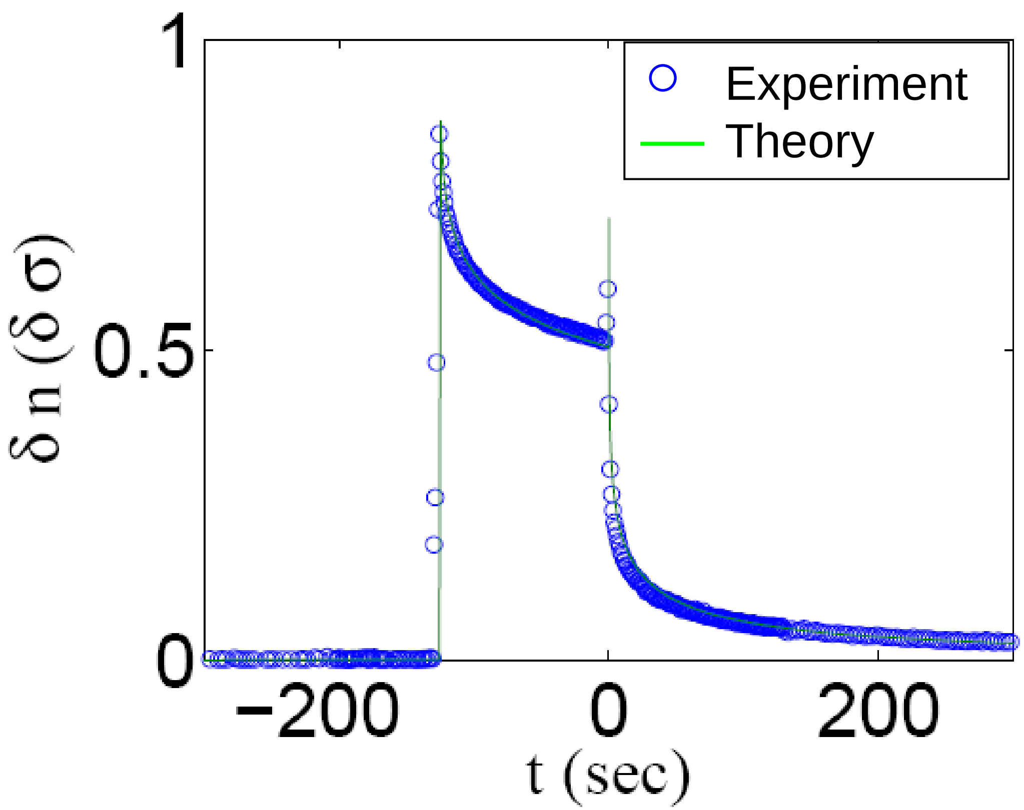
Step II

V_g is changed, for a time of t_w.

Throughout the experiment

Conductance is measured as a function of time.



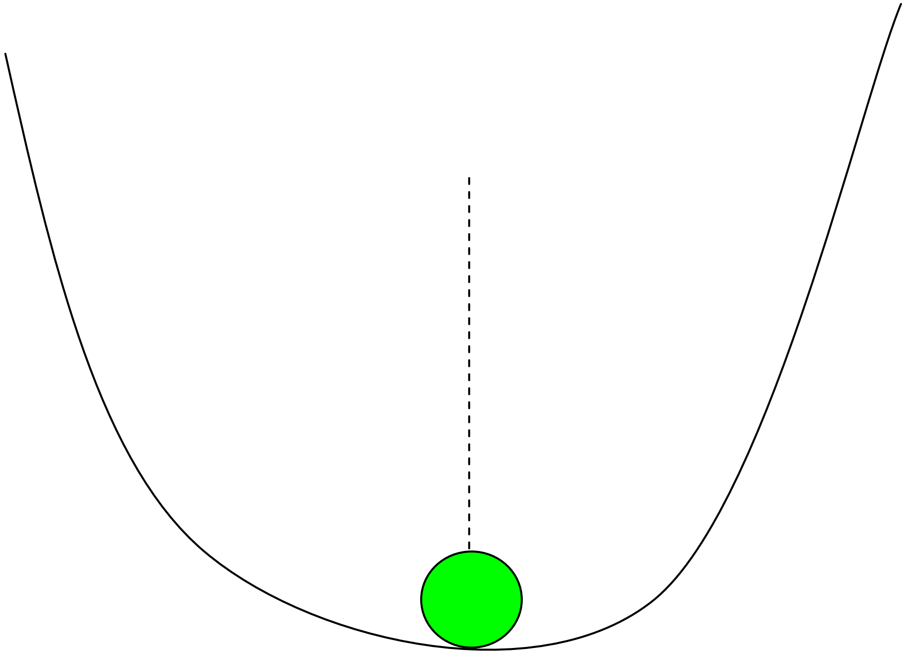


Data: *Ovadyahu et al.*

Aging – physical picture

Assume a parameter of the system is slightly modified (e.g: V_g)

After time t_w it is changed back. What is the repsonse?

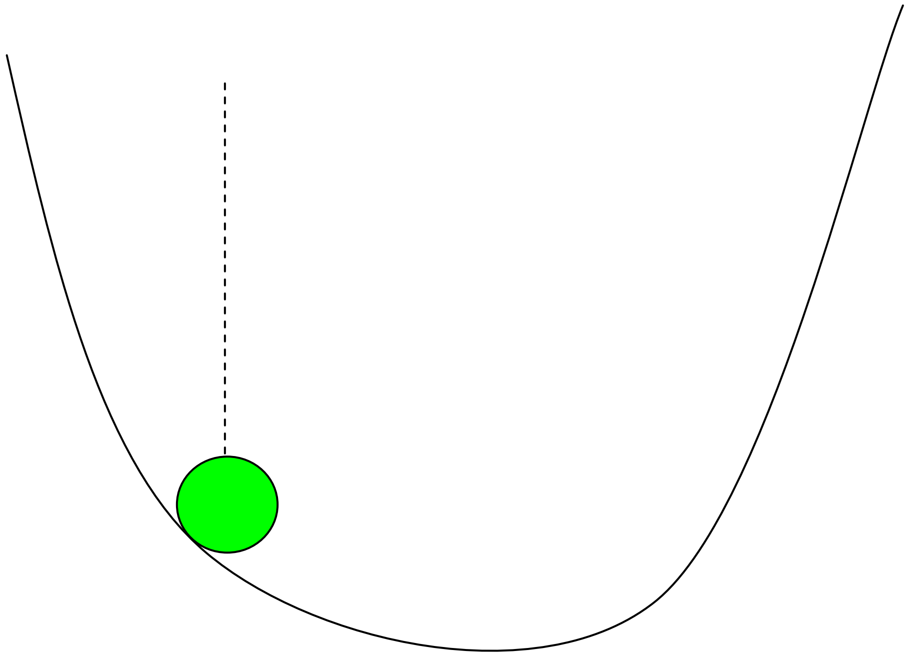


Initially, system is at some local minimum

Aging – physical picture

Assume a parameter of the system is slightly modified (e.g: V_g)

After time t_w it is changed back. What is the repsonse?



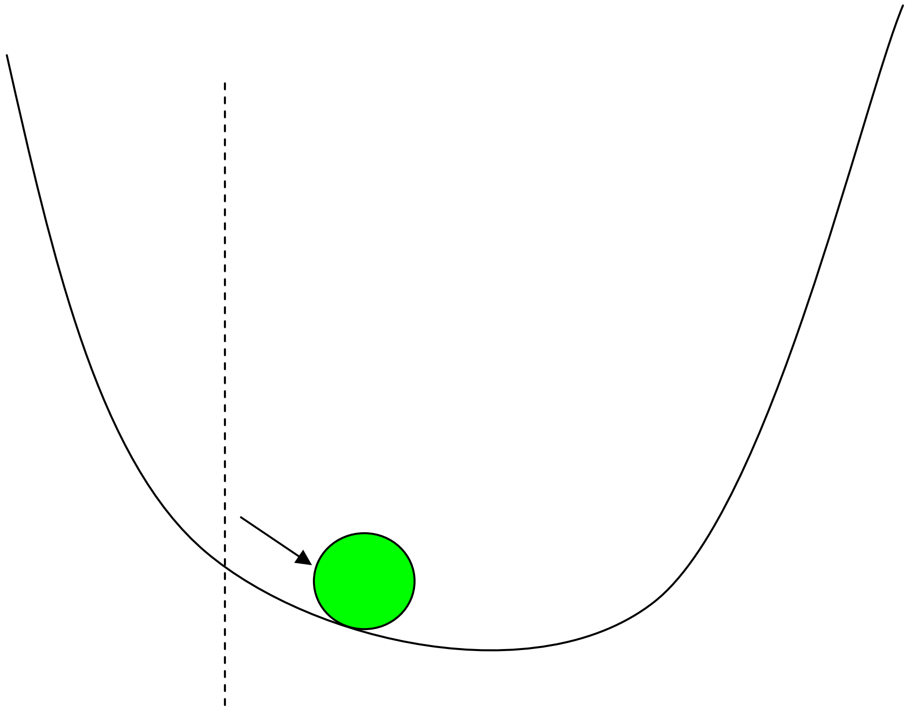
At time t=0 the potential changes,

and the system begins to roll towards the new minimum

Aging – physical picture

Assume a parameter of the system is slightly modified (e.g: V_g)

After time t_w it is changed back. What is the repsonse?

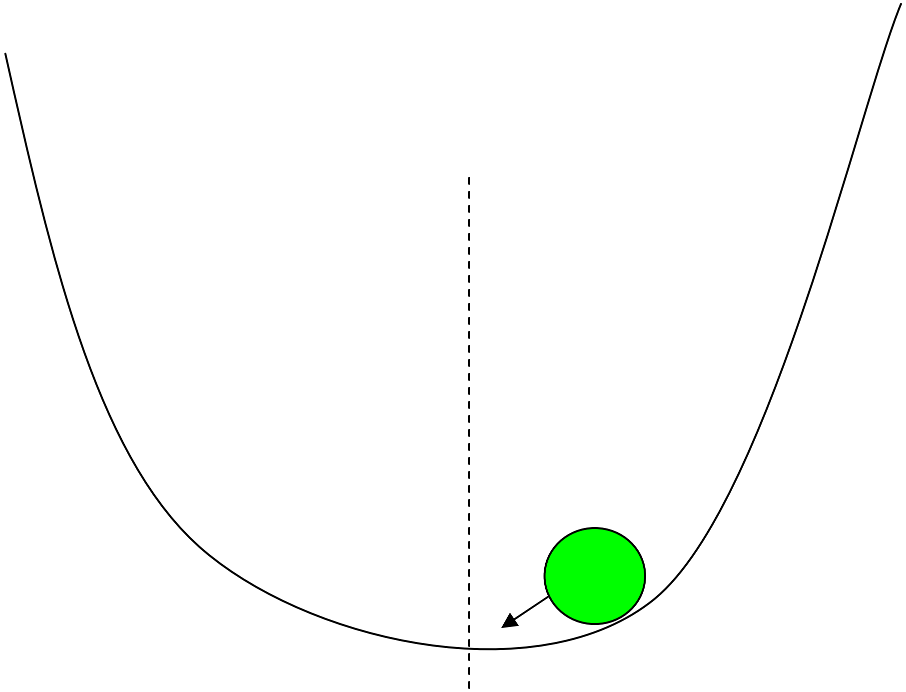


At time t_w the system reached some new configuration

Aging – physical picture

Assume a parameter of the system is slightly modified (e.g: V_g)

After time t_w it is changed back. What is the repsonse?



Now the potential is changed back to the initial form-

the particle is not in the minimum!

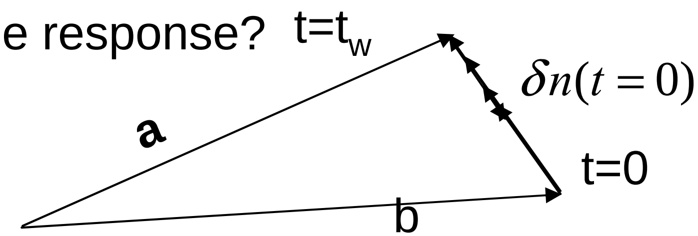
The longer t_w, the further it got away from it.

It will begin to roll down the hill.

Aging – Analysis

Assume a parameter of the system is slightly modified (e.g: V_g)

After time t_w it is changed back. What is the response?



Sketch of calculation

If a and b configurations are close enough in phase space:

$$\delta n(t=t_w) \sim \sum_{\text{eigenmodes}_\alpha} \chi_\alpha e^{-\lambda_\alpha t_w} \left| V_\alpha \right\rangle \Rightarrow \sum_{\text{eigenmodes}_\alpha} e^{-\lambda_\alpha t_w} =$$

modes are independent and contribute uniformly

Logarithmic relaxation during step II

Time *t* after the perturbation is switched off:

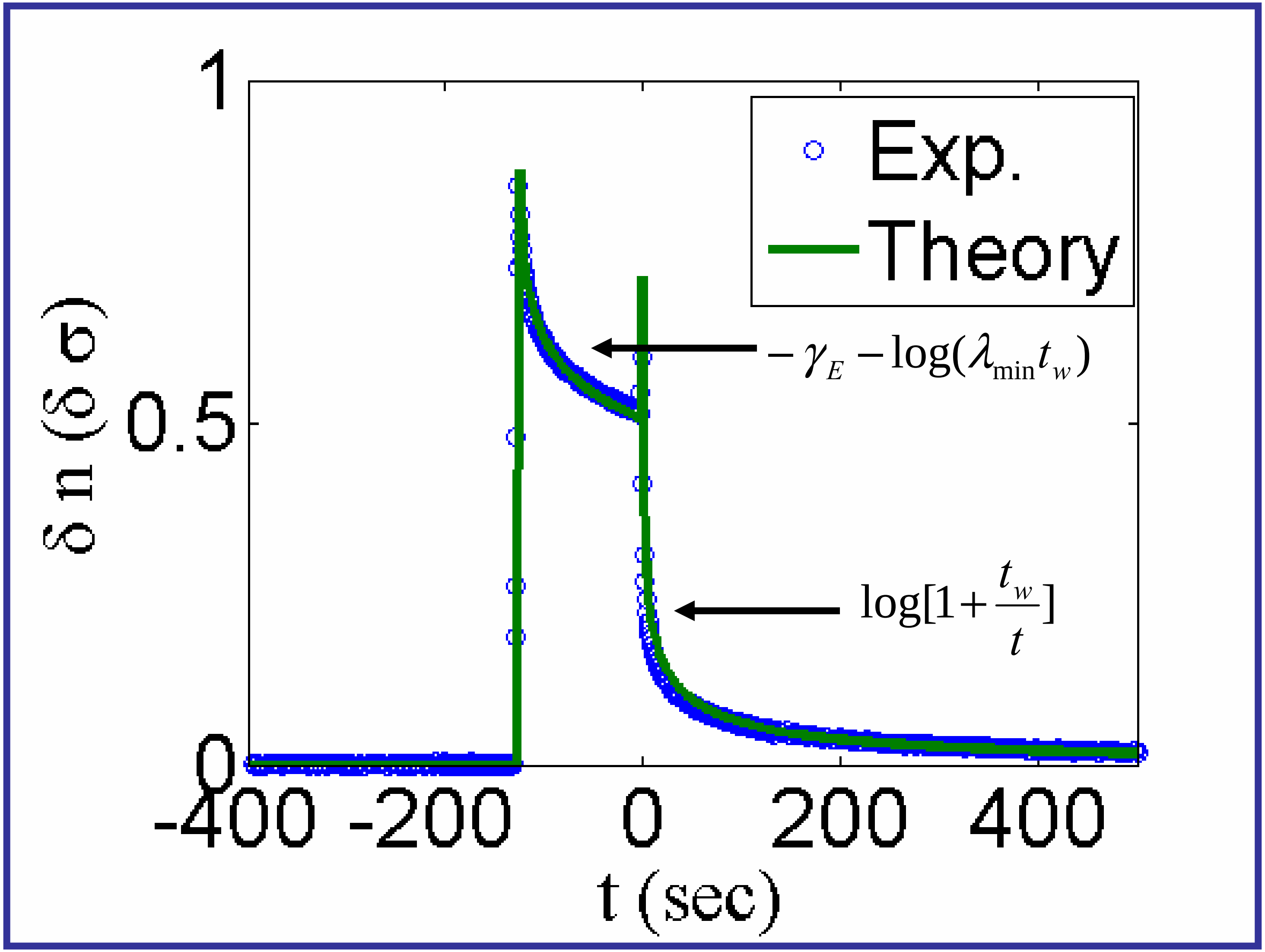
$$\delta n(t) \sim \sum_{eigen\,\text{modes}\ \alpha} \chi_\alpha (1 - e^{-\lambda_\alpha t_w}) e^{-\lambda_\alpha t} \,| V_\alpha > = f(t+t_w) - f(t)$$

Full aging

Only $1/\lambda$ distribution yields full aging!

See also: T. Grenet et al. Eur. Phys. J B 56, 183 (2007)

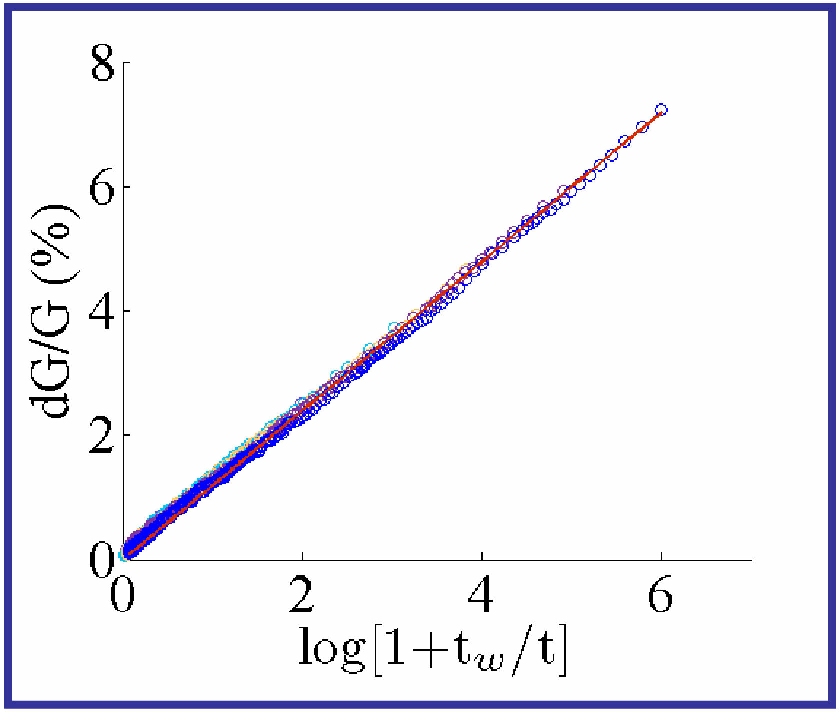
Aging Protocol - Results

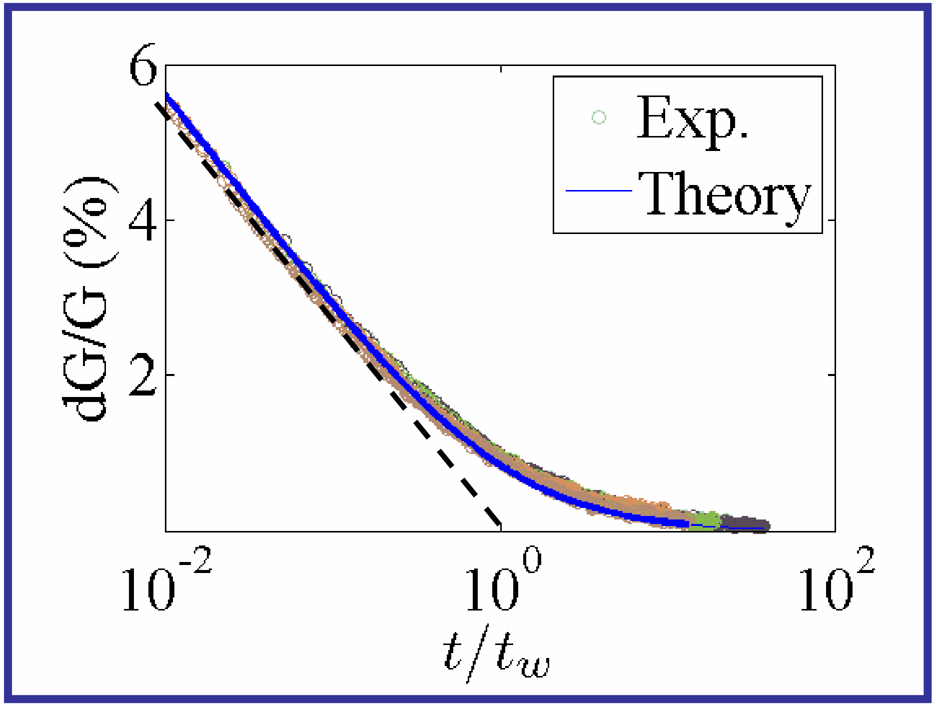


Amir, Oreg and Imry, *PRL* 2009

Detailed fit to experimental data

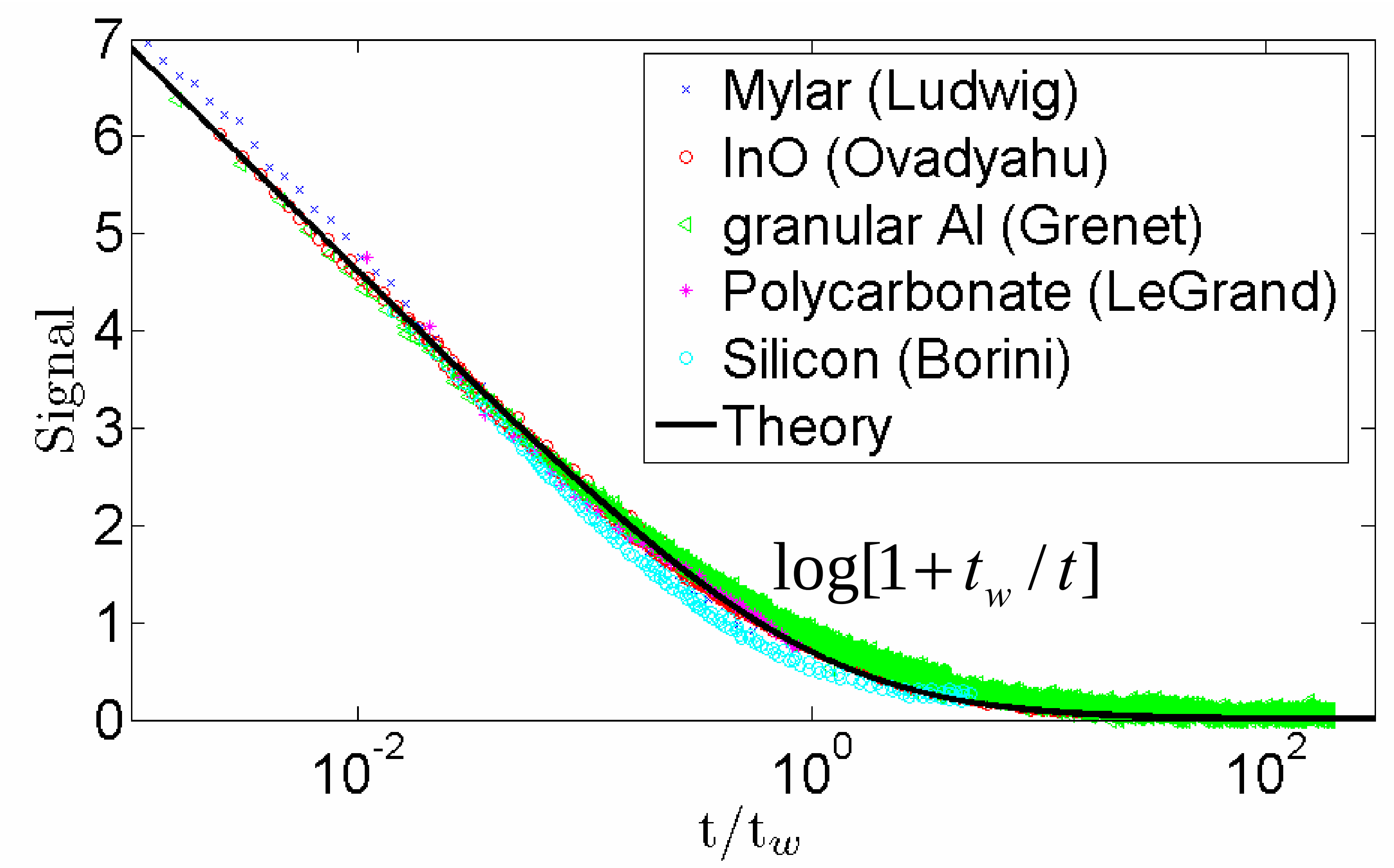
- Full aging
- Deviations from logarithm start at t/t_w





Amir, Oreg and Imry, *PRL* (2009)

Full aging and universality



Amir, Oreg and Imry, to be published

Conclusions

• Statics: Coulomb gap

• Dynamics near locally stable point: many slow *localized* modes, $\sim {}^{1}\!/_{\lambda}$ distribution.

How universal? **We believe: a very relevant RMT class.**

• One obtains full aging, with relaxation approximately of the form :

$$\delta\sigma \sim \log[1+\frac{t_w}{t}]$$

More details:

Phys. Rev. B 77, 1, 2008 (local mean-field model)

Phys. Rev. Lett. 103, 126403 (2009) (aging properties)

Phys. Rev. B 80, 245214 2009 (variable-range hopping)

Ann. Phys. 18, 12, 836 (2009) (*1/f* noise)

arxiv: 1002.2123, soon in Phys. Rev. Lett. (exponential matrices – solution)

Electron glass dynamics – Review (soon online)