Weak chaos in the disordered nonlinear Schrödinger chain: destruction of Anderson localization by Arnold diffusion

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Thanks to: I. Aleiner, B. Altshuler, S. Flach, O. Yevtushenko

Instead of introduction

$$H = \sum_{n=-\infty}^{\infty} \omega_n \left(\frac{p_n^2}{2m\omega_n} + \frac{m\omega_n}{2} x_n^2 \right) + \sum_{n=-\infty}^{\infty} \frac{g}{2} \left(\frac{p_n^2}{2m\omega_n} + \frac{m\omega_n}{2} x_n^2 \right)^2$$

anharmonic oscillators

$$- \sum_{n=-\infty}^{\infty} \Omega \left(\frac{p_n p_{n+1}}{2m\sqrt{\omega_n \omega_{n+1}}} + \frac{m\sqrt{\omega_n \omega_{n+1}}}{2} x_n x_{n+1} \right)$$

nearest-neighbor coupling

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nearest-neighbor coupling

equations of motion

$$\frac{dp_n}{dt} = -\frac{\partial H}{\partial x_n}, \qquad \frac{dx_n}{dt} = \frac{\partial H}{\partial p_n}$$

change of variables

$$\psi_n = \frac{p_n}{\sqrt{2m\omega_n}} + \sqrt{\frac{m\omega_n}{2}}\, i x_n$$

dimensionality: $|\psi|^2$ = action

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discrete nonlinear Schrödinger equation with disorder

$$i\frac{d\psi_n}{dt} = \underbrace{\omega_n \psi_n - \Omega(\psi_{n+1} + \psi_{n-1})}_{\text{Anderson localization}} + g\psi_n^* \psi_n^2$$

nonlinearity

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nonlinearity

$$H(i\psi^*, \psi) \;=\; \sum_{n=-\infty}^{\infty} \omega_n \psi_n^* \psi_n + \sum_{n=-\infty}^{\infty} \frac{g}{2}\, \psi_n^* \psi_n^* \psi_n \psi_n$$

anharmonic oscillators

$$- \sum_{n=-\infty}^{\infty} \Omega \left(\psi_n^* \psi_{n+1} + \psi_{n+1}^* \psi_n\right)$$

nearest-neighbor coupling

action-angle variables: $\psi_n = \sqrt{I_n}\, e^{-i\phi_n}$

$$H(I, \phi) \;=\; \sum_{n=-\infty}^{\infty} \omega_n I_n + \sum_{n=-\infty}^{\infty} \frac{g}{2} I_n^2 - \sum_{n=-\infty}^{\infty} \Omega \sqrt{I_n I_{n+1}}\, 2\cos(\phi_n - \phi_{n+1})$$

Disordered nonlinear 1D systems

Stationary solutions of NLSE: Iomin, Fishman (2007); Fishman *et al.* (2008); Bodyfelt *et al.* (2010)

Transmission of a finite sample: Gredeskul, Kivshar (1992); Paul *et al*. (2005); Paul *et al.* (2007);
Tietsche, Pikovski (2008); Paul *et al.* (2009)

Dipole oscillations in a trap: Albert *et al.* (2008)

Wave packet spreading in discrete NLSE: Shepelyansky (1993); Molina, Tsironis (1994);
Molina (1998); Kopidakis *et al.* (2008); Pikovsky, Shepelyansky (2008); Fishman *et al.* (2008);
Bourgain, Wang (2008); Wang, Zhang (2009); Flach *et al.* (2009); Skokos *et al.* (2009);
Fishman *et al.* (2009); Veksler *et al.* (2009); Krivolapov *et al.* (2009); Veksler *et al.* (2010);
Iomin (2010); Skokos, Flach (2010); Flach (2010); Mulansky, Pikovsky (2010);
Laptyeva *et al.* (2010)

Wave packet spreading in other nonlinear disordered 1D systems: Fröhlich *et al.* (1986);
Kopidakis *et al.* (2008); Flach *et al.* (2009); Skokos *et al.* (2009);
Garcia-Mata, Shepelyansky (2009); Krimer *et al.* (2009); Flach (2010); Laptyeva *et al.* (2010)

Thermalization in NLSE and other nonlinear disordered 1D systems: Dhar, Lebowitz (2008);
Dhar, Saito (2008); Oganesyan *et al.* (2009); Mulansky *et al.* (2009);
Pikovsky, Fishman (2010)

Numerical integration: wave packet spreads as a power law

Wang, Zhang + Fishman *et al*.: slower than any power law

Given

1. Strong localization
2. Weak nonlinearity

$$\left.\begin{array}{l}\text{1. Strong localization}\\ \text{2. Weak nonlinearity}\end{array}\right\} \rightarrow \text{ worst conditions for transport}$$

3. Arbitrary initial condition with extensive norm and energy

Question: will the system equilibrate at long distances, and how?

Answer: yes, by normal nonlinear diffusion:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x}$$

Mechanism: CHAOS

- Arnold diffusion in the space of actions
- driven by rare local chaotic spots
- which migrate along the chain

(as seen by Oganesyan, Pal, Huse, 2009)

Assumptions

$$i\frac{d\psi_n}{dt} = \omega_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

disorder

$$-\frac{\Delta}{2} \le \omega_n \le \frac{\Delta}{2}$$

"tunnelling"

nonlinearity

$$g > 0$$

1. Strong localization: $\frac{\Omega}{\Delta} \equiv \tau \ll 1$

assumption about the Hamiltonian

2. Weak nonlinearity: $\frac{g|\psi_n|^2}{\Delta} \sim \rho \ll 1$

(nonlinear frequency shift << disorder)

note the invariance under $\psi_n \to C\psi_n, \quad g \to C^{-2}g$

3. Single action scale: $-\Delta\sum_n |\psi_n|^2 \ll H < 0$

(all oscillators are excited more or less equally; thermodynamic relations have a simple form)

assumptions about the initial conditions

Thermalization and transport

Two conserved quantities: total energy H, total action $I = \sum |\psi_n|^2$

Local equilibration
in a **finite** time → $\mathcal{P}(\{\psi_n\}) \propto e^{-\beta(H-\mu I)}, \qquad \beta \equiv 1/T$

Global equilibration: **transport of the conserved quantities**

Macroscopic action density: $\rho(x) = \frac{1}{L^*} \sum_{n=x-L^*/2}^{x+L^*/2} \frac{g|\psi_n|^2}{\Delta} \approx \frac{gT}{|\mu|\Delta}$

get rid of the energy density thanks to $|H| \ll I\Delta$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \underset{\text{finite!}}{D(\rho)} \frac{\partial \rho}{\partial x}$$

Diffusion coefficient

$$D(\rho) \sim \exp\left(-\mathcal{C} \ln^2 \frac{1}{\tau^p \rho} \ln \frac{1}{\rho}\right) \qquad \frac{1}{2} \leq p \leq 3$$

stronger than any
power law

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stronger than any
power law

$$\frac{1}{3}\left[1 + \ln\left(1 + \frac{\ln(1/\rho)}{\ln(1/\tau)}\right)\right]^{-2} \leq \mathcal{C} \leq 8\left[1 + \ln\left(1 + \frac{\ln(1/\rho)}{\ln(1/\tau)}\right)\right]^{-2}$$

double logarithm ~ constant

D^{-1} is self-averaging at distances $L^* \gg \exp\left(\mathcal{C} \ln^2 \frac{1}{\tau^p \rho}\right)$

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double logarithm ~ constant

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A finite-norm wave packet spreads as size $\sim \exp\left[(\ln t)^{1/3}\right]$

Off-resonant coupling

$$H = \boxed{\omega_1|\psi_1|^2+\frac{g}{2}|\psi_1|^4}+\boxed{\omega_2|\psi_2|^2+\frac{g}{2}|\psi_2|^4}-\boxed{\tau\Delta(\psi_1^*\psi_2+\psi_2^*\psi_1)}+\dots$$

nonlinear frequency shifts: $\phi_n = (\omega_n + g|\psi_n^0|^2)t$

perturbative correction from tunnelling: $$\psi_1(t) = \psi_1^0 e^{-i(\omega_1+g|\psi_1^0|^2)t} - \frac{\tau\Delta\,\psi_2^0 e^{-i(\omega_2+g|\psi_2^0|^2)t}}{\omega_2 + g|\psi_2^0|^2 - \omega_1 - g|\psi_1^0|^2}$$

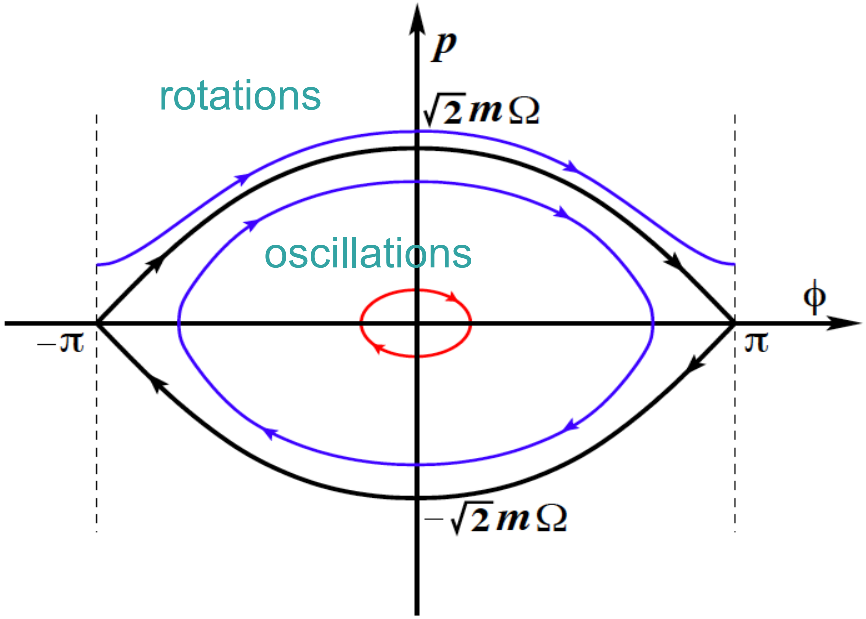
The correction to $|\psi_1|^2$ is small at all times unless the denominator → 0

Theorem of Kolmogorov, Arnold, & Moser:

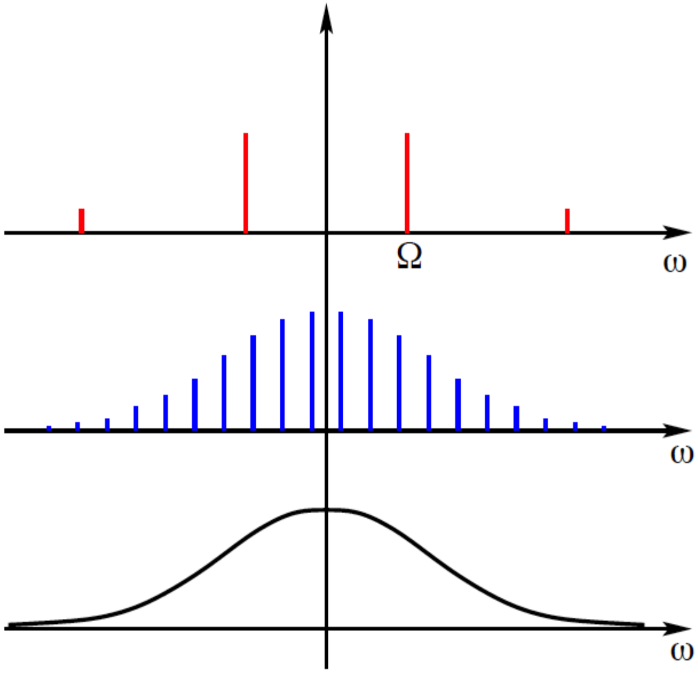
in most of the phase space the perturbed trajectories are small deformations of the unperturbed trajectories

Pendulum: $H(p,\phi) = \frac{p^2}{2m} - m\Omega^2 \cos\phi$

Phase space:



Spectrum $\int \phi(t)\, e^{i\omega t}\, dt$



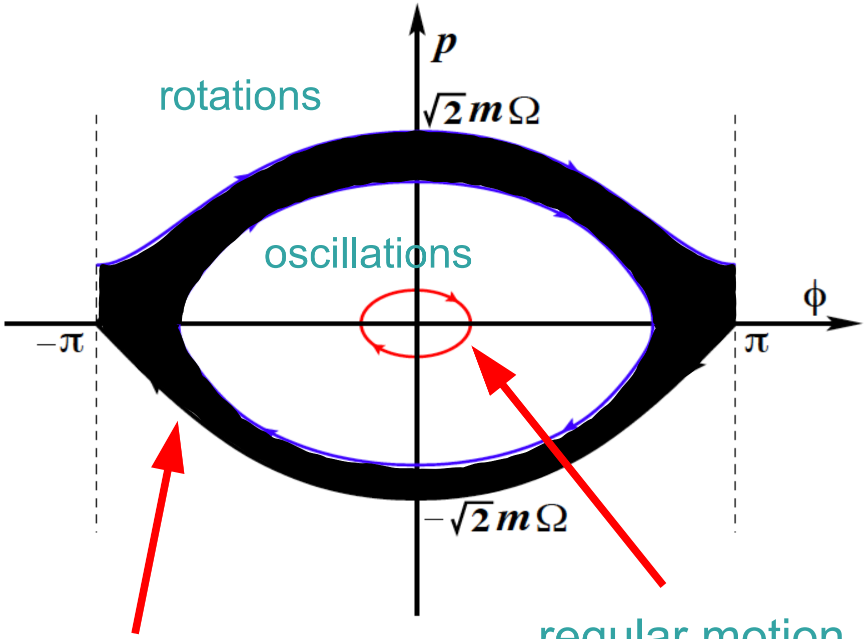
the period diverges
at the separatrix

⟺

the separatrix motion
has a continuous spectrum

Perturbed pendulum:

$$H(p,\phi,t) = \frac{p^2}{2m} - m\Omega^2 \cos\phi - V\cos(\phi - \omega t)$$



ergodic trajectories
within
the stochastic layer

regular motion
survives

Stochastic layer area:

$$W_s \equiv \int_{\text{layer}} \frac{dp\,d\phi}{2\pi} \sim \frac{V}{\Omega} e^{-|\omega|/\Omega}$$

Melnikov-Arnold integral

$$|\omega| \gg \Omega$$

Continuous spectrum
of the chaotic motion:

$$\left\langle e^{i\phi(t)} e^{-i\phi(t')} \right\rangle_\omega \sim \frac{1}{\Omega} e^{-|\omega|/\Omega}$$

review: B. Chirikov (1979)

Making a pendulum out of oscillators

two-oscillator Hamiltonian:

$$H = \boxed{\omega_1 I_1 + \frac{gI_1^2}{2}} + \boxed{\omega_2 I_2 + \frac{gI_2^2}{2}} - \boxed{2\tau\Delta\sqrt{I_1 I_2}\cos(\phi_1 - \phi_2)}$$

canonical transformation:

$$I = I_1 + I_2, \quad \phi = \frac{\phi_1 + \phi_2}{2}, \quad \tilde{I} = \frac{I_1 - I_2}{2}, \quad \tilde{\phi} = \phi_1 - \phi_2$$

$$H = \underline{H_0(I)} + \underline{(\omega_1 - \omega_2)\tilde{I} + g\tilde{I}^2} - 2\tau\Delta\sqrt{\underline{I^2/4 - \tilde{I}^2}}\cos\tilde{\phi}$$

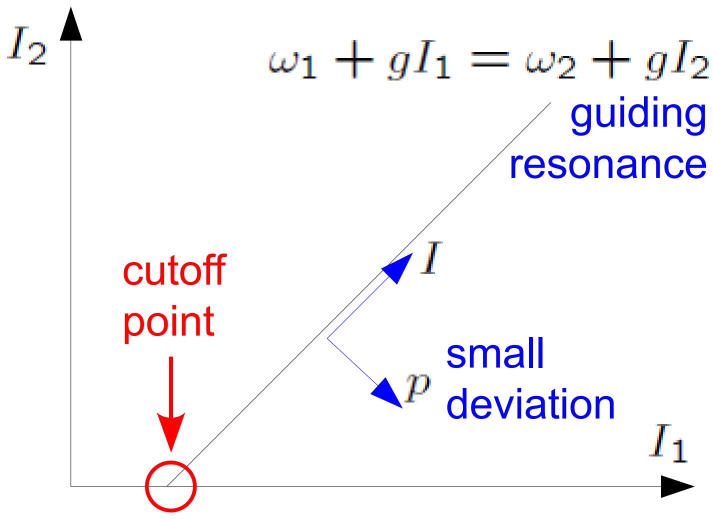
constant

shift

$$\tilde{I} = \frac{\omega_2 - \omega_1}{2g} + p$$

almost constant

$$\tau \ll 1$$



A third oscillator:

$$2\tau\Delta\sqrt{I_2 I_3}\cos(\phi_2 - \phi_3)$$

perturbation of the pendulum

Three oscillators are sufficient to generate chaos

The price of making a pendulum

To find a separatrix:

the shift is possible only if $I_1, I_2 > 0$

cutoff point

$$\omega_1 + gI_1 = \omega_2 + gI_2$$

$$\frac{H - \mu I}{T} > \underbrace{\frac{|\mu||\omega_1 - \omega_2|}{gT}}_{\sim \frac{1}{\rho}}$$

unless $|\omega_1 - \omega_2| \ll \Delta$

Look for a resonance
or
pay the thermal exponential

(guiding resonance)

To create the stochastic layer:

the pendulum frequency $\Omega \sim \sqrt{\tau \Delta g I}$

$$\frac{|\omega_2 - \omega_3|}{\Omega} \sim \frac{1}{\sqrt{\tau \rho}}$$

unless $|\omega_2 - \omega_3| \ll \Delta$

Look for another resonance
or
pay the Melnikov-Arnold exponential

(layer resonance)

Chaotic oscillators are rare:

density $\sim \min\{\tau\rho, \rho^2\}$

Making a pendulum out of more oscillators

$$-\tau\Delta(\psi_1^*\psi_2+\psi_2^*\psi_1)+\omega_2\psi_2^*\psi_2-\tau\Delta(\psi_2^*\psi_3+\psi_3^*\psi_2)$$

↳ effective coupling 1 ↔ 3: $\frac{(\tau\Delta)^2}{\omega_1-\omega_2}(\psi_1^*\psi_3+\psi_3^*\psi_1)$

works when $\omega_1 \approx \omega_3 \neq \omega_2$

Tunnelling + nonlinearity → effective couplings of the form

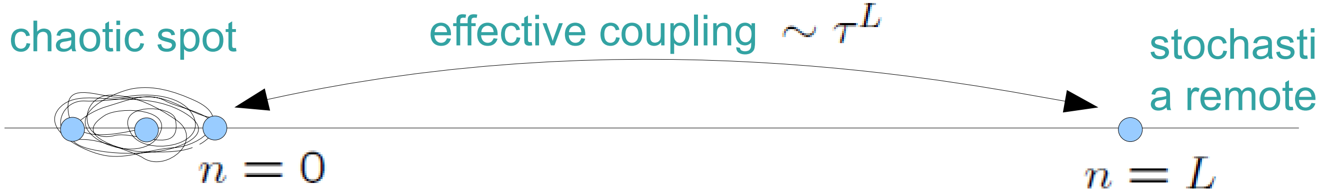
$$\psi_1^*\psi_2^*\psi_3^*\psi_4\psi_5\psi_6 \rightarrow \cos(\phi_1+\phi_2+\phi_3-\phi_4-\phi_5-\phi_6)$$

Guiding and layer resonances can be generated in high orders of the perturbation theory

Competition: **<u>number of combinations ↔ power of the coupling constants</u>**

size of a chaotic spot << distance between chaotic spots

Arnold diffusion



chaotic spot

effective coupling $\sim \tau^L$

stochastic force on a remote oscillator

$n = 0$

$n = L$

Effective coupling $2V_{m_1 \ldots m_N} \cos(m_1\phi_1 + \ldots + m_N\phi_N)$

action conservation: $m_1 + \ldots + m_N = 0$

Change in actions due to the stochastic force after time t

$$\langle \delta I_n \delta I_{n'} \rangle \sim t\, V^2_{m_1 \ldots m_N} \frac{m_n m_{n'}}{\Omega} \exp\left(-\frac{|m_1\omega_1 + \ldots + m_N\omega_N|}{\Omega} \right)$$

probability = $\underbrace{f(\{I_n\})}_{\text{distribution function}}\, W_s\, g|\vec{m}^{g}|^2\, \delta\big(\underbrace{m_1^{g}(\omega_{n_1^g} + gI_{n_1^g}) + \ldots + m_N^{g}(\omega_{n_N^g} + gI_{n_N^g})}_{\text{guiding resonance}}\big) \prod_n dI_n$

stochastic layer area

$$W_s \frac{\partial f}{\partial t} = \sum_{n,n'} \frac{\partial}{\partial I_n} W_s D_{nn'} \frac{\partial f}{\partial I_{n'}}$$

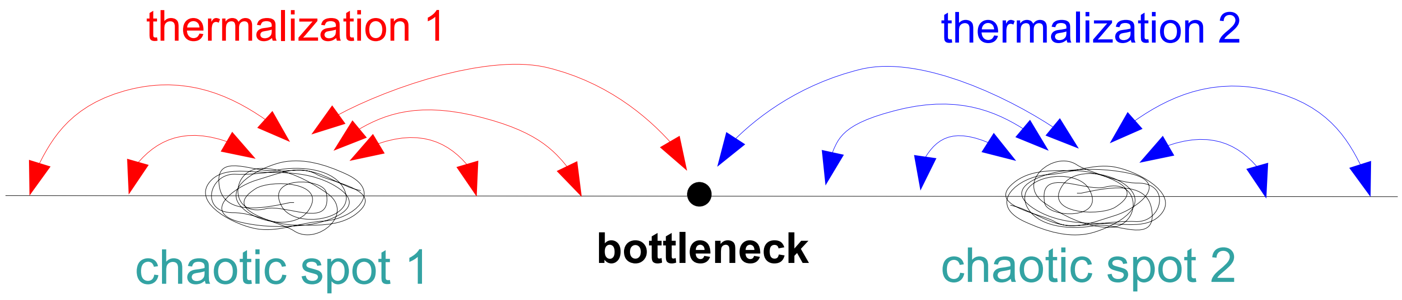
Constraints on the diffusion:

1. Total action is conserved

2. Total energy is conserved

3. The system stays on the guiding resonance

Long-distance relaxation



Typical density of chaotic spots $\sim \rho^2$

Coupling between the chaotic spots and the bottleneck $\sim \tau^{1/\rho^2}$

worse than activation ($\rho \propto T$)

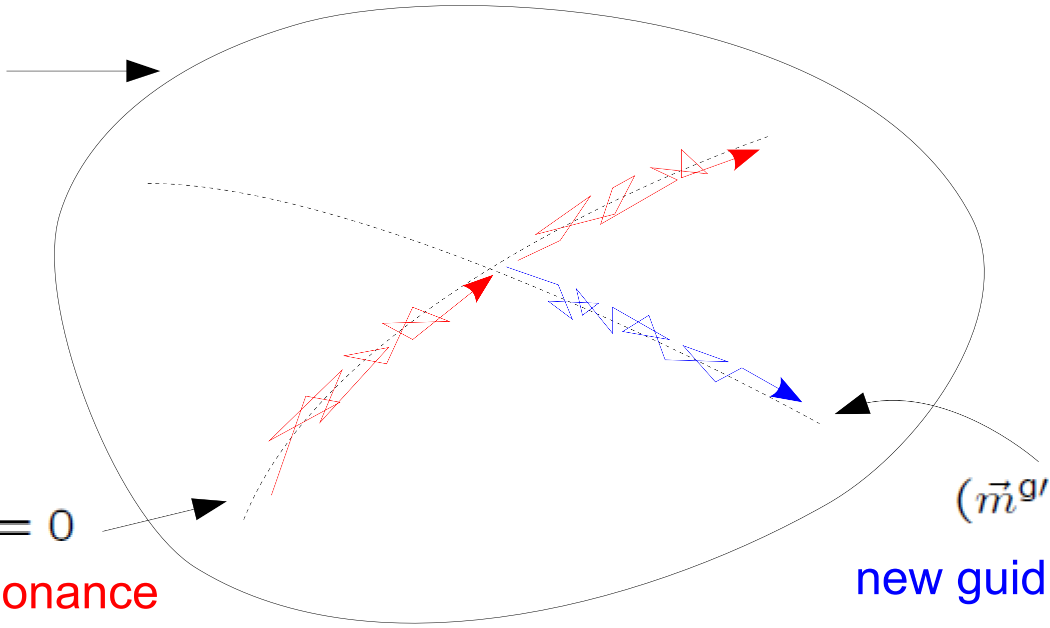
Look for a better mechanism!

Changing the guiding resonance

1. **Time needed to create another chaotic spot at a distance L**

   ~ (thermalization time at the distance L) $\times e^{\text{activation}}$

2. **One of the two chaotic spots is quickly quenched**



hypersurface
$H = \text{const}$
$I = \text{const}$

$(\vec{m}^{\mathrm{g}}, \vec{\omega}+g\vec{I}) = 0$

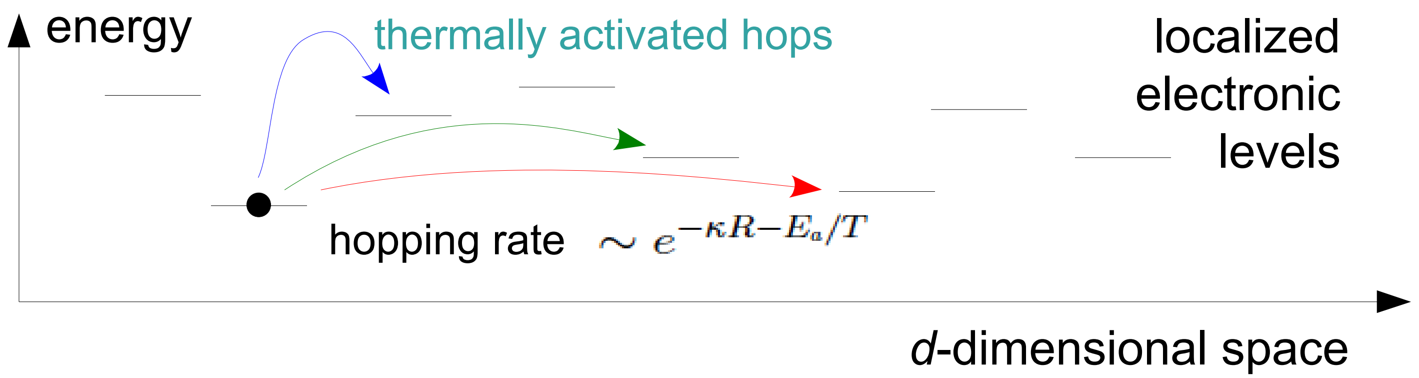
old guiding resonance

$(\vec{m}^{\mathrm{g}'}, \vec{\omega}+g\vec{I}) = 0$

new guiding resonance

Chaotic spots can randomly migrate along the chain

Variable-range hopping of electrons


energy

thermally activated hops

localized electronic levels

hopping rate $\sim e^{-\kappa R - E_a/T}$

d-dimensional space

To find a low level one should explore large distances $E_a^{min} \sim \dfrac{1}{\nu R^d}$

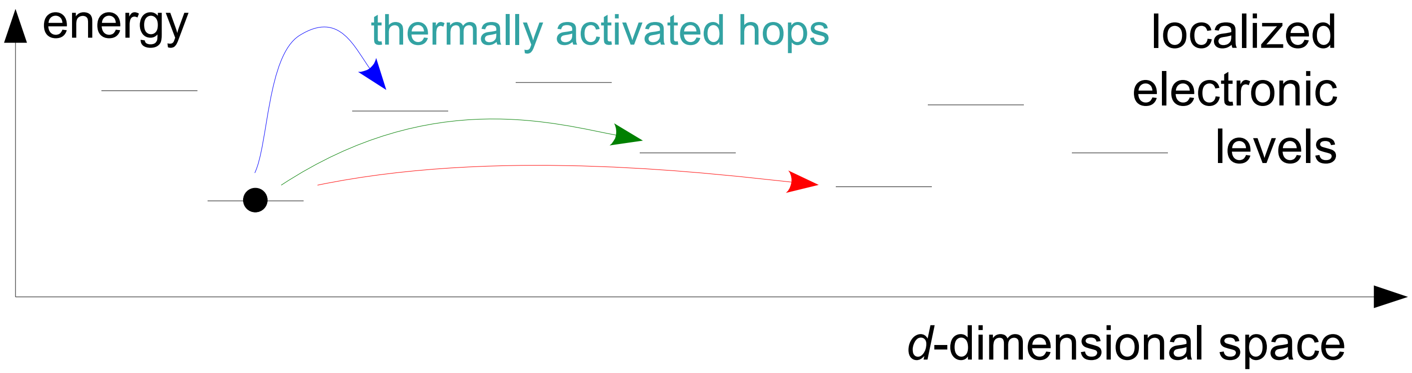
Competition between $e^{-\kappa R}$ and $e^{-E_a/T}$

$$\sigma(T) \propto \max_R e^{-\kappa R-(\nu R^d)^{-1}/T} = \exp\left[-\frac{d+1}{d}\left(\frac{\kappa^d}{\nu T}\right)^{1/(d+1)}\right]$$

Mott (1969)

stretched exponential after optimization

Variable-range hopping of electrons



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~~Mott (1969)~~

stretched exponential ~~after optimization~~

One dimension:

$$\sigma(T) \sim \exp\left(-\frac{\kappa}{2\nu T}\right)$$

Kurkijarvi (1973)

Rare “bad” regions block the transport in 1D

“Breaks”

Chaotic fraction w_n

$$w_n = \frac{1}{Z} \int_{\text{chaotic}(n)} e^{-(H-\mu I)/T} \prod_n \frac{dI_n\, d\phi_n}{2\pi}, \qquad Z \equiv \int e^{-(H-\mu I)/T} \prod_n \frac{dI_n\, d\phi_n}{2\pi}$$

all guiding resonances whose leftmost oscillator is *n*

$w = e^{-\lambda}$ for chaotic spots ⟺ $e^{-E_a/T}$ for electrons

thermal exponential

Melnikov-Arnold exponential

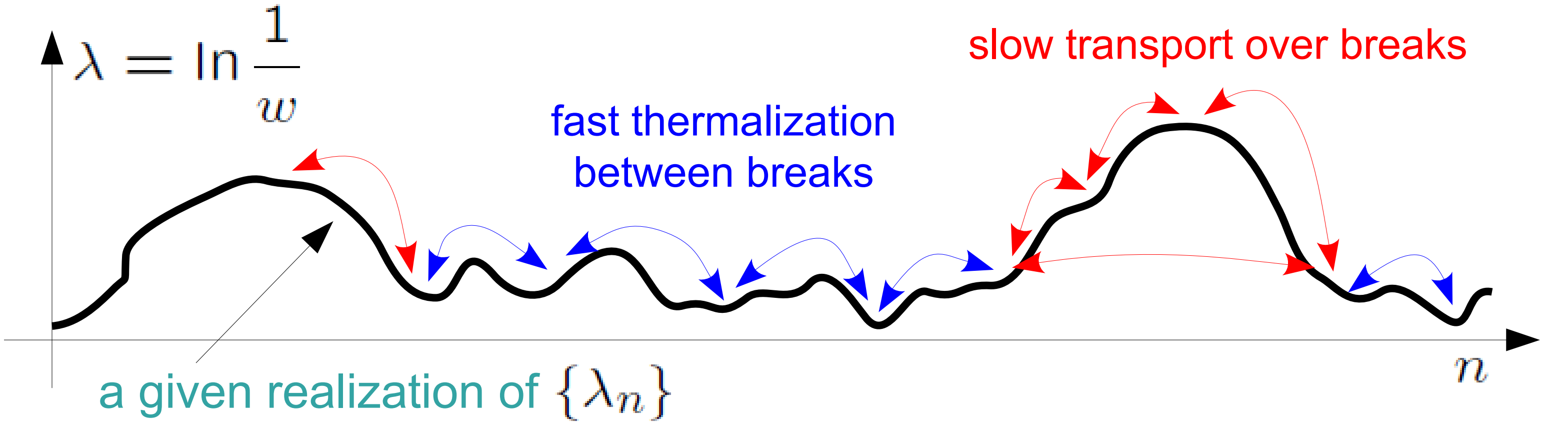
coupling constants

w is a random quantity determined by the disorder

Probability distribution:

$$\mathcal{P}\left\{w < w_0\right\} = \exp\left(-\mathcal{C}_1 \rho\, w_0^{1/[\mathcal{C}\ln^2(1/\tau^p \rho)]}\right)$$

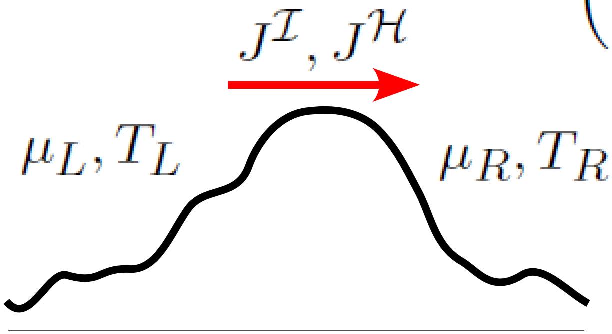
From λ to σ : break resistance



slow transport over breaks

fast thermalization between breaks

a given realization of $\{\lambda_n\}$



1. Definition of the current

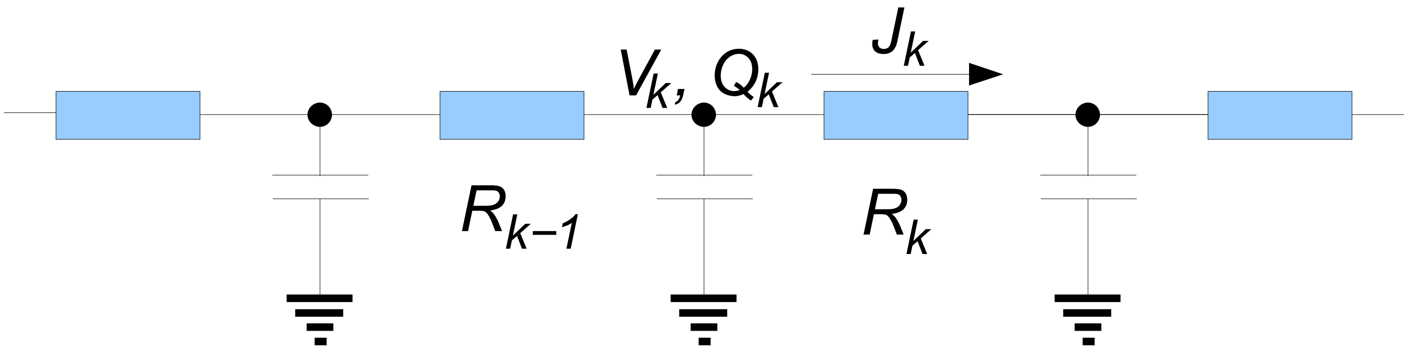
$$\begin{pmatrix} J^{\mathcal{I}} \\ J^{\mathcal{H}} \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} \langle I \rangle_R \\ \langle H \rangle_R \end{pmatrix}$$

2. The diffusion equation for actions

$$\begin{pmatrix} J^{\mathcal{I}} \\ J^{\mathcal{H}} \end{pmatrix} = R_b^{-1} \begin{pmatrix} \mu_L/T_L - \mu_R/T_R \\ 1/T_R - 1/T_L \end{pmatrix}$$

Each break can be characterized by its “resistance”

From λ to σ: resistors in series



$$\left\{\begin{aligned} \frac{dQ_k}{dt} &= J_{k-1} - J_k \\ J_k &= R_k^{-1}(V_k - V_{k+1}) \\ Q_k &= Q_k(V_k) \end{aligned}\right.$$

definition of the current

"resistance" of the break

thermodynamics

⇩

$$\left\{\begin{aligned} \frac{\partial Q}{\partial t} &= \frac{\partial}{\partial x}\sigma(V)\frac{\partial V}{\partial x} \\ Q &= Q(V) \end{aligned}\right.$$

macroscopic "charge density" $Q = \frac{1}{L}\sum Q_k$

macroscopic "conductivity" $\sigma = \left(\frac{1}{L}\sum R_k\right)^{-1}$

Optimal breaks

$$\sigma^{-1} = \frac{1}{L} \sum_{\text{breaks} \in L} R_b$$

self-averaging at long distances

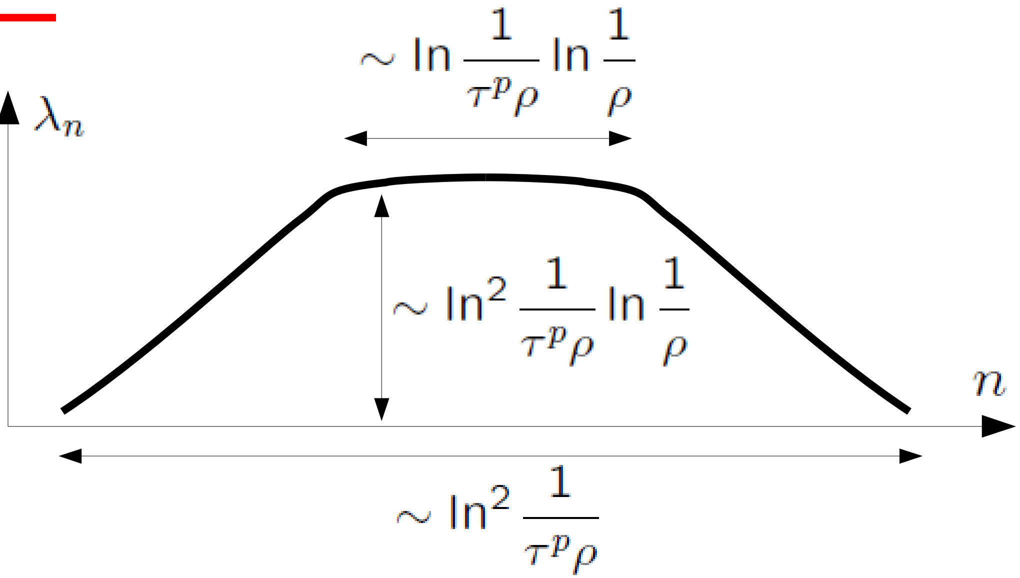
$$\sigma^{-1} = \int R_b(\{\lambda_n\})\, \underline{dP(\{\lambda_n\})}$$

probability measure per unit length

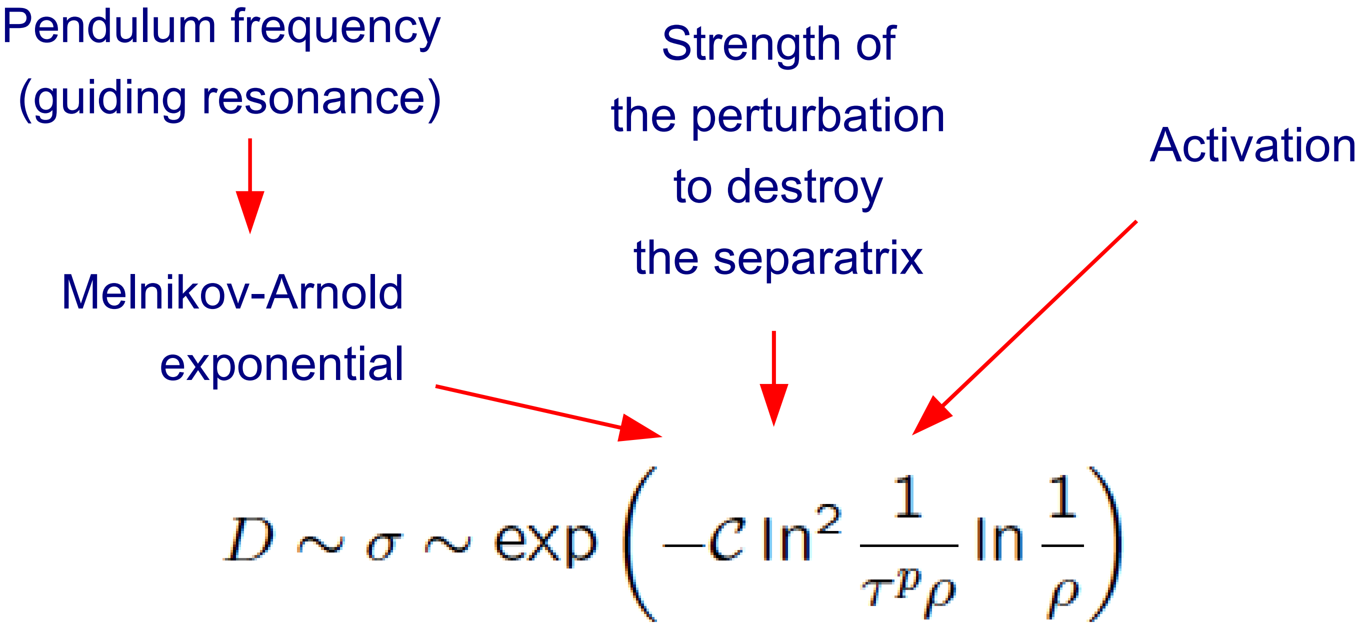
increasing

decreasing

The integral is dominated by configurations close the optimal one



Macroscopic diffusion coefficient: three logarithms



Pendulum frequency (guiding resonance)

Melnikov-Arnold exponential

Strength of the perturbation to destroy the separatrix

Activation

$$D \sim \sigma \sim \exp\left(-\mathcal{C} \ln^2 \frac{1}{\tau^p \rho} \ln \frac{1}{\rho}\right)$$

Macroscopic length scale

$$L^* \sim \exp\left(\mathcal{C} \ln^2 \frac{1}{\tau^p \rho}\right)$$

distance between the optimal breaks

Conclusions

1. Anderson localization + weak nonlinearity → weak chaos
2. Rare chaotic spots play the role of a bath
3. They induce relaxation by driving the Arnold diffusion
4. They migrate along the chain
5. In 1D the transport of conserved quantities is determined by rare breaks

D. M. Basko, arXiv:1005.5033