C-theorems and entanglement entropy

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Plan of the talk

What is a c-theorem. What do we need?

Zamolodchikov theorem in 1+1

Entanglement entropy in QFT: properties

Entropic c-theorem in 1+1

Comparizon between c-functions

Proposals for generalization to higher dimensions

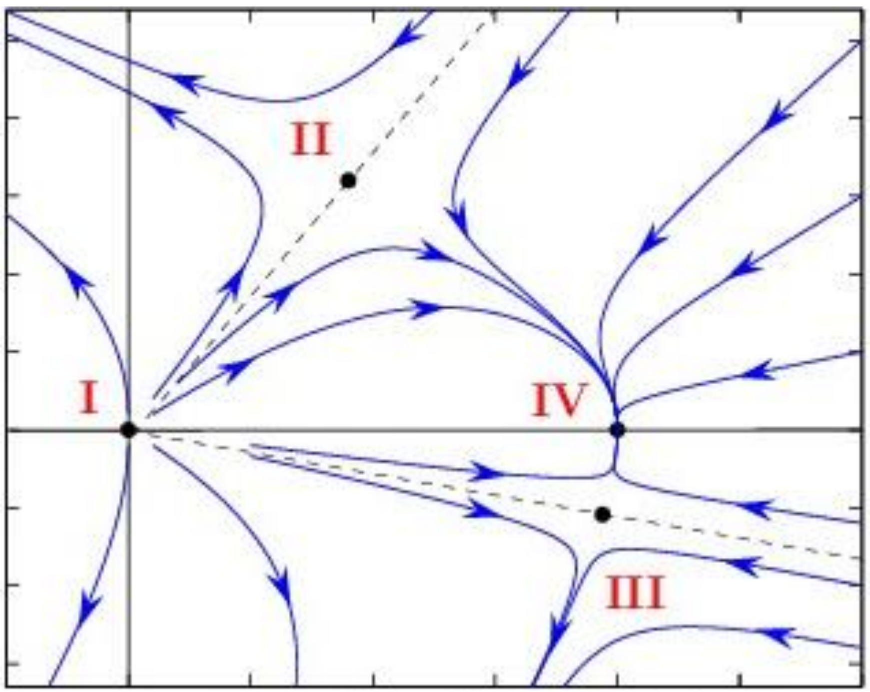
Entropic and free energy versions

Entropic F-theorem (d=2+1)

Subtleties: mutual information

Conclusions

Renormalization group flow in the space of QFT



$$\tau \frac{dg_i}{d\tau} = \beta_i(\{g(\tau)\})$$

Change in the physics with scale through the change of coupling constants with the RG flow. At fixed points there is scale invariance: the theory looks the same at all scales. The RG flow Interpolates between UV (short distance) to IR (large distance) fix points.

Are there any general constraints on these RG flows?

C-theorem: General constraint for the renormalization group: Ordering of the fixed points.

C – Theorem: What is needed?

1) A regularization independent quantity C, well defined in the space of theories.

2) C dimensionless and finite at the fix points. C partially characterizes the fix points.

3) C decreases along the renormalization group trajectories. In particular $C_{\rm UV} \geq C_{\rm IR}$

A universal dimensionless decreasing function C(r) of some length scale r will do the job

small size $C(r) \longrightarrow C_{\rm UV}$

large size $C(r) \longrightarrow C_{\rm IR}$

$$\left.\begin{array}{r} \tau \dfrac{\partial}{\partial \tau} C = -\sum_i \beta_i(g) \dfrac{\partial}{\partial g_i} C \\ \left(r\dfrac{\partial}{\partial r} - \tau \dfrac{\partial}{\partial \tau}\right) C = 0 \end{array}\right\} \; r\frac{dC(r)}{dr} = -\sum \beta_i(g) \frac{\partial}{\partial g_i} C$$

Zamolodchikov's C-theorem in 1+1 dimensions (1986)

C is the central charge of the conformal field theory fix point.
Can be extracted from the two point function of the stress tensor at the fix point

$$\langle T_{\mu\nu}(0) T_{\alpha\beta}(x)\rangle = \frac{C}{|x|^4} I_{\mu\nu,\alpha\beta}(\vec{x})$$

Using conservation of the stress tensor and Lorentz symmetry an interpolating function can be constructed

$$C(r) = \frac{3}{4\pi}\int_r^\infty d^2x \; x^2 \, \langle \Theta(0)\Theta(x)\rangle \qquad \Theta(x) = T_\mu^\mu(x)$$

$$C'(r) = -\frac{3}{2} r^3 \, \langle \Theta(0)\Theta(x)\rangle \leq 0 \longrightarrow$$

Reflection positivity: unitarity in the Euclidean correlation functions

$$\langle 0| \int dx\, \alpha^*(x)\Theta(x) | \int dy\, \alpha(y)\Theta(y)|0\rangle \geq 0$$

Needs unitarity and Lorentz invariance.
Stress tensor: we expect to have one for every theory!
Θ is zero for CFT and it drives the C-function out of the fix point: C cannot remain constant when there is a RG flow

Entanglement entropy: universaly defined for any theory

Reduced density matrix $\rho_V = \mathrm{tr}_{-V} |0\rangle \langle 0| \longrightarrow S(V) = -\mathrm{tr}\rho_V \log \rho_V$

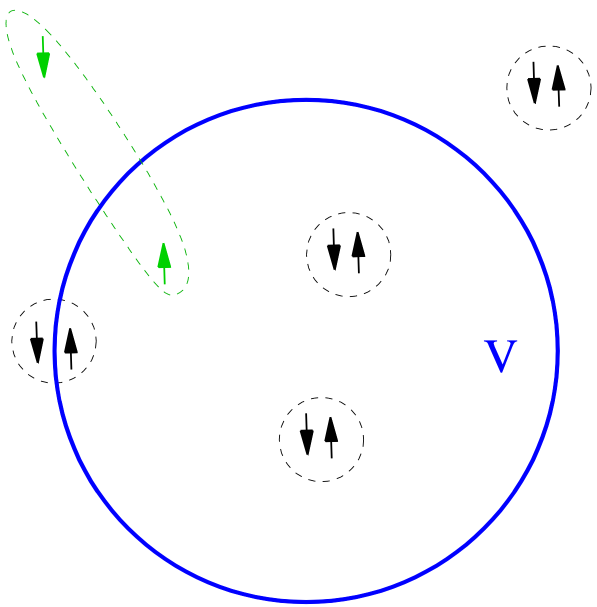
S(V) meassures the entropy in vacuum fluctuations

Entanglement entropy

Structure of divergences:

$$S(V) = \underbrace{g_{d-1}[\partial V]\epsilon^{-(d-1)}}_{\text{Area law}} + \cdots + g_1[\partial V]\epsilon^{-1} + g_0[\partial V]\log\epsilon + S_0(V)$$

The functions g are local and extensive on the boundary due to UV origin of divergences.

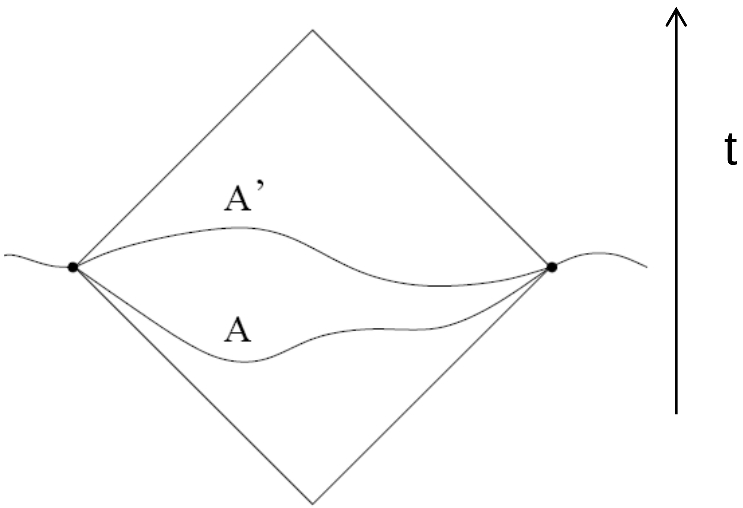


Large ammount of short distance entanglement and little, but very important large distance entanglement.
We always want to get rid of short distance entanglement

Properties:

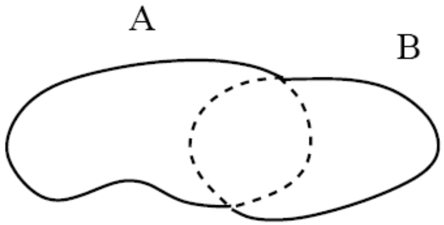
Causality $S(A) = S(A') \quad (\rho_A = \rho_{A'})$

S is a function of the "diamond shaped region" of equivalently the region boundary (vacuum state on an operator algebra)



Strong subadditivity

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

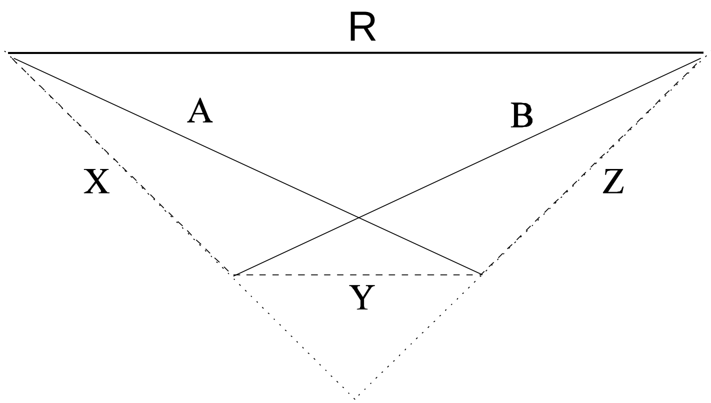


Conditions for use of SSA in spacetime
Cauchy (spatial) surface passing through a A and B.
Boundaries must be spatial to each other

Entanglement entory universally defined and has a nice inequality…

Entropic C-theorem from SSA in d=1+1

H.C., M. Huerta, 2004



$$S(XY)+S(YZ)\geq S(Y)+S(XYZ)$$

$$XY\equiv A\,,\qquad YZ\equiv B\,,\qquad XYZ\equiv R$$

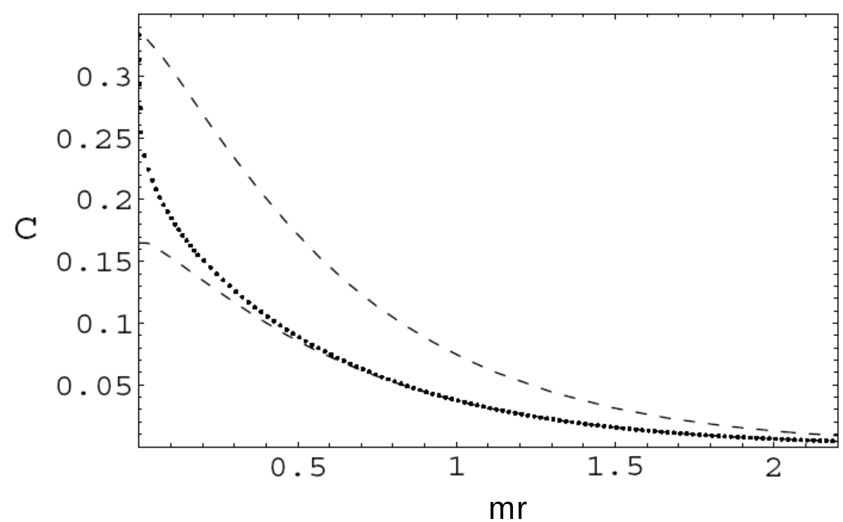
$$2\,S(\sqrt{rR})\geq S(R)+S(r)\,.$$

$$rS''(r)+S'(r)\leq 0\,.\qquad C(r)\;=\;rS'(r)\;\longrightarrow\;C'(r)\;\leq\;0$$

C(r) dimensionless, well defined, decreasing. At conformal points:

$$S(r)=\frac{c}{3}\log(r/\epsilon)+c_0\;\longrightarrow\;\text{C(r)=c/3}$$

The central charge of the uv conformal point must be larger than the central charge at the ir fixed point: the same result than Zamolodchikov c-theorem but different interpolating function



Again Lorentz symmetry and unitarity are used but in a different way

Different c-functions, does it matter?

Is there a best one?
Are they related to each other?
Non-stationarity of entropic c-function

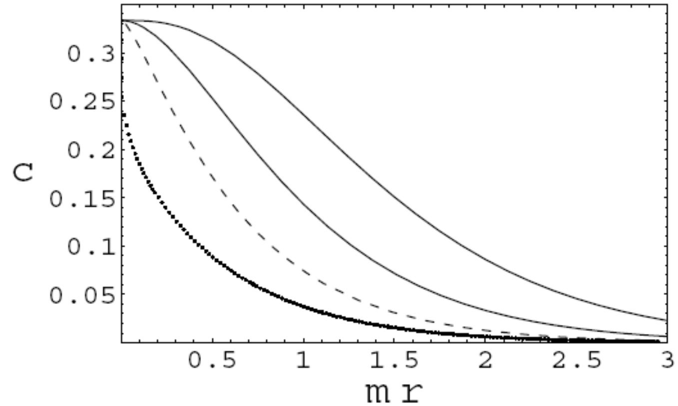


Figure 2. From top to bottom: one third of the Zamolodchikov c-functions for a real scalar and a Dirac field, and entropic c-functions for a Dirac (dashed curve) and a real scalar field (dotted curve).

$$c_D(t) \sim \frac{1}{3} - \frac{1}{3} t^2 \log^2(t) \quad \text{for } t \ll 1$$
$$c_S(t) \sim \frac{1}{3} + \frac{1}{2\log(t)} \quad \text{for } t \ll 1 \; ;$$

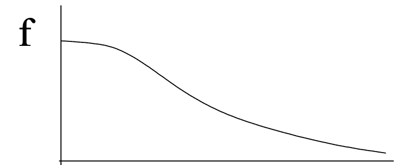
Once we have one c-function we can construct infinitely many other by convoluting with a numerical function. They are highly non-unique.

Cardy (1988)
Cappelli, Friedan, Latorre (1991)
infinitely many c-functions exploiting positivity of spectral density

Spectral decomposition of the θ correlator

$$\langle \Theta(0)\Theta(x)\rangle = \frac{\pi}{3}\int_0^\infty d\mu\ \rho(\mu)\,\Box^2 G_0(x,\mu)\,, \qquad \rho(\mu)\geq 0$$

$$\tilde{c}(r) = \int d\mu\ \rho(\mu)\, f(\mu r)\,, \qquad f(x)>0,\ f'(x)<0,\ f(0)=1,\ f(\infty)=0$$



$$\rho_{\text{scalar}} = \rho_{\text{Dirac}} + \frac{1}{2}\partial_\mu(\mu\,\rho_{\text{Dirac}}) \longrightarrow \tilde{C}_{\text{scalar}}(r) = \tilde{C}_{\text{Dirac}}(r) - \frac{1}{2}\, r\,\partial_r\, \tilde{C}_{\text{Dirac}}(r)$$

The entropic c-functions do not satisfy this relation. Contain different information than the two point function of θ. No simple direct relation between strong subadditivity and correlator positivity.

C-theorem in more dimensions?

Proposal for even dimensions: coefficient of the Euler density term in the trace anomaly at the fixed point, Cardy (1988).

d=2 $\langle \Theta \rangle = -cR/12 \longrightarrow c = -\frac{3}{\pi}\int_{S^2} \langle \Theta \rangle \sqrt{g}\, d^2x$

general d $\langle \Theta(x) \rangle = \frac{(-1)^{d/2}}{2} a_d E(x) + \text{other polynomials of order d/2 in the curvatura tensor}$

d=4: $E(x) \sim R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$

$$\longrightarrow C = (-1)^{d/2} a_d \int_{S^d} d^dx \sqrt{g} \langle \Theta \rangle$$

As θ meassures the variation of the effective action under scaling this number is proportional to the logarithmically divergent term in log Z on a d-dimensional sphere

Proved by Komargodski and Schwimmer for d=4 (2011) (a-theorem) using the effective action for the dilaton coupled to the theory and a sum rule = unitarity of the S-Matrix.

Odd dimensions? No trace anomaly in odd dimensions

Myers-Sinha (2010) Holographic c-theorems

$$ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}^2_{d-1}) + dr^2 \qquad A(r) = \text{cons} \quad r$$ at fixed points (AdS space)

Higher curvature gravity lagrangians:
a(r) function of A(r) and coupling constants
$a(r) = a^* = \text{constant}$ at fixed points

$$a'(r) \sim (T^t_t - T^r_r) \geq 0$$

null energy condition

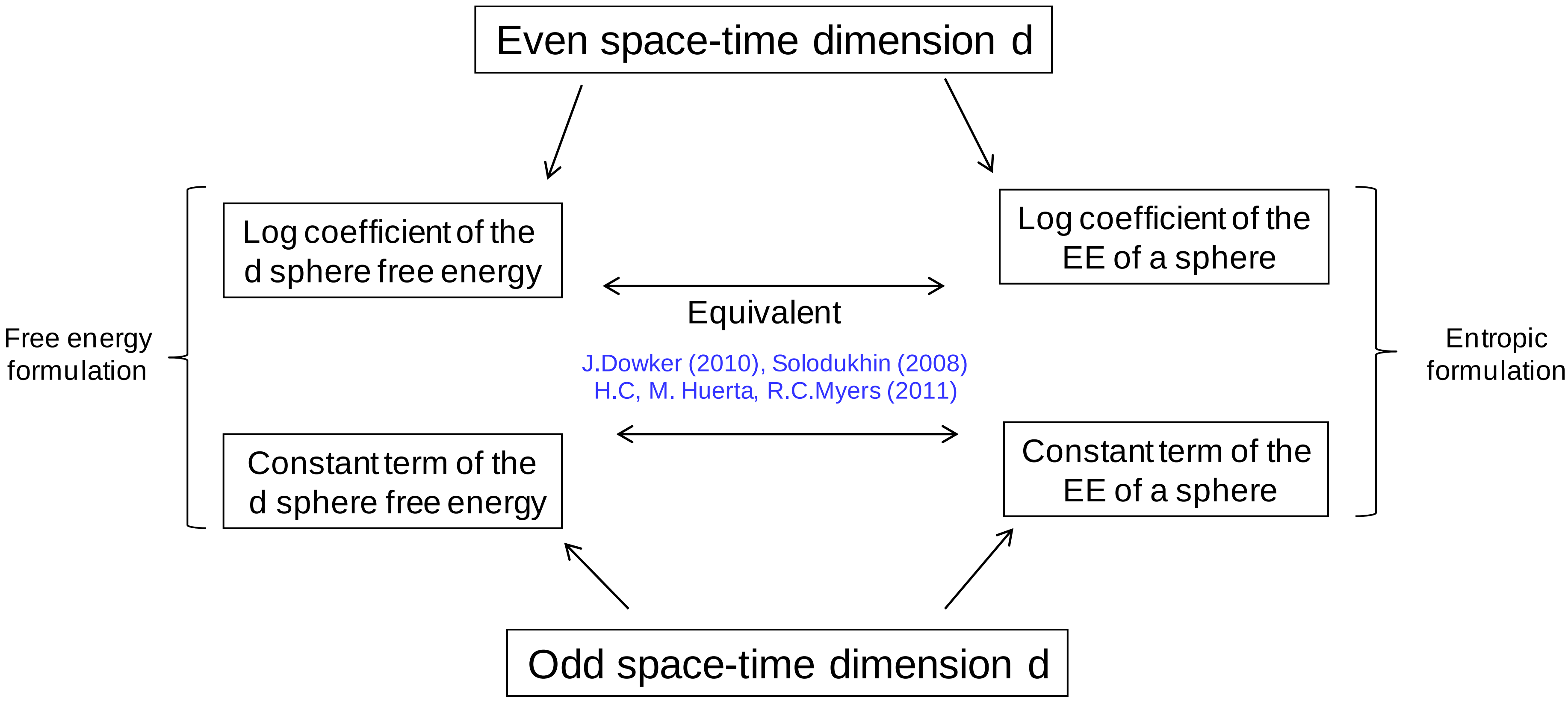
$$a^*_{uv} \geq a^*_{ir}$$

QFT interpretation: For even spacetime dimensions a^* is the coefficient of the Euler term in the trace anomaly (coincides with Cardy proposal for the c-theorem)

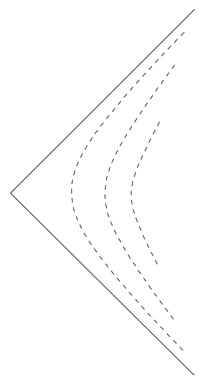
For odd dimensions the constant term of the sphere entanglement entropy is proportional to a^* (by interpreting entanglement entropy in the boundary as BH entropy in the bulk)

F-theorem (Jafferis, Klebanov, Pufu, Safdi (2011)): propose finite term in the free energy F=–log(Z) of a three sphere decreases between fix points under RG. Non trivial tests for supersymmetric and non-susy theories (Explicit computations of F for interacting theories by localization)

Relation between EE of spatial (d-2) entangling sphere / partition function on euclidean d-sphere

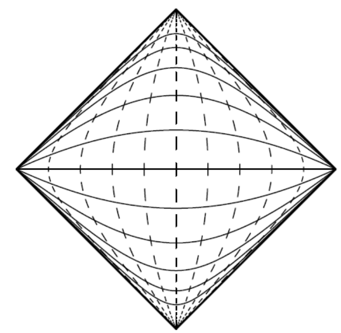


H.C, M. Huerta, R.C.Myers (2011)



Conformal mapping

$\longleftrightarrow$



S=entanglement entropy

Causal domain of dependence of a sphere in Minkowski

Rindler Wedge
Unruh temperature
(any QFT)

$$\rho \sim e^{2\pi K}$$

Conformal mapping

$\updownarrow$

Static patch in de Sitter space

$$ds^2 = -\left(1-\frac{\hat{r}^2}{R^2}\right)d\tau^2 + \frac{d\hat{r}^2}{1-\frac{\hat{r}^2}{R^2}} + \hat{r}^2\, d\Omega_{d-2}^2 \longrightarrow$$

Thermalized space is a three sphere

$$T = 1/(2\pi R)\,, \qquad \langle T^\mu_\nu \rangle = \kappa\, \delta^\mu_\nu = 0 \quad \longleftrightarrow \quad S = \beta E + \log(Z) = \log(Z)$$

odd dimensions

Hence Myers-Sinha and F-theorem (Jafferis,Klebanov,Pufu,Sadfi) proposals coincide for the monotonic quantity at fix points.
It extends Cardy proposal to odd dimensions

Is the choice of constant term in Log(Z) on a sphere natural in odd dimensions? Dimensionaly continued c-theorem for free fields: some numerology

Normalize the c-charge to the scalar c-charge in any dimension. For the Dirac field we have for the ratio of c-charge to number of field degrees freedom

d	2	4	6	8	10	12	14
$\frac{c[\mathrm{Dirac}]}{2^{d/2}c[\mathrm{scalar}]}$	$\frac{1}{2}$	$\frac{11}{4}$	$\frac{191}{40}$	$\frac{2497}{368}$	$\frac{73985}{8416}$	$\frac{92427157}{8562368}$	$\frac{257184319}{20097152}$
approx.	0.5	2.75	4.775	6.7853	8.7909	10.7946	12.7971

Fitting as $$\frac{C[\mathrm{Dirac}]}{2^{d/2}C[\mathrm{scalar}]} = (d-2) + k_0 + \frac{k_1}{d} + \frac{k_2}{d^2} + \ldots$$

Fitting with 100 dimensions gives for d=3 $$\frac{C[\mathrm{Dirac}]}{2^{[d/2]}C[\mathrm{scalar}]} \to 1.7157936606$$

The correct value $$\frac{C[\mathrm{Dirac}]}{2^{[d/2]}C[\mathrm{scalar}]} = \frac{\frac{\log(2)}{4} - \frac{3\zeta(3)}{8\pi^2}}{\frac{\log(2)}{4} + \frac{3\zeta(3)}{8\pi^2}} = 1.71579366494\ldots$$

Reason? The ratios of free energies on the sphere in zeta regularization $$F = -\frac{1}{2}\lim_{s\to 0}\left[\mu^{2s}\zeta'(s) + \zeta(s)\log(\mu^2)\right]$$

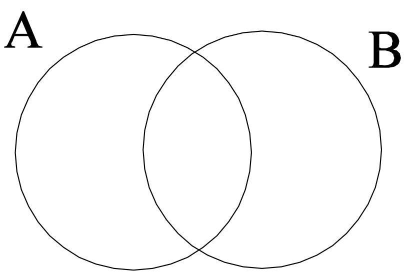
Ratio of C charges is always (for d odd or even)

$$\lim_{s\to 0}\frac{\zeta^1(s)}{\zeta^2(s)} = \begin{cases} \dfrac{\zeta^1(0)}{\zeta^2(0)} & \text{even dimensions} \\[2ex] \dfrac{\zeta^1{}'(0)}{\zeta^2{}'(0)} & \text{odd dimensions} \end{cases}$$

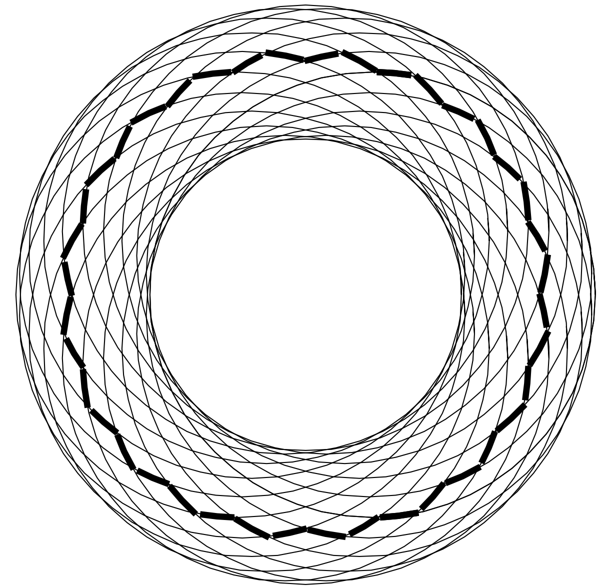
The same could be expected for the ratios of the entropies of spheres (taking out power-like divergent terms)

C-theorem in d=3 from SSA for circles

H.C, M. Huerta 2012



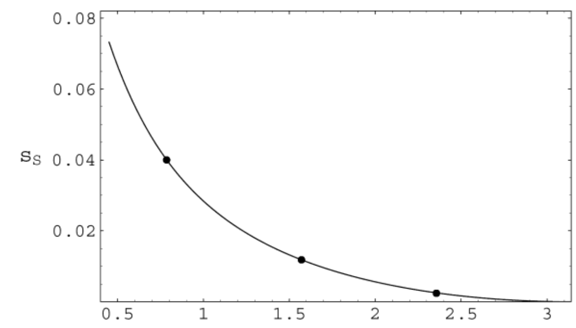
Two problems: different shapes and log divergent angle contributions. Use many rotated regions for first problem



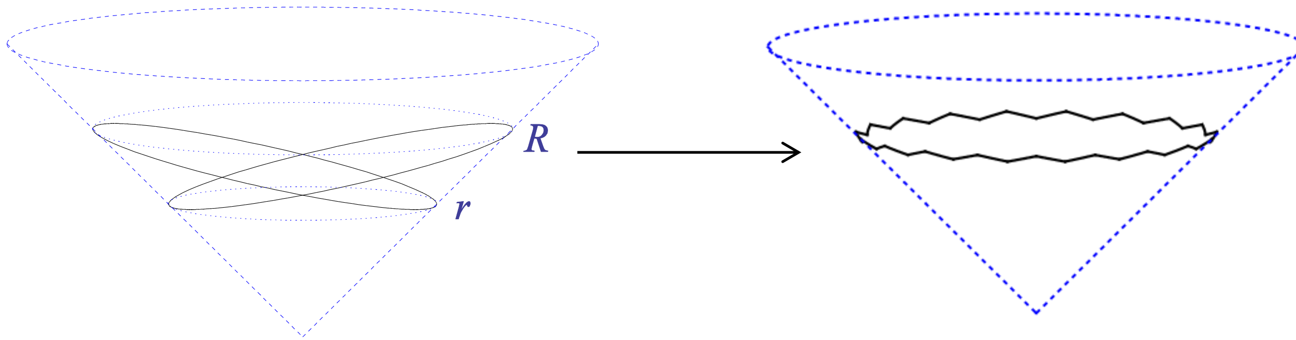
From SSA:

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i)$$

Log divergent terms cannot appear for «angles» on a null plane since the feature does not have any local geometric measure



Coefficient of the logarithmically divergent term for a free scalar field



$$S(\sqrt{Rr}) \geq \frac{1}{\pi}\int_0^\pi dz\, S\left(\frac{2rR}{R+r-(R-r)\cos(z)}\right) \Rightarrow S'' \leq 0$$

$$c_0(r) = rS'(r) - S(r) \Rightarrow c_0(r)' \leq 0$$

Dimensionless and decreasing
C-function proposed by H.Liu and M. Mezei (2012)
Based on holografic and QFT analysis

At fixed points $S(R) = c_1 R - c_0$

$c_0(r) = c_0$ Is the constant term of the entropy of the circle

Running of area term: allways decreases towards the infrared

At fix points $S(R) = R\left(\frac{k_1}{\epsilon} + k_0\right) - c_0$

Away from fix points $S''(R) < 0$

$$
\begin{aligned}
c_0^{UV} - c_0^{IR} &= -\int_0^\infty dR\, R\, S''(R) \geq 0\,, \\
\mu = k_0^{IR} - k_0^{UV} &= \int_0^\infty dR\, S''(R) \leq 0\,.
\end{aligned}
$$

Sum rule for variation of area term

V.Rosenhaus,M.Smolkin (2014),
H.C., D.Mazzitelli,E.Teste (2014)

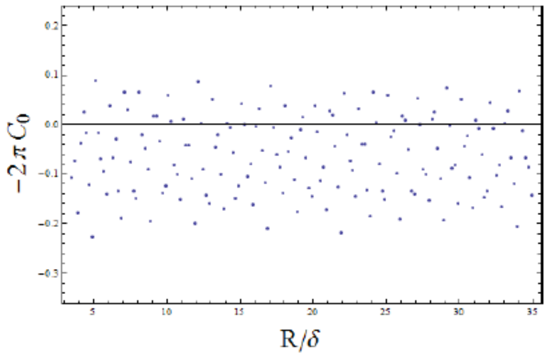
$$\mu = -\frac{\pi}{6}\int d^3x\; x^2\,\langle 0|\Theta(0)\Theta(x)|0\rangle$$

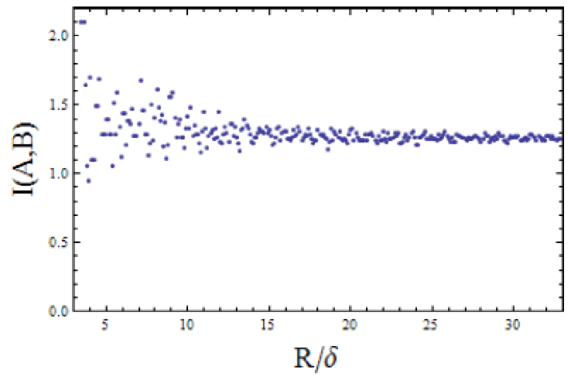
Area term drives constant term.
Implies C_0 necesarilly changes with RG running, as in Zamolodchikov's theorem

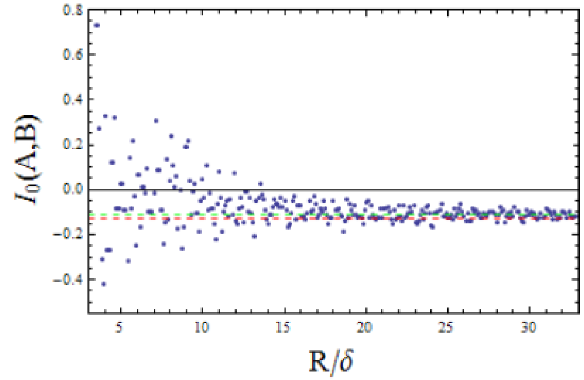
Change of area term can diverge for perturbations of the UV fix point with operator with dimension $3 > \Delta > 5/2$ but still change of C_0 can remain finite.

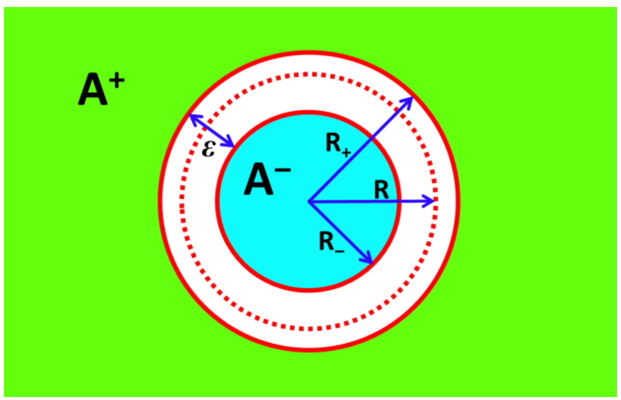
Subtleties in defining C_0 at fix points: Mutual information clarify this issue

H.C., M. Huerta, R.C. Myers, A. Yale









$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

The local divergences cancel in I(A,B) which is finite and well defined in QFT

Mutual information as a geometric regulator for EE: all coefficients on the expansion are universal and well defined

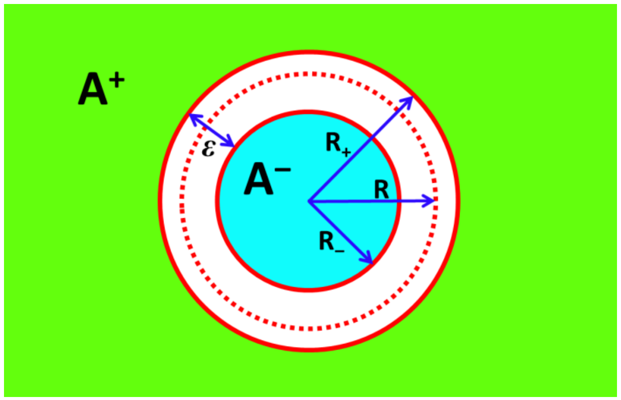
$$S(R) = \left(\frac{a}{\delta} + b\right) R - C_0 \rightarrow I(A^+, A^-) = 2\left(\frac{\tilde{a}}{\epsilon} + \tilde{b}\right) R - 2C_0$$

Locality+symmetry argument

(similar to Liu-Mezei 2012, Grover, Turner,Vishwanath 2011)

C charge well defined through mutual information
IR , UV values depend only on the CFT
This is a physical quantity calculable with any regularization, including lattice
The constant term coincides with the one in the entropy of a circle for «good enough» regularizations.

Topological entanglement entropy



For a topological model we have zero mutual information unless ϵ cross the scale of the gap.
ϵ must cross all mass scales in the theory to prove the c-theorem.

In that case we expect $I(A^+,A^-)=2\left(\frac{\tilde{a}}{\epsilon}+\tilde{b}\right)R-2C_0$

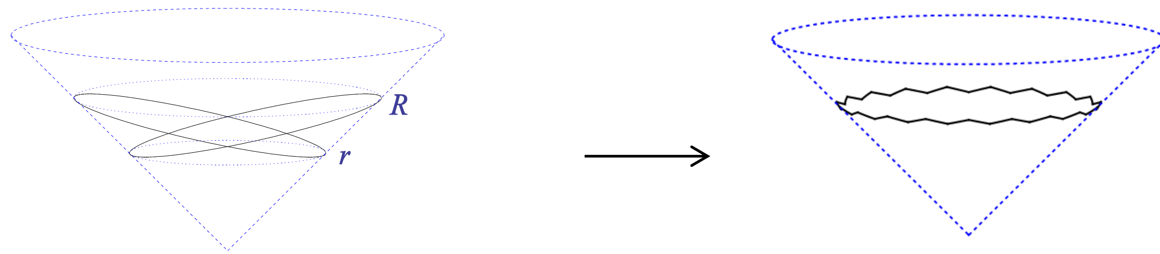
With $C_0=\gamma$ the topological entanglement entropy

Local degrees of freedom in the UV needed to properly define entanglement entropy

Does topological entanglement entropy play a role in the UV?

Entropic proof in more dimensions?

Symmetric configuration of boosted spheres in the limit of large number of spheres



Divergent terms do not cancel, trihedral angles, curved dihedral angles. Wiggly spheres not converge to smooth spheres: mismatch between curvatures.

More generally:

a) Strong subadditivity always gives inequalities for second derivatives

b) This inequality should give C'<0. Then C is constructed with S and S'

c) C has to be cutoff independent. But at fix points

$$S(r) = c_2 \frac{r^2}{\epsilon^2} + c_{\log} \log(r/\epsilon) + c_0$$

It is not possible to extract the coeff. of the logarithmic term with S and S'.

New inequalties for the entropy? Some posibilities have been discarded holografically (H.Liu and M. Mezei (2012)) c-theorem in 1+1 and 2+1 dimensions for relativistic theories have been found using entanglement entropy and strong subadditivity. No proof has been found yet for 2+1 that does not use entanglement entropy. (Difficult to construct C_0 from correlators if it contains topological information. How to uniquely define the theory on the sphere from the one in flat space?)

Why a c-theorem should exist?: loss of d.o.f along RG not a good reason (in a direct way). It can be a relativistic QFT theorem as CPT, spin-statistics, etc., or there is a deeper information theoretical explanation (suggested by entanglement entropy)

Is C a measure of «number of field degrees of freedom»?

C is not an anomaly in d=3. It is a small universal non local term in a divergent entanglement entropy. It is very different from a «number of field degrees of freedom»: Topological theories with no local degree of freedom can have a large C (topological entanglement entropy)! C does measure some form of entanglement that is lost under renormalization, but what kind of entanglement?

Is there some loss of information interpretation?

Even if the theorem applies to an entropic quantity, there is no known interpretation in terms of some loss of information. Understanding this could tell us whether there is a version of the theorem that extends beyond relativistic theories.