Many-body Entanglement Witness & Branching MERA

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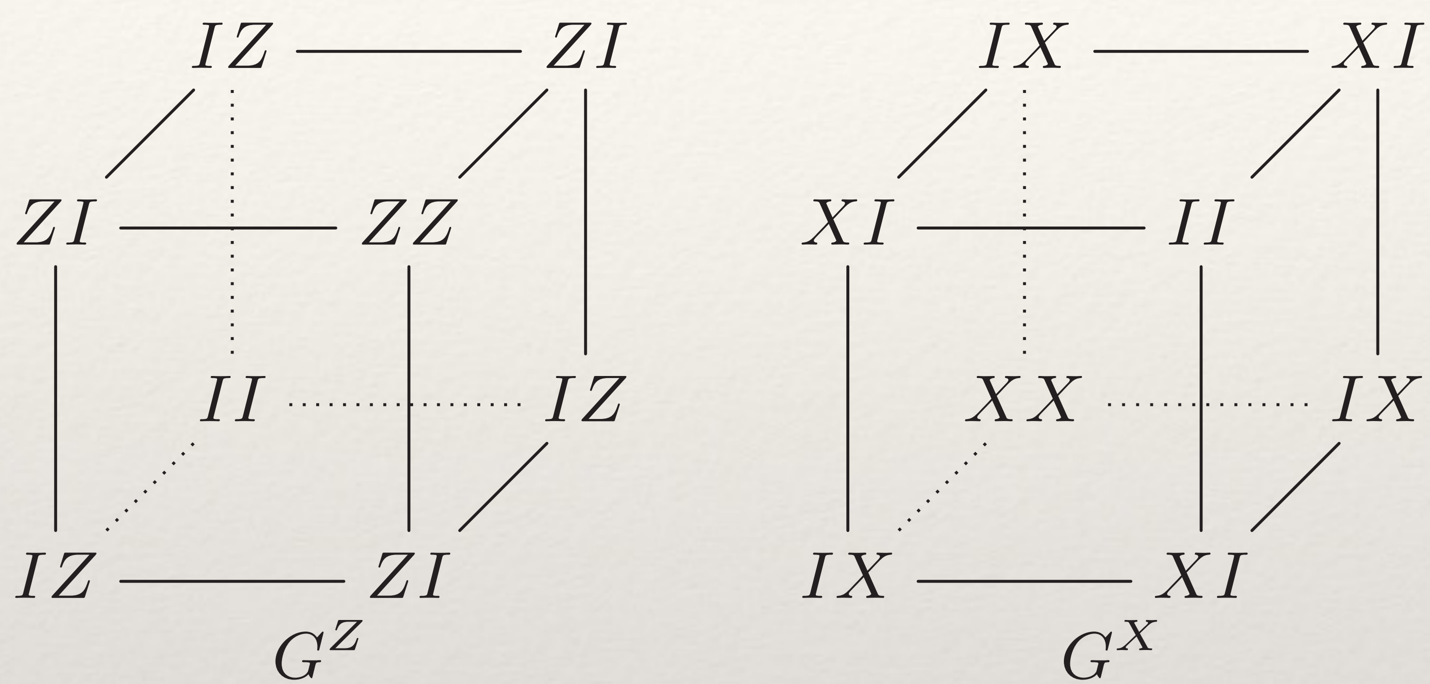
Cubic code



$$H = -J \sum_{i \in \Lambda} \Big(\sigma^x_{i,1} \sigma^x_{i,2} \sigma^x_{i+\hat{x},1} \sigma^x_{i+\hat{y},1} \sigma^x_{i+\hat{z},1} \sigma^x_{i+\hat{y}+\hat{z},2} \sigma^x_{i+\hat{z}+\hat{x},2} \sigma^x_{i+\hat{x}+\hat{y},2} + \sigma^z_{i,1} \sigma^z_{i,2} \sigma^z_{i-\hat{x},2} \sigma^z_{i-\hat{y},2} \sigma^z_{i-\hat{z},2} \sigma^z_{i-\hat{y}-\hat{z},1} \sigma^z_{i-\hat{z}-\hat{x},1} \sigma^z_{i-\hat{x}-\hat{y},1} \Big)$$

JH, 1101.1962

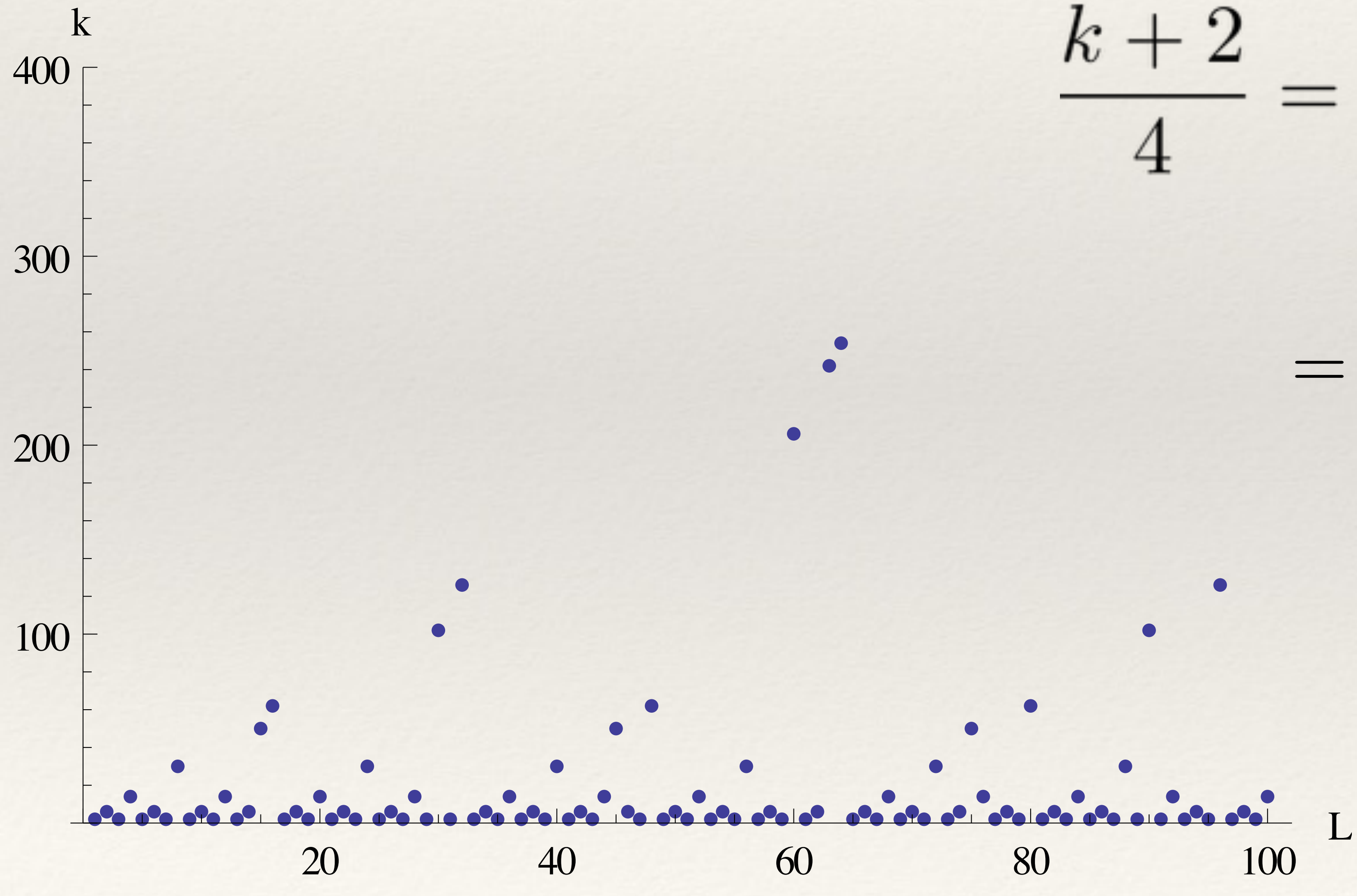
Cubic code

- Ground state is degenerate
  under periodic boundary conditions.
- No local operator is capable of lifting the degeneracy.
- **Topological Excitations** are point-like and immobile.
- Immobility is robust against perturbations
  [Isaac Kim, JH].
- **Branching MERA description.**

Degeneracy

Under **periodic** boundary conditions

$$2^k = \text{degeneracy}$$



$$\frac{k+2}{4} = \deg \gcd \left\{ \begin{array}{c} 1+(1+t)^L, \\ 1+(1+\omega t)^L, \\ 1+(1+\omega^2 t)^L \end{array} \right\}$$

$$= \begin{cases} L & \text{if } L = 2^p \\ 1 & \text{if } L = 2^p + 1 \end{cases}$$

JH, 1204.1063

Entanglement is an Invariant

$$|\psi\rangle = \sum_r \sqrt{\lambda_r} |\phi_r^A\rangle |\phi_r^B\rangle$$

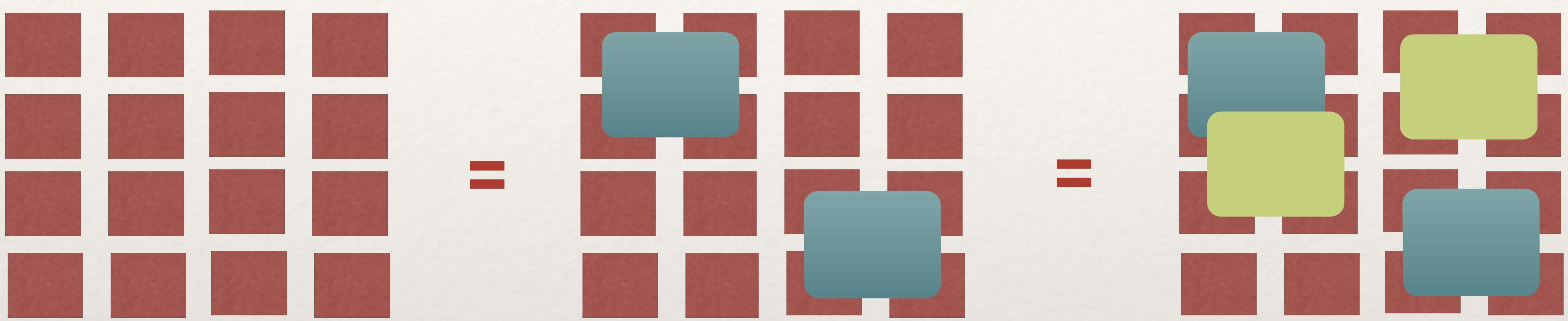
- Entanglement = invariant under local unitaries.
- Schmidt coefficients = the complete set of **invariants**

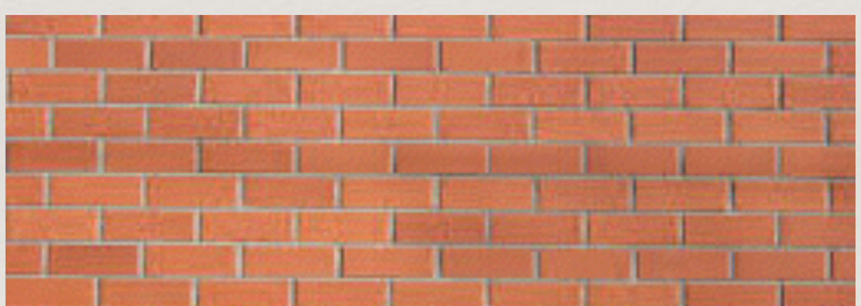
$$S = -\sum_r \lambda_r \log \lambda_r$$

- Information, thermodynamic entropy

Many-body Entanglement

- Local entanglement is washed away by local unitaries.



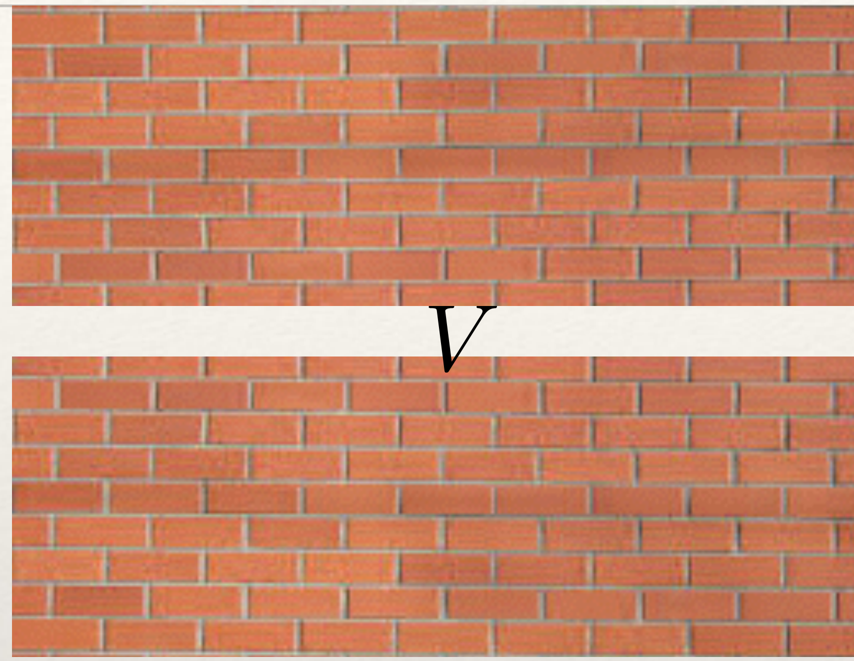


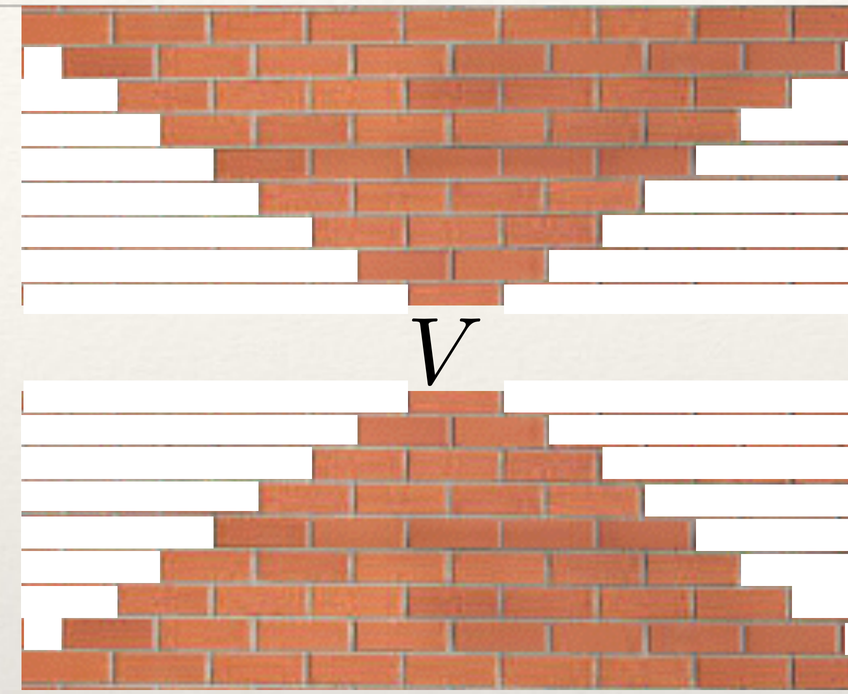
= Unentangled state

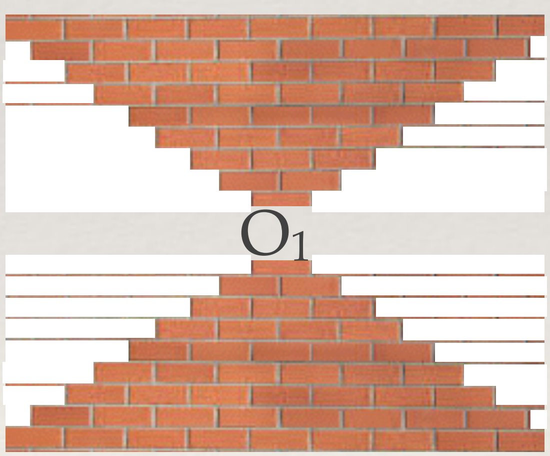
$$|01001\cdots011\rangle$$

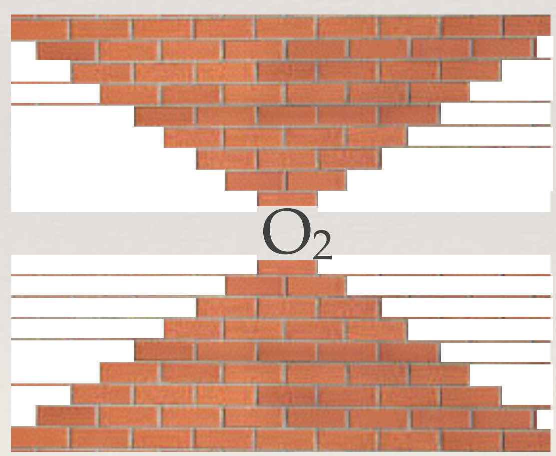
- What is the characterization? What are the representatives?

Q. Circuits and Correlation

$UVU^\dagger =$ 

$=$ 





$$Cor_{|\psi\rangle}(O_1, O_2) \sim 0$$
$$\Leftrightarrow Cor_{W|\psi\rangle}(O_1, O_2) \sim 0$$

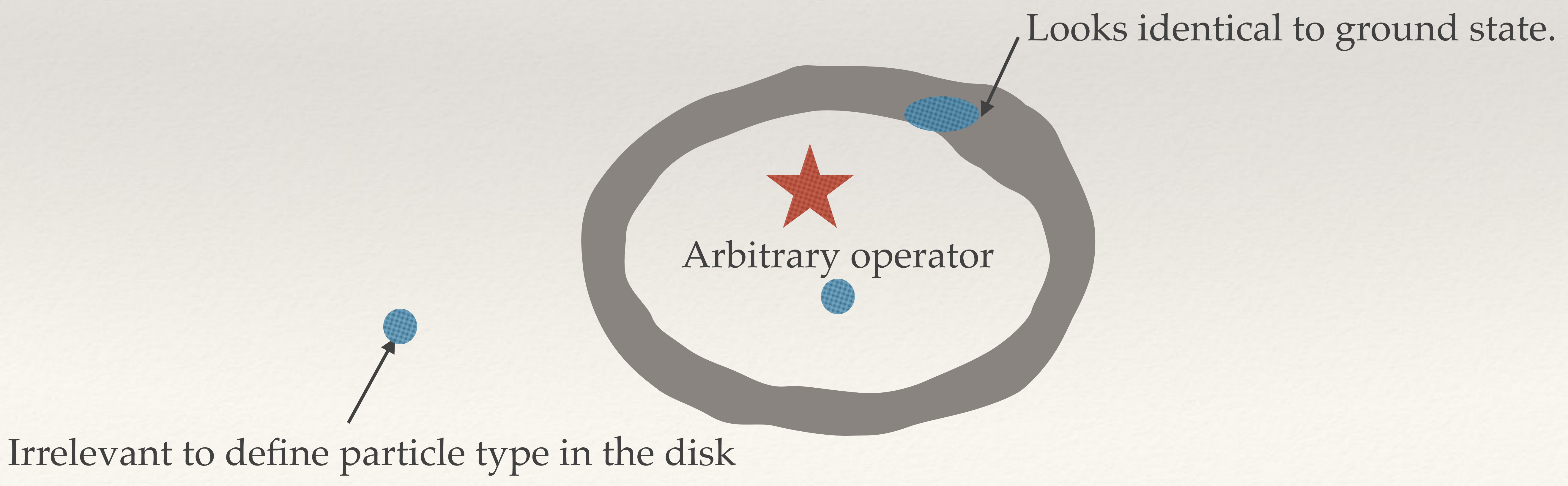
Topological Order

$\nLeftarrow$

- Long-range correlation $\Rightarrow$ Deep Q. circuit required

Topological Charge

- A set of states related by local operators, not necessarily unitary.
- No symmetry constraint.



Recall Spin

$$[J_x, J_y] = iJ_z$$

$$J_x^2 + J_y^2 + J_z^2 = j(j+1)$$

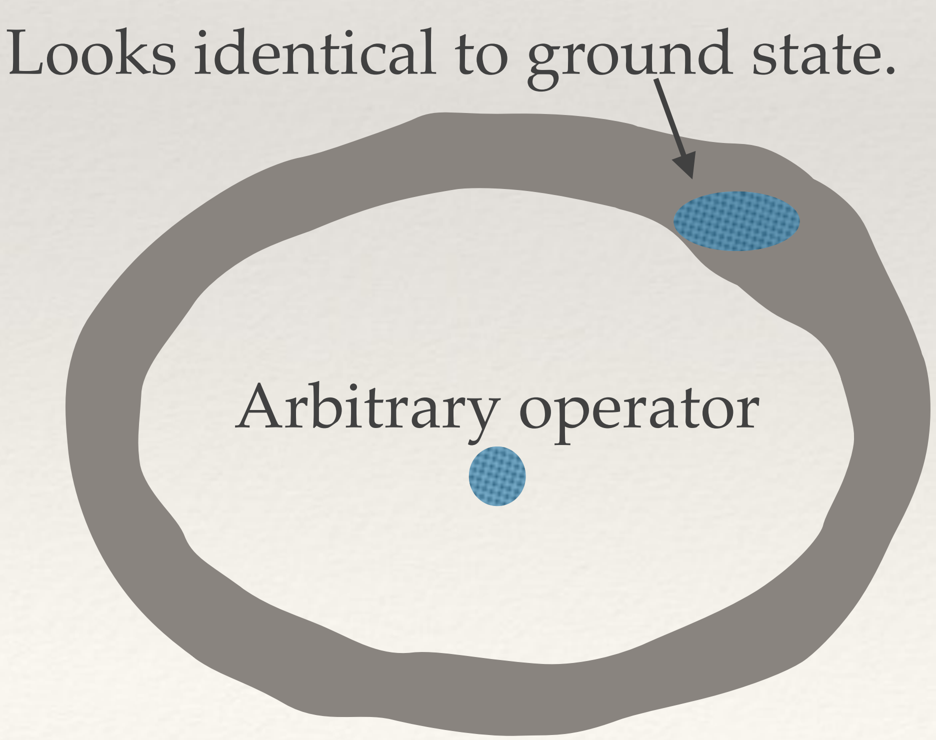
- Transformations
- Operator in the center of the operator algebra.
- Eigenvalue of the central operator = Spin

Topological Charge

$$Mat(D) \otimes \mathcal{A}/\mathcal{N}$$

Operator on grey that annihilates the state

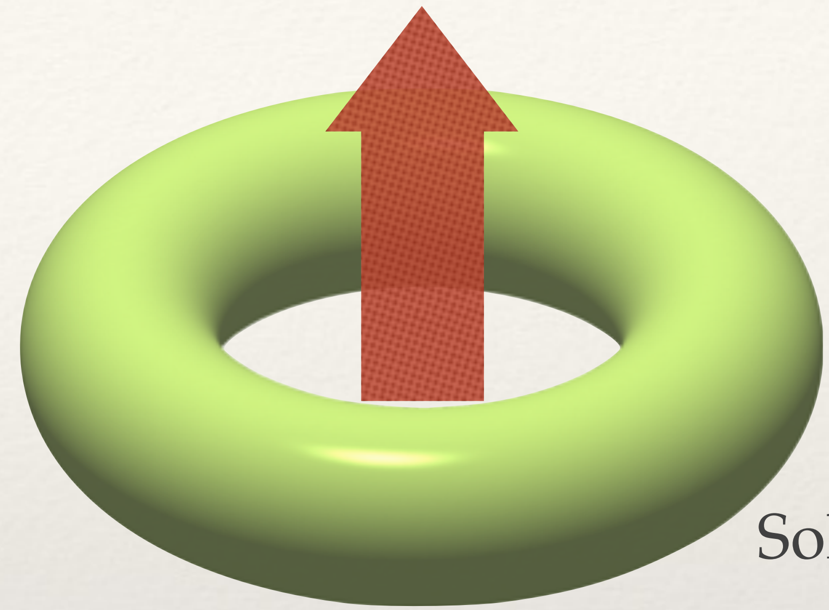
Any local term of H should commute



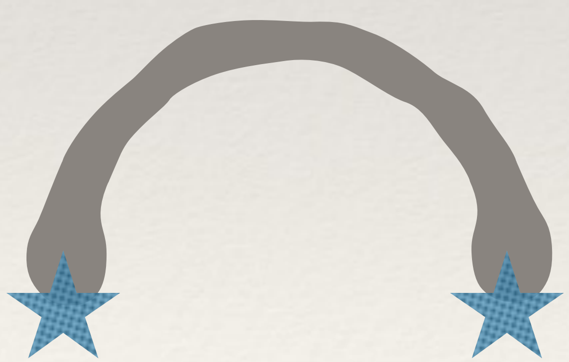
- Transformations
- Find an operator in the center of the operator algebra.
- Eigenvalue of the central operator = particle type

In any dimensions

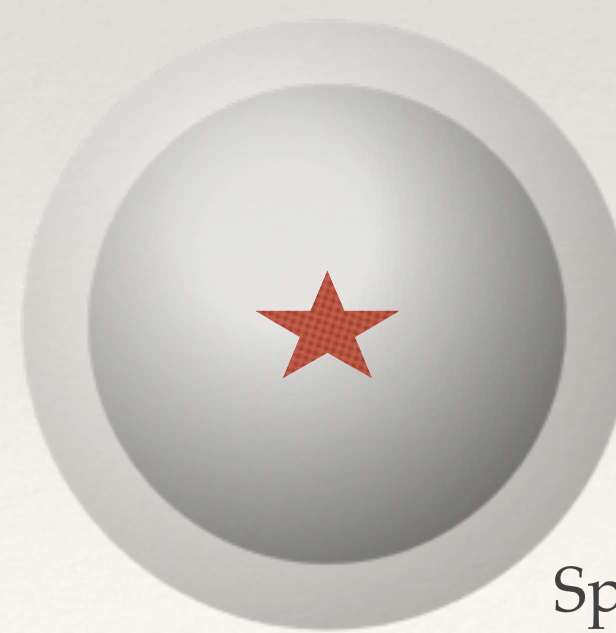
$\mathcal{A}/\mathcal{N}$



Solid Torus



It is a trajectory of excitations.



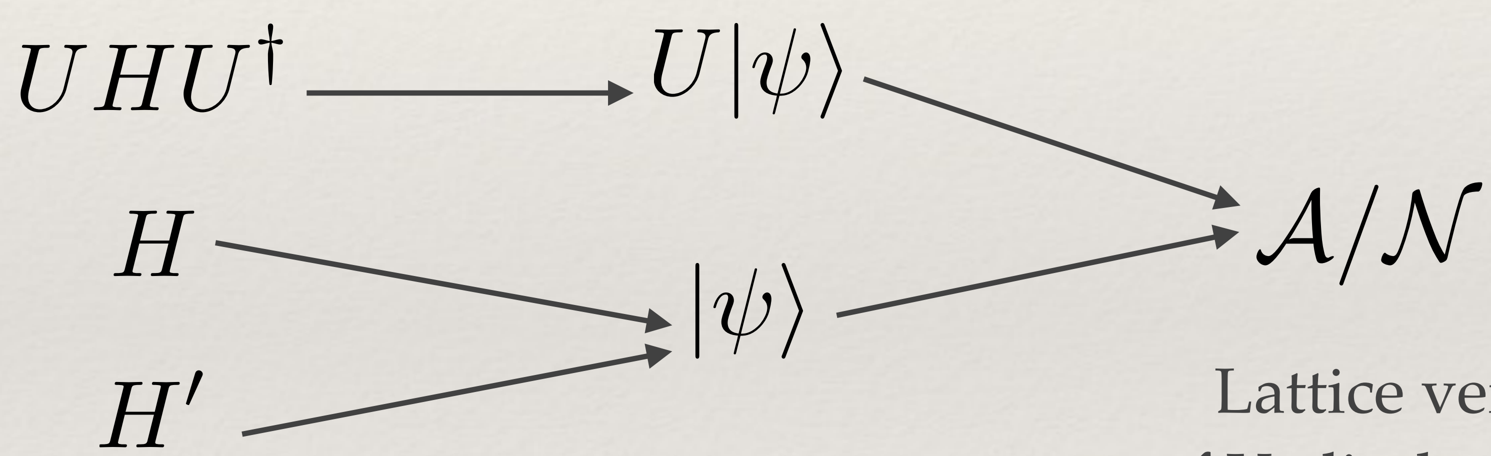
Sphere with Wall

Topological Order/Entanglement Witness

If it is independent of thickness,

is in fact independent of Hamiltonians*

is invariant under small-depth Q. circuit.



Lattice version
of Verlinde algebra

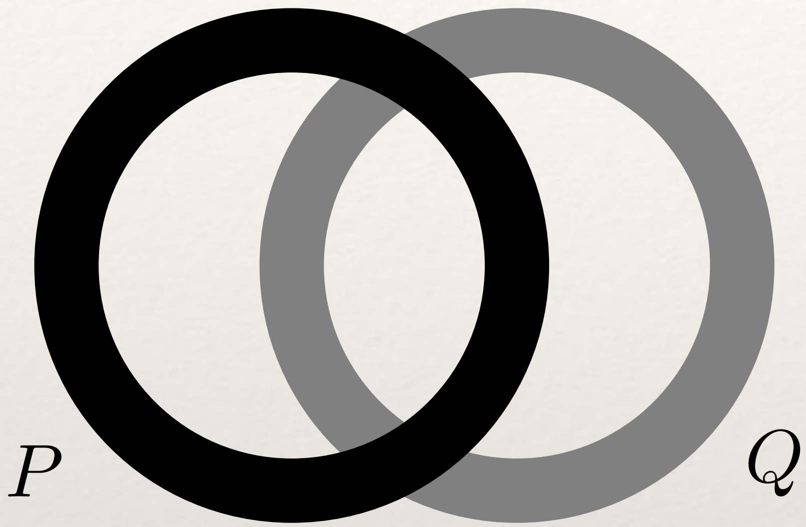
- Derived S, T matrices are properties of a **bulk** patch.

Hamiltonian* = commuting projectors, local TQO

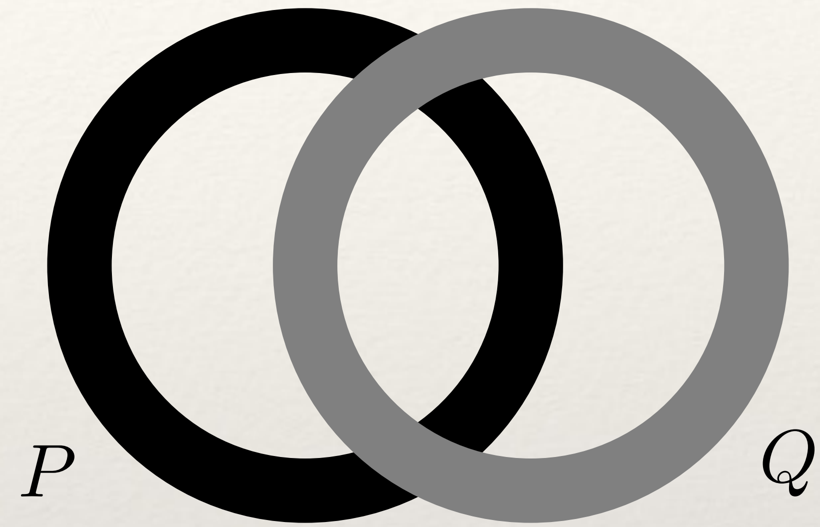
JH, 1407.2926

Extracting Numbers

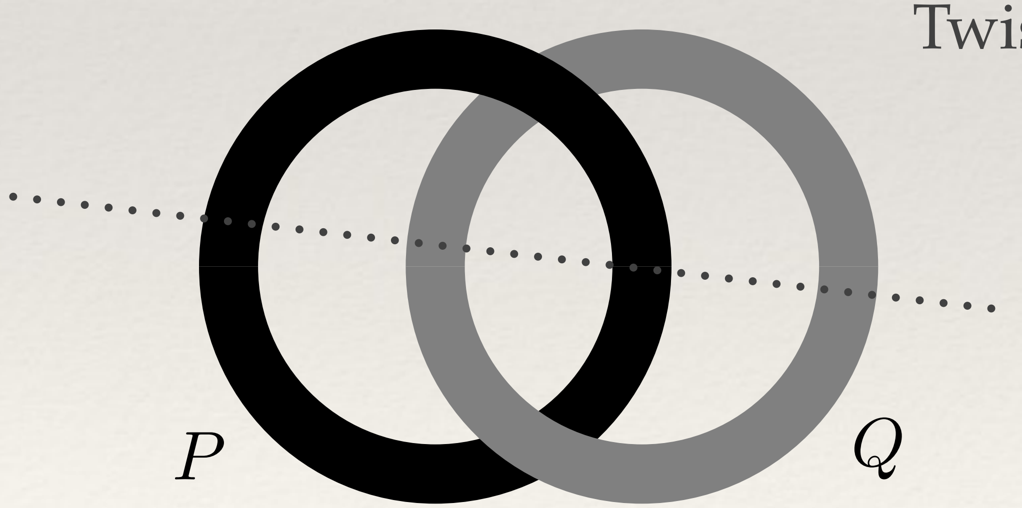
Ordinary product PQ



Ordinary product QP



Twist Product



$$\sum_{ij} P^{(i)}_{\text{up}} Q^{(j)}_{\text{up}} \otimes Q^{(j)}_{\text{down}} P^{(i)}_{\text{down}}$$

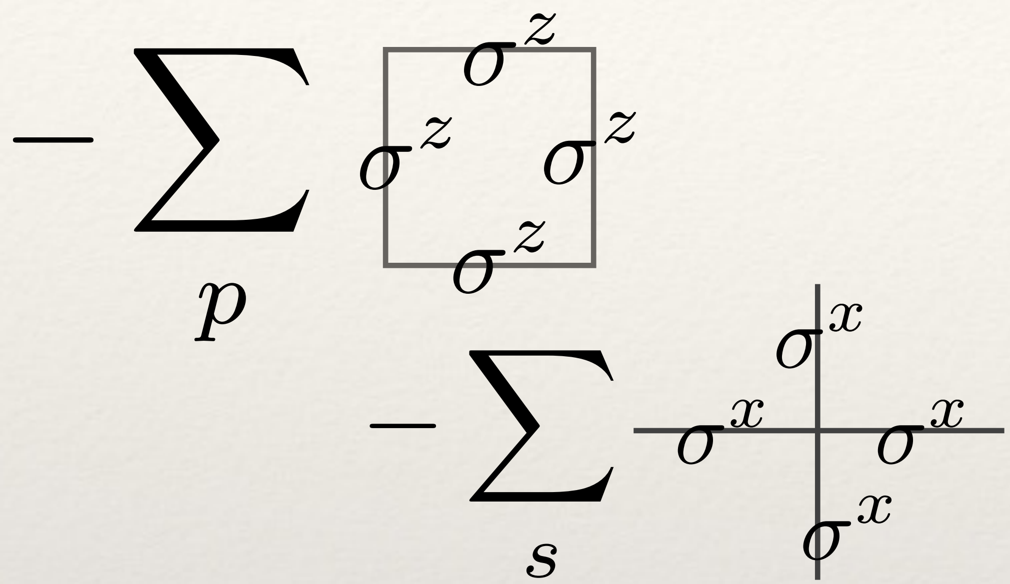
Well-defined as long as intersection is separated.

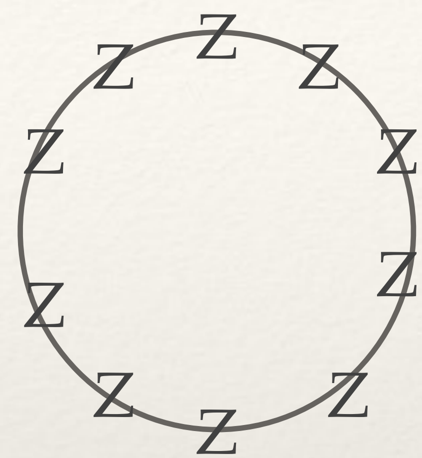
Good & Bad

- Simple definition — No need to go through TQFT
- S, T (analogues) matrices are properties of a bulk patch;
  They can be computed very simply.
- No false-positive answers for long-range entanglement (topological order).
  Triangular lattice cluster state under prevalent numerics gives
  a false-positive answer in topological entanglement entropy.
  [JH, Zou, Senthil, unpublished]
- Easy to give an algorithm — Linear algebra on the space of matrices.

- Inefficient algorithm
- Rigorous scope is limited. (Commuting++)
- No error analysis w.r.t. perturbations / finite correlation length

e.g. Toric code



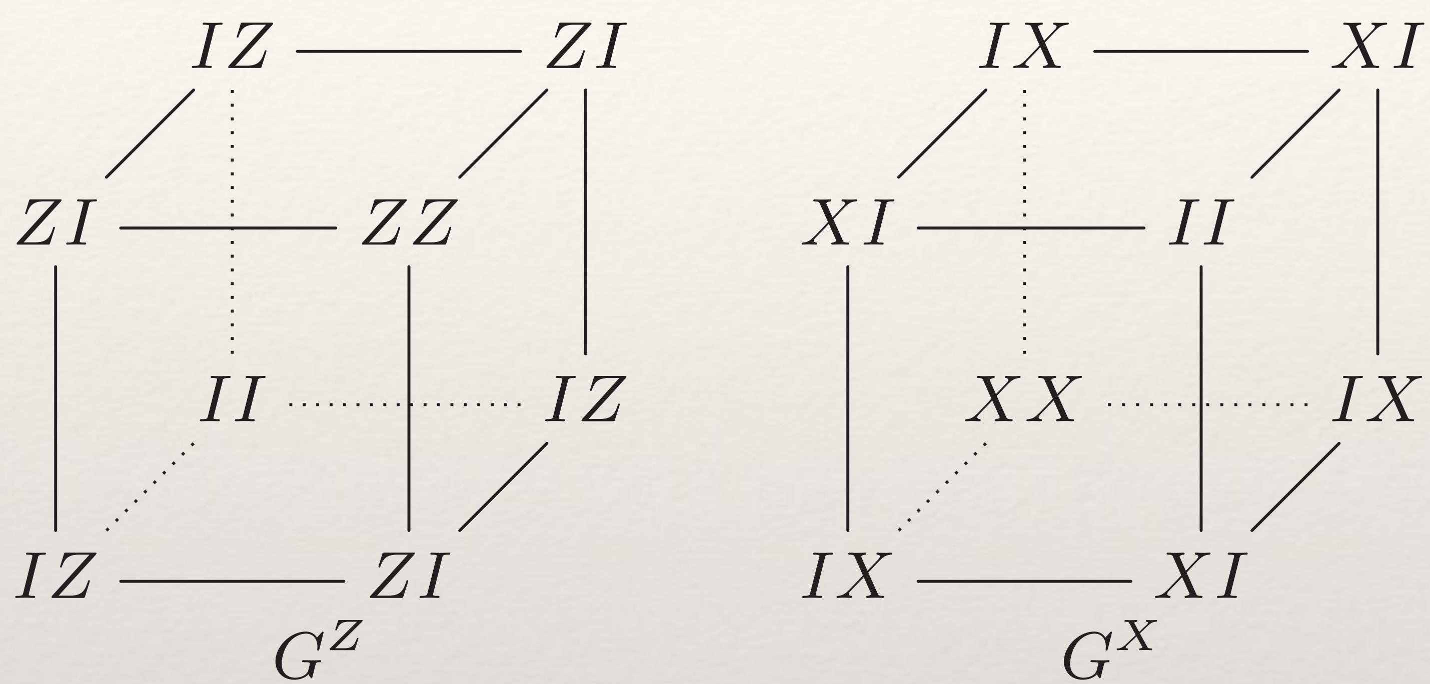


$\mathcal{A}/\mathcal{N}$

- 4-dimensional algebra = 4-types of topological charge

Kitaev 1997

e.g. Cubic code



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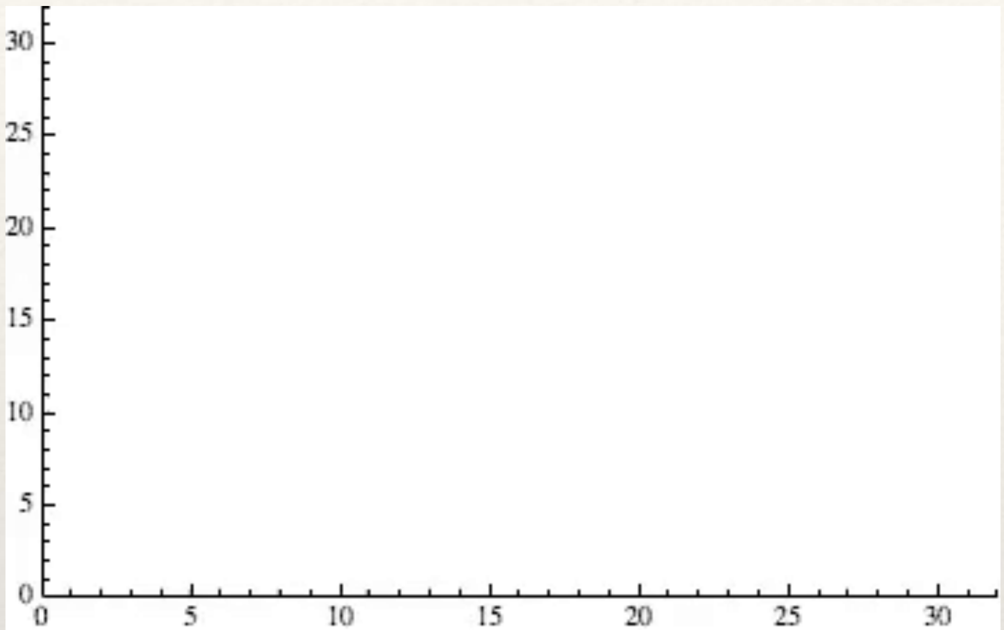
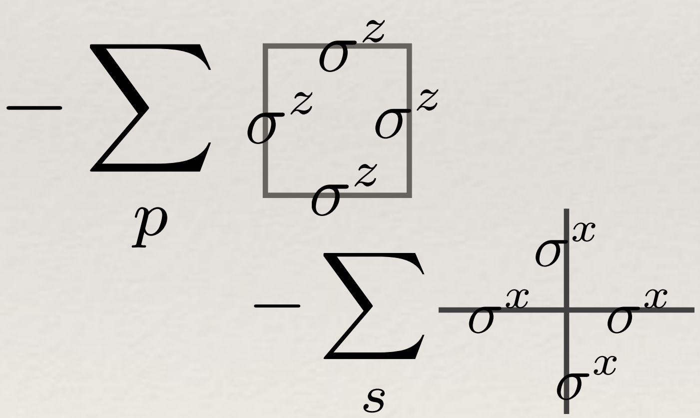
$$+ \sigma^z_{i,1}\sigma^z_{i,2}\sigma^z_{i-\hat{x},2}\sigma^z_{i-\hat{y},2}\sigma^z_{i-\hat{z},2}\sigma^z_{i-\hat{y}-\hat{z},1}\sigma^z_{i-\hat{z}-\hat{x},1}\sigma^z_{i-\hat{x}-\hat{y},1} \Big)$$

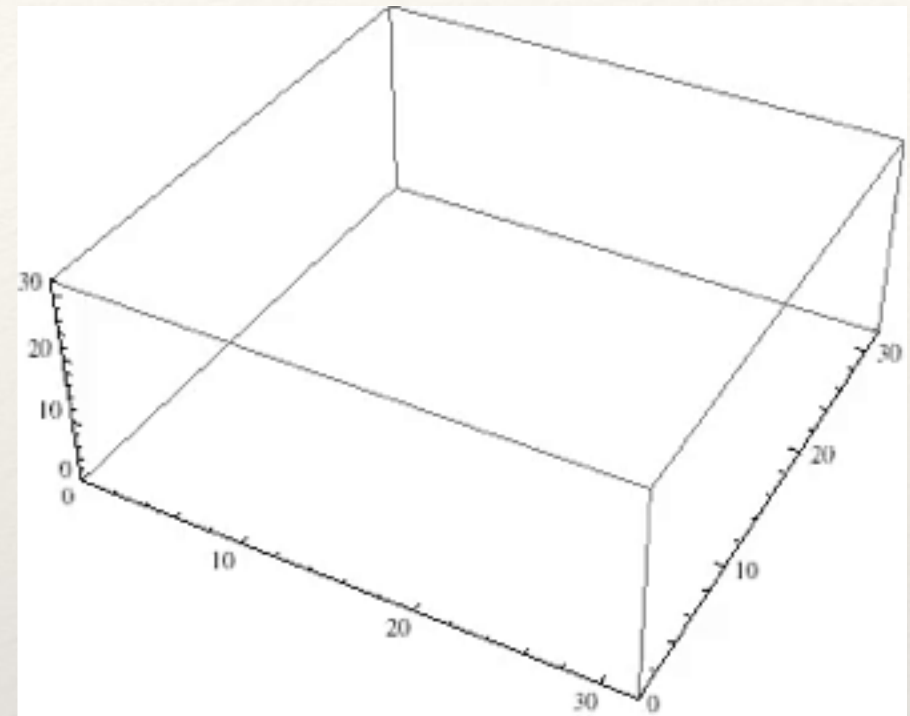
JH, 1101.1962

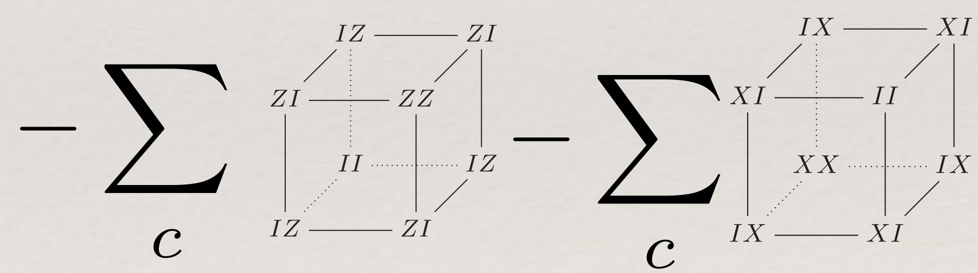
Isolating a charge





- String can be arbitrarily extended.
- Excitations can be moved arbitrarily





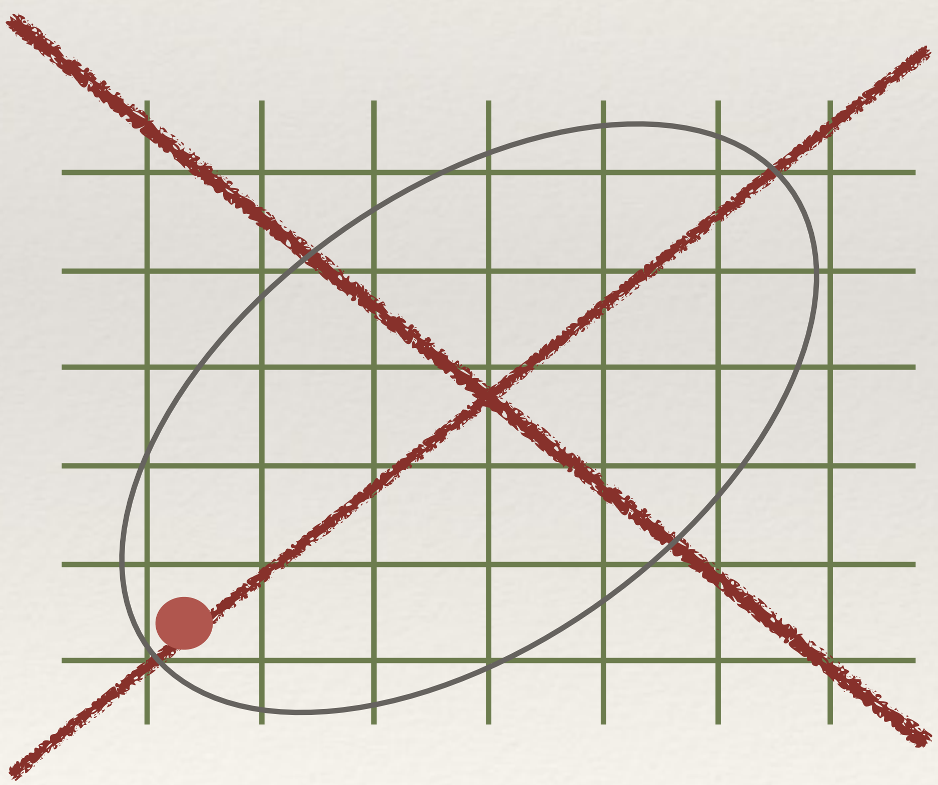
- Self-similar pyramids can be extended in a special way.
- A single excitation **cannot** be moved to a nearby position

Bravyi, JH, 1105.4159

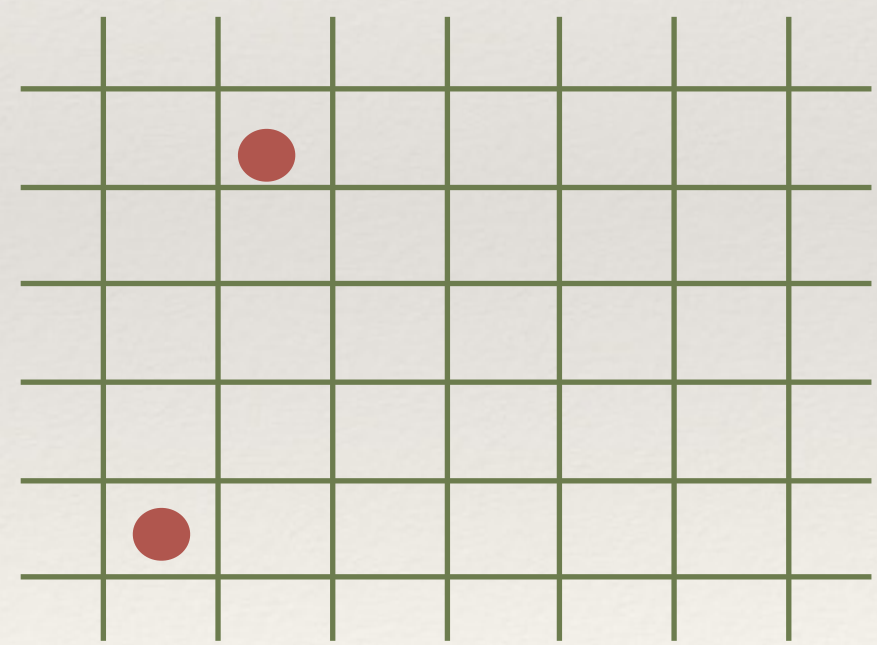
Immobile Excitations

$$\langle \psi_1 | O T^n | \psi_1 \rangle = 0 \quad n \geq 1$$

T is a translation along any direction.
O is supported on a ball that does not touch the boundary of the system

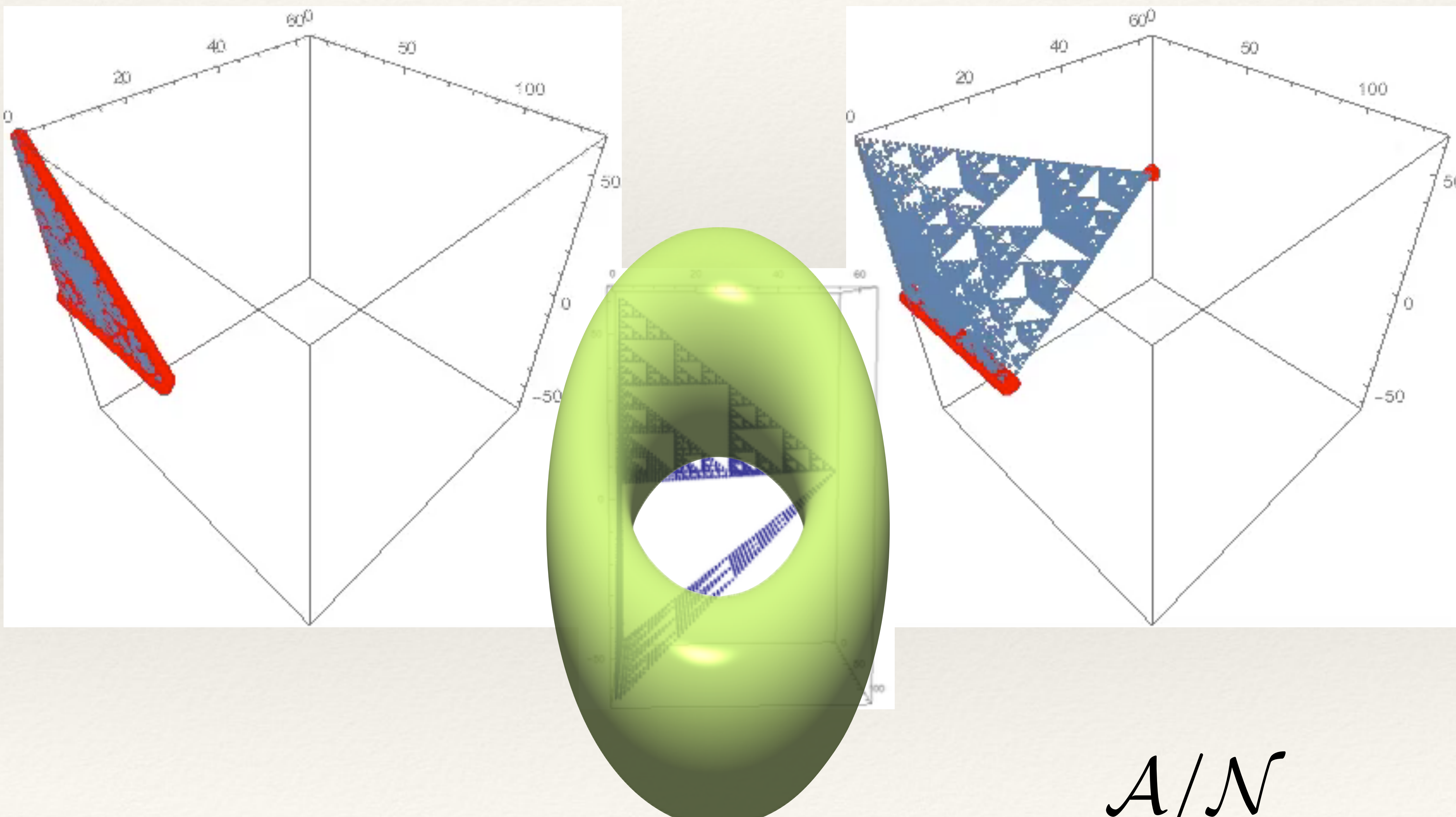


perhaps allowed



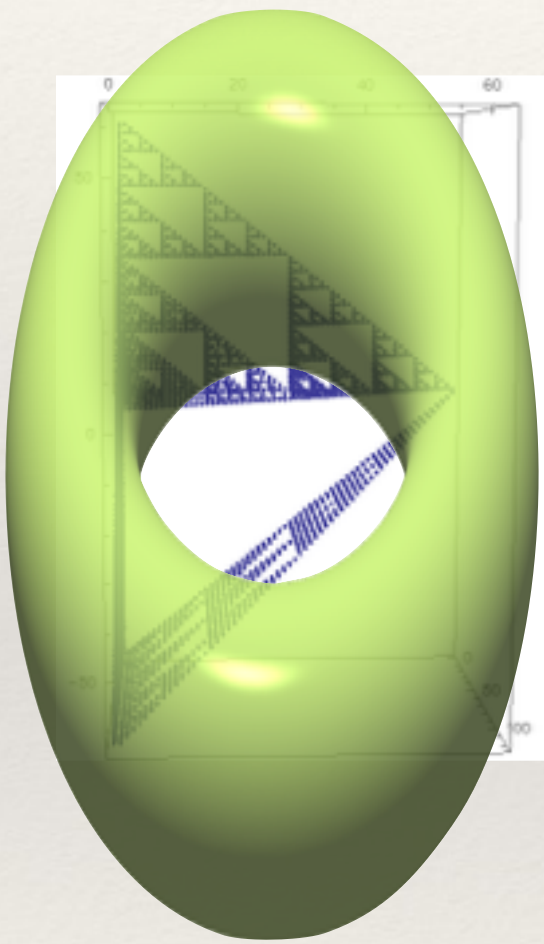
- Interaction-driven localization [Kim, JH, 1505.01480]

Braiding of extended charges



$\mathcal{A}/\mathcal{N}$

Braiding of extended charges

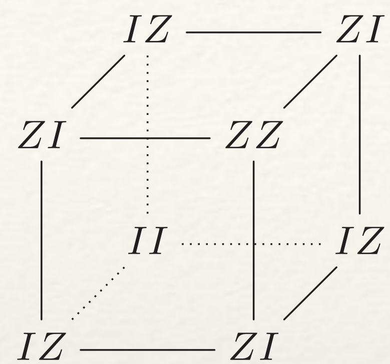


$\mathcal{A}/\mathcal{N}$

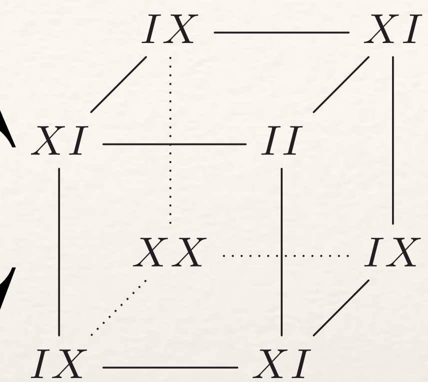
- A generalized notion of braiding; it must fatten at some point.
- Non-trivial algebra,
  hence long-range entangled / topologically ordered.
- exp(R)-dimensional algebra. Hamiltonian-independence proof doesn't apply.

Wave function

$$-\sum_{c}$$



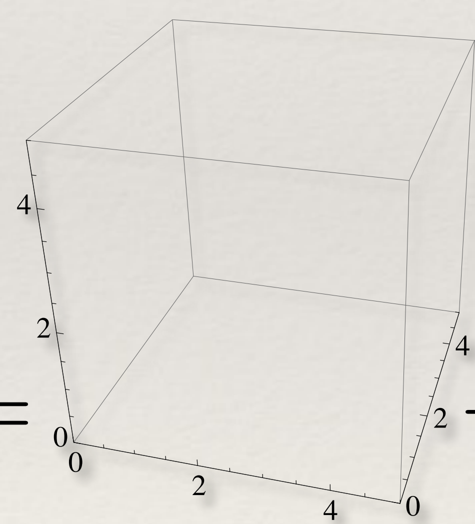
$$-\sum_{c}$$



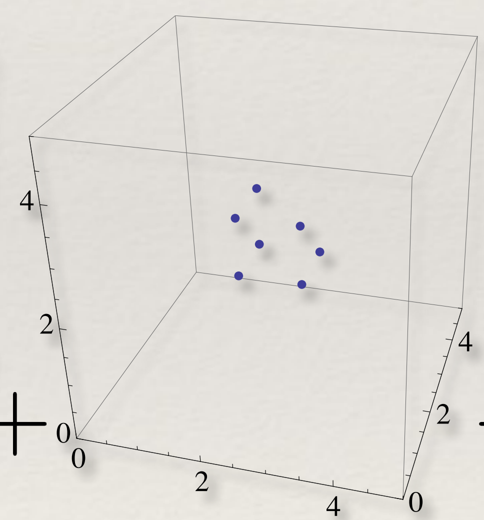
σ^x = +

σ^x = -

$$|\psi\rangle =$$



$+$



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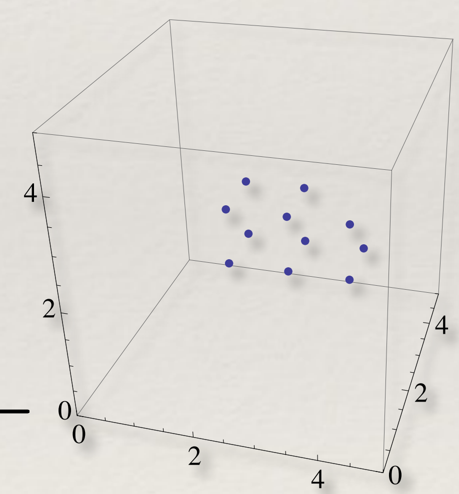

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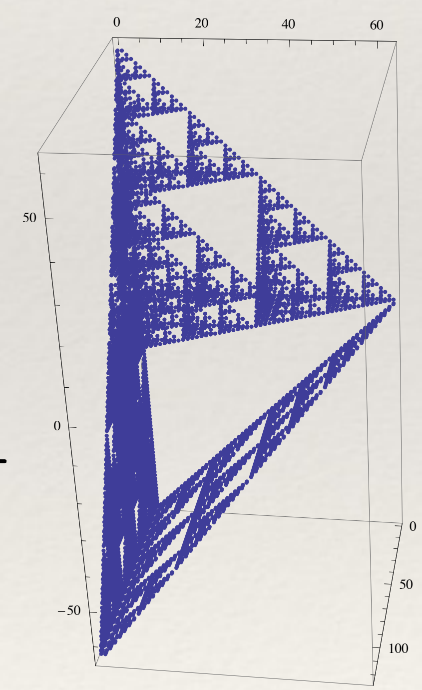
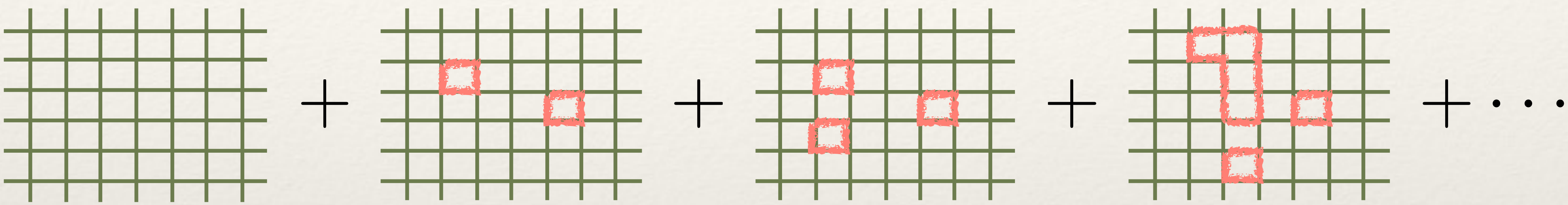


$+\cdots$

Ground state is a condensate of “fractals” or “objects.”

Generating Circuit/Entanglement RG

$|\psi\rangle =$ 

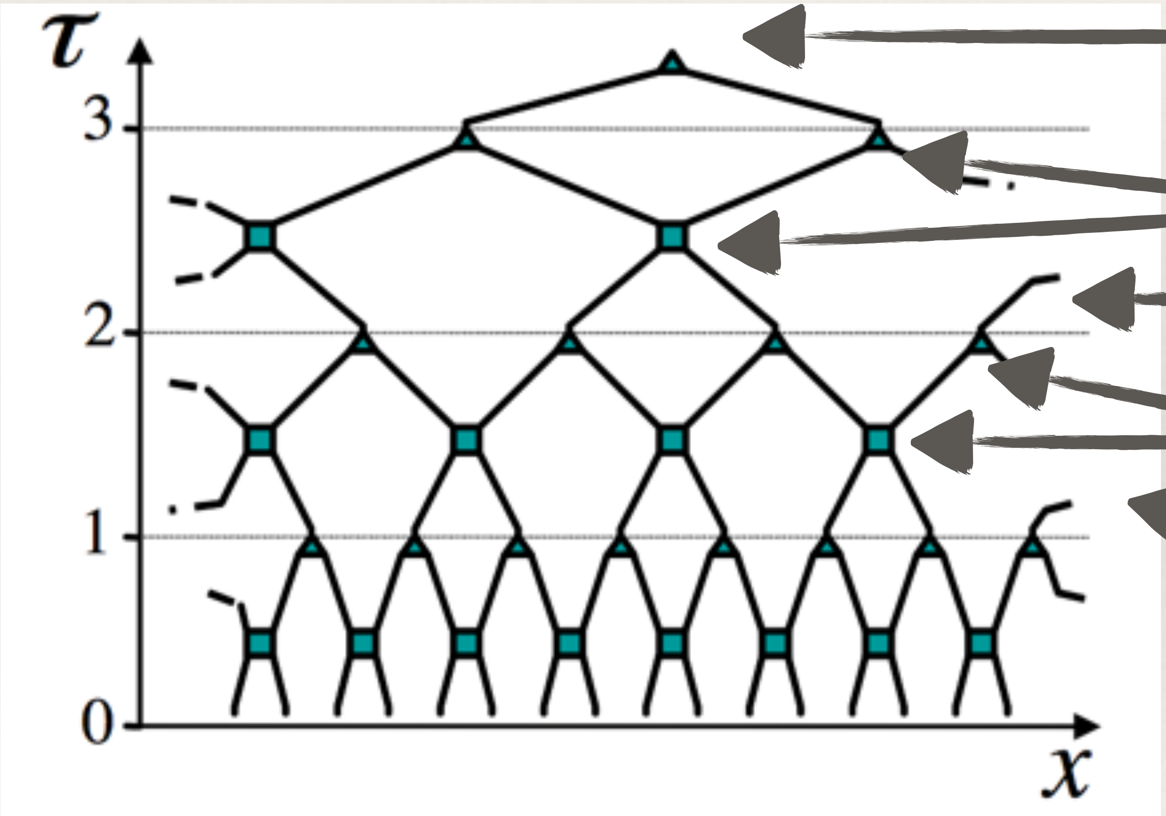
- RG = Coarse-graining by eliminating smallest loops
- Entanglement RG = Disentangling then Discarding

  Exactly solvable models of topological order have ground states
  that are entanglement RG fixed-points

Aguado, Vidal; Gu, Levin, Swingle, Wen (2008)

MERA

Multi-scale Entanglement Renormalization Ansatz (Vidal 2006)



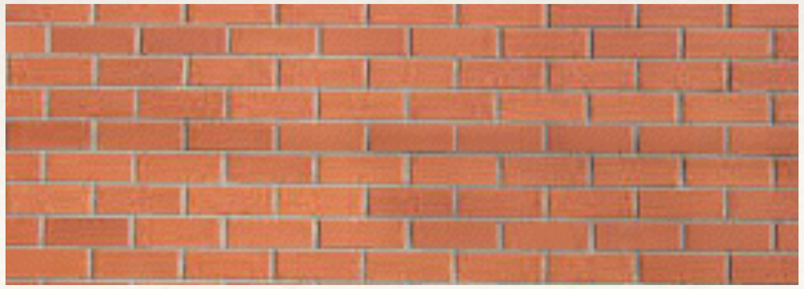
Start with a qubit

Apply quantum circuit of constant depth

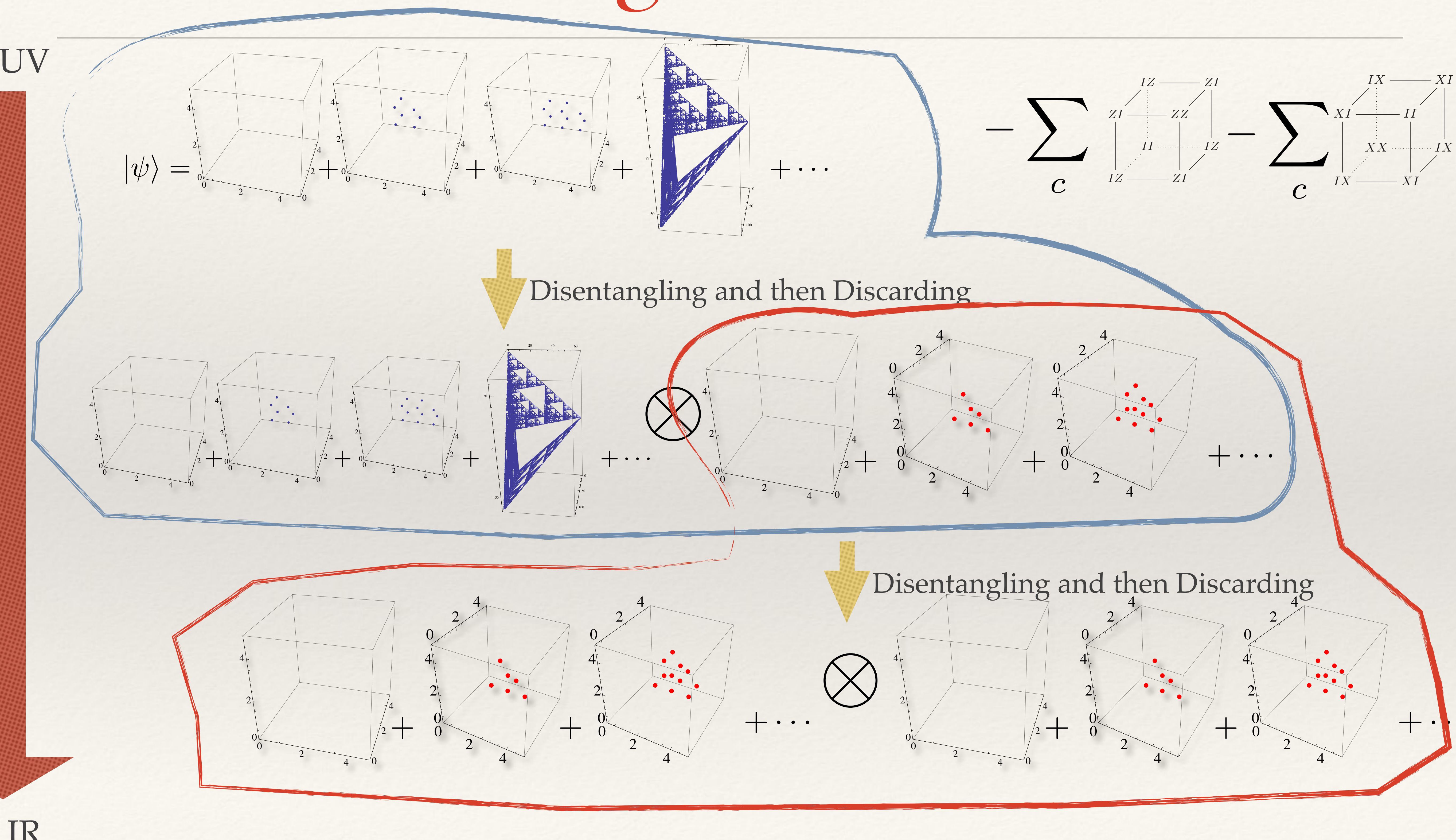
Expand the lattice

Apply quantum circuit of constant depth

Expand the lattice

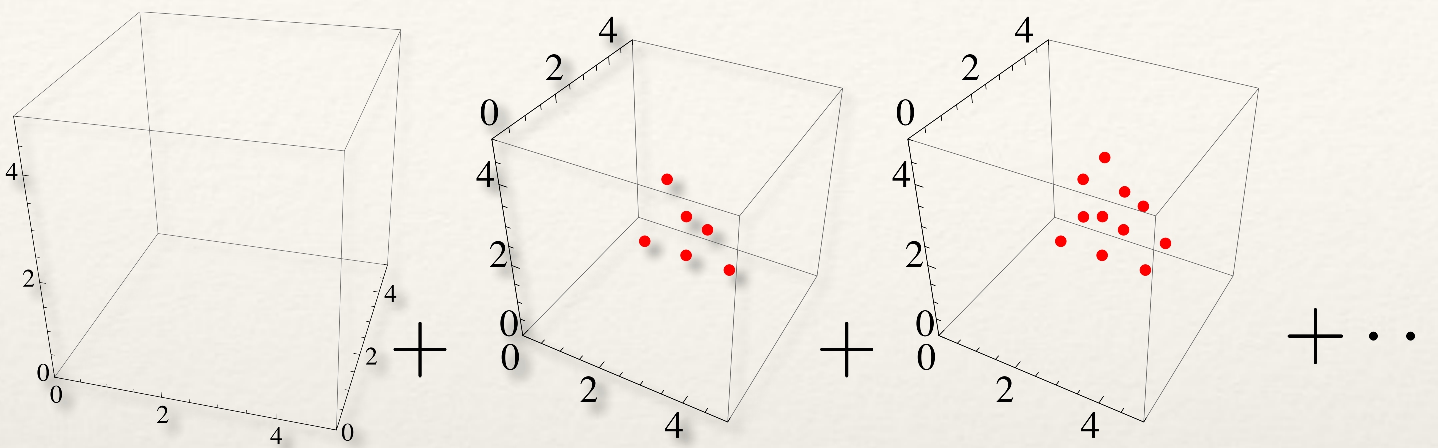


Entanglement RG



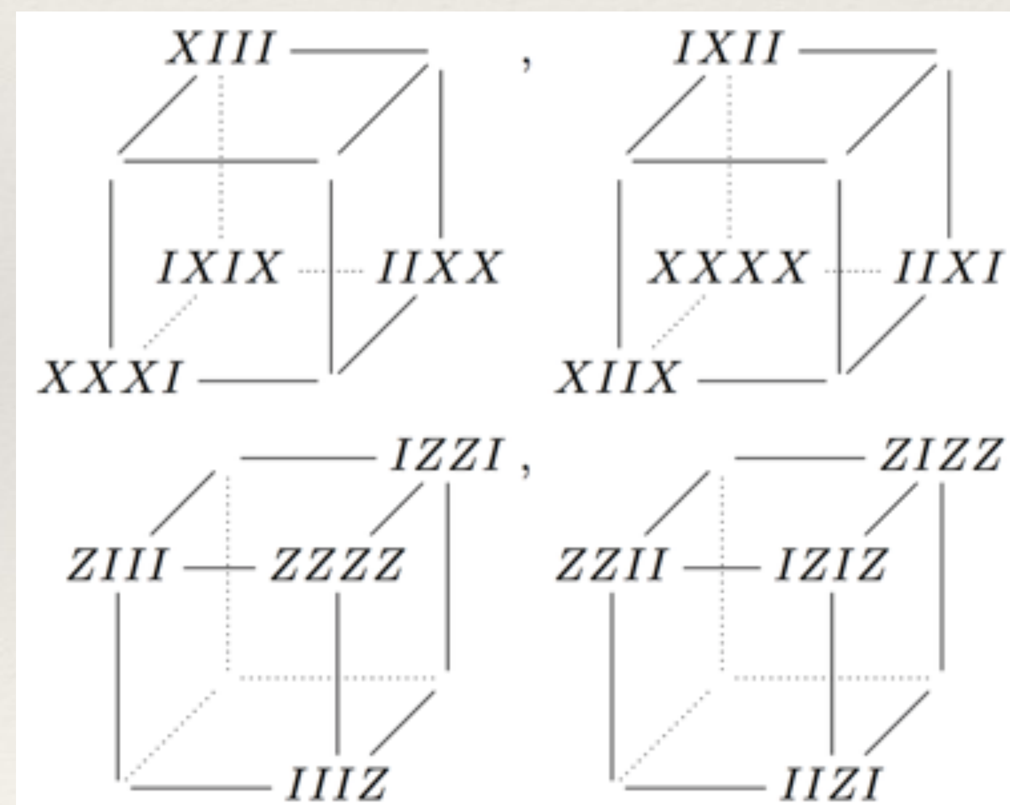
JH, 1310.4507

Entanglement RG fixed point



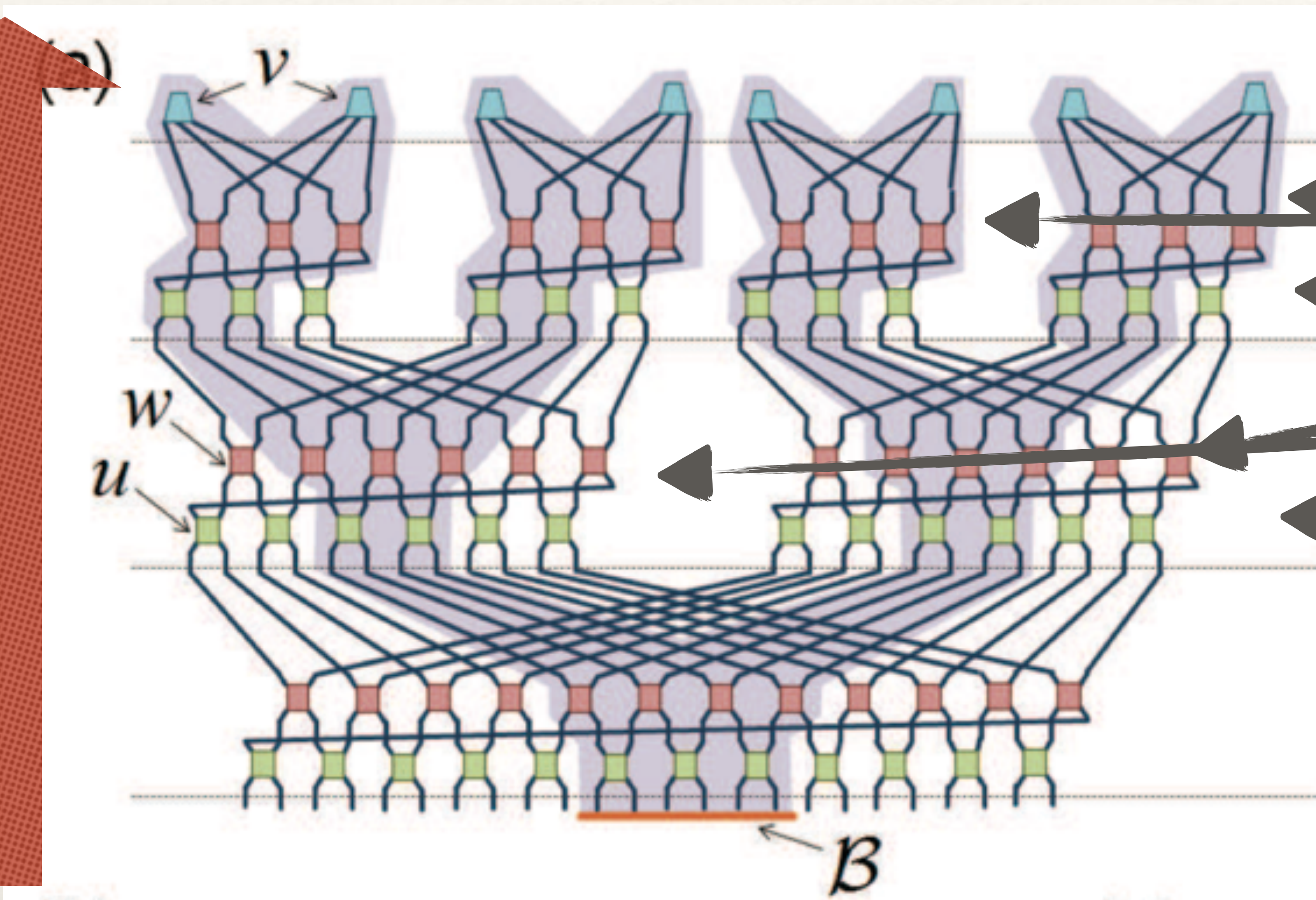
is a ground state of

$$H = -\sum_c$$



Branching MERA

IR



UV

Start with qubits

Apply quantum circuit of constant depth

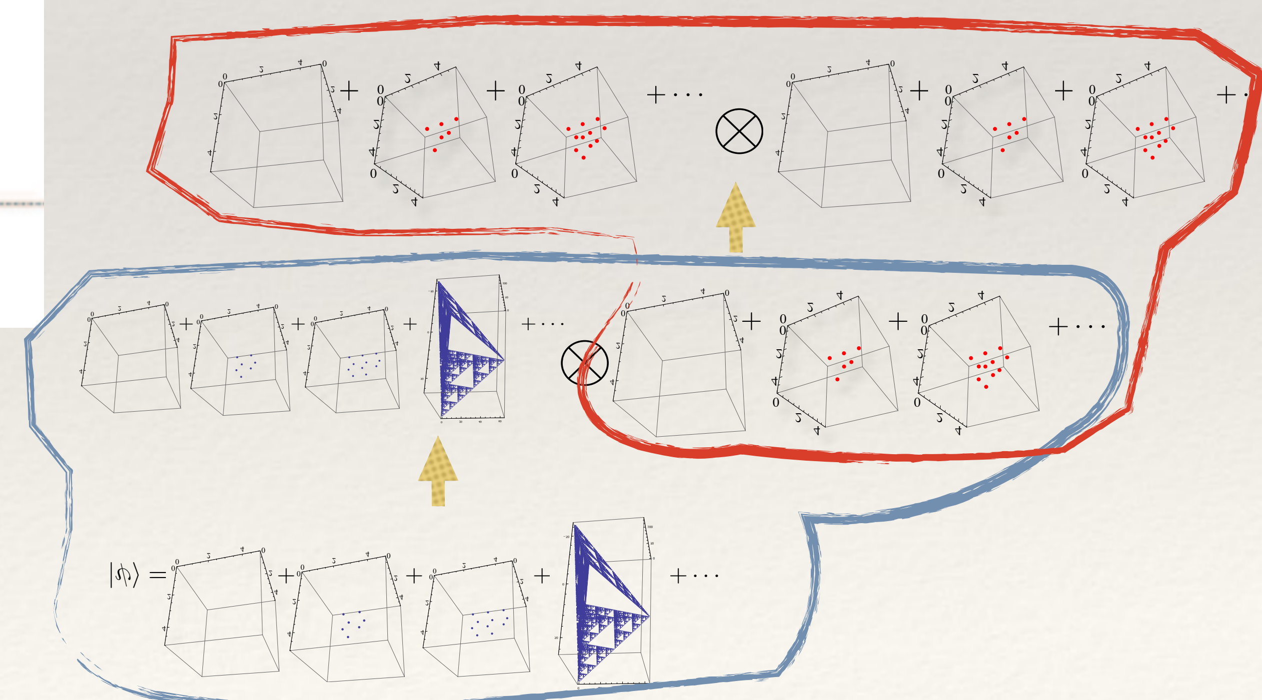
Expand the lattice

Apply quantum circuit of constant depth

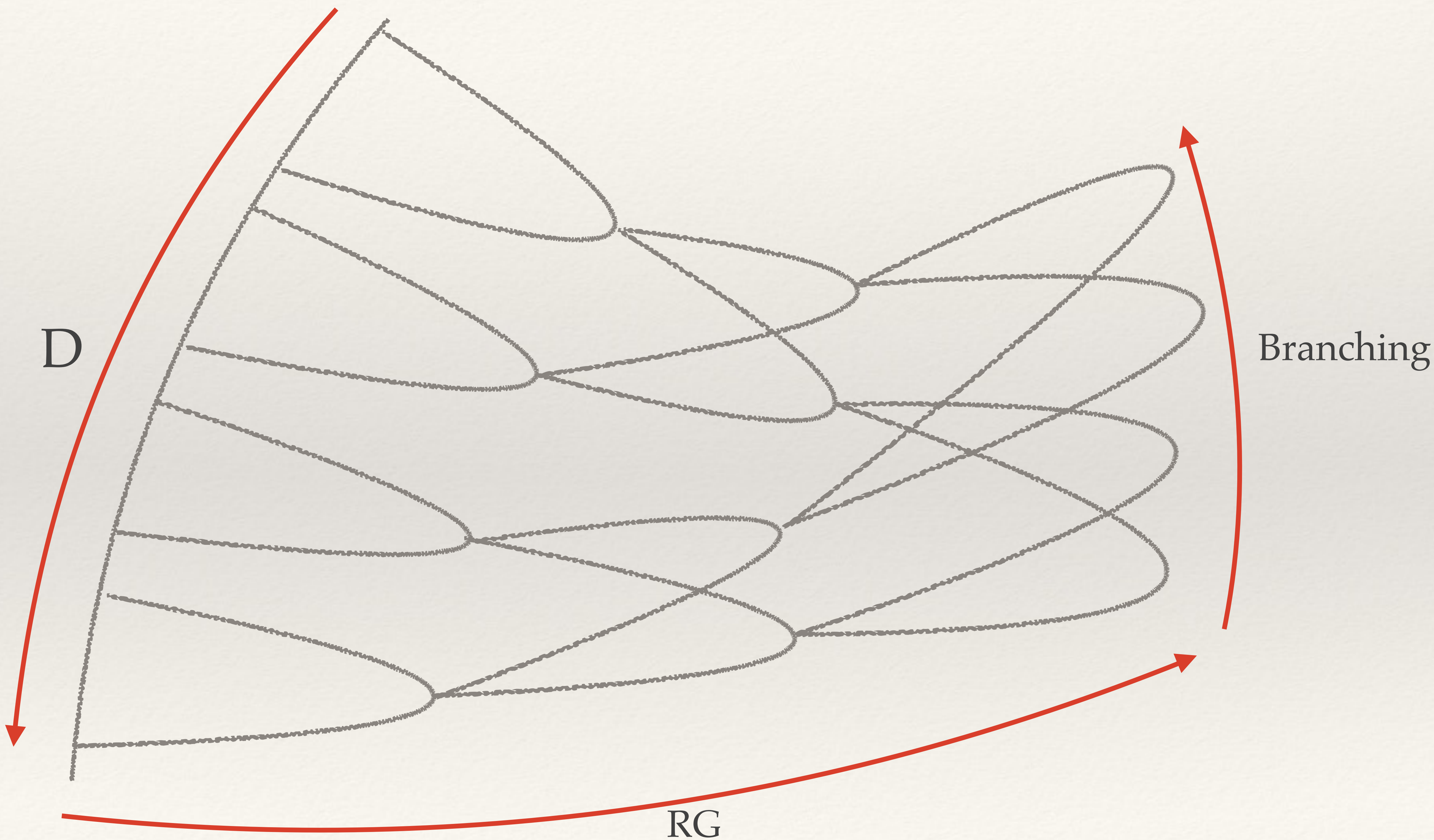
Expand the lattice

Evenbly, Vidal (1310.8372)

$$S = \begin{cases} \mathcal{O}(L^{d-1}) & \text{if } b < 2^{d-1} \\ \mathcal{O}(L^{d-1}\log L) & \text{if } b = 2^{d-1} \\ \mathcal{O}(L^{d-1+\log_2(b/2^{d-1})}) & \text{if } b > 2^{d-1} \end{cases}$$

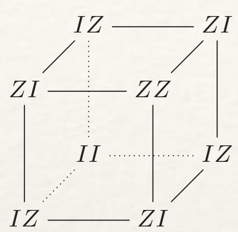


D+2

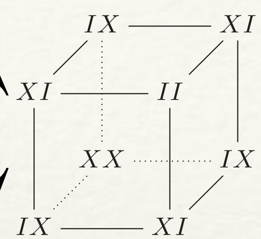


Calculation

$$-\sum_c$$



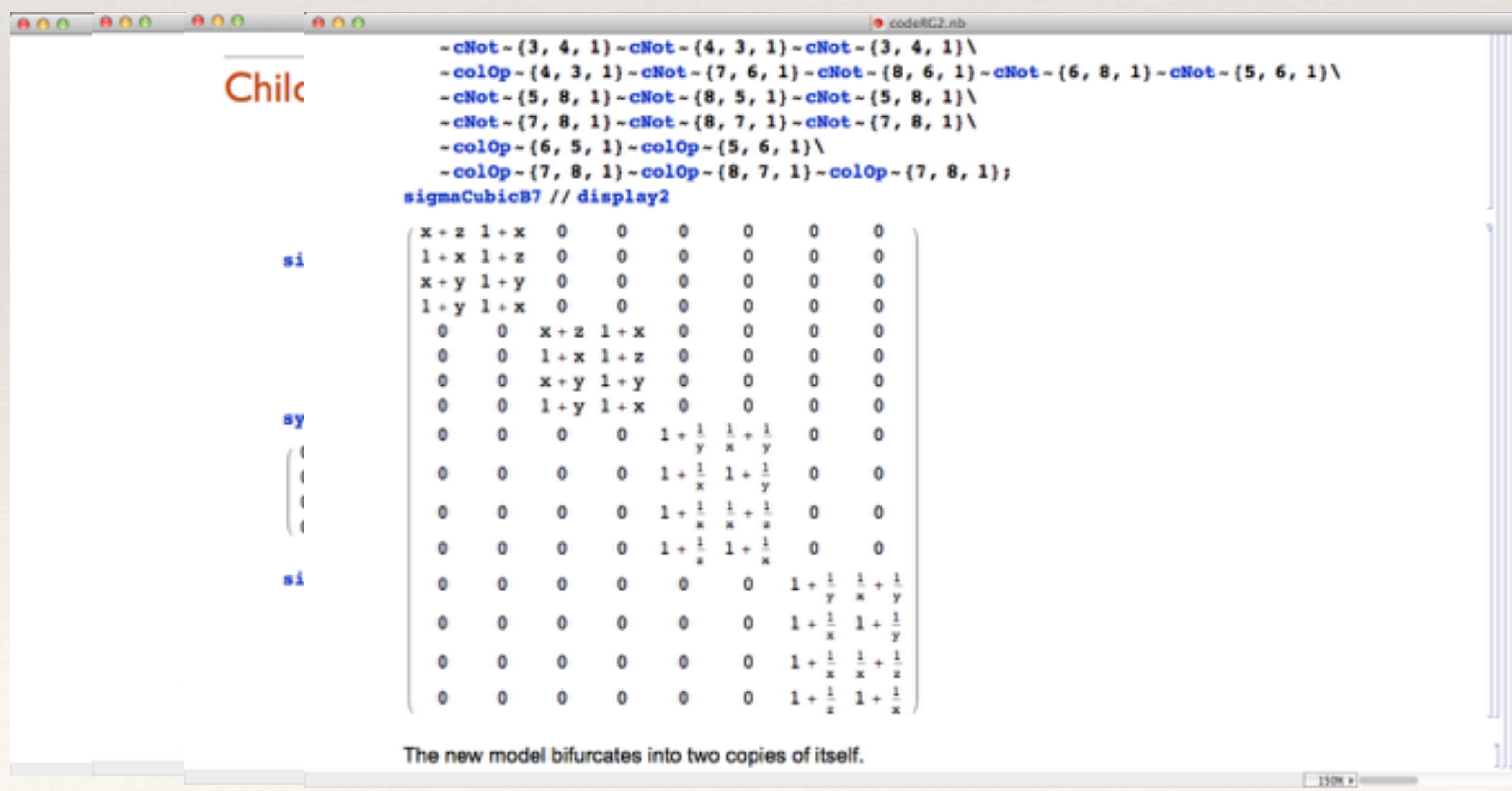
$$-\sum_c$$



Constraint Hamiltonian $S|\psi\rangle = |\psi\rangle$

$$(USU^\dagger)U|\psi\rangle = U|\psi\rangle$$

$$\sigma_i^z|\psi\rangle = |\psi\rangle \quad \Longrightarrow \quad |\psi\rangle = |0_i\rangle \otimes |\psi_{\text{rest}}\rangle$$



Summary

- Topological charges are defined as irreps of an algebra, and are manifestation of long-range entanglement.
- The cubic code model has many localized topological charges, appearing at the tip of some fractal operator.
- Unlike usual topological models, its ground state admits branching MERA (first, so far only example).