Entanglement in Holography

Veronika Hubeny

Durham University

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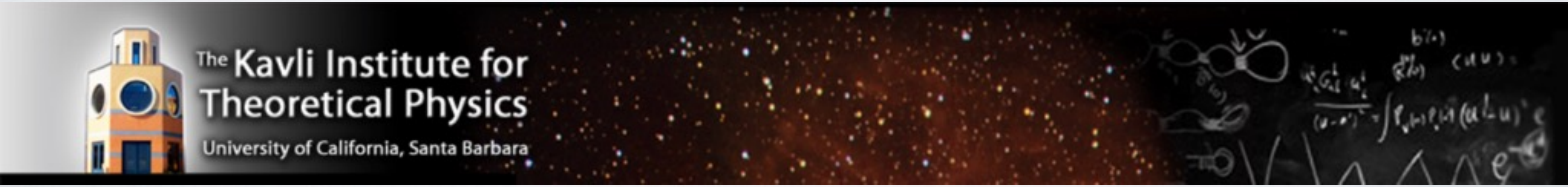
Closing the entanglement gap:

Quantum information, quantum matter, and quantum fields



Entanglement

- Most non-classical manifestation of quantum mechanics
  - "Best possible knowledge of a whole does not include best possible knowledge of its parts — and this is what keeps coming back to haunt us" [Schrodinger '35]
- New quantum resource for tasks which cannot be performed using classical resources [Bennet '98]
- Plays a central role in wide-ranging fields
  - quantum information (e.g. cryptography, teleportation, …)
  - quantum many body systems
  - quantum field theory
- Hints at profound connections to geometry…

Entanglement Entropy (EE)

Suppose we only have access to a subsystem A of the full system = A + B. The amount of entanglement is characterized by Entanglement Entropy S_A :

- reduced density matrix $\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$

  (more generally, for a mixed total state, $\rho_A = \mathrm{Tr}_B \rho$)

- EE = von Neumann entropy $S_A = -\mathrm{Tr}\, \rho_A \, \log \rho_A$

Defined if we can divide a quantum system into a subsystem A and its complement B, such that the Hilbert space decomposes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Entanglement Entropy (EE)

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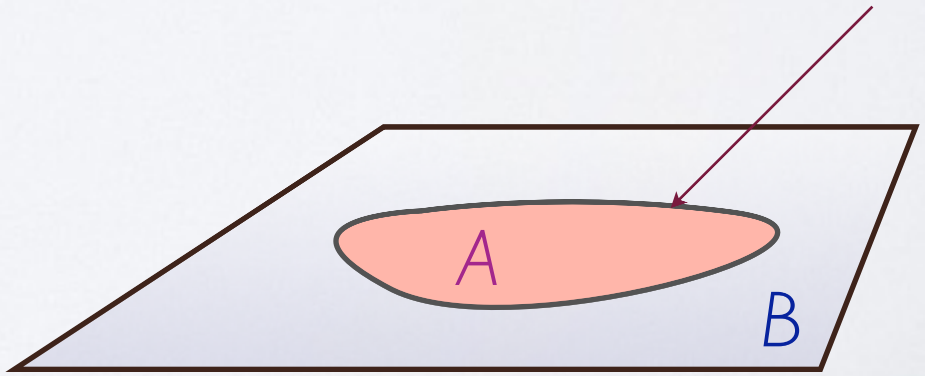
- e.g. in local QFT:

  A and B can be spatial regions, separated by a smooth entangling surface



The good news & the bad news

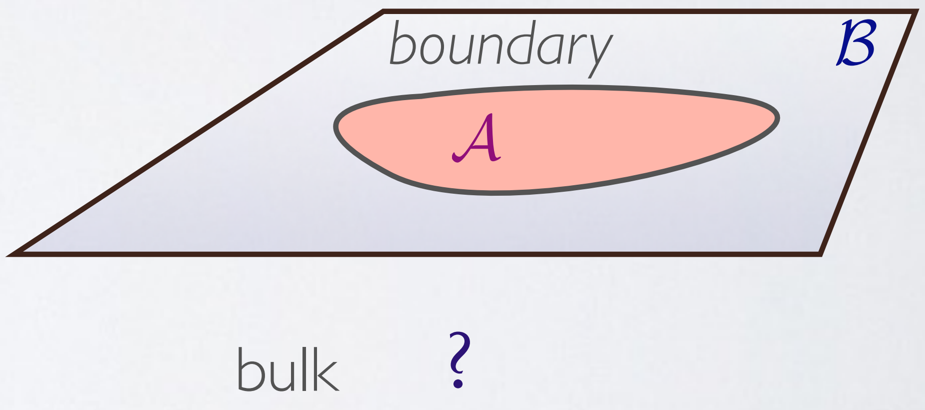
- But EE is hard to deal with…
  - non-local quantity, intricate & sensitive to environment
  - difficult to measure
  - difficult to calculate

… especially in strongly-coupled quantum systems

- AdS/CFT to the rescue?

~ Is there a natural bulk dual of EE?

(= "Holographic EE")



Yes! - described geometrically…

OUTLINE

- Entanglement ✔
- Holography
  - AdS/CFT Correspondence
  - Essential elements of the AdS/CFT dictionary
- Holographic Entanglement Entropy
  - RT & HRT prescriptions
- Utility of Geometry
  - Easy-to-prove properties of EE
  - Entanglement plateaux
  - Dynamics (example: quench & thermalization)
  - Curious features of EE
- Summary & Outlook

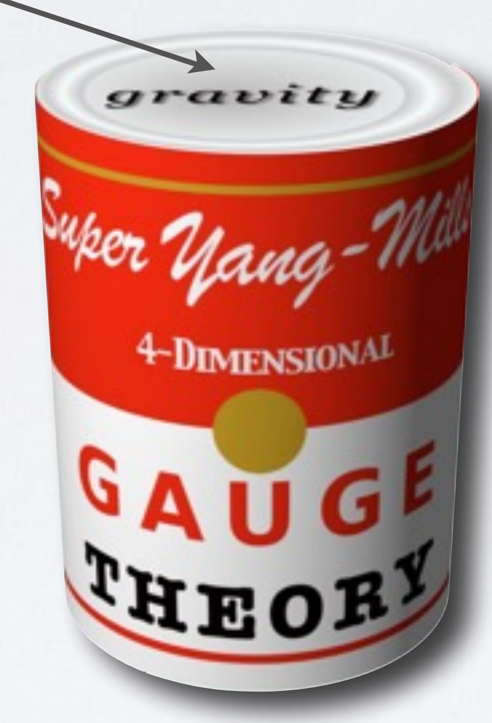
AdS/CFT correspondence

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS × S *"on boundary"*

[Maldacena, '97]

'soup can' diagram of AdS:



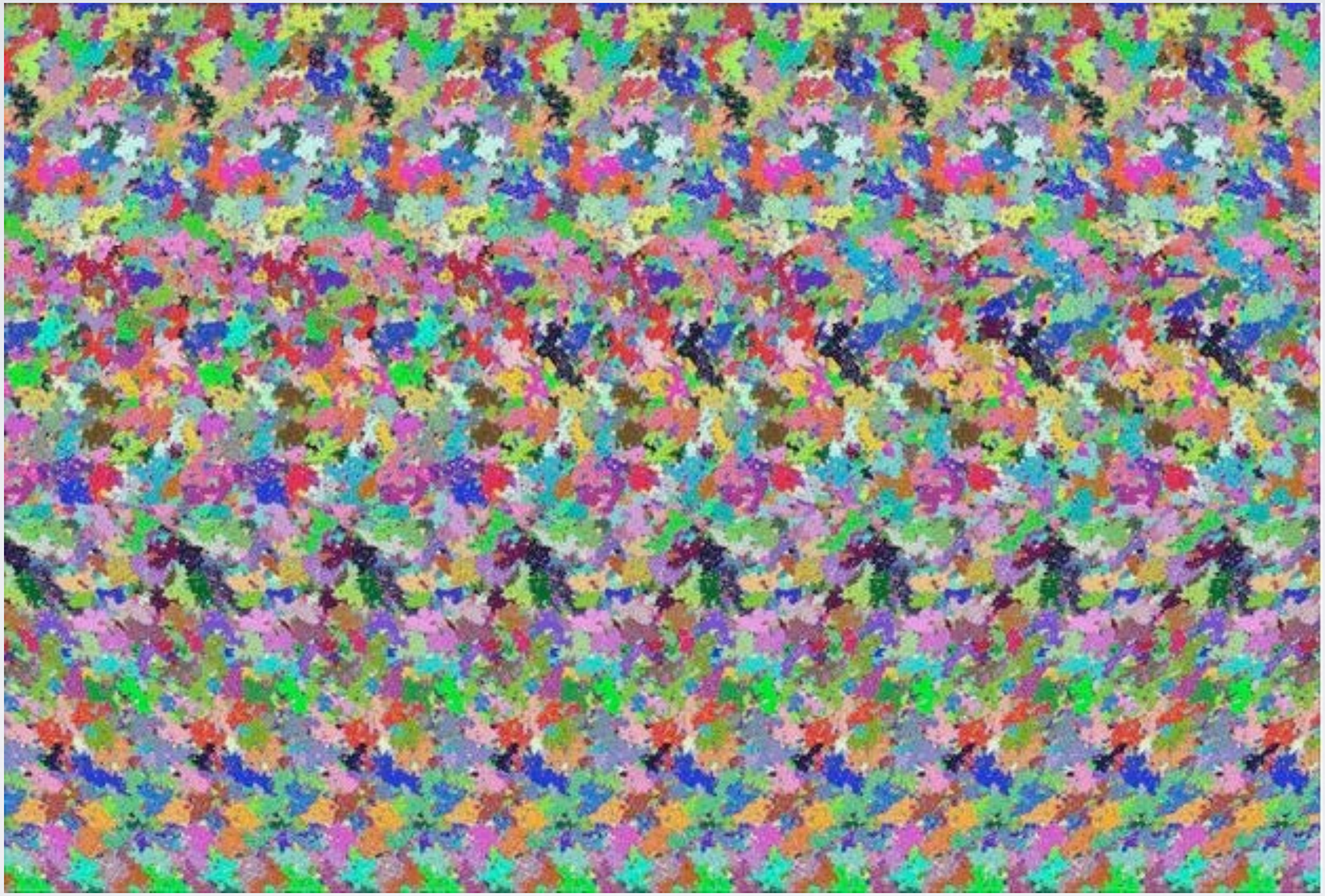
here label is everything...

Key aspects:

- ✶ Gravitational theory maps to non-gravitational one!
- ✶ *Holographic*: gauge theory lives in fewer dimensions.

AdS/CFT correspondence

- better analogy: stereogram...



...but infinitely more complicated

AdS/CFT correspondence

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS × K *"on boundary"*

Specific example:

IIB string theory on $AdS_5 \times S^5$:

g_s , $\dfrac{\ell}{\ell_s}$

string coupling: g_s

AdS size: ℓ

string length: ℓ_s

$\mathcal{N} = 4$ SU(N) SYM:

N , $\lambda \equiv g_{\text{YM}}^2 N$

rank of gauge group: N

't Hooft coupling: λ

where $g_s \sim \dfrac{\lambda}{N}$ and $\dfrac{\ell}{\ell_S} \sim \lambda^{1/4}$

- large λ $\Rightarrow$ small stringy corrections
- large N $\Rightarrow$ small quantum corrections
- Hence $N \gg \lambda \gg 1$ $\Rightarrow$ classical gravity on $AdS_5 \times S^5$

AdS/CFT correspondence

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

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Key aspects:

- ✳ Gravitational theory maps to non-gravitational one!
- ✳ *Holographic*: gauge theory lives in fewer dimensions.
- ✳ Strong/weak coupling duality.

Invaluable tool to:

~ Use gravity on AdS to learn about strongly coupled field theory

(as successfully implemented in e.g. AdS/QCD & AdS/CMT programs)

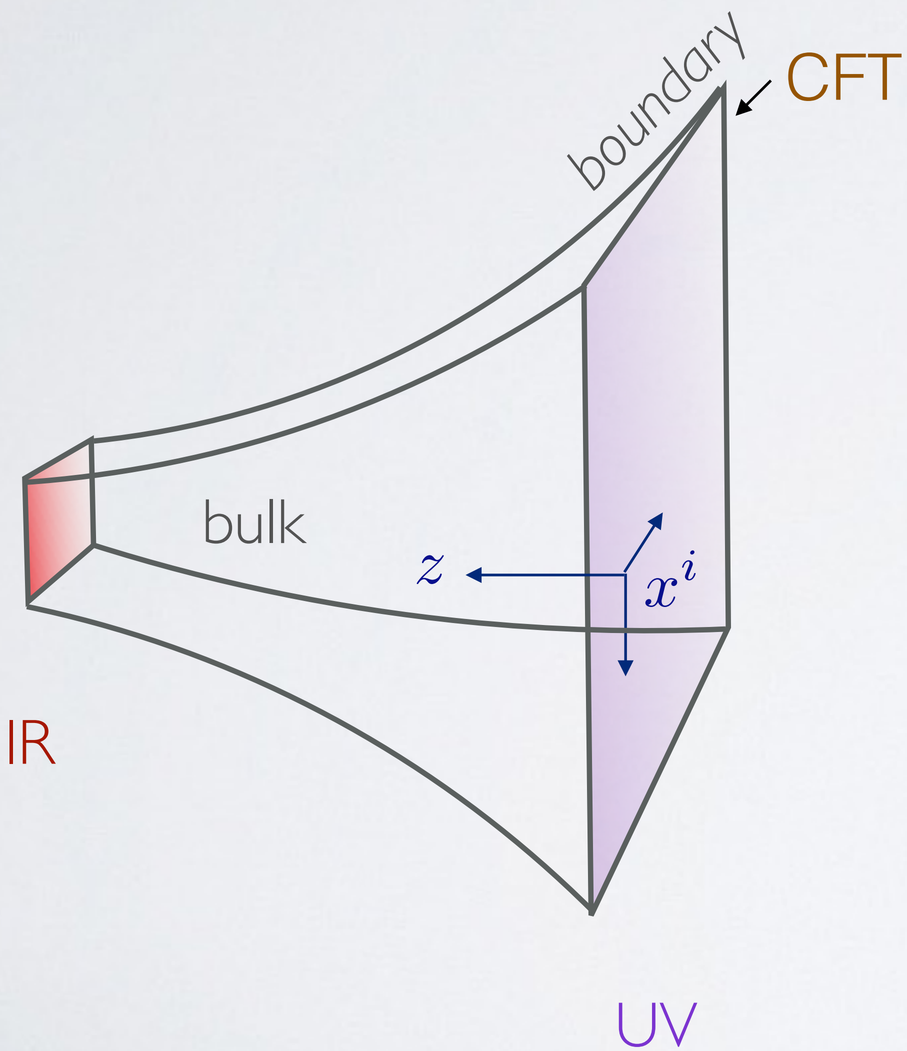
~ Use the gauge theory to define & study quantum gravity in AdS

Pre-requisite: Understand the AdS/CFT 'dictionary'...

Geometry of AdS

Poincare AdS:

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i\, dx^i + dz^2\right)$$



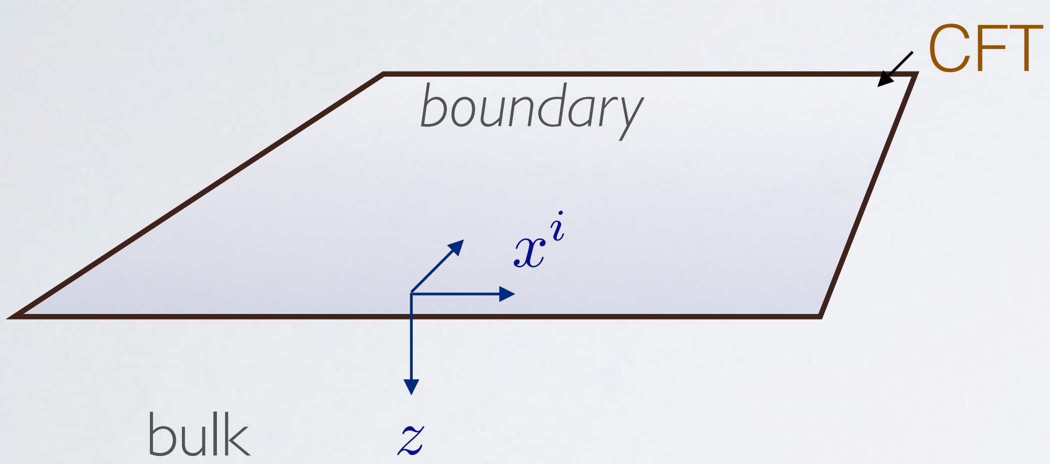
Geometry of AdS

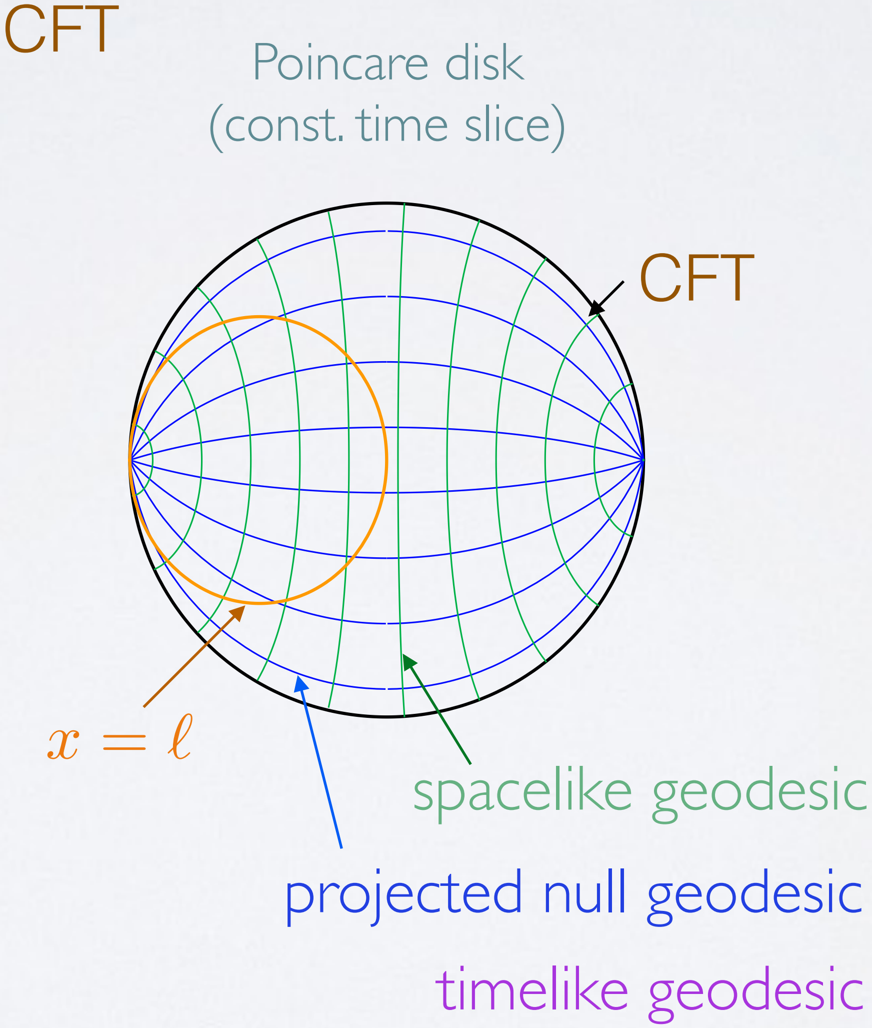
Poincare AdS:

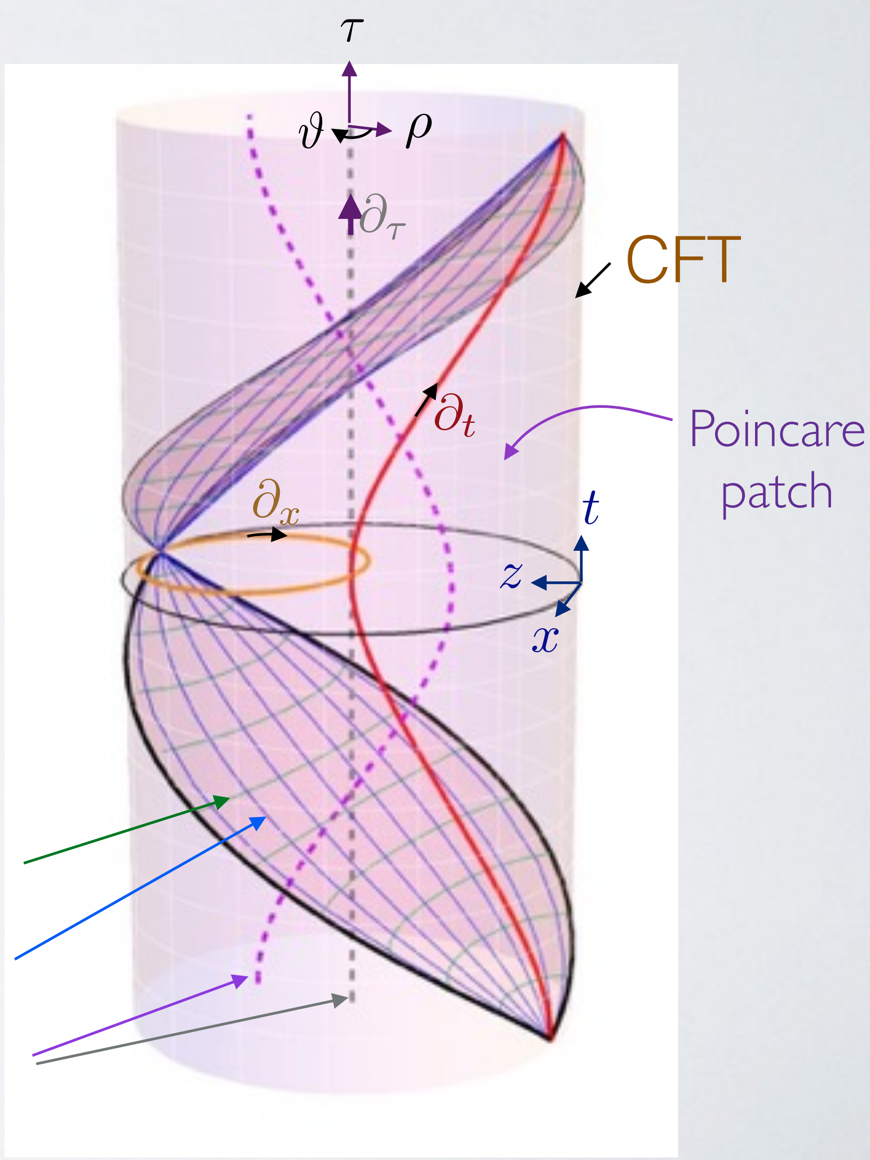
$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i\, dx^i + dz^2\right)$$

Global AdS:

$$ds^2 = -\left(\frac{\rho^2}{\ell^2}+1\right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{\ell^2}+1\right)} + \rho^2\, d\Omega_3^2$$







Scale/radius duality

What CFT quantity encodes the extra bulk direction?

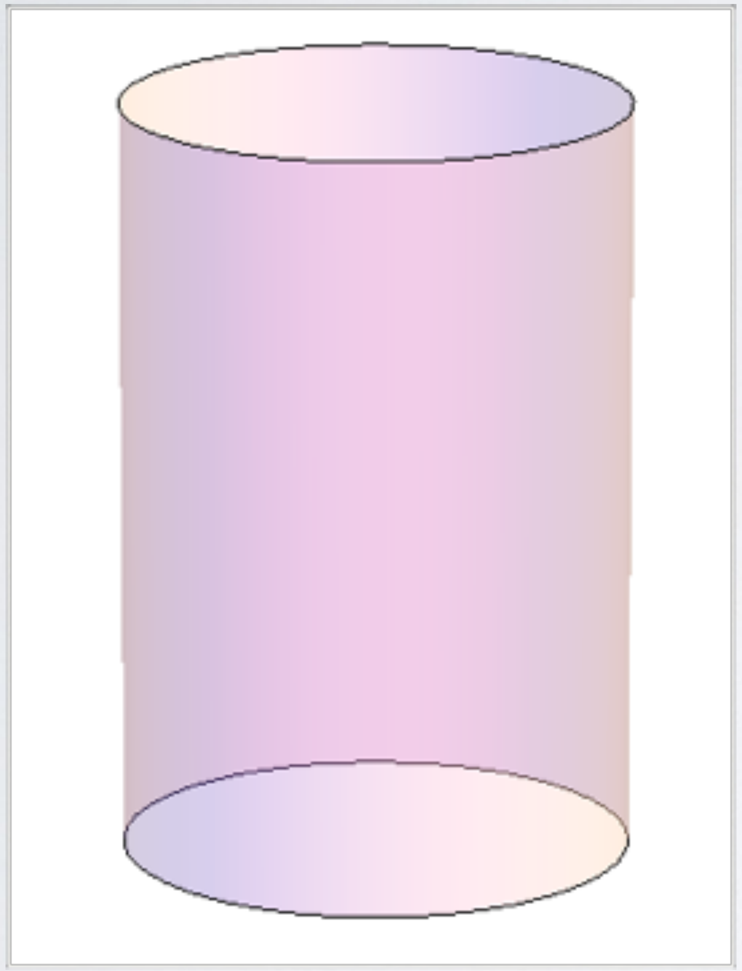
- Scale/radius (or UV/IR) duality:
  - UV (small scale) in CFT ⇝ IR (large radius) in AdS
  - Local bulk excitation at radial position *z* in AdS is manifested by CFT excitation at scale *L*~*z*. [Susskind & Witten]
  - Follows from AdS geometry…
  - Provides useful intuition: e.g. object falling into a black hole ⇝ CFT excitation spreads & thermalizes [Banks, Douglas, Horowitz, Martinec]

- Asymptotic fall-off of bulk fields ⇝ Expectation values of local gauge-invariant operators in CFT

Bulk geometries and CFT states

different bulk geometries $\longleftrightarrow$ different states in CFT

(asymptotically AdS)



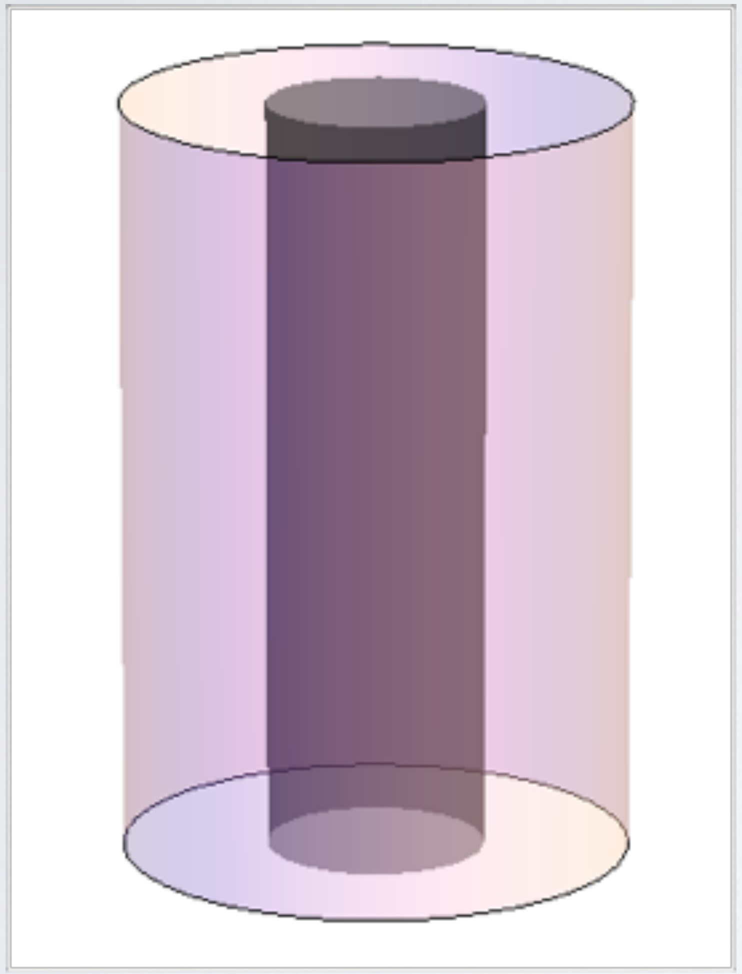
- Pure AdS $\leftrightsquigarrow$ vacuum state in CFT

Finite-mass deformations
of the bulk geometry result in
non-zero boundary stress tensor

Bulk geometries and CFT states

different bulk geometries $\longleftrightarrow$ different states in CFT

(asymptotically AdS)

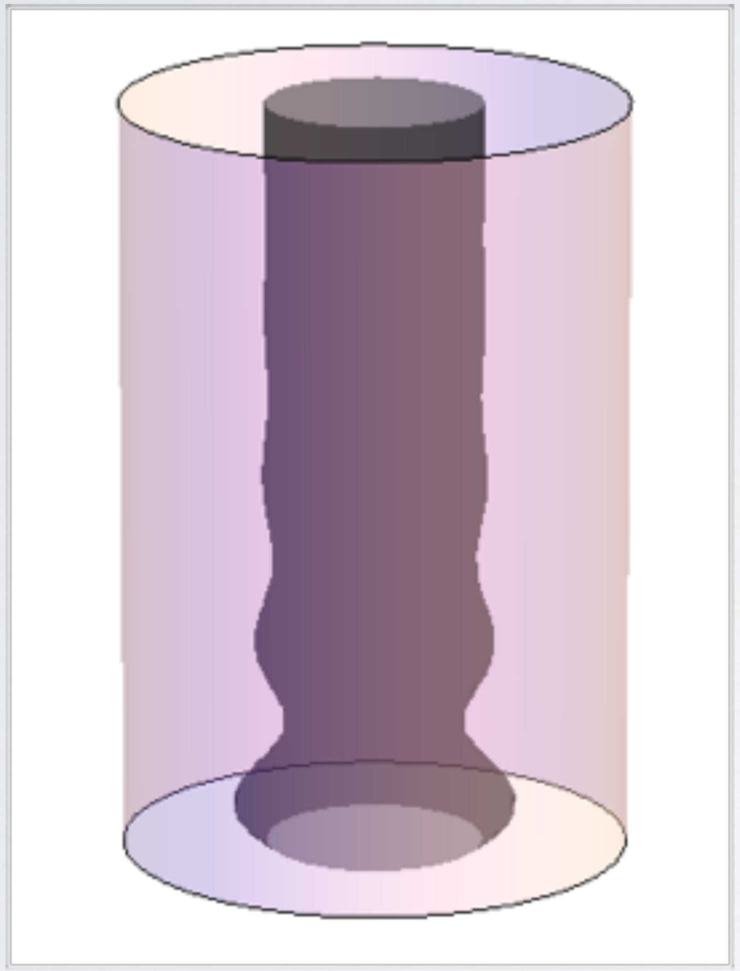


- Pure AdS $\leftrightsquigarrow$ vacuum state in CFT
- Black hole $\leftrightsquigarrow$ thermal state in CFT

(large BH $\leftrightsquigarrow$ high temperature)

Bulk geometries and CFT states

evolving bulk geometries $\longleftrightarrow$ corresponding dynamics



- Pure AdS $\leftrightsquigarrow$ vacuum state in CFT
- Black hole $\leftrightsquigarrow$ thermal state in CFT
- quasinormal modes of perturbed black hole $\leftrightsquigarrow$ approach to thermal equilibrium

  [Horowitz & VH]
- horizon response properties $\leftrightsquigarrow$ CFT transport coefficients

  [Kovtun, Son, Starinets]

- at non-linear level, in hydro regime (large BHs) $\rightarrow$ fluid/gravity correspondence

  [Bhattacharyya, VH, Minwalla, Rangamani]

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Paths to Holographic EE

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS $\times$ K *"on boundary"*

Applied AdS/CFT:

- study specific system via its dual
- e.g. AdS/QCD, AdS/CMT, …

Fundamentals of AdS/CFT:

- why/how does the duality work
- map between the 2 sides

Holographic Entanglement Entropy

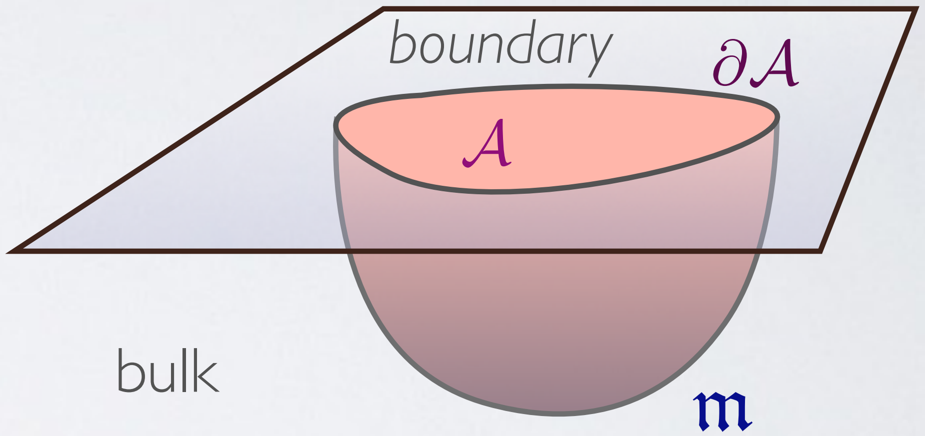
Quantum Gravity

Holographic Entanglement Entropy

Proposal [RT = Ryu & Takayanagi, '06] for *static* configurations:

In the bulk, EE $S_{\mathcal{A}}$ is captured by the area of minimal co-dimension 2 bulk surface $\mathfrak{m}$ *(at constant t)* anchored on $\partial\mathcal{A}$.

$$S_{\mathcal{A}} = \min_{\partial\mathfrak{m}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{m})}{4\,G_N}$$



Remarks:

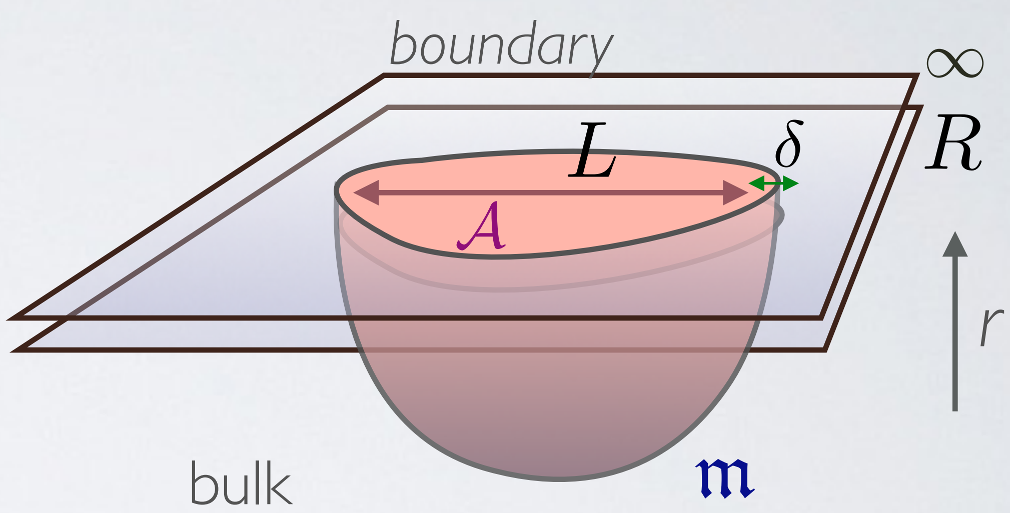
- Large body of evidence, culminating in [Lewkowycz & Maldacena]
- cf. black hole entropy…
- Minimal surface "hangs" into the bulk due to large distances near bdy.
- Note that both LHS and RHS are in fact infinite…

Area-law divergence of HEE

Short-distance cutoff δ in the CFT translates to large-radius cutoff R in AdS$_{d+1}$

with $\delta = \dfrac{\ell^2}{R}$ (cf. UV/IR duality)

Bulk area reproduces the correct divergence structure:



$$S_{\mathcal{A}} = c_0 \left(\frac{L}{\delta}\right)^{d-2} + c_1 \left(\frac{L}{\delta}\right)^{d-4} + \cdots$$

cutoff-dependent coefficients

$$+ \begin{cases} c_{d-2} \log\left(\frac{L}{\delta}\right) + \cdots , & d \ \text{even} \\ c_{d-2} + \cdots , & d \ \text{odd} \end{cases}$$

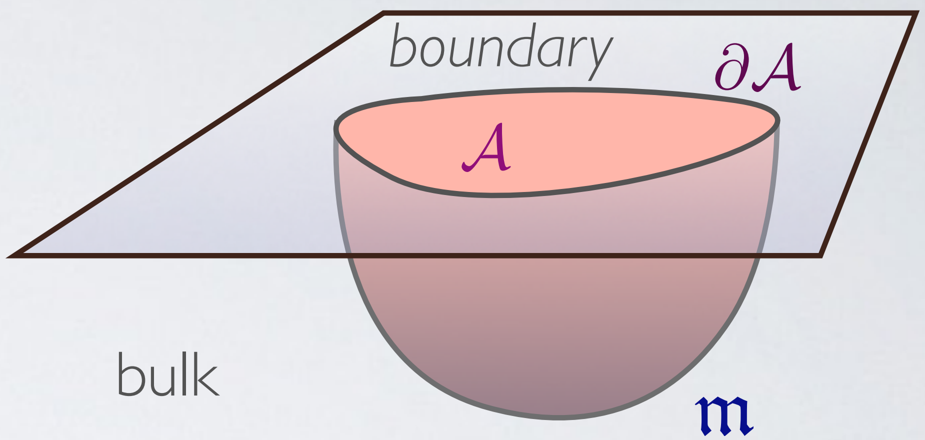
universal coefficients

We can regulate EE by e.g. background subtraction.

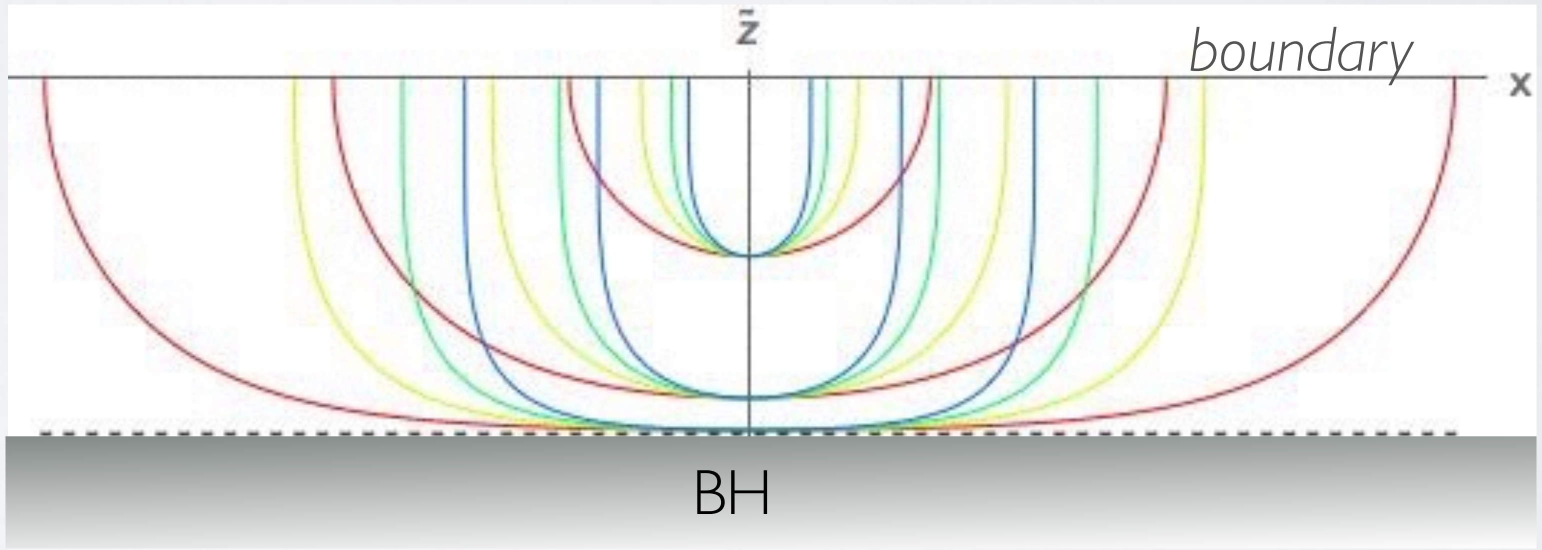
Area-law vs. Volume-law

The leading divergence of $S_{\mathcal{A}}$ necessarily scales with the area of the entangling surface $\partial\mathcal{A}$

→ Area-law

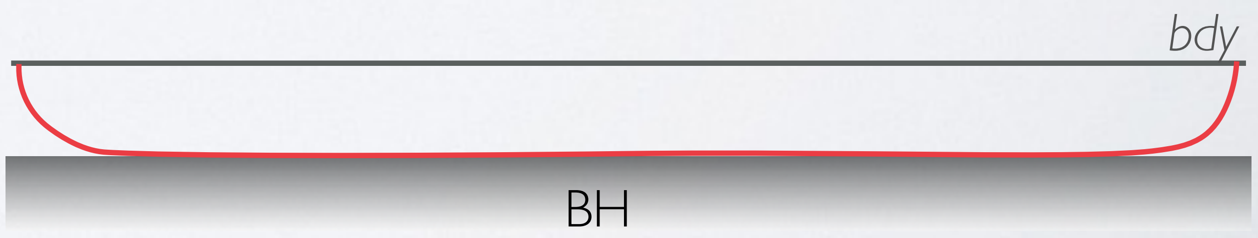


But in the presence of a (static) black hole, extremal surface get repelled by the horizon.



At high temperature ($T\,L \gg 1$), the finite part of $S_{\mathcal{A}}$ scales with the volume of the region $\mathcal{A}$

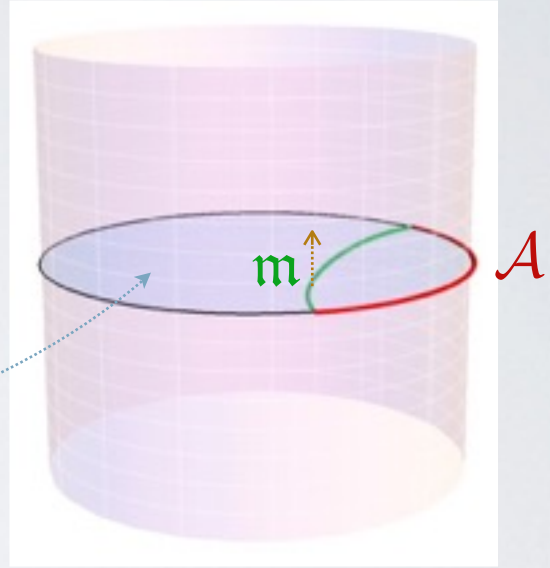
→ Volume-law



Covariant Holographic EE

But the RT prescription is not well-defined outside the context of static configurations:

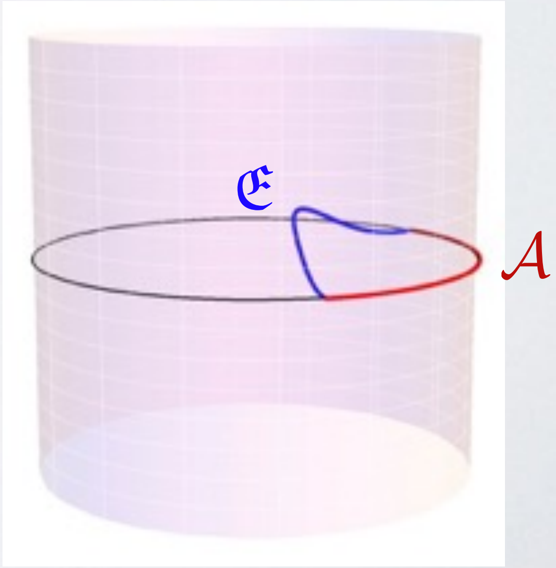
- In Lorentzian geometry, we can decrease the area arbitrarily by timelike deformations
- In time-dependent context, no natural notion of "const. t" slice…



In *time-dependent* situations, RT prescription must be covariantized:

Simplest candidate:

minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk

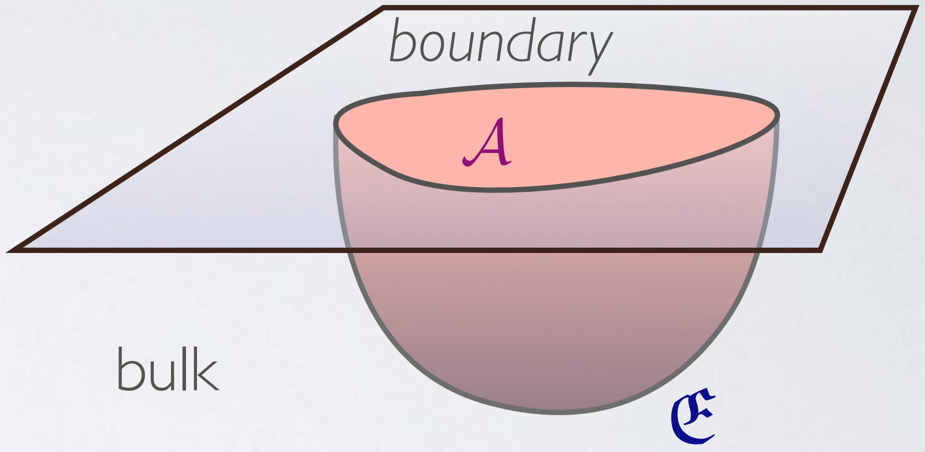


[HRT = VH, Rangamani, Takayanagi '07]

Covariant Holographic EE

HRT Prescription: [VH, Rangamani, Takayanagi]

In the bulk EE $S_{\mathcal{A}}$ is captured by the area of extremal co-dimension 2 bulk surface $\mathfrak{E}$ anchored on $\partial\mathcal{A}$ & homologous to $\mathcal{A}$



$$S_{\mathcal{A}} = \min_{\partial\mathfrak{E}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{E})}{4\,G_N}$$

Equivalently:

- $\mathfrak{E}$ is the surface with zero null expansions; (cf. light sheet construction [Bousso])
- maximin construction: maximize over minimal-area surface on a spacelike slice [Wall]

This gives a well-defined quantity in any (arbitrarily time-dependent asymptotically AdS) spacetime ⇒ equally robust as in CFT

But we can't use Euclidean techniques / minimization for proofs…

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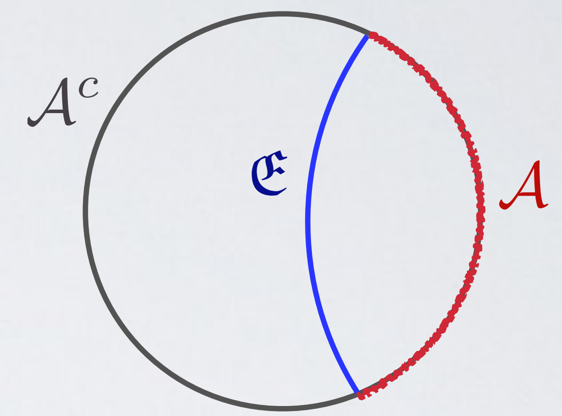
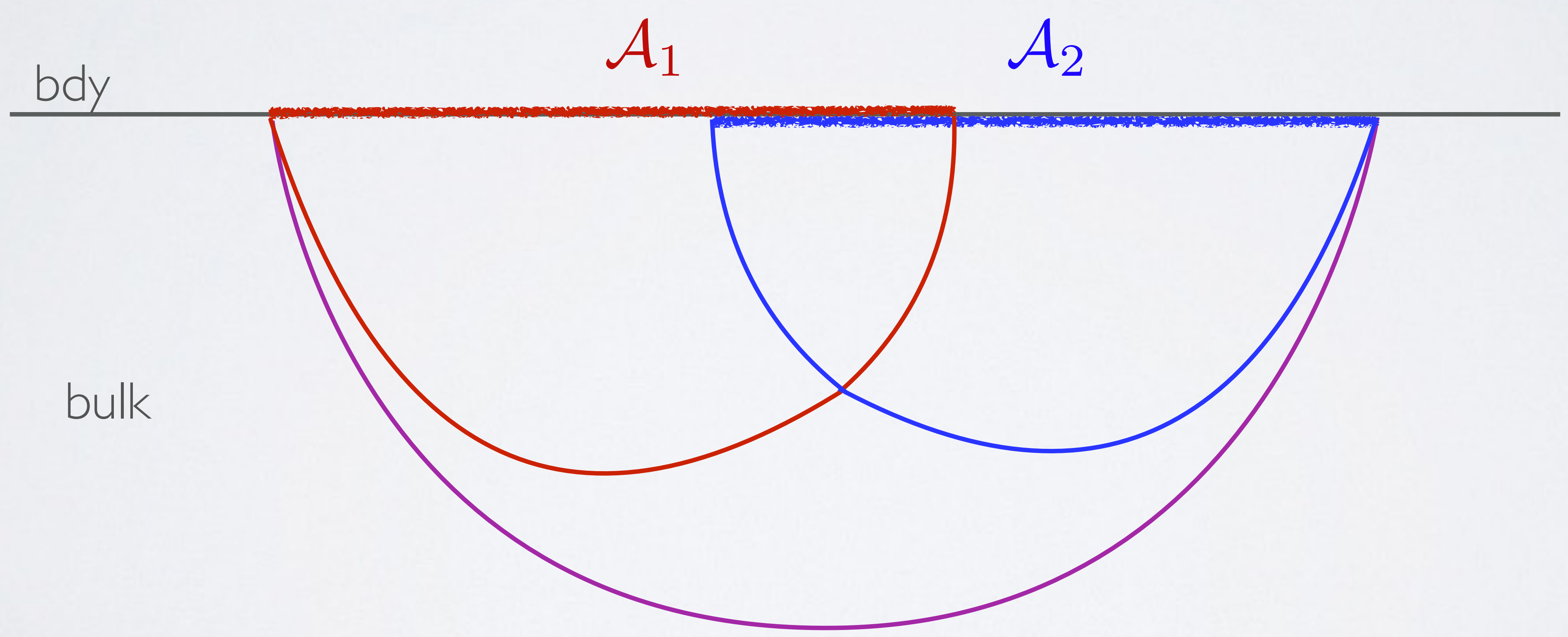
Manifest properties of EE



- For pure states $S_{\mathcal{A}} = S_{\mathcal{A}^c}$
- Positivity: $S_{\mathcal{A}} \geq 0$
- Subadditivity: $S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2}$



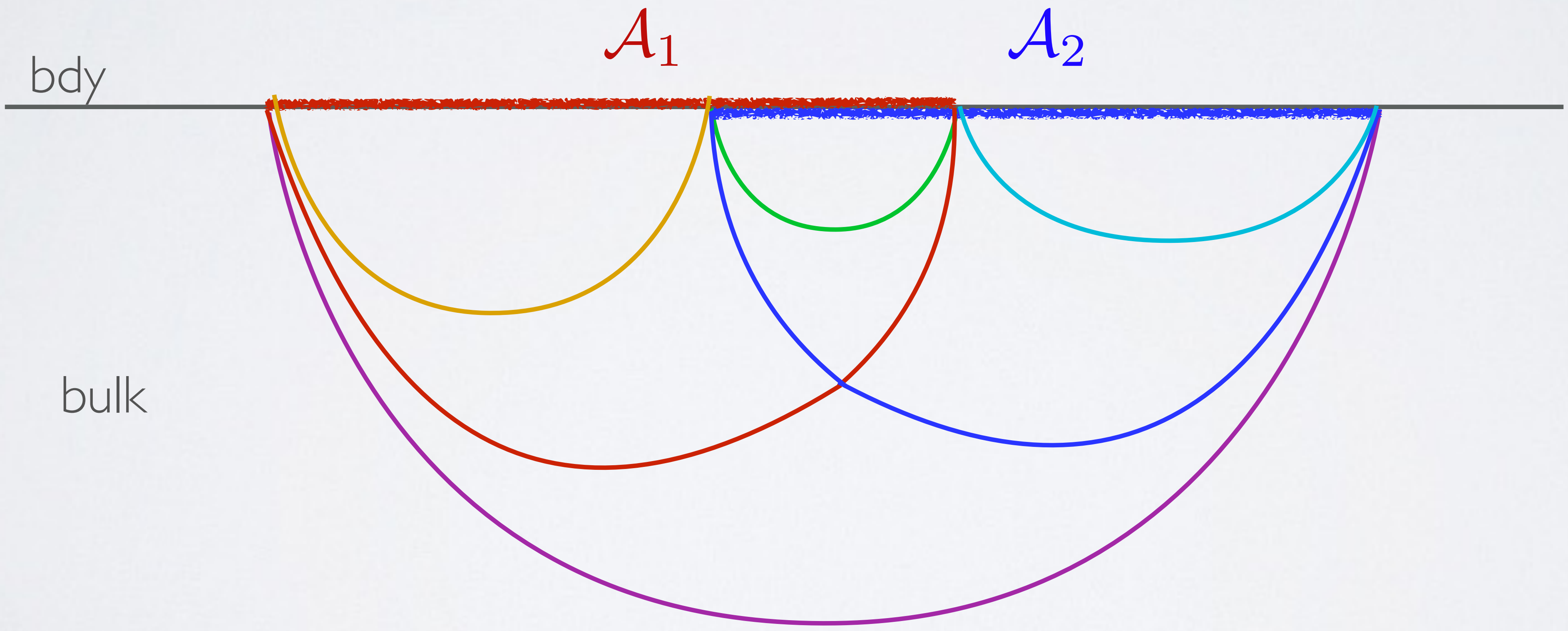
- Implies positivity of mutual information: $I(\mathcal{A}_1, \mathcal{A}_2) = S_{\mathcal{A}_1} + S_{\mathcal{A}_2} - S_{\mathcal{A}_1 \cup \mathcal{A}_2}$

Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \setminus \mathcal{A}_2} + S_{\mathcal{A}_2 \setminus \mathcal{A}_1}$$

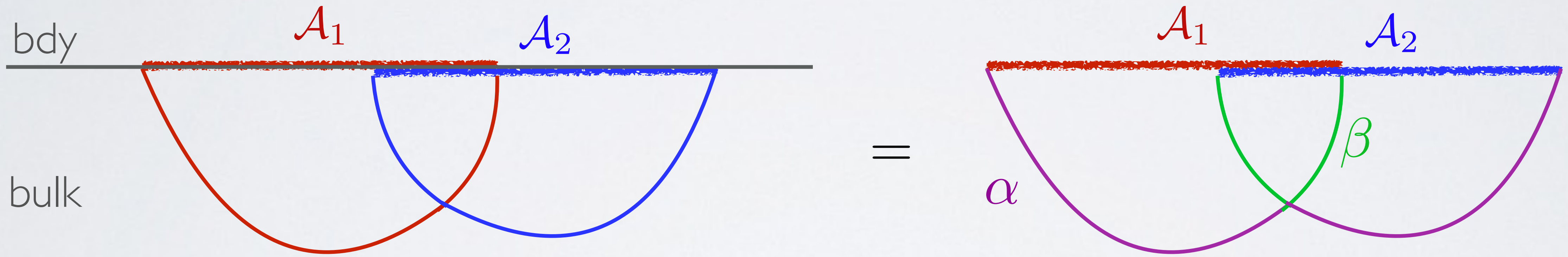


Proof of Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

- proof in static configurations [Headrick & Takayanagi]



$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \alpha + \beta$$

Proof of Strong Subadditivity

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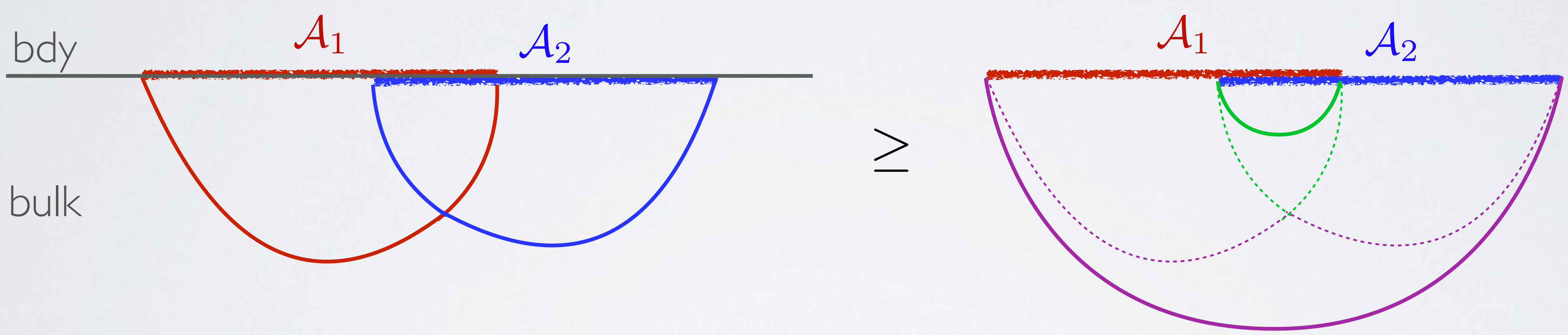


$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \alpha + \beta \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

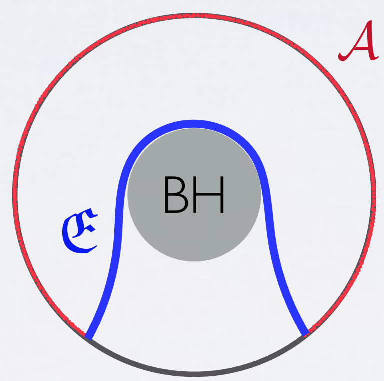
- Similarly prove monogamy of mutual information [Hayden, Headrick, Maloney] valid in holography but not in general: $S_A + S_B + S_C + S_{ABC} \leq S_{AB} + S_{BC} + S_{AC}$
- general proof uses properties of null geodesics + energy conditions [Wall]

Araki-Lieb inequality

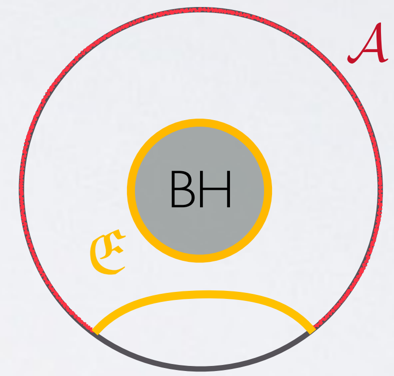
- for system in a mixed state (density matrix) ρ_Σ on spatial slice $\Sigma = \mathcal{A} \cup \mathcal{A}^c$
- in general $S_\mathcal{A} \neq S_{\mathcal{A}^c}$, due to the homology constraint

[Headrick&Takayanagi]

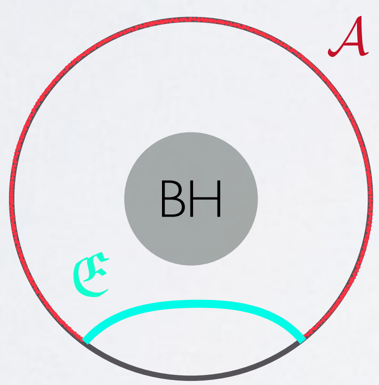
E.g. for thermal state on S[1], dual to eternal (BTZ) BH, both of the following satisfy the homology constraint:



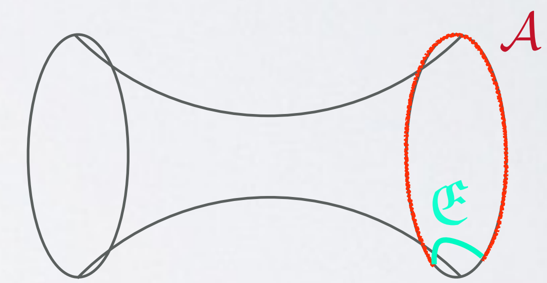
or



but NOT



since $\nexists$ interpolating region whose only boundaries are $\mathcal{A}$ and $\mathfrak{E}$:



- but EE satisfies: $\underbrace{|S_\mathcal{A} - S_{\mathcal{A}^c}|}_{\delta S_\mathcal{A}} \leq S_{\rho_\Sigma} \leq S_\mathcal{A} + S_{\mathcal{A}^c}$

Araki-Lieb

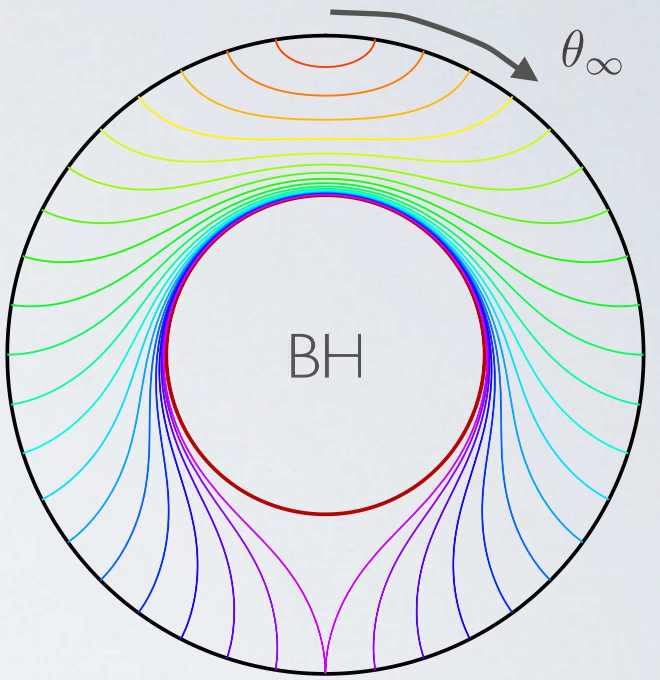
subadditivity ✔

Entanglement plateaux

- Naively, Araki-Lieb appears in danger of being violated; E.g. for thermal state on compact space S^1

Bulk dual: BTZ black hole:

$$(S_{\mathcal{A}})_{\text{naive}} = \frac{c}{3} \log\left(\frac{2\, r_\infty}{r_+} \sinh(r_+\, \theta_\infty)\right)$$



- Resolution is supplied by the minimality condition:

$$S_{\mathcal{A}} = \min_{\partial\mathfrak{E} = \partial\mathcal{A}} \frac{\text{Area}(\mathfrak{E})}{4\, G_N}$$

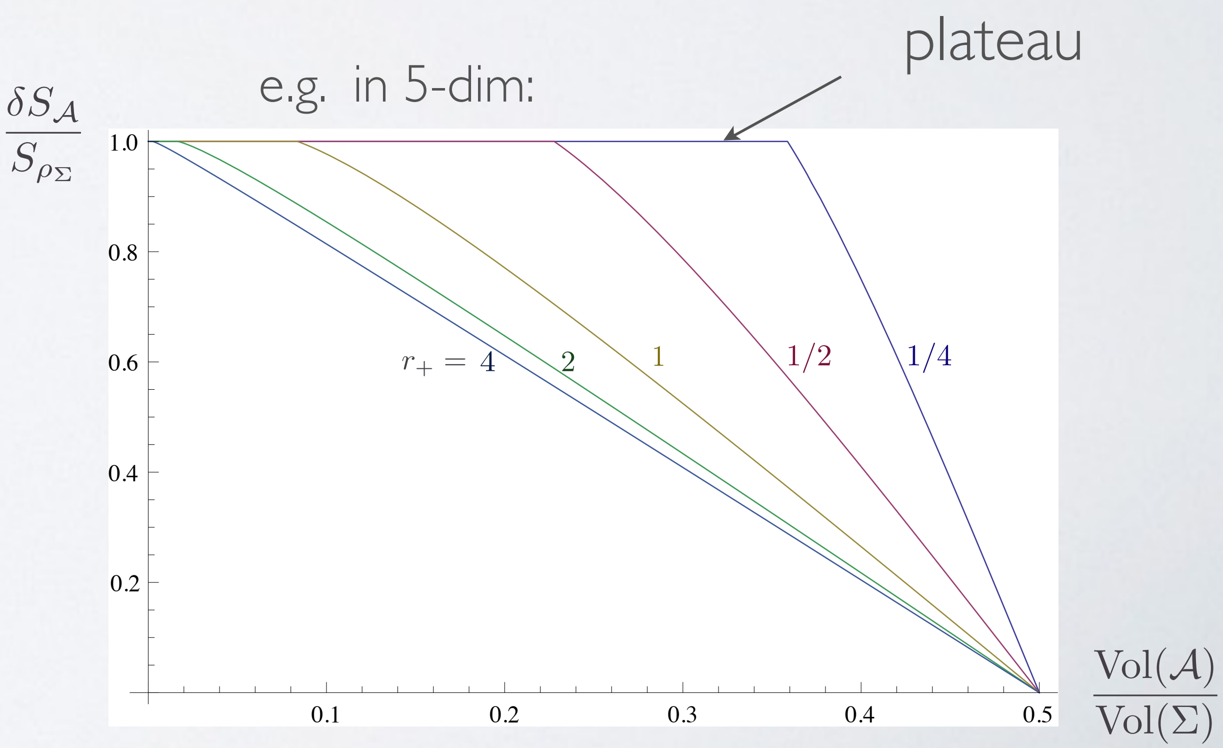
for $\mathcal{A}$ large enough, the disjoint configuration dominates:

$$\Rightarrow \quad S_{\mathcal{A}} = S_{\mathcal{A}^c} + S_{\text{BH}}$$

⇝ entanglement plateau

[VH, Maxfield, Rangamani, Tonni]

e.g. in 5-dim:

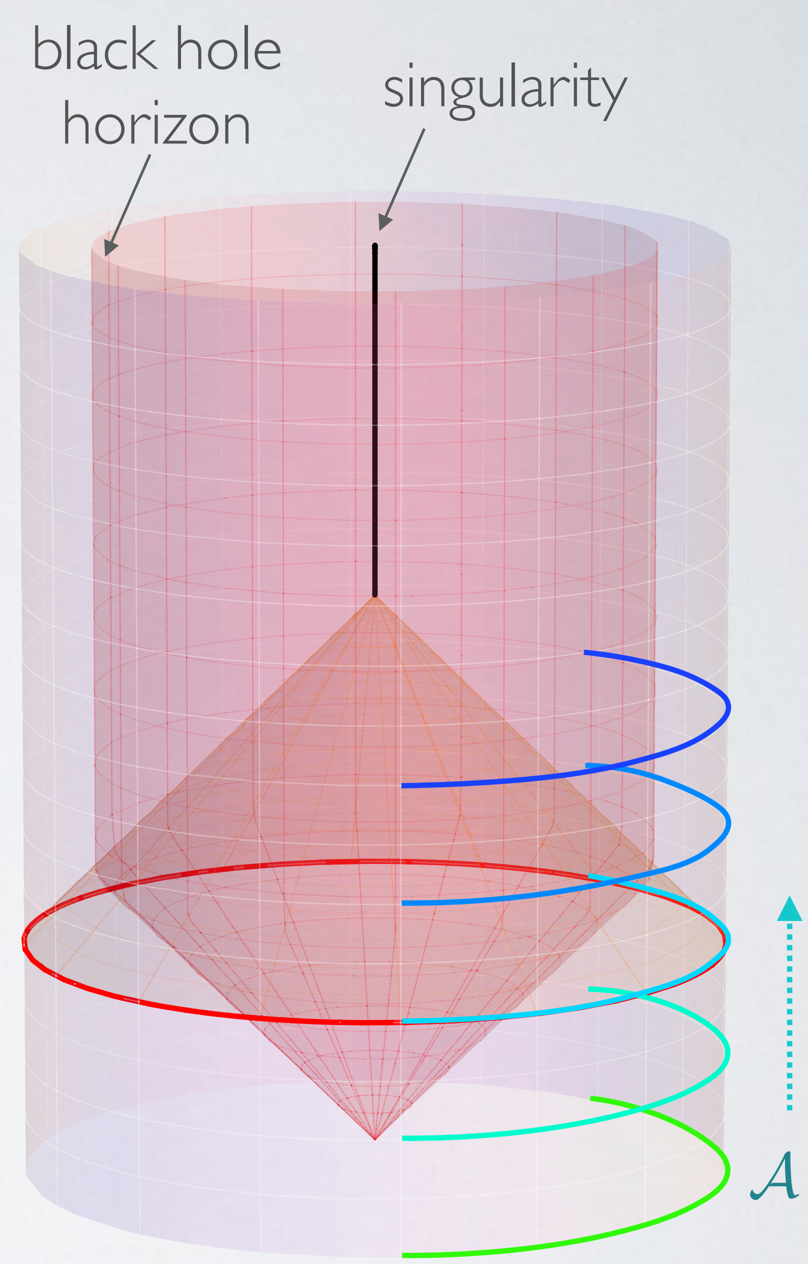


Utility for dynamics

- For strongly time-dependent situations, holography is often the best tool presently available…
- E.g. consider quantum quench & thermalization
  - prepare a system in ground state of Hamiltonian H_0
  - at t=0 deform the Hamiltonian to a new Hamiltonian H by sudden change in some parameter
  - let the system evolve (~ thermalize) with the new H
- Global quench: change to H is homogeneous
  - in the bulk dual, we can model this by Vaidya-AdS corresponding to collapse of a null shell which forms a black hole.

Building up Vaidya-AdS

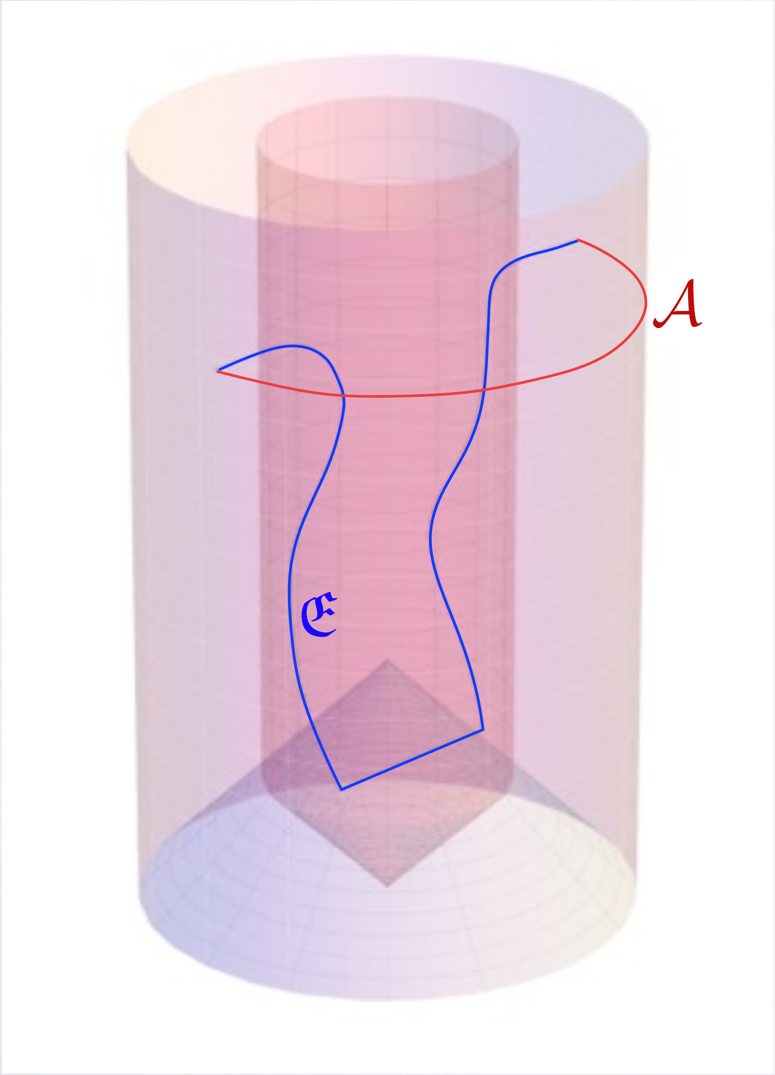
- start with vacuum state in CFT
  = pure AdS in bulk
- at t=0, create a short-duration disturbance in the CFT (global quench)
- this will excite a pulse of matter (shell) in AdS which implodes under evolution
- gravitational backreaction: collapse to a black hole ⇒ CFT 'thermalizes'
- large CFT energy ⇒ large BH

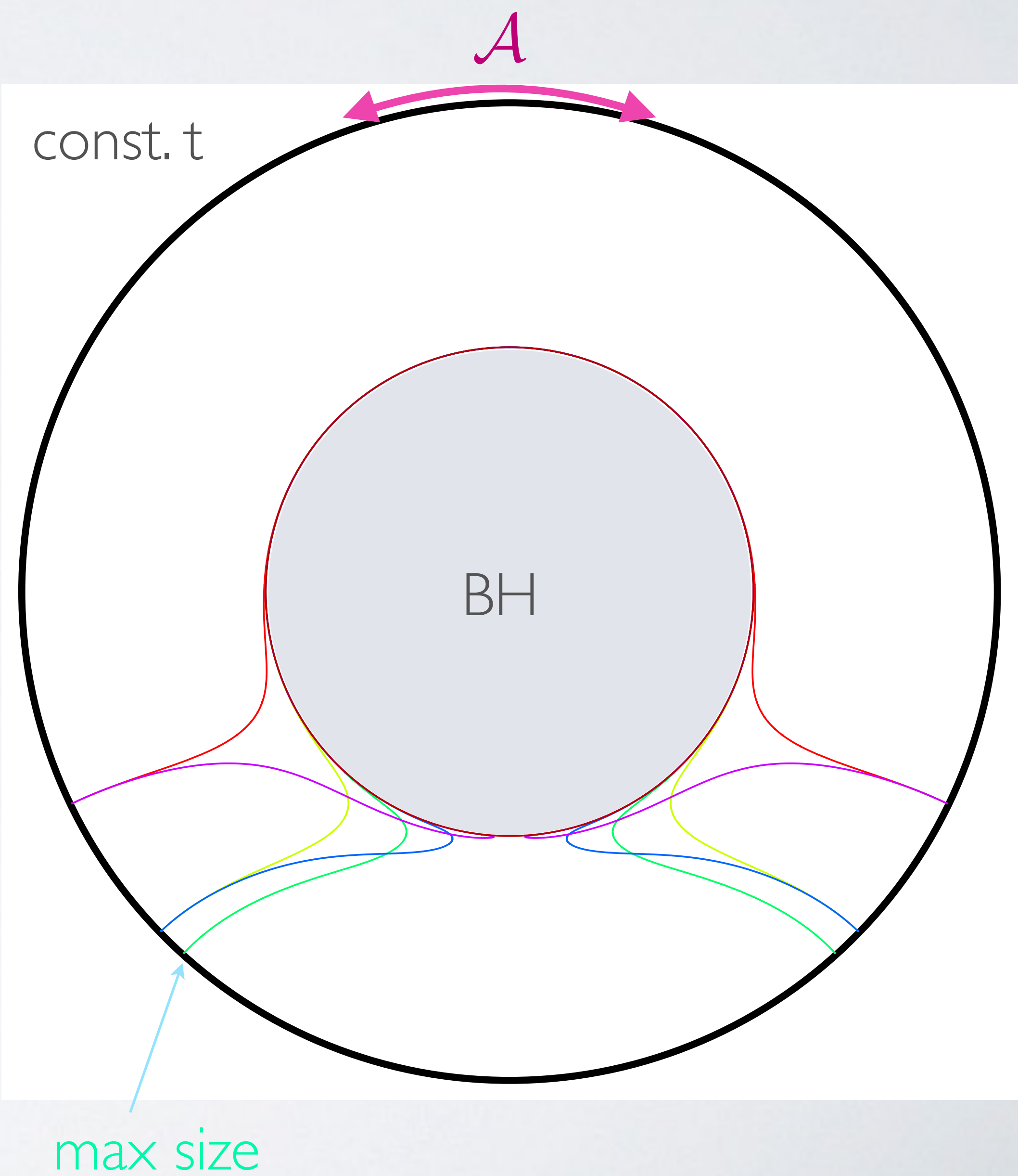


- To study thermalization of $S_{\mathcal{A}}$, fix spatial extent of $\mathcal{A}$ and vary the time…

Thermalization

- Results: ∃ rather exotic extremal surfaces for d>3, e.g. ones
  - penetrating the BH from arbitrarily late boundary time
  - having arbitrarily many ‘folds’ (even for the static BH)





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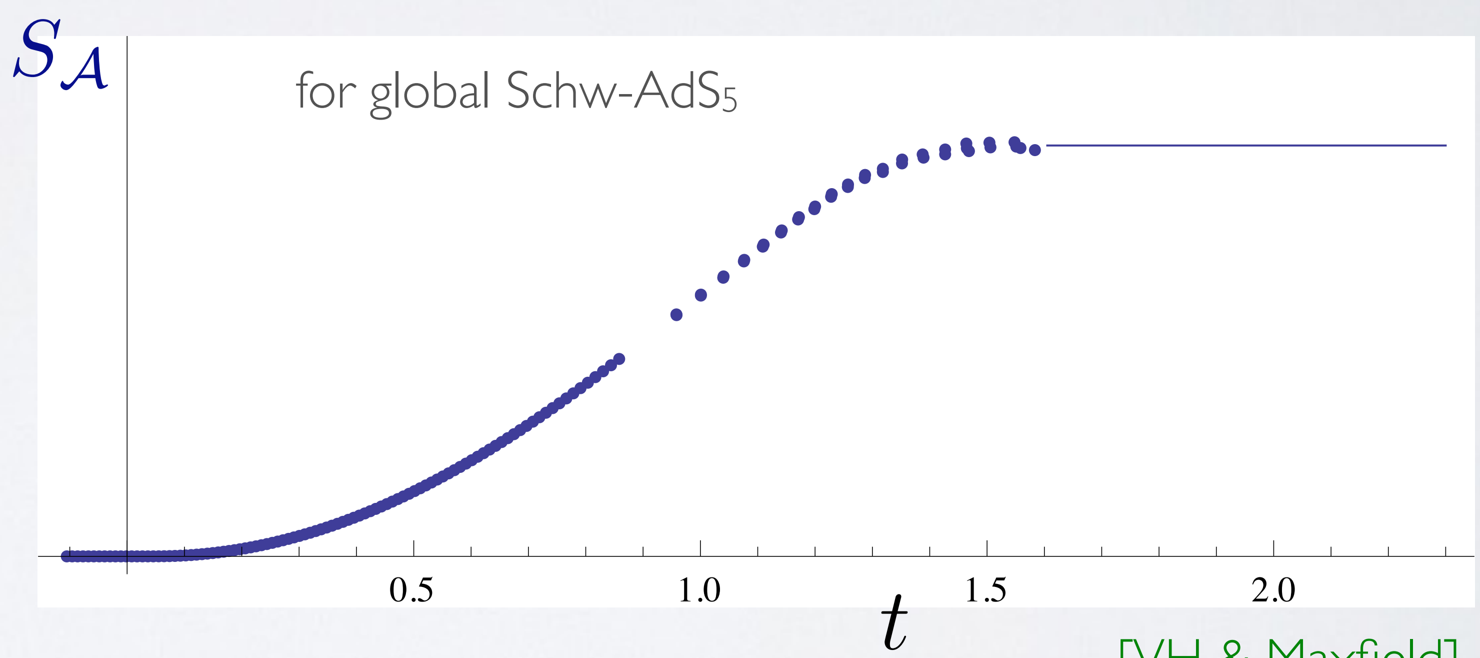
  - penetrating the BH from arbitrarily late boundary time
  - having arbitrarily many 'folds' (even for the static BH)

- nevertheless, minimality seems to ensure that $S_{\mathcal{A}}$ 'thermalizes' continuously and monotonically:



[VH & Maxfield]

  - cf. 'entanglement tsunami' [Liu & Suh] for planar Vaidya-AdS
  - (Note: a-priori, monotonicity & continuity wasn't guaranteed)

Power of covariant constructs

- 'Natural' geometrical constructs (defined for general bulk spacetimes, independent of coordinates) provide useful candidates for dual of 'natural' quantities in CFT
- e.g. dual of $\rho_{\mathcal{A}}$? [Bousso, Leichenauer, Rosenhaus; Czech, Karczmarek, Nogueira, Van Raamsdonk;...]
- In generic Lorentzian spacetime, null congruences which define a causal set provide useful characterization of 'natural' bulk regions.

2 options:

...starting from bdy:

$D[\mathcal{A}] \rightsquigarrow$ Causal Wedge: $\blacklozenge_{\mathcal{A}}$

= future and past causally-separated from bdy region determined by $\rho_{\mathcal{A}}$

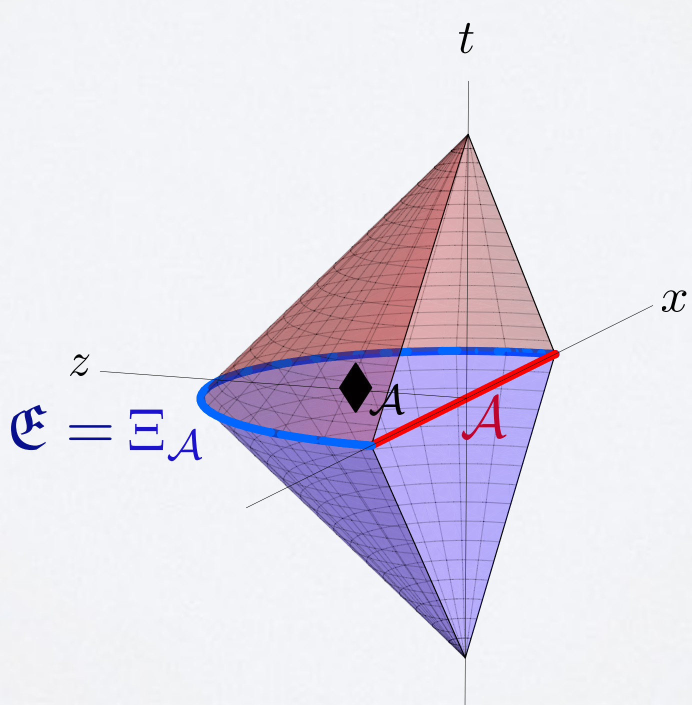
[VH & Rangamani]

...starting from bulk:

$\mathfrak{E} \rightsquigarrow$ Entanglement Wedge: $\mathcal{W}_E[\mathcal{A}]$

= spacelike-separated (toward $\mathcal{A}$) from $\mathfrak{E}$

[Headrick, VH, Lawrence, Rangamani]

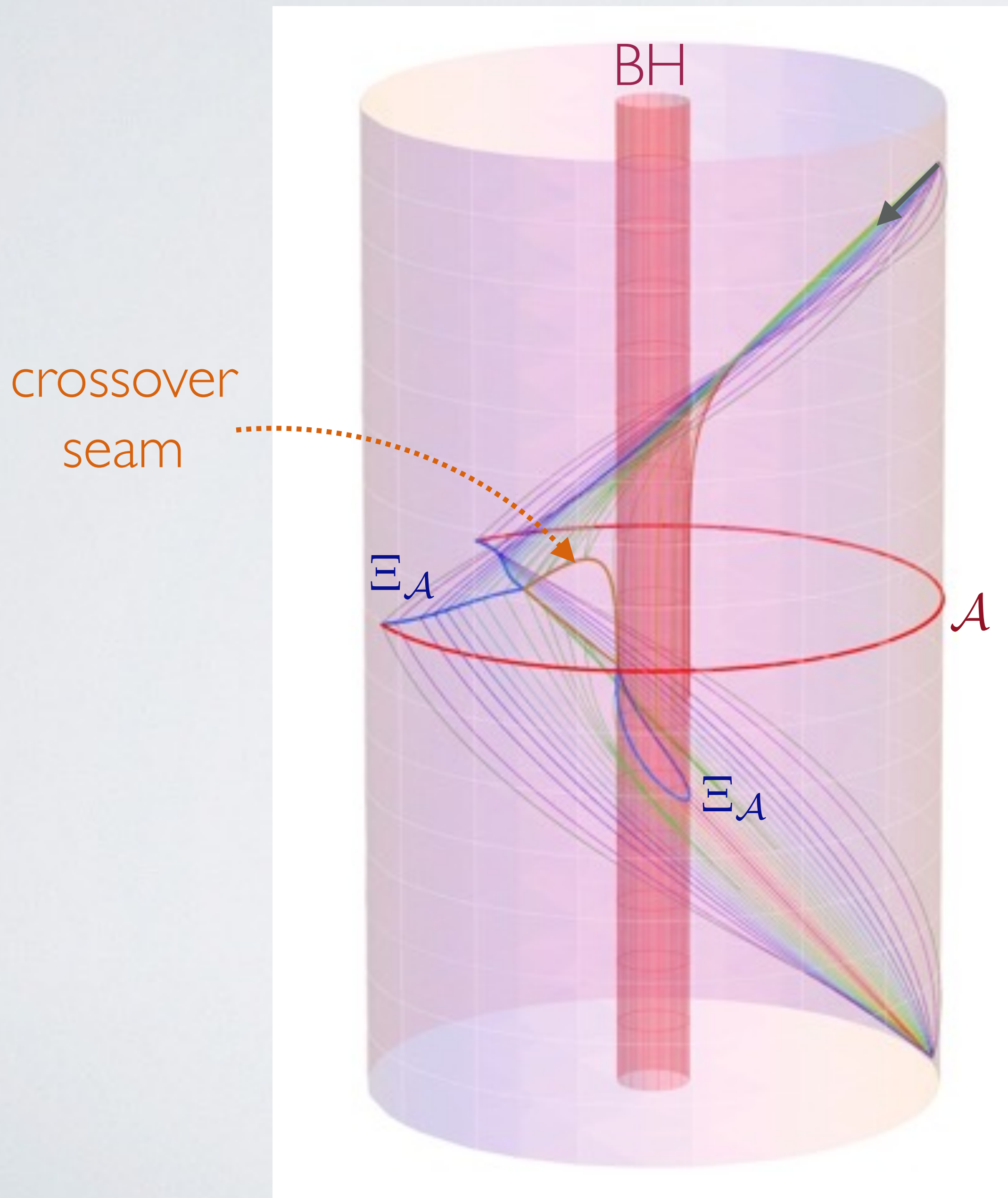


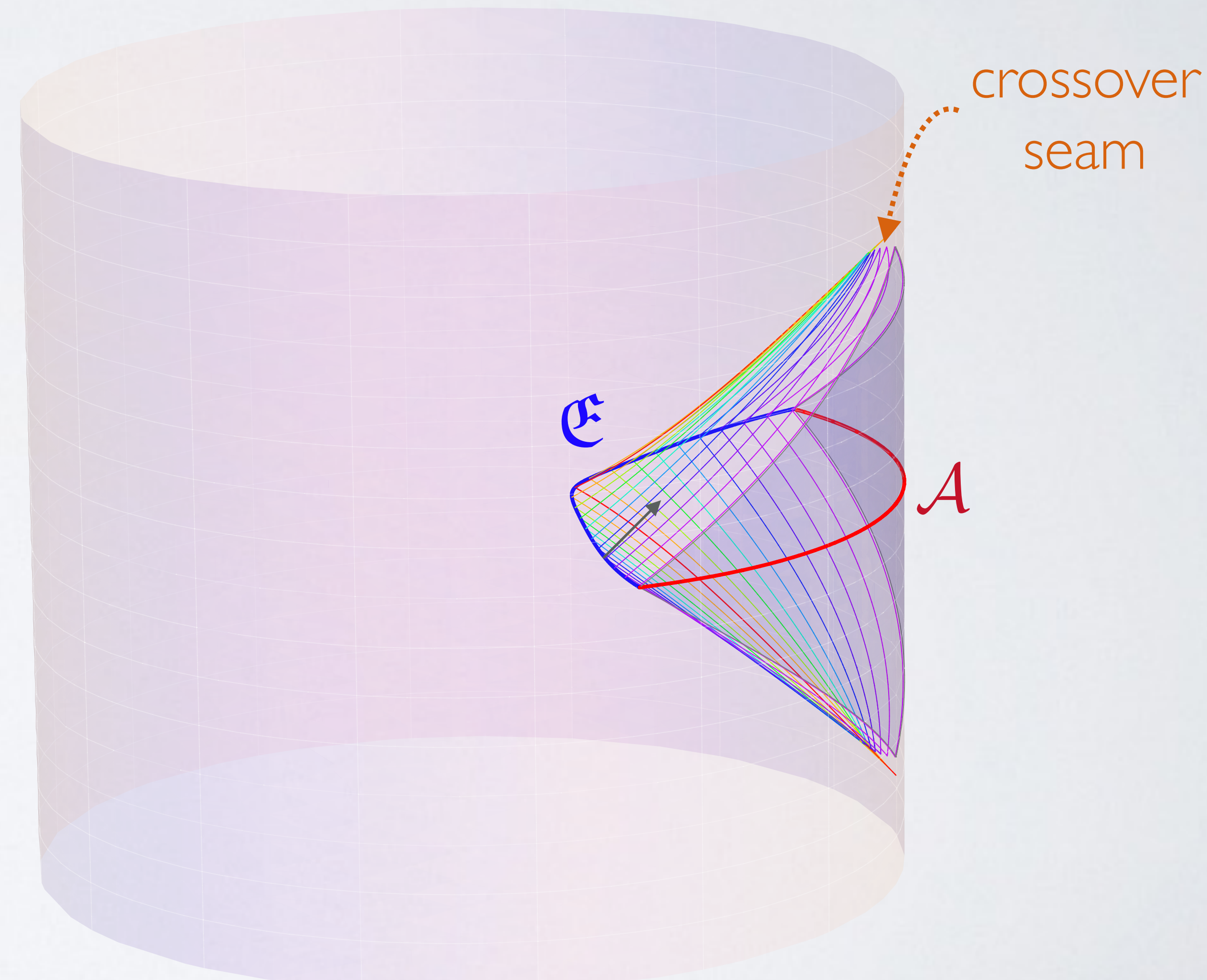
NB: in pure AdS, & for spherical $\mathcal{A}$, these coincide: $\blacklozenge_{\mathcal{A}} = \mathcal{W}_E[\mathcal{A}]$ (but not in general)

Causal wedge vs. Entanglement wedge

$D[\mathcal{A}] \rightsquigarrow$ Causal Wedge: $\blacklozenge_{\mathcal{A}}$

$\mathfrak{E} \rightsquigarrow$ Entanglement Wedge: $\mathcal{W}_E[\mathcal{A}]$





Power of covariant constructs

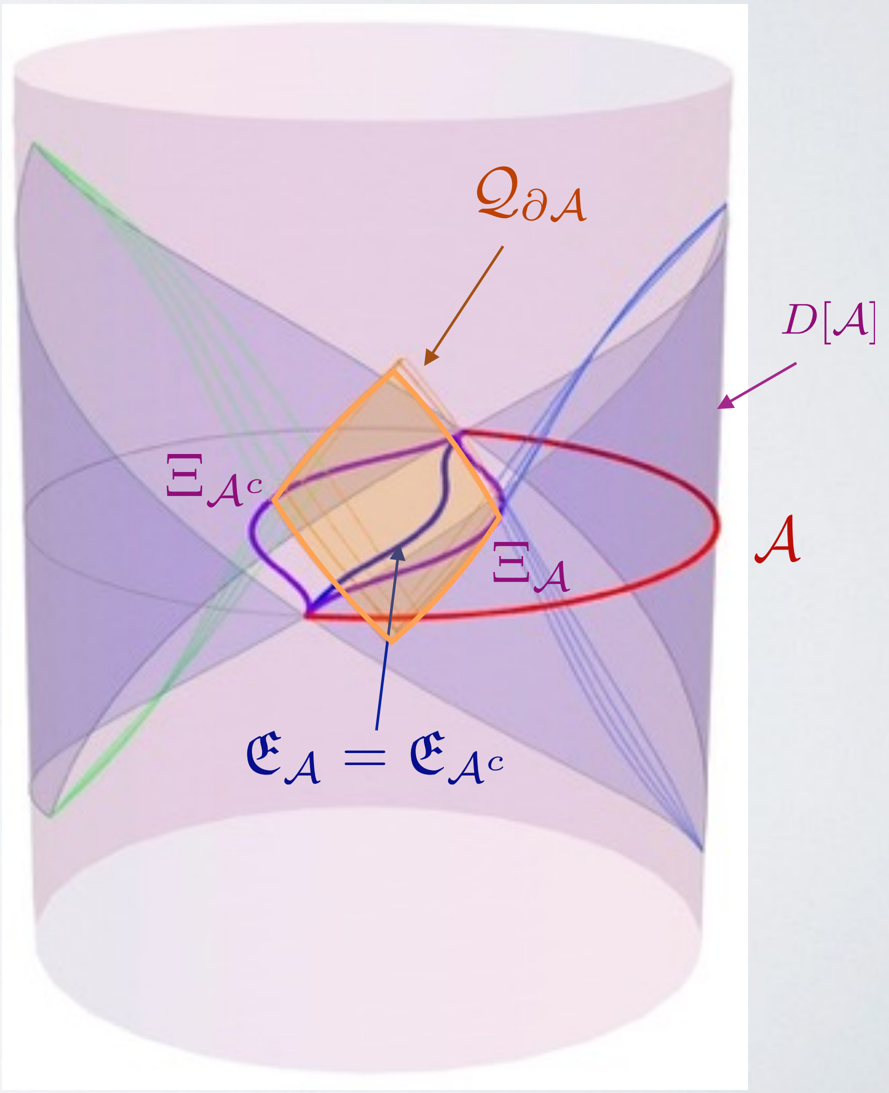
$D[\mathcal{A}] \rightsquigarrow$ Causal Wedge: $\blacklozenge_{\mathcal{A}}$ $\mathfrak{E} \rightsquigarrow$ Entanglement Wedge: $\mathcal{W}_E[\mathcal{A}]$

…continued past Ξ: $\rightsquigarrow$ Causal Shadow $\mathcal{Q}_{\partial\mathcal{A}}$

- We can prove the inclusion property [Headrick, VH, Lawrence, Rangamani; Wall]

$$\blacklozenge_{\mathcal{A}} \subset \mathcal{W}_E[\mathcal{A}]$$

or equivalently, $\mathfrak{E} \subset \mathcal{Q}_{\partial\mathcal{A}}$



- Consequences:
  - HRT is consistent with CFT causality (= non-trivial check of HRT)
  - Entanglement plateaux
  - Entanglement wedge can reach deep inside a black hole!

Curious features of EE:

- Extremal surfaces can have intricate behavior:
  - $S_{\mathcal{A}}$ can have discontinuous jumps under smooth variations of $\mathcal{A}$

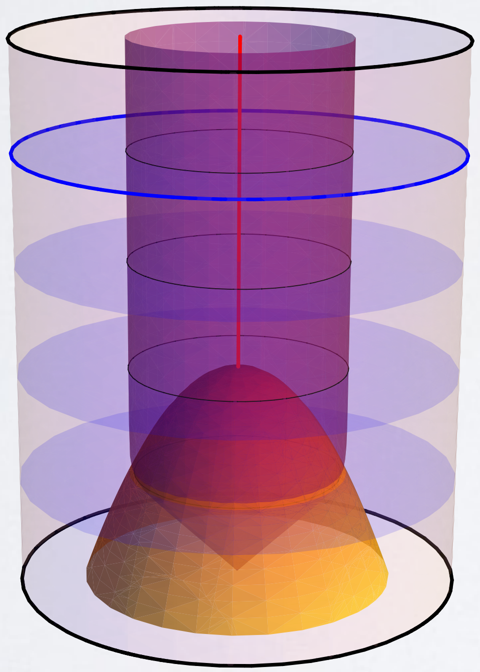
    → phase transitions in EE
  - $\mathfrak{E}$ can be topologically nontrivial even for simply-connected regions $\mathcal{A}$
- Holographic EE seems too local:
  - sharply-specified both on boundary and in bulk
  - but: → we can reconstruct the bulk metric (modulo caveats) solely from the set $\{S_{\mathcal{A}}\}$ for a suitable set of $\{\mathcal{A}\}$
- Holographic EE seems too non-local:
  - global minimization condition + homology constraint makes $S_{\mathcal{A}}$ sensitive to arbitrarily distant regions in the bulk…

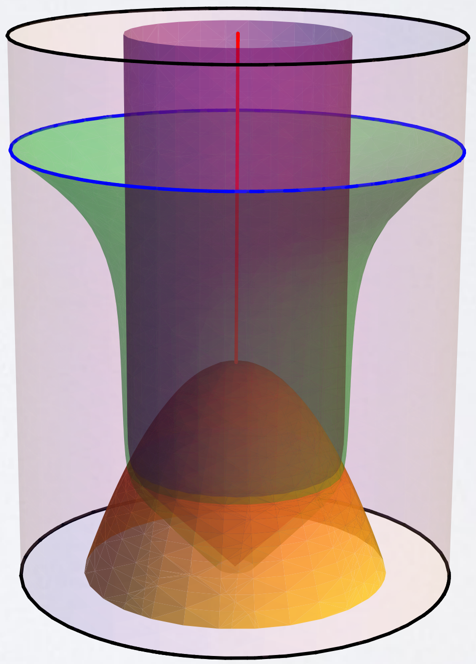
EE is fine-grained observable!

Example: black hole formed from a collapse

- In contrast to the static (i.e. eternal) black hole, for a collapsed black hole, there is no non-trivial homology constraint on extremal surfaces.

[cf. Takayanagi & Ugajin]





- Hence we always have $S_{\mathcal{A}} = S_{\mathcal{A}^c}$ as for a pure state.

Summary & Outlook

- Holography conveniently geometrizes entanglement
  - Finding bulk extremal surfaces and their area is (relatively) easy!
  - Useful in proving important properties!
  - Can we prove HRT directly?
  - Why is EE related to geometry so simply?
  - Duals of other measures of entanglement?
- General covariance is a powerful guiding principle
  - Motivated entanglement wedge, causal wedge, …
  - How is bulk geometry encoded in $\rho_{\mathcal{A}}$?
  - (In what sense) is entanglement wedge the 'dual' of $\rho_{\mathcal{A}}$?
  - What is the CFT dual of causal wedge (from first principles)?
- Relation between spacetime (gravity) and entanglement?

Space Ref (Harvard-Smithsonian CfA) soihub.org/itschool