Entanglement Negativity in Conformal Field Theories

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P. Calabrese, J. Cardy and E.T.	**[1206.3092]**	**PRL**
	[1210.5359]	**JSTAT**
	[1408.3043]	**JPA**
C. De Nobili, A. Coser and E.T.	**[1501.04311]**	**JSTAT** (to appear)
A. Coser, E.T. and P. Calabrese	**[1503.09114]**	

Closing the entanglement gap:
Quantum information, quantum matter and quantum fields

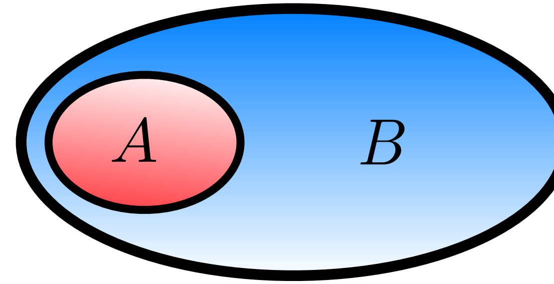
KITP, June 2015

Outline

- Introduction & some motivations
- Entanglement in 2D CFT:
  - Entanglement negativity: definitions and replica limit
  - Entanglement entropies for disjoint intervals
  - Entanglement negativity for adjacent and disjoint intervals
  - Entanglement negativity at finite temperature
  - Partial transpose in the XY spin chain
- Conclusions & open issues

Mutual Information & Entanglement Negativity

- Ground state $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



Reduced density matrix $\boxed{\rho_A = \mathrm{Tr}_B \rho}$

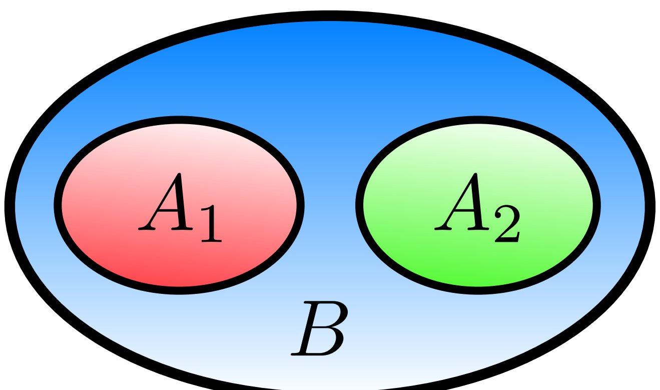
Rényi entropies

Entanglement entropy

$$S_A \equiv -\mathrm{Tr}(\rho_A \log \rho_A) = \lim_{n\to 1} \boxed{\frac{\log(\mathrm{Tr}\rho_A^n)}{1-n}} = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\,\rho_A^n$$

- $S_A = S_B$ for pure states

- *Tripartite* system $\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B$ $\boxed{\rho_{A_1\cup A_2} \text{ is mixed}}$



- $S_{A_1\cup A_2}$: entanglement between $A_1 \cup A_2$ and B

Entanglement between A_1 and A_2?

- The mutual information $S_{A_1} + S_{A_2} - S_{A_1\cup A_2}$ gives an upper bound
- A *computable* measure of the entanglement is the *logarithmic negativity*

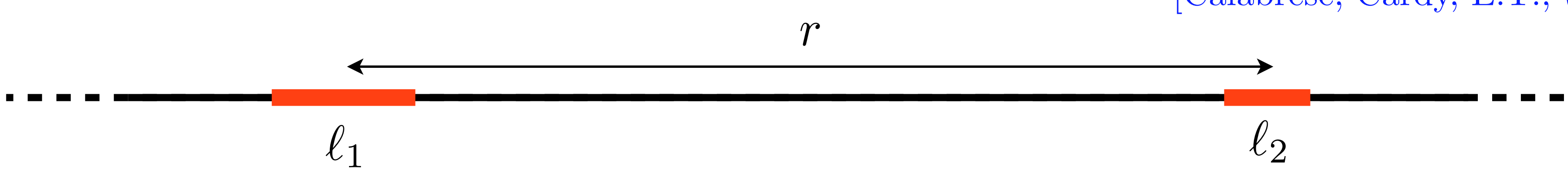
Why disjoint intervals?

- One interval on the infinite line at $T = 0$
  [Holzhey, Larsen, Wilczek, (1994)]
  [Calabrese, Cardy, (2004)]

$$S_A = \frac{c}{3} \log \frac{\ell}{a} + \text{const}$$

- Two intervals A_1 and A_2: $\text{Tr}\rho^n_{A_1 \cup A_2}$ for small intervals w.r.t. to other characteristc lengths of the system

[Headrick, (2010)]
[Calabrese, Cardy, E.T., (2011)]



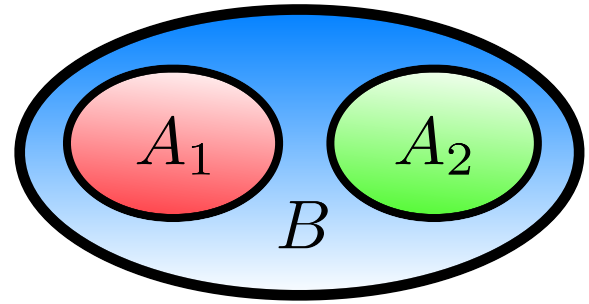
$$\text{Tr}\rho^n_A = c_n^2 (\ell_1 \ell_2)^{-c/6(n-1/n)} \sum_{\{k_j\}} \left(\frac{\ell_1 \ell_2}{n^2 r^2} \right)^{\sum_j (\Delta_j + \bar{\Delta}_j)} \langle \prod_{j=1}^{n} \phi_{k_j} \left(e^{2\pi i j/n} \right) \rangle^2_{\mathbf{C}}$$

$\text{Tr}\rho^n_A$ for disjoint intervals contains *all* the data of the CFT
(conformal dimensions and OPE coefficients)

- Generalization to higher dimensions [Cardy, (2013)]

Entanglement between disjoint regions: Negativity

- $\rho = \rho_{A_1 \cup A_2}$ is a mixed state



ρ^{T_2} is the *partial transpose* of ρ

$$\boxed{\langle e_i^{(1)} e_j^{(2)} |\, \rho^{T_2} \,| e_k^{(1)} e_l^{(2)} \rangle \; = \; \langle e_i^{(1)} e_l^{(2)} |\, \rho \,| e_k^{(1)} e_j^{(2)} \rangle}$$

$(|e_i^{(k)}\rangle$ base of $\mathcal{H}_{A_k})$

[Peres, (1996)] [Zyczkowski, Horodecki, Sanpera, Lewenstein, (1998)] [Eisert, (2001)] [Plenio, (2005)] [Vidal, Werner, (2002)]

- *Trace norm*

$$\boxed{||\rho^{T_2}|| = \mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = 1 - 2 \sum_{\lambda_i < 0} \lambda_i}$$

λ_j eigenvalues of ρ^{T_2}

$\mathrm{Tr}\, \rho^{T_2} = 1$

Logarithmic negativity

$$\boxed{\mathcal{E}_{A_2} \; = \; \ln ||\rho^{T_2}|| \; = \; \ln \mathrm{Tr}|\rho^{T_2}|}$$

- *Bipartite* system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ in *any* state ρ $\longrightarrow$ $\mathcal{E}_1 = \mathcal{E}_2$

Replica approach to Negativity

[Calabrese, Cardy, E.T., (2012)]

- A *parity effect* for $\mathrm{Tr}(\rho^{T_2})^n$

$$\mathrm{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e}$$

$$\mathrm{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o}$$

- Analytic continuation on the *even* sequence $\mathrm{Tr}(\rho^{T_2})^{n_e}$ (*make 1 an even number*)

$$\boxed{\mathcal{E} = \lim_{n_e\to 1} \log\left[\mathrm{Tr}(\rho^{T_2})^{n_e}\right]} \qquad \lim_{n_o\to 1} \mathrm{Tr}(\rho^{T_2})^{n_o} = \mathrm{Tr}\,\rho^{T_2} = 1$$

- Pure states $\rho = |\Psi\rangle\langle\Psi|$ and *bipartite* system $(\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2)$

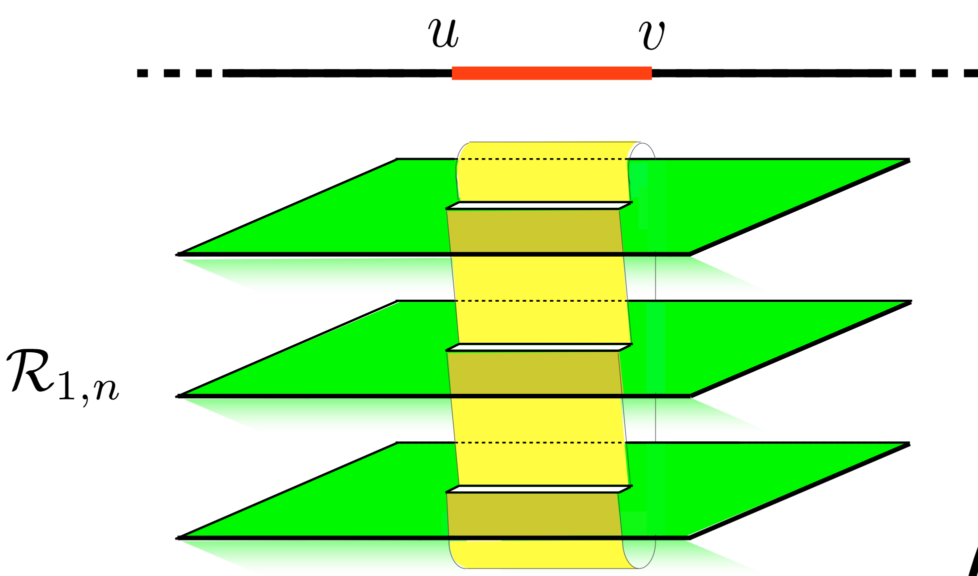
Schmidt decomposition

$$\boxed{\mathrm{Tr}(\rho^{T_2})^n = \begin{cases} \mathrm{Tr}\,\rho_2^n & n = n_o \quad \text{odd} \\ \left(\mathrm{Tr}\,\rho_2^{n/2}\right)^2 & n = n_e \quad \text{even} \end{cases}}$$

- Taking $n_e \to 1$ we have $\boxed{\mathcal{E} = 2\log \mathrm{Tr}\rho_2^{1/2}}$ (Renyi entropy 1/2)

2D CFT: Renyi entropies as correlation functions

- One interval ($N = 1$): the Renyi entropies can be written as a two point function of *twist fields* on the sphere [Calabrese, Cardy, (2004)]



[Holzhey, Larsen, Wilczek, (1994)]

$$S_A = \frac{c}{3}\log\frac{\ell}{\epsilon} + c_1'$$

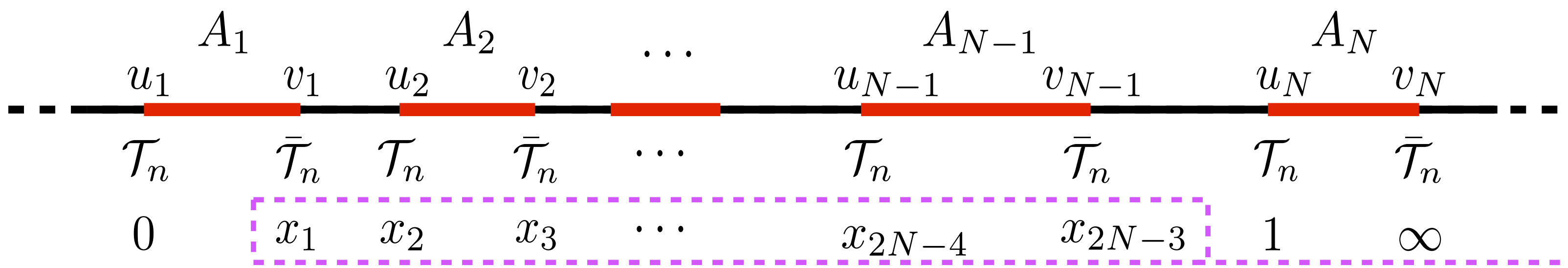
$$\boxed{\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_{1,n}}{\mathcal{Z}^n} = \langle \mathcal{T}_n(u)\bar{\mathcal{T}}_n(v)\rangle = \frac{c_n}{|u-v|^{2\Delta_n}}}$$

$$\boxed{\Delta_n = \frac{c}{12}\left(n - \frac{1}{n}\right)}$$

- Twist fields have been largely studied in the 1980s
  [Zamolodchikov, (1987)] [Dixon, Friedan, Martinec, Shenker, (1987)]
  [Knizhnik, (1987)] [Bershadsky, Radul, (1987)]
- Integrable field theories [Cardy, Castro-Alvaredo, Doyon, (2008)] [Doyon, (2008)]

2D CFT: Renyi entropies for many disjoint intervals

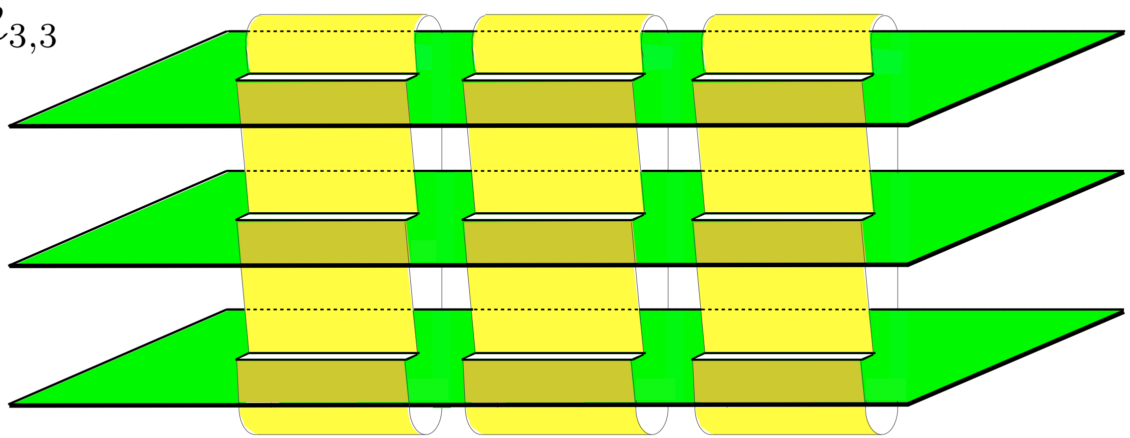
- N disjoint intervals $\Longrightarrow$ $2N$ point function of twist fields



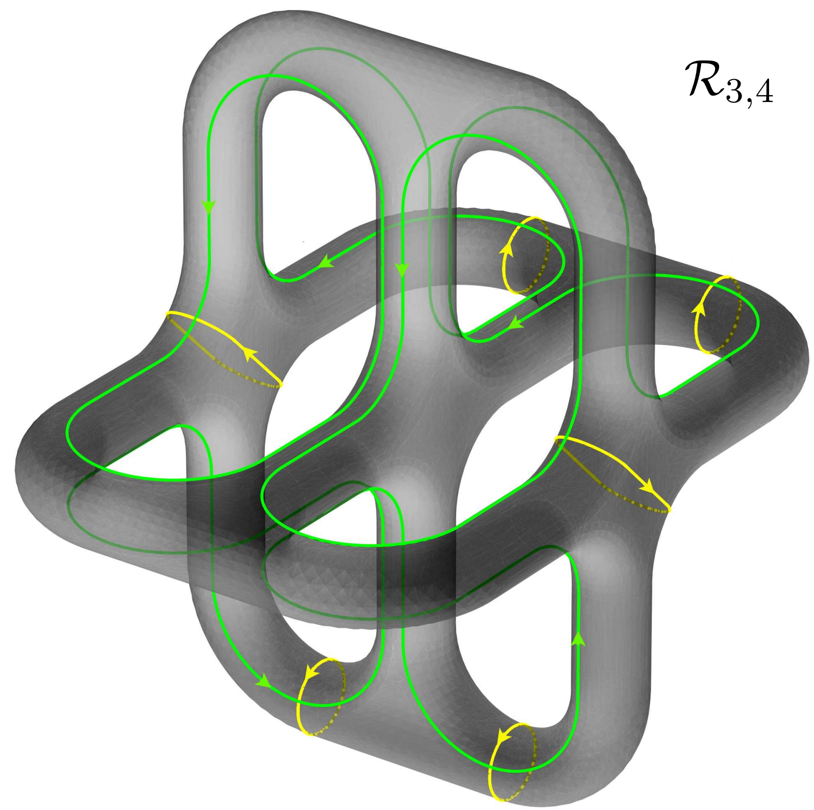
$$\mathrm{Tr}\rho_A^n = \frac{\mathcal{Z}_{N,n}}{\mathcal{Z}^n} = \langle \prod_{i=1}^{N} \mathcal{T}_n(u_i)\bar{\mathcal{T}}_n(v_i)\rangle = c_n^N \left| \frac{\prod_{i<j}(u_j-u_i)(v_j-v_i)}{\prod_{i,j}(v_j-u_i)} \right|^{2\Delta_n} \mathcal{F}_{N,n}(\boldsymbol{x})$$

- $\mathcal{Z}_{N,n}$ partition function of $\mathcal{R}_{N,n}$, a particular Riemann surface of genus $g=(N-1)(n-1)$ obtained through replication

$\mathcal{R}_{3,3}$



$\mathcal{R}_{3,4}$



N intervals: free compactified boson & Ising model

- $\mathcal{R}_{N,n}$ is $\boxed{y^n = \prod_{\gamma=1}^{N} (z - x_{2\gamma-2}) \left[\prod_{\gamma=1}^{N-1} (z - x_{2\gamma-1}) \right]^{n-1}}$ $g = (N-1)(n-1)$ [Enolski, Grava, (2003)]

- Partition function for a generic Riemann surface studied long ago in string theory [Zamolodchikov, (1987)] [Alvarez-Gaume, Moore, Vafa, (1986)] [Dijkgraaf, Verlinde, Verlinde, (1988)]

  Riemann theta function with characteristic
  $$\Theta[\boldsymbol{e}](\mathbf{0}|\Omega) = \sum_{\boldsymbol{m} \in \mathbb{Z}^p} \exp\left[\mathrm{i}\pi (\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \Omega \cdot (\boldsymbol{m}+\boldsymbol{\varepsilon}) + 2\pi \mathrm{i} (\boldsymbol{m}+\boldsymbol{\varepsilon})^{\mathrm{t}} \cdot \boldsymbol{\delta} \right]$$

- Free compactified boson $(\eta \propto R^2)$ [Coser, Tagliacozzo, E.T., (2013)]

$$\boxed{\mathcal{F}_{N,n}(\boldsymbol{x}) = \frac{\Theta(\mathbf{0}|T_\eta)}{|\Theta(\mathbf{0}|\tau)|^2} \qquad T_\eta = \begin{pmatrix} \mathrm{i}\,\eta\,\mathcal{I} & \mathcal{R} \\ \mathcal{R} & \mathrm{i}\,\mathcal{I}/\eta \end{pmatrix}}$$

$\tau = \mathcal{R} + \mathrm{i}\,\mathcal{I}$ period matrix $(g \times g)$

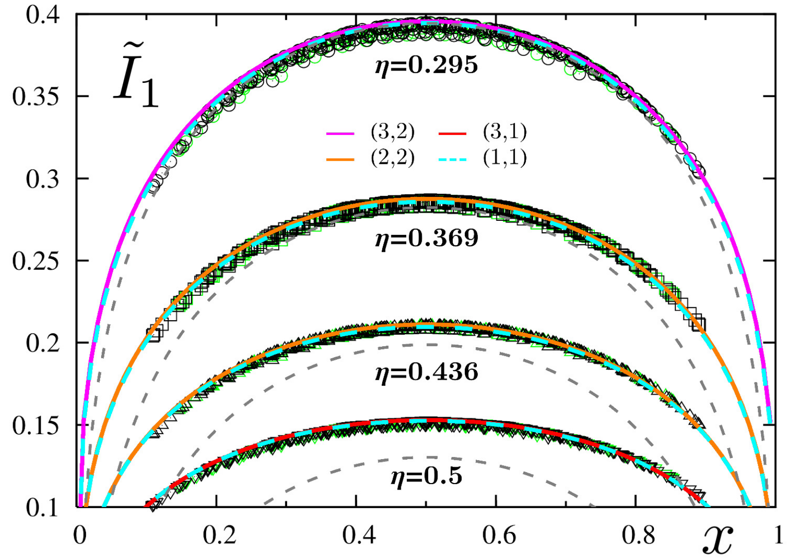
- Ising model $\boxed{\mathcal{F}^{\text{Ising}}_{N,n}(\boldsymbol{x}) = \frac{\sum_{\boldsymbol{e}} |\Theta[\boldsymbol{e}](\mathbf{0}|\tau)|}{2^g\, |\Theta(\mathbf{0}|\tau)|}}$

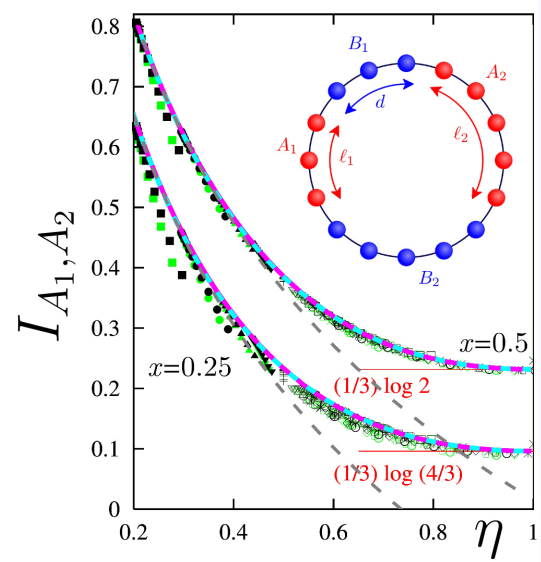
Nasty n dependence

- Two intervals case: [Caraglio, Gliozzi, (2008)] [Furukawa, Pasquier, Shiraishi, (2009)] [Calabrese, Cardy, E.T., (2009), (2011)] [Fagotti, Calabrese, (2010)] [Alba, Tagliacozzo, Calabrese, (2010), (2011)]

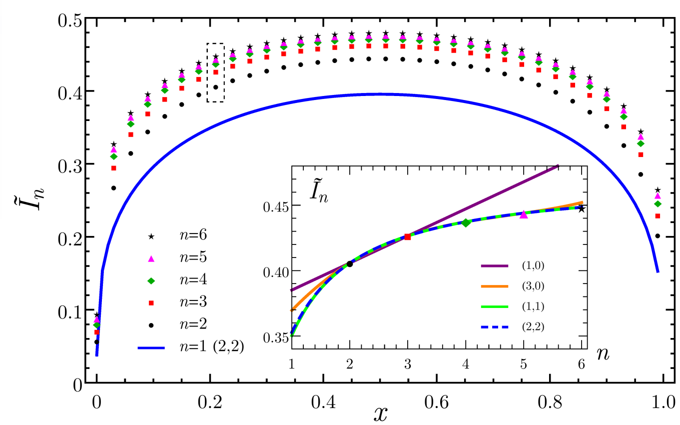
Two disjoint intervals

- Mutual information in XXZ model
  (exact diagonalization) [Furukawa, Pasquier, Shiraishi, (2009)]

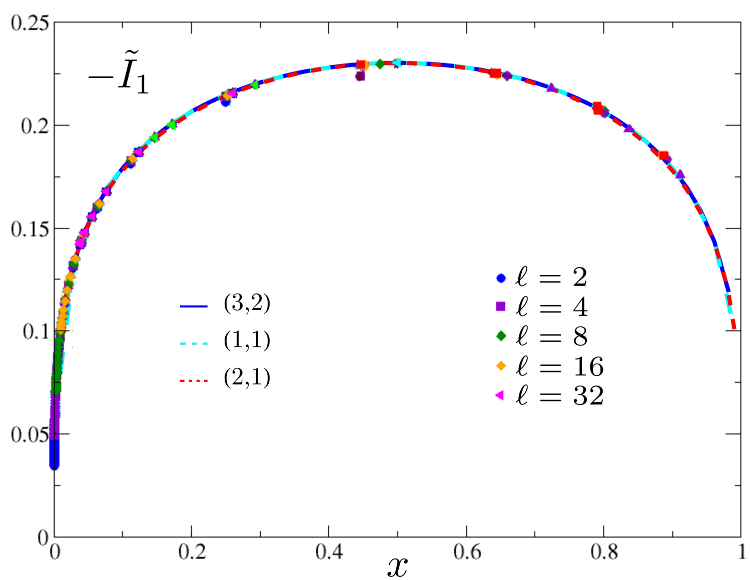




Rational interpolation:
an example



- Mutual information in critical Ising chain
  (Tree Tensor Network) [Alba, Tagliacozzo, Calabrese, (2010)]



- Rational interpolation:

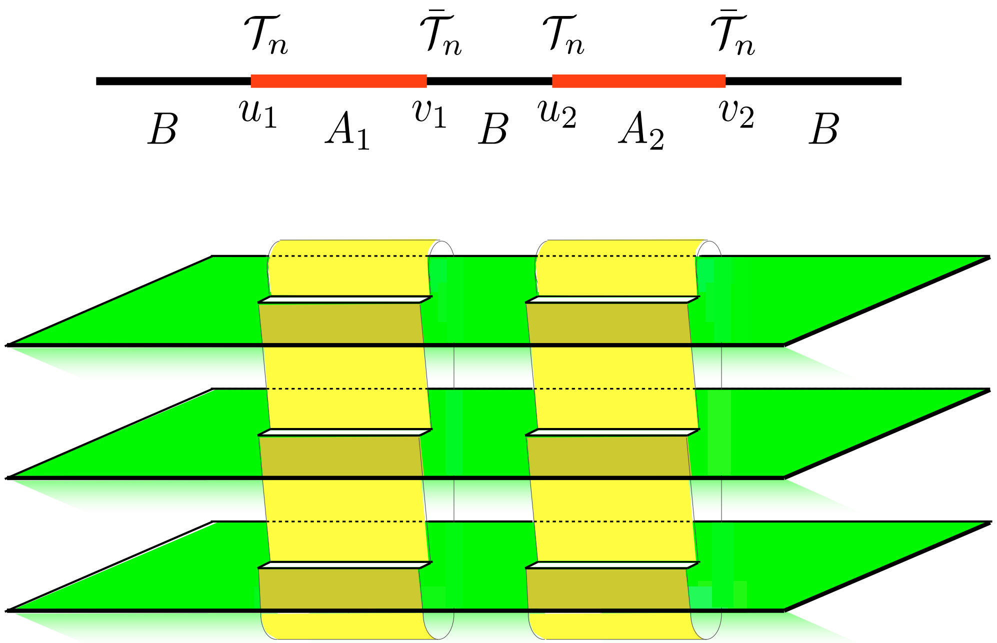
[De Nobili, Coser, E.T., (2015)]

$$W^{(n)}_{(p,q)}(x) \equiv \frac{a_0(x) + a_1(x)n + \cdots + a_p(x)n^p}{b_0(x) + b_1(x)n + \cdots + b_q(x)n^q}$$

Method first employed for Riemann theta functions in $2+1$ dimensions [Agón, Headrick, Jafferis, Kasko, (2014)]

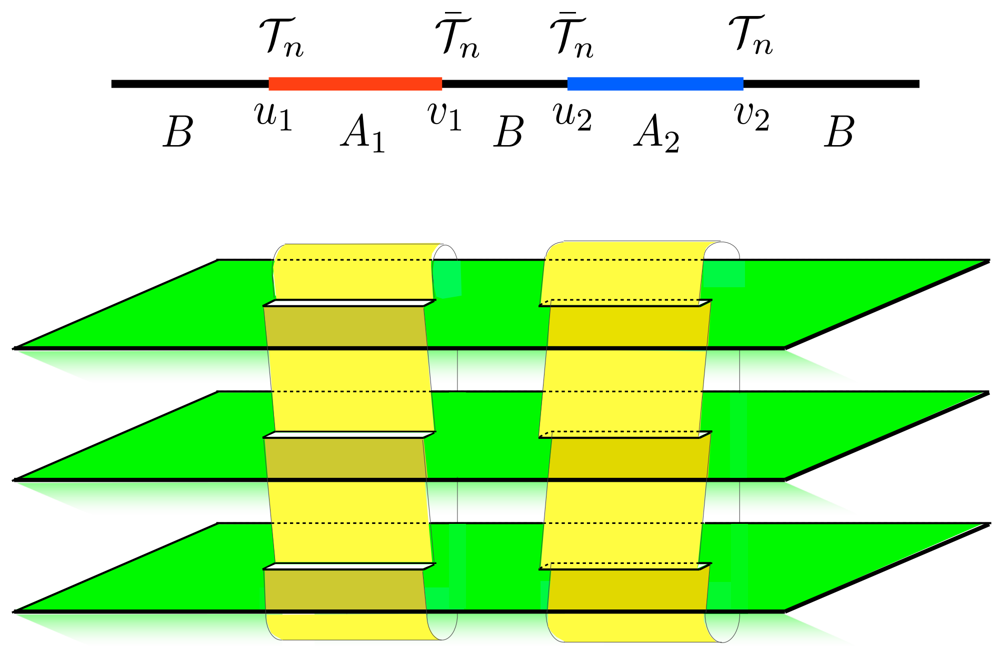
Partial transposition: two disjoint intervals

$$\mathrm{Tr}\rho^n_{A_1\cup A_2}$$



$$\mathrm{Tr}\rho^n_A = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$

$$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$$



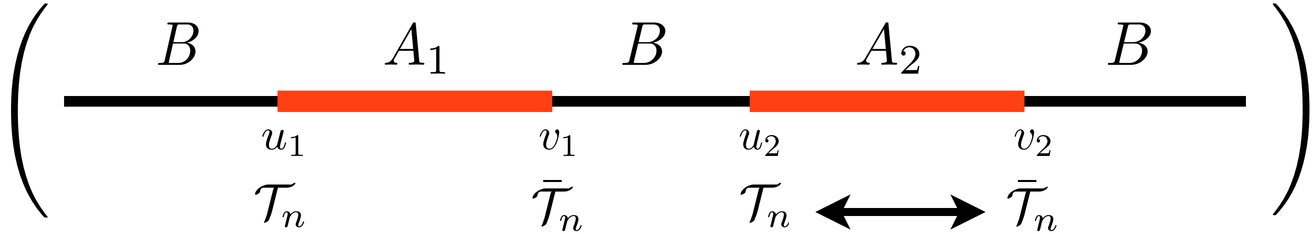
$$\boxed{\mathrm{Tr}(\rho^{T_2}_A)^n = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle}$$

- The partial transposition exchanges $\mathcal{T}_n$ and $\bar{\mathcal{T}}_n$

[Calabrese, Cardy, E.T., (2012)]

Partial Transposition for bipartite systems: pure states

$\mathcal{H} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$

- $\lim_{B \to \emptyset}$ 

$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u_2) \bar{\mathcal{T}}_n^2(v_2) \rangle$$

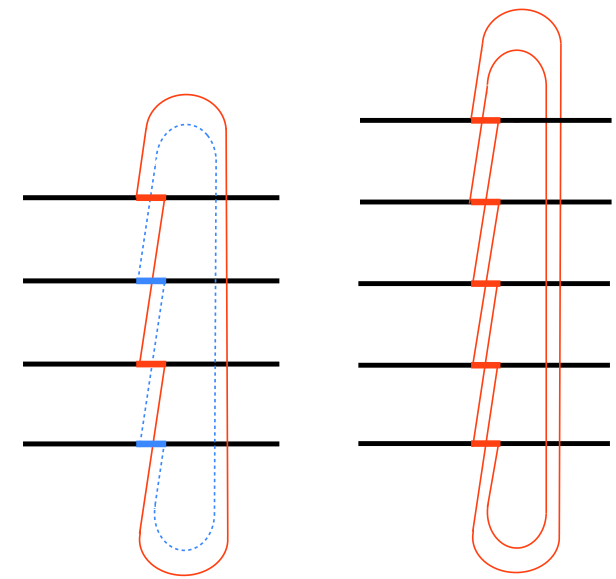
Partial Transposition = exchange two twist fields

- $\mathcal{T}_n^2$ connects the j-th sheet with the $(j+2)$-th one

  Even $n = n_e \implies$ decoupling

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} = \left(\langle \mathcal{T}_{n_e/2}(u_2) \bar{\mathcal{T}}_{n_e/2}(v_2) \rangle\right)^2 = \left(\mathrm{Tr}\, \rho_{A_2}^{n_e/2}\right)^2$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} = \langle \mathcal{T}_{n_o}(u_2) \bar{\mathcal{T}}_{n_o}(v_2) \rangle = \mathrm{Tr}\, \rho_{A_2}^{n_o}$$



$n = 4$ $\qquad$ $n = 5$

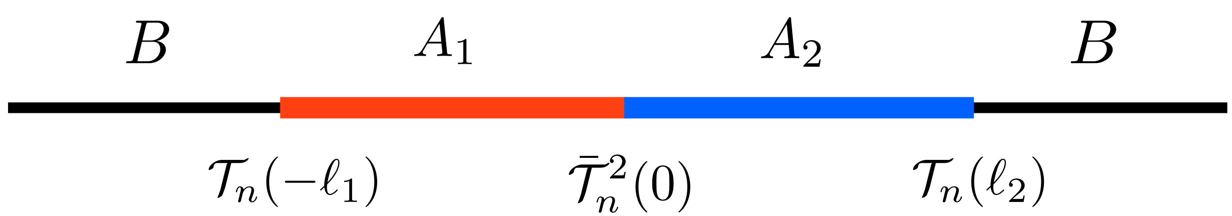
- Two dimensional CFTs

$$\Delta_{\mathcal{T}^2_{n_o}} = \frac{c}{12}\left(n_o - \frac{1}{n_o}\right) = \Delta_{\mathcal{T}_{n_o}}$$

$$\Delta_{\mathcal{T}^2_{n_e}} = \frac{c}{6}\left(\frac{n_e}{2} - \frac{2}{n_e}\right)$$

$$\mathcal{E} = \frac{c}{2} \ln \ell + \mathrm{const}$$

Partial Transpose in 2D CFT: two adjacent intervals



- Three point function

$$\boxed{\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1)\bar{\mathcal{T}}_n^2(0)\mathcal{T}_n(\ell_2)\rangle}$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} \propto (\ell_1\ell_2)^{-\frac{c}{6}\left(\frac{n_e}{2}-\frac{2}{n_e}\right)}(\ell_1+\ell_2)^{-\frac{c}{6}\left(\frac{n_e}{2}+\frac{1}{n_e}\right)}$$

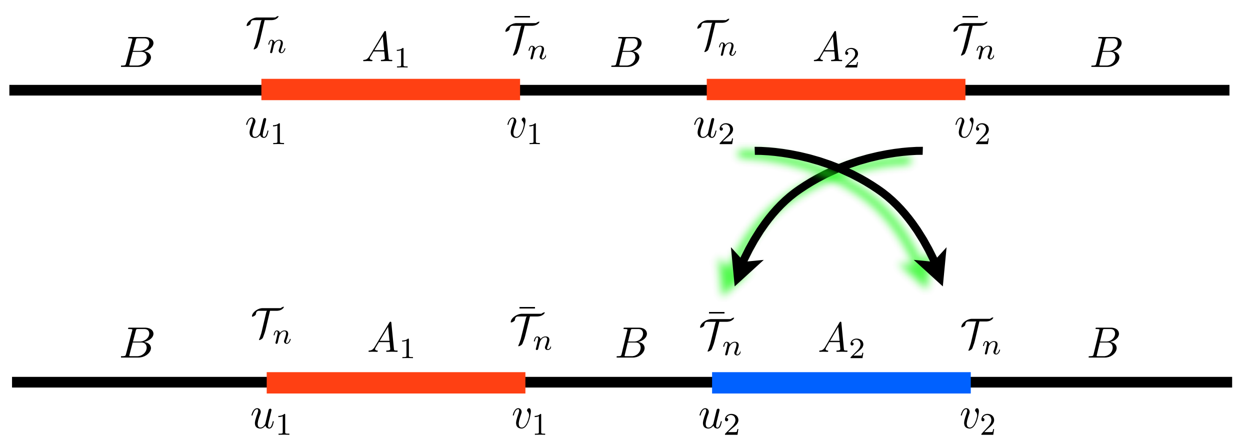
$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} \propto \left(\ell_1\ell_2(\ell_1+\ell_2)\right)^{-\frac{c}{12}\left(n_o-\frac{1}{n_o}\right)}$$

- $$\boxed{\mathcal{E} = \frac{c}{4}\ln\left(\frac{\ell_1\ell_2}{\ell_1+\ell_2}\right) + \mathrm{const}}$$

Partial Transpose in 2D CFT: two disjoint intervals

$\mathrm{Tr}\rho^n_{A_1\cup A_2}$

$\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$



$$\boxed{\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n = c_n^2\left[\ell_1\ell_2(1-y)\right]^{-\frac{c}{6}\left(n-\frac{1}{n}\right)}\mathcal{G}_n(y)}$$

- $\mathrm{Tr}\left(\rho^{T_2}_{A_1\cup A_2}\right)^n$ is obtained from $\mathrm{Tr}\rho^n_{A_1\cup A_2}$ by exchanging two twist fields

$$\mathcal{G}_n(y) = (1-y)^{\frac{c}{3}\left(n-\frac{1}{n}\right)}\mathcal{F}_n\left(\frac{y}{y-1}\right)$$

- $$\boxed{\mathcal{E}(y) = \lim_{n_e\to 1}\mathcal{G}_{n_e}(y) = \lim_{n_e\to 1}\left[\mathcal{F}_n\left(\frac{y}{y-1}\right)\right]}$$

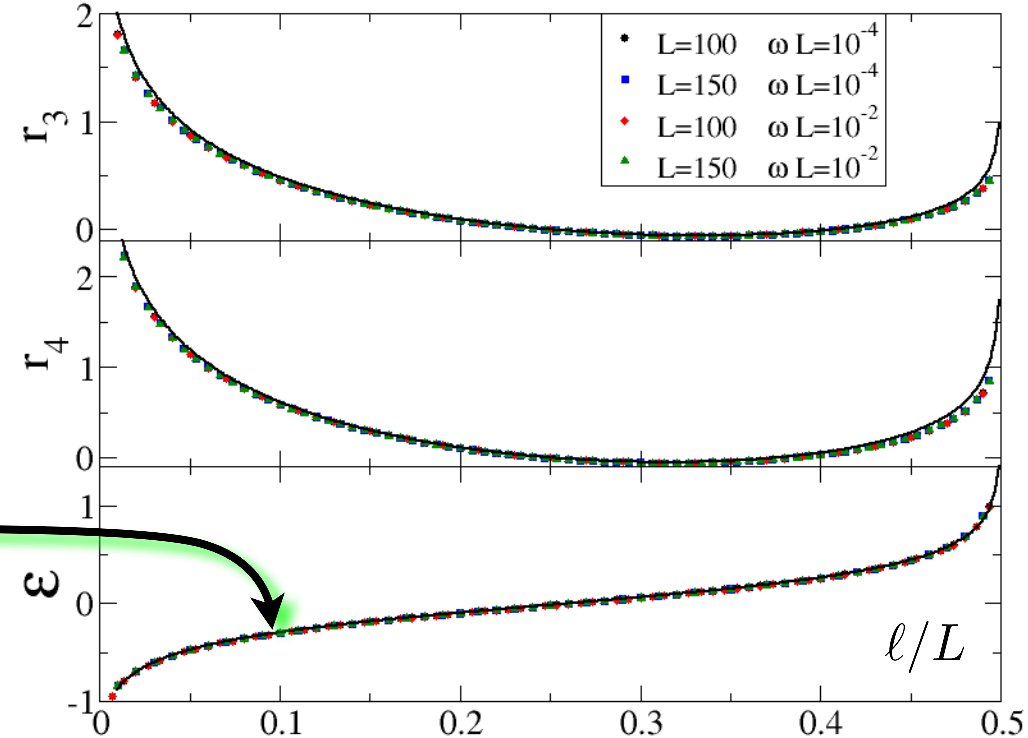
Two adjacent intervals: harmonic chain & Ising model

- Critical periodic harmonic chain

Finite system: $\ell \longrightarrow (L/\pi)\sin(\pi\ell/L)$

$$r_n = \ln \frac{\mathrm{Tr}(\rho_A^{T_{A_2=\ell}})^n}{\mathrm{Tr}(\rho_A^{T_{A_2=L/4}})^n}$$

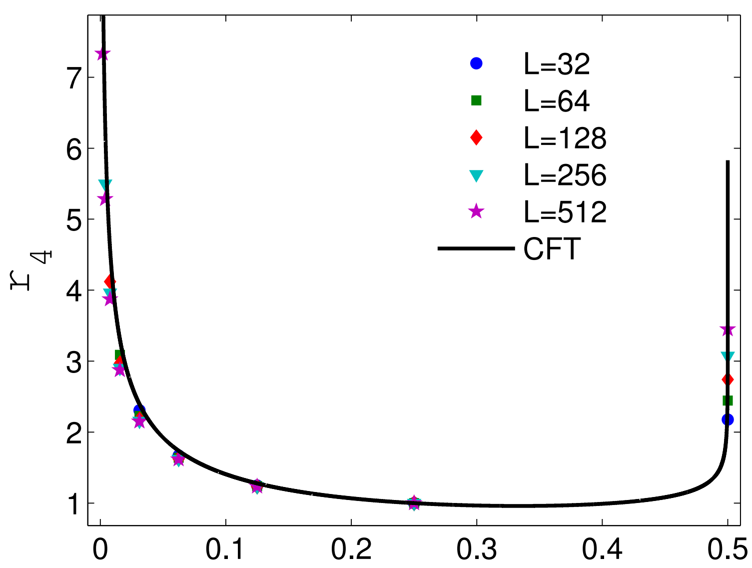
$$\boxed{\frac{1}{4}\log\frac{\sin(\pi\ell_1/L)\sin(\pi\ell_2/L)}{\sin(\pi[\ell_1+\ell_2]/L)} + \mathrm{cnst}}$$

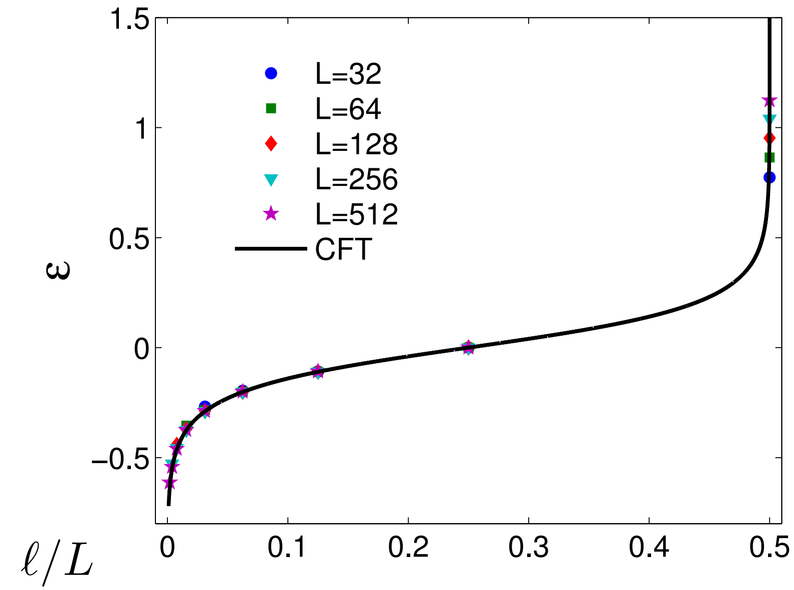


- Ising model:

Monte-Carlo analysis [Alba, (2013)]

Tree Tensor Network [Calabrese, Tagliacozzo, E.T., (2013)]





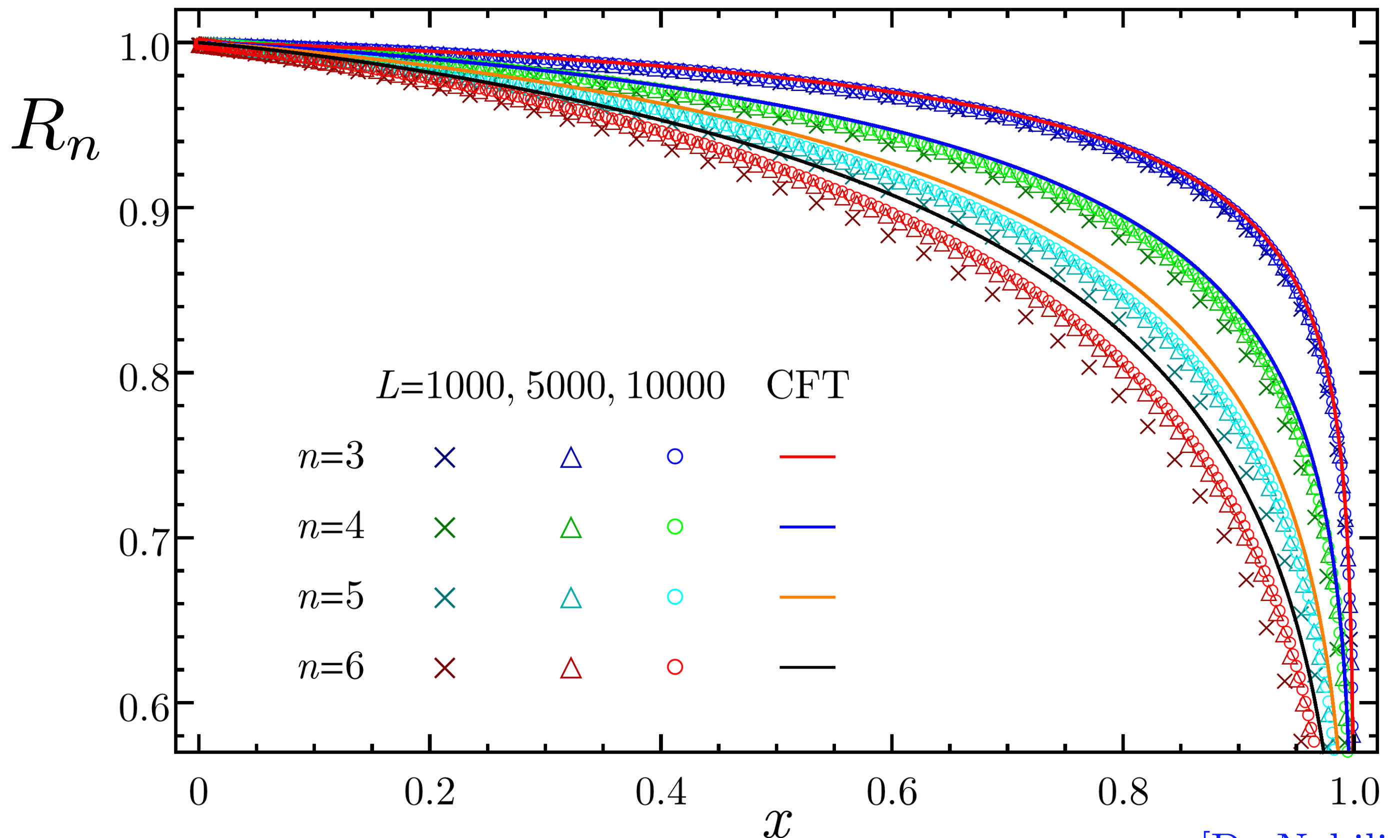
Two disjoint intervals: periodic harmonic chains

- Previous numerical results for $\mathcal{E}$: Ising (DMRG) and harmonic chains [Wichterich, Molina-Vilaplana, Bose, (2009)] [Marcovitch, Retzker, Plenio, Reznik, (2009)]

- Non compact free boson [Calabrese, Cardy, E.T., (2012)]

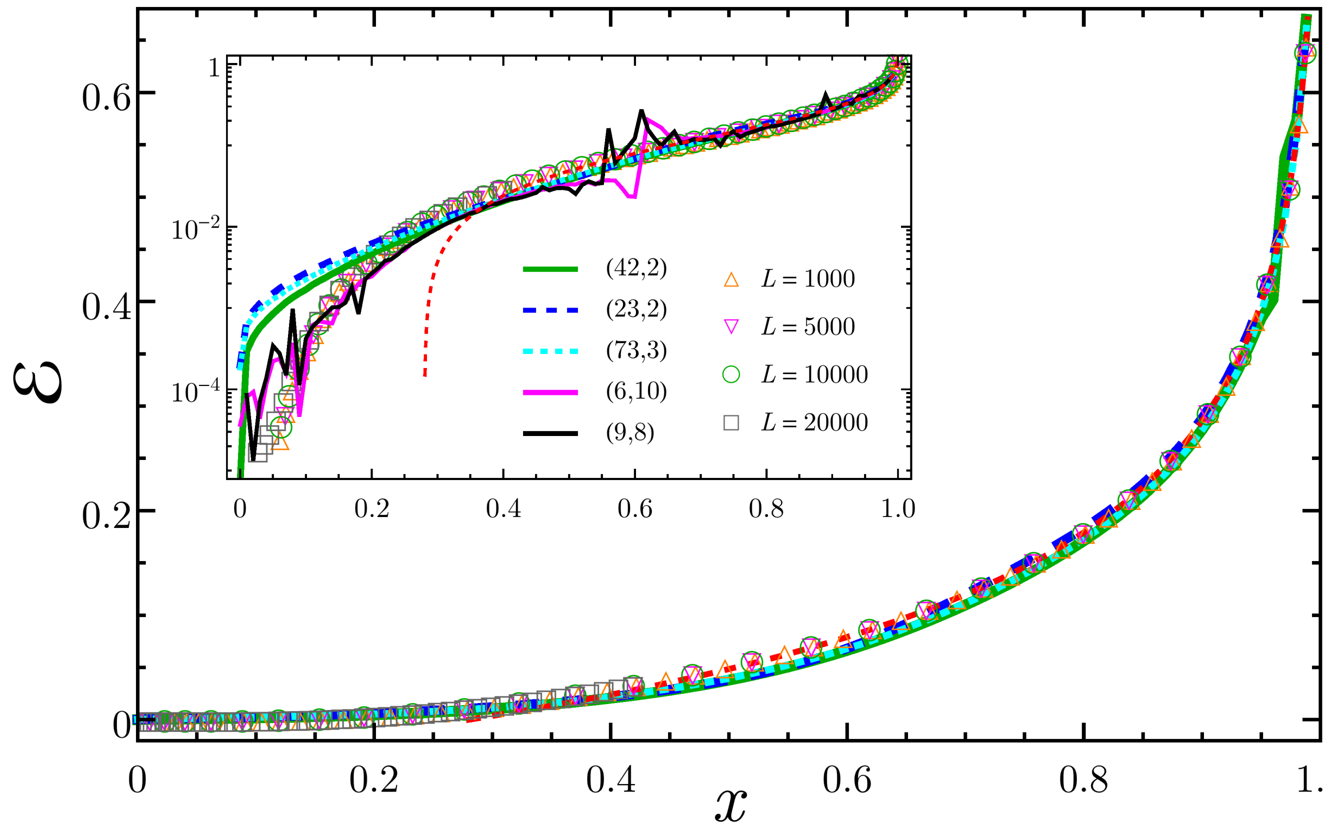
$$\boxed{R_n = \frac{\mathrm{Tr}(\rho_A^{T_2})^n}{\mathrm{Tr}\,\rho_A^n}}$$

$$R_n = \left[\frac{(1-x)^{\frac{2}{3}(n-\frac{1}{n})}\prod_{k=1}^{n-1}F_{\frac{k}{n}}(x)\,F_{\frac{k}{n}}(1-x)}{\prod_{k=1}^{n-1}\mathrm{Re}\left(F_{\frac{k}{n}}\left(\frac{x}{x-1}\right)\bar{F}_{\frac{k}{n}}\left(\frac{1}{1-x}\right)\right)}\right]^{\frac{1}{2}}$$



[De Nobili, Coser, E.T., (2015)]

Two disjoint intervals: periodic harmonic chains



- Analytic continuation for $x \sim 1$
  [Calabrese, Cardy, E.T., (2012)]

$$\mathcal{E} = -\frac{1}{4}\log(1-x) + \log K(x) + \text{cnst}$$

- Analytic continuation $n_e \to 1$ for $0 < x < 1$ not known
- $\mathcal{E}(x)$ for $x \sim 0$ vanishes faster than any power

- Numerical extrapolations (rational interpolation method) [De Nobili, Coser, E.T., (2015)]

Two disjoint intervals: Ising model

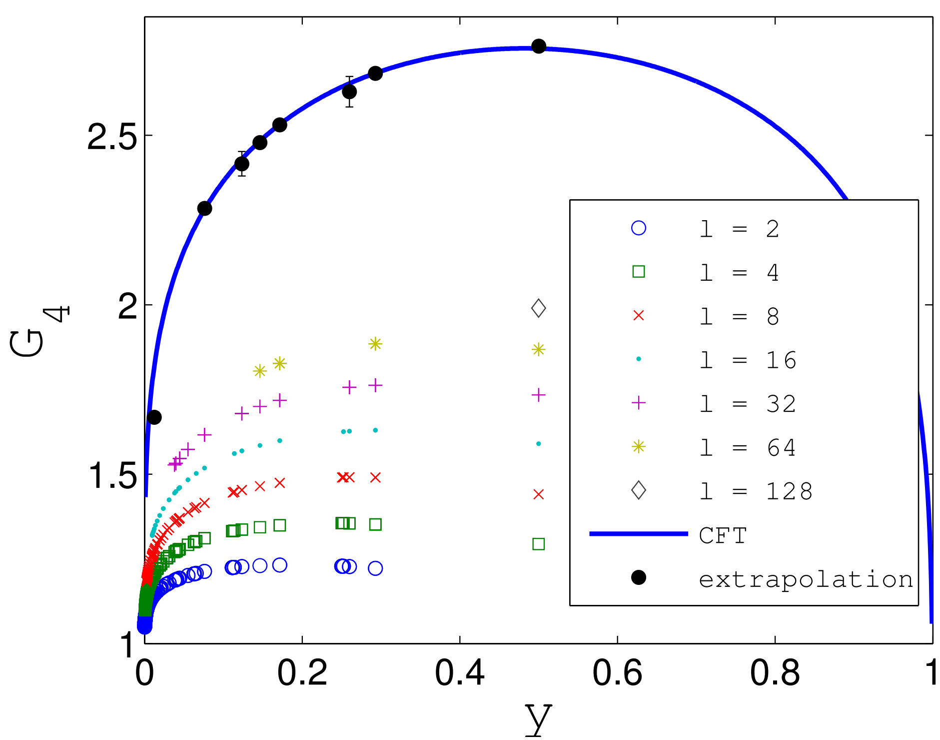
[Alba, (2013)] [Calabrese, Tagliacozzo, E.T., (2013)]

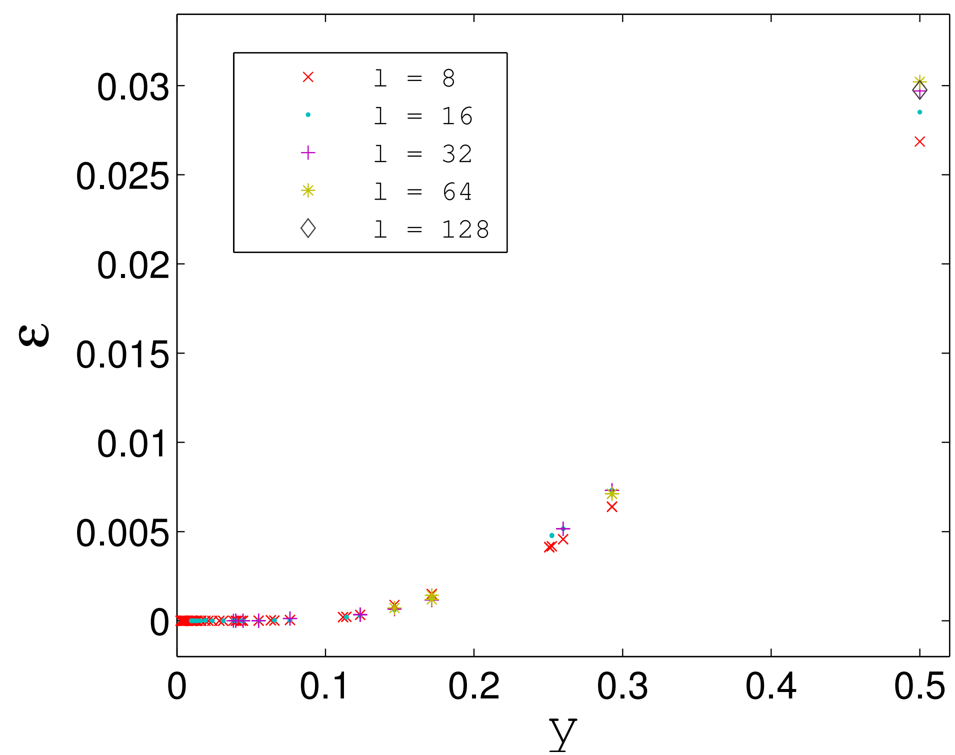
- CFT

$$\mathcal{G}_n(y) = (1-y)^{(n-1/n)/6} \frac{\sum_{\mathbf{e}} |\Theta[\mathbf{e}](\mathbf{0}|\tau(\frac{y}{y-1}))|}{2^{n-1}\prod_{k=1}^{n-1} |F_{k/n}(\frac{y}{y-1})|^{1/2}} \qquad 0 < y < 1$$

- Tree tensor network:

[Calabrese, Tagliacozzo, E.T., (2013)]





One interval at finite temperature: a naive approach

[Calabrese, Cardy, E.T., (2014)]

- Logarithmic negativity $\mathcal{E}$ of one interval at finite $T = 1/\beta$

- A naive approach: compute $\langle \mathcal{T}_n^2(u) \bar{\mathcal{T}}_n^2(v) \rangle_\beta$ through the conformal map relating the cylinder to the complex plane

$$\mathcal{E}_{\text{naive}} = \frac{c}{2} \ln \left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta} \right) + 2 \ln c_{1/2}$$

Problems:

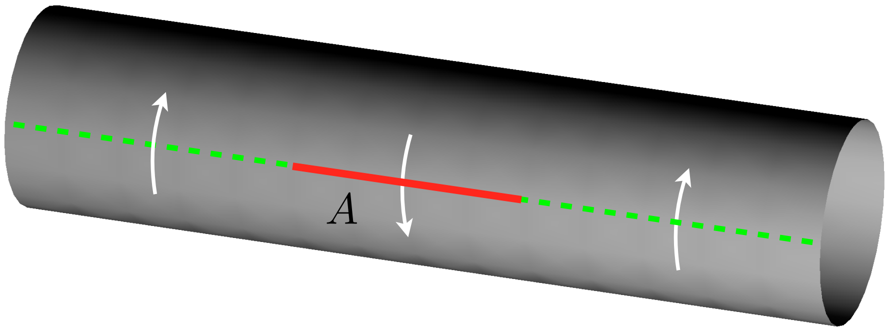
→ The Rényi entropy $n = 1/2$ is not an entanglement measure at finite T

→ $\mathcal{E}_{\text{naive}}$ is an increasing function of T, linearly divergent at high T

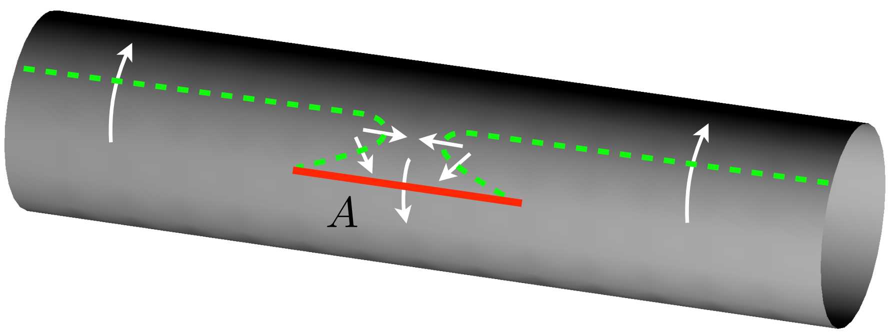
Entanglement should decrease as the system becomes classical

One interval at finite temperature in the infinite line

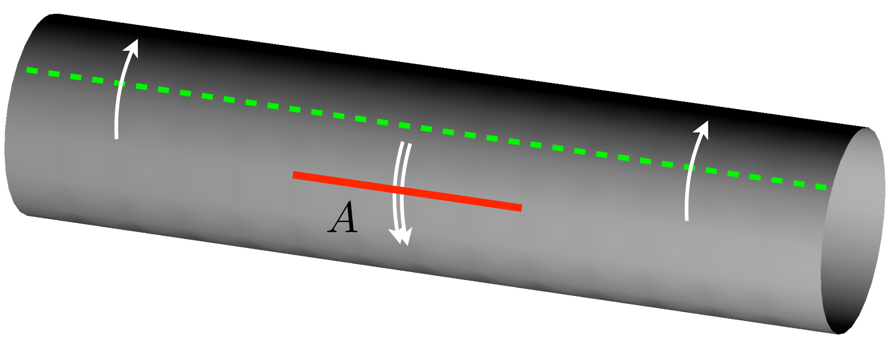
(connection to the $(j+1)$-th cylinder following the arrows)



Single copy of $\rho_\beta^{T_A} \quad \Longrightarrow \quad \mathrm{Tr}\big(\rho_\beta^{T_A}\big)^n$



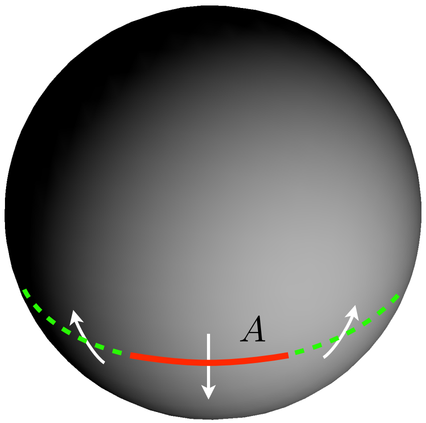
Deformation of the cut along B



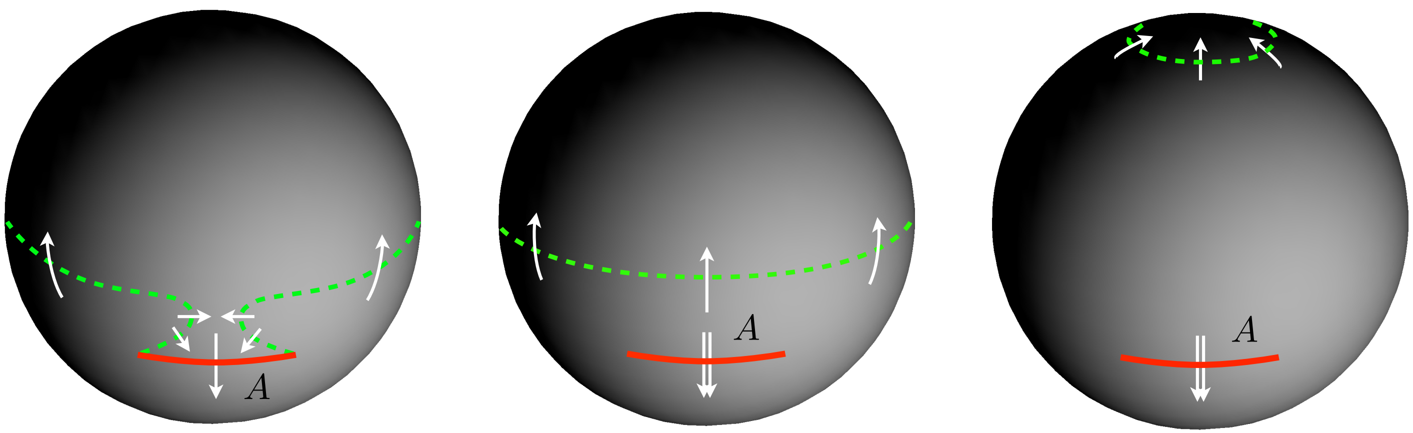
A cut remains connecting
consecutive copies
$\Longrightarrow$ No factorization for even n

(The double arrow indicates the connection to the $(j+2)$-th copy)

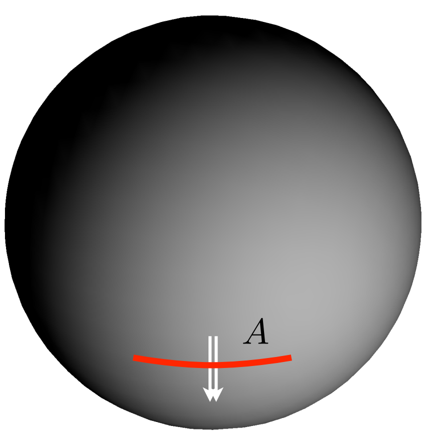
Deforming the cut at zero temperature



Single copy of $\left(|\psi\rangle\langle\psi|\right)^{T_A} \implies \mathrm{Tr}\left[\left(|\psi\rangle\langle\psi|\right)^{T_A}\right]^n$



Deformation of the cut along B

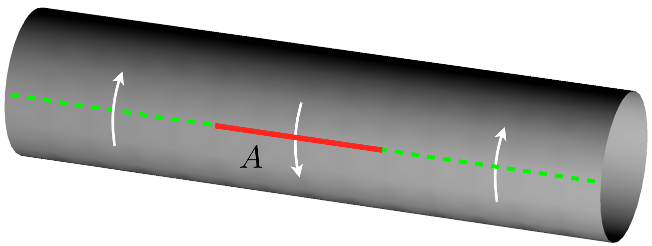


The cut connecting consecutive copies shrinks to a point

Only the connection to the $j \pm 2$ copies along A remains

$\implies$ Factorization for even n

One interval at finite temperature in the infinite line



Two auxiliary twist fields at $\mathrm{Re}(w) = \pm L$, then $L \to \infty$

$$\boxed{\mathcal{E}_A = \lim_{L\to\infty} \lim_{n_e\to 1} \ln \langle \mathcal{T}_{n_e}(-L)\bar{\mathcal{T}}^2_{n_e}(-\ell)\mathcal{T}^2_{n_e}(0)\bar{\mathcal{T}}_{n_e}(L)\rangle_\beta}$$

- Conformal map the cylinder into the plane $z = e^{2\pi w/\beta}$

$$\langle \mathcal{T}_n(z_1)\bar{\mathcal{T}}^2_n(z_2)\mathcal{T}^2_n(z_3)\bar{\mathcal{T}}_n(z_4)\rangle = \frac{c_n\, c_n^{(2)}}{z_{14}^{2\Delta_n}\, z_{23}^{2\Delta_n^{(2)}}}\frac{\mathcal{F}_n(x)}{x^{\Delta_n^{(2)}}} \qquad \mathcal{F}_n(1) = 1 \qquad \mathcal{F}_n(0) = \frac{C^2_{\mathcal{T}_n\bar{\mathcal{T}}^2_n\bar{\mathcal{T}}_n}}{c_n^{(2)}}$$

$x \to e^{-2\pi\ell/\beta}$ when $L \to \infty$

$$\boxed{f(x) \equiv \lim_{n_e\to 1} \ln[\mathcal{F}_{n_e}(x)]}$$

$$\boxed{\mathcal{E}_A = \frac{c}{2}\ln\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi\ell}{\beta}\right)\right] - \frac{\pi c\ell}{2\beta} + f(e^{-2\pi\ell/\beta}) + 2\ln c_{1/2}}$$

→ $\mathcal{E}_A = \mathcal{E}_B$

→ $\mathcal{E}$ depends on the full operator content of the model

→ large T linear divergence of $\mathcal{E}_{\text{naive}}$ is canceled

→ semi infinite systems $\mathrm{Re}(w) < 0$ (BCFT) have been also studied

XY spin chain: two disjoint blocks

- XY spin chain
  with periodic b.c.

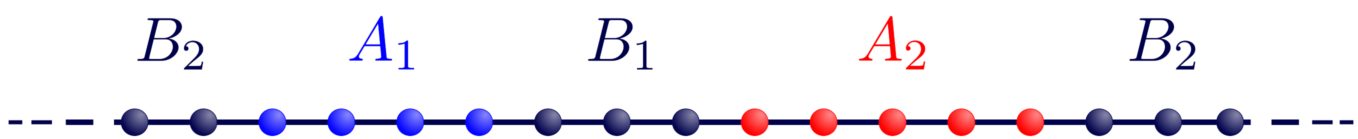
$$H_{XY} = -\frac{1}{2}\sum_{j=1}^{L}\left(\frac{1+\gamma}{2}\,\sigma^x_j\sigma^x_{j+1} + \frac{1-\gamma}{2}\,\sigma^y_j\sigma^y_{j+1} + h\,\sigma^z_j\right)$$

Ising model in a transverse field for $\gamma = 1$, XX spin chain for $\gamma = 0$

- Jordan-Wigner transformation $\quad c_j = \left(\prod_{m<j}\sigma^z_m\right)\frac{\sigma^x_j - \mathrm{i}\sigma^z_j}{2} \qquad c^\dagger_j = \left(\prod_{m<j}\sigma^z_m\right)\frac{\sigma^x_j + \mathrm{i}\sigma^z_j}{2}$

then introduce Majorana fermions $a_{2j} = c_j + c^\dagger_j$ and $a_{2j-1} = \mathrm{i}(c_j - c^\dagger_j)$.

- Two disjoint blocks



➡ The string $P_{B_1} \equiv \prod_{j\in B_1}(\mathrm{i}a_{2j-1}a_{2j})$ enters in a crucial way
[Alba, Tagliacozzo, Calabrese, (2010)] [Igloi, Peschel, (2010)] [Fagotti, Calabrese, (2010)]

➡ Rényi entropies can be written through 4 fermionic Gaussian operators
[Fagotti, Calabrese, (2010)]

$$\mathrm{Tr}\rho^n_A = \mathrm{Tr}\left(\frac{\rho^{\mathbf{1}}_A + P_{A_2}\rho^{\mathbf{1}}_A P_{A_2}}{2} + \langle P_{B_1}\rangle\,\frac{\rho^{B_1}_A - P_{A_2}\rho^{B_1}_A P_{A_2}}{2}\right)^n \qquad \rho^{B_1}_A \equiv \frac{\mathrm{Tr}_B\left(P_{B_1}|\Psi\rangle\langle\Psi|\right)}{\langle P_{B_1}\rangle}$$

XY spin chain: partial transpose of two disjoint blocks

- Free fermion: $\rho_A^{T_2}$ is a sum of 2 fermionic Gaussian operators [Eisler, Zimboras, 1502.01369]
- XY spin chain: $\mathrm{Tr}\big(\rho_A^{T_2}\big)^n$ can be written in terms of 4 fermionic Gaussian operators [Coser, E.T., Calabrese, 1503.09114]

$$\mathrm{Tr}\big(\rho_A^{T_2}\big)^n = \mathrm{Tr}\left(\frac{\tilde{\rho}_A^{\mathbf{1}} + P_{A_2}\tilde{\rho}_A^{\mathbf{1}}P_{A_2}}{2} + \langle P_{B_1}\rangle\,\frac{\tilde{\rho}_A^{B_1} - P_{A_2}\tilde{\rho}_A^{B_1}P_{A_2}}{2\mathrm{i}}\right)^n$$

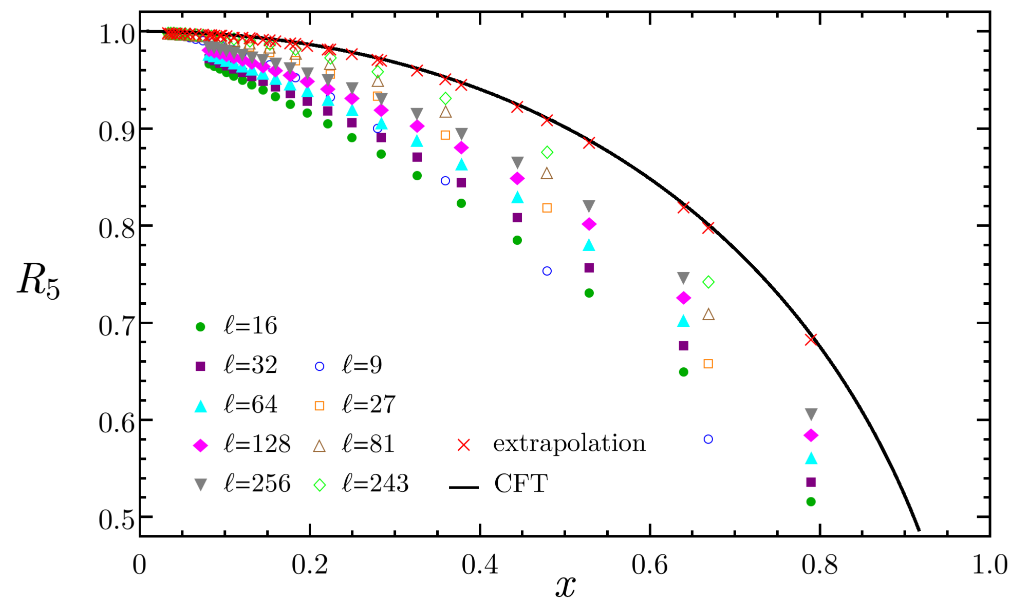
- CFT predictions of $\mathrm{Tr}\big(\rho_A^{T_2}\big)^n$ have been checked for:
  Ising chain, XX chain and tight binding model at half filling (free fermion)

E.g.: free fermion

$\tilde{\tau} = \tau(x/(x-1))$

$$\mathcal{G}_n(x) = \frac{(1-x)^{\frac{1}{3}(n-\frac{1}{n})}}{2^{n-1}\,\Theta(\tilde{\tau})^2}\sum_{\boldsymbol{\delta}} q_n(\boldsymbol{\delta})\,\Theta\begin{bmatrix}\mathbf{0}\\ \boldsymbol{\delta}\end{bmatrix}(\tilde{\tau})^2$$

$q_n(\boldsymbol{\delta})$ known expression.



Conclusions & open issues

- Entanglement for mixed states.
  Entanglement negativity in QFT (1+1 CFTs): $\mathrm{Tr}(\rho^{T_2})^n$ and $\mathcal{E}$
  - → free boson, Ising model, free fermion
  - → finite temperature

- Negativity. Some recent analysis:
  - → topological systems (toric code) [Lee, Vidal, (2013)] [Castelnovo, (2013)]
  - → results for holographic models [Rangamani, Rota, (2014)] [Kulaxizi, Parnachev, Policastro, (2014)]
  - → evolution after a quantum quench [Eisler, Zimboras, (2014)] [Coser, E.T., Calabrese, (2014)] [Hoogeveen, Doyon, (2014)] [Wen, Chang, Ryu, (2015)]

- Some open issues:
  - → Analytic continuations
  - → Higher dimensions
  - → Interactions
  - → Negativity in AdS/CFT

Thank you!