# Entanglement Hamiltonian and spectra from quantum Monte Carlo simulations of interacting fermions.

Fakher F. Assaad (KITP, June 23rd 2015 )

Organization

- Introduction: Entanglement Hamiltonian and spectra for free electrons.
- QMC

  Weak coupling approaches → Gaussian representation of reduced density matrix

  Strong coupling approaches → Replica
- Conclusions

F. Parisen Toldin

T. Lang

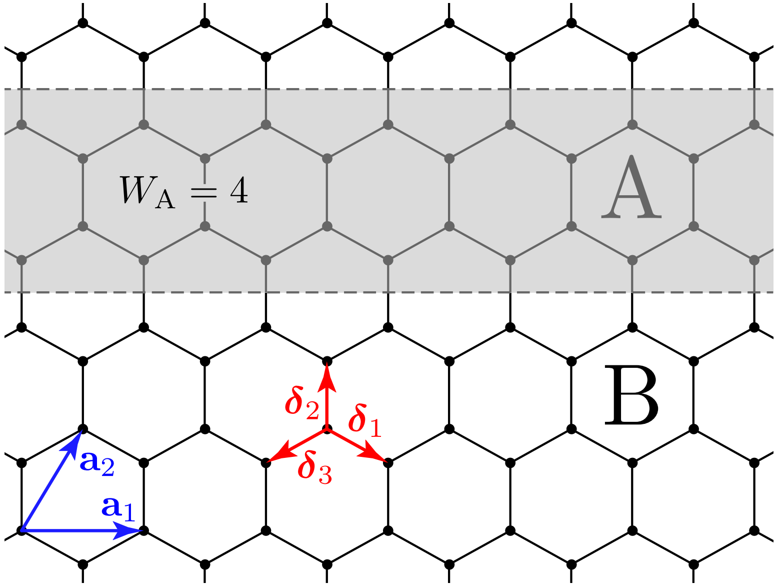
Discussions with Tarun Grover

Julius-Maximilians-UNIVERSITÄT WÜRZBURG

FFA, T. C. Lang, and F. Parisen Toldin Phys. Rev. B, 89, 125121 (2014)
FFA, PRB 91, 125146 (2015)

## Entanglement Hamiltonian for **non-interacting** systems

Partitioning $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$




$$\hat{\rho}_A = \mathrm{Tr}_B\, \hat{\rho} = e^{-\hat{H}_E}, \quad \hat{\rho} = \frac{e^{-\beta \hat{H}_0}}{Z}$$

$$\hat{\rho}_A = \det\left[1 - G_A\right] e^{-\hat{\mathbf{a}}^\dagger \ln\left[G_A^{-1} - 1\right]\hat{\mathbf{a}}}$$

$G_A$ Single particle equal time Green function matrix in $\mathcal{H}_A$

$\forall \hat{O} \in \mathcal{H}_A$ Wick's theorem leads to

$\mathrm{Tr}_A\, \hat{\rho}_A\, \hat{O} = \mathrm{Tr}\, \hat{\rho}\, \hat{O}$ since

$$\mathrm{Tr}_A\left[\hat{\rho}_A\, \hat{a}_x \hat{a}_y^\dagger\right] \overset{!}{=} \mathrm{Tr}\left[\hat{\rho}\hat{a}_x \hat{a}_y^\dagger\right] = \left[G_A\right]_{xy}$$

$$\tfrac{1}{2} - G_A = \frac{1}{2}\tanh\left(H_E / 2\right)$$


I. Peschel, Journal of Physics A 36, L205 (2003)

L. Fidkowski. Phys. Rev. Lett., 104, 130502, (2010)

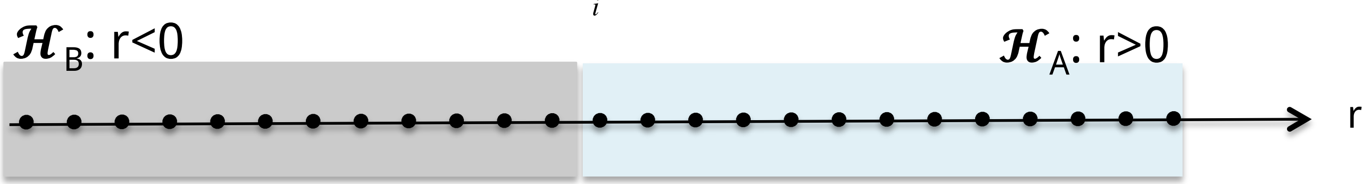
A. M. Turner, Y. Zhang, and A. Vishwanath
Phys. Rev. B, 82, 241102, (2010)


Insulating states. Band flattening.

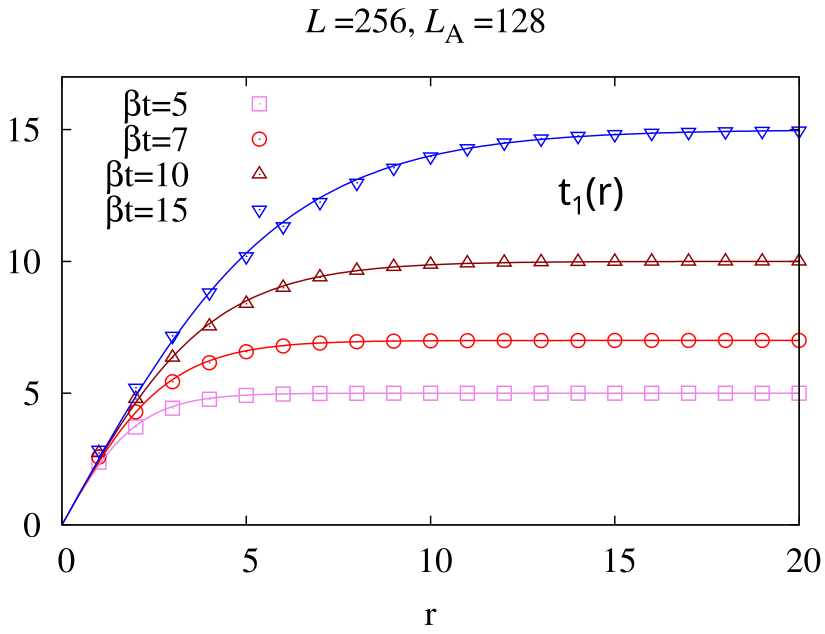
½ - G and $H_0$ are adiabatically connected
→ Have the same topological properties

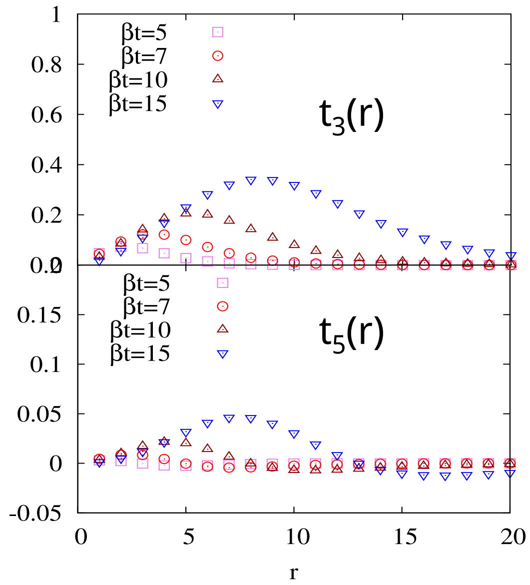
# Entanglement Hamiltonian of a tight binding chain

Partitioning $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ $\quad H_0 = t\sum_i \left(\hat{c}_i^\dagger \hat{c}_{i+1} + h.c.\right)$




$$H_E = \sum_{n,r>0} t_n(r)\left(\hat{c}_r^\dagger \hat{c}_{r+n} + h.c.\right) \cong \sum_{r>0} t\beta \tanh\left(\frac{\alpha}{\beta} r\right)\left(\hat{c}_r^\dagger \hat{c}_{r+1} + h.c.\right), \quad \alpha \cong 2.5/t$$




$$T = 0: \quad H_E \cong \sum_{r>0} t\alpha\, r \left(\hat{c}_r^\dagger \hat{c}_{r+1} + h.c.\right)$$

$$T > 0: \quad H_E \cong \sum_{r>0} t\beta \left(\hat{c}_r^\dagger \hat{c}_{r+1} + h.c.\right)$$

## Entanglement Hamiltonian of a tight binding chain

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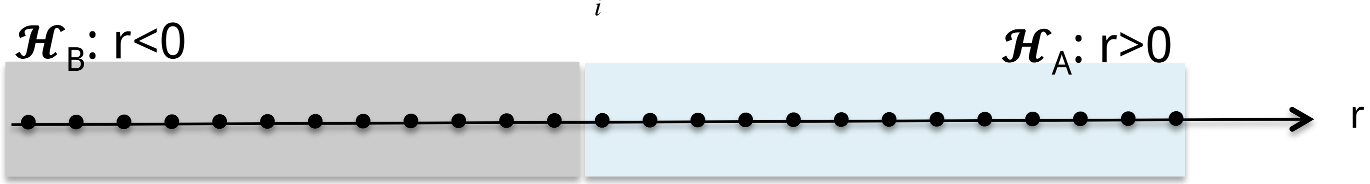


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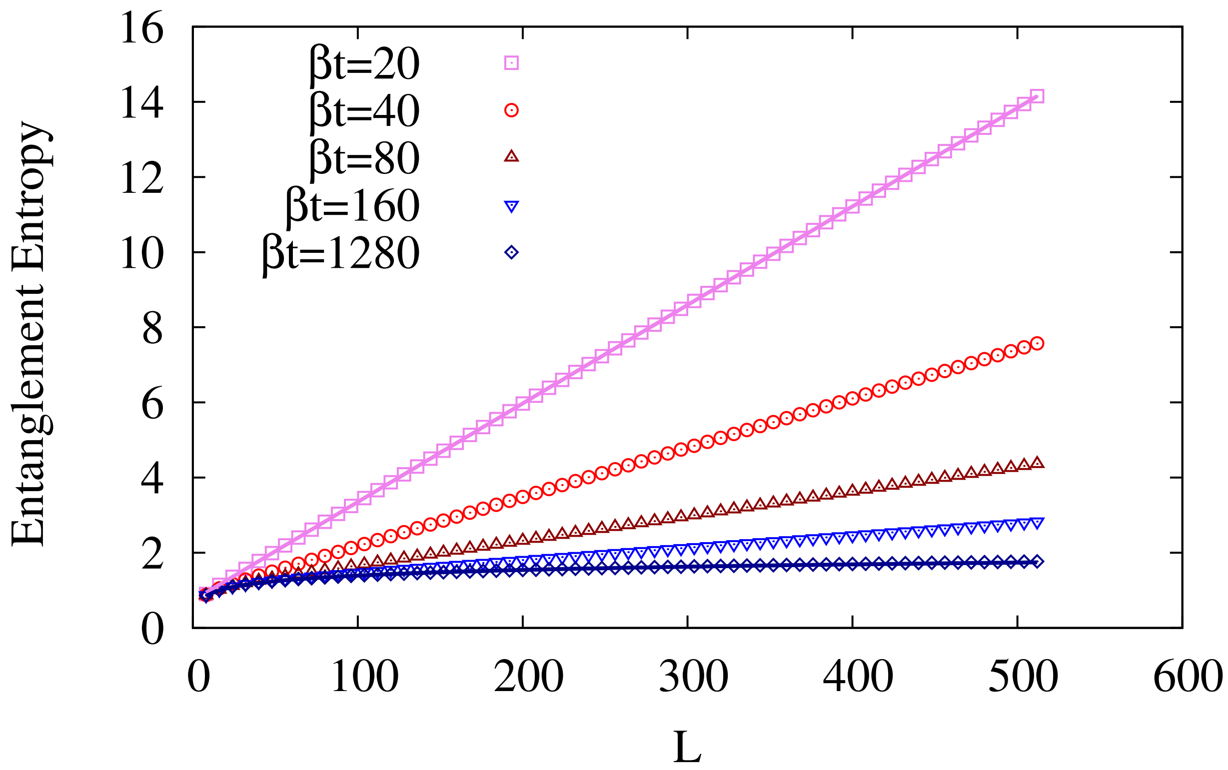
Entanglement entropy = Von Neumann entropy of $H_E$




T>0: a + b L

Volume law

T=0: a + b log(L)

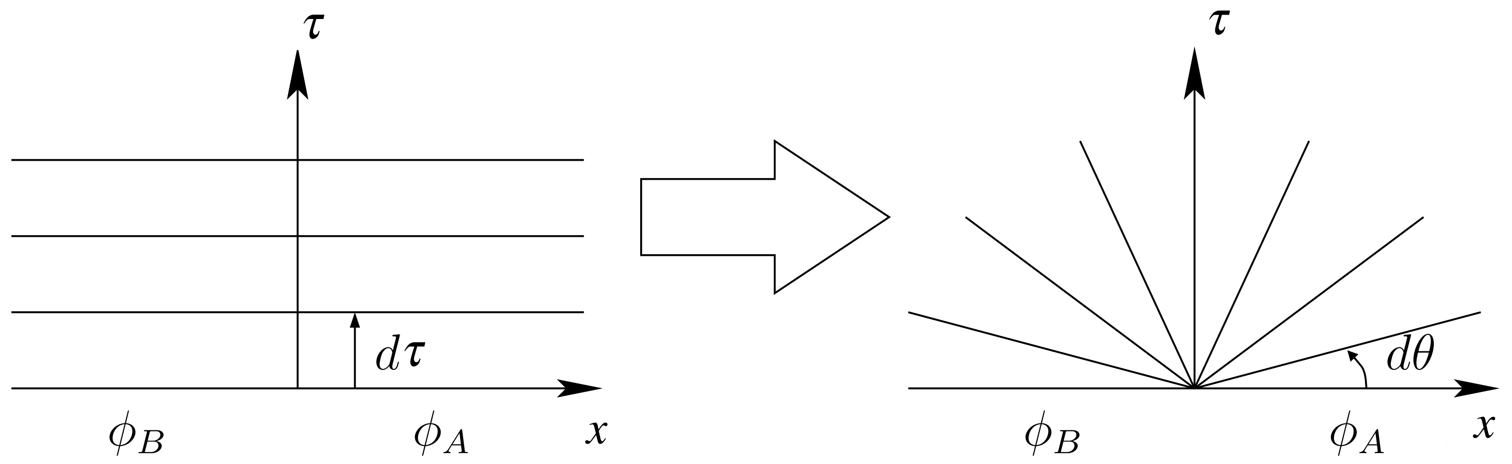
Area law with additive log correction.

Pasquale Calabrese and John Cardy
J. Stat. Mech.: Theor. Exp. (2004) P06002

## The Rindler Hamiltoninan

$$\langle \psi_0 | \phi_B \phi_A \rangle = \lim_{\beta\to\infty} \langle \psi_T | e^{-\beta \hat{H}} | \phi_B \phi_A \rangle = \langle \phi_B | e^{-\pi \hat{H}_R} | \phi_A \rangle$$

Lorentz invariance required!




$$\hat{U}(\tau + d\tau, \tau) = e^{-d\tau \hat{H}}, \qquad \hat{H} = \int dx\, \hat{h}(x)$$

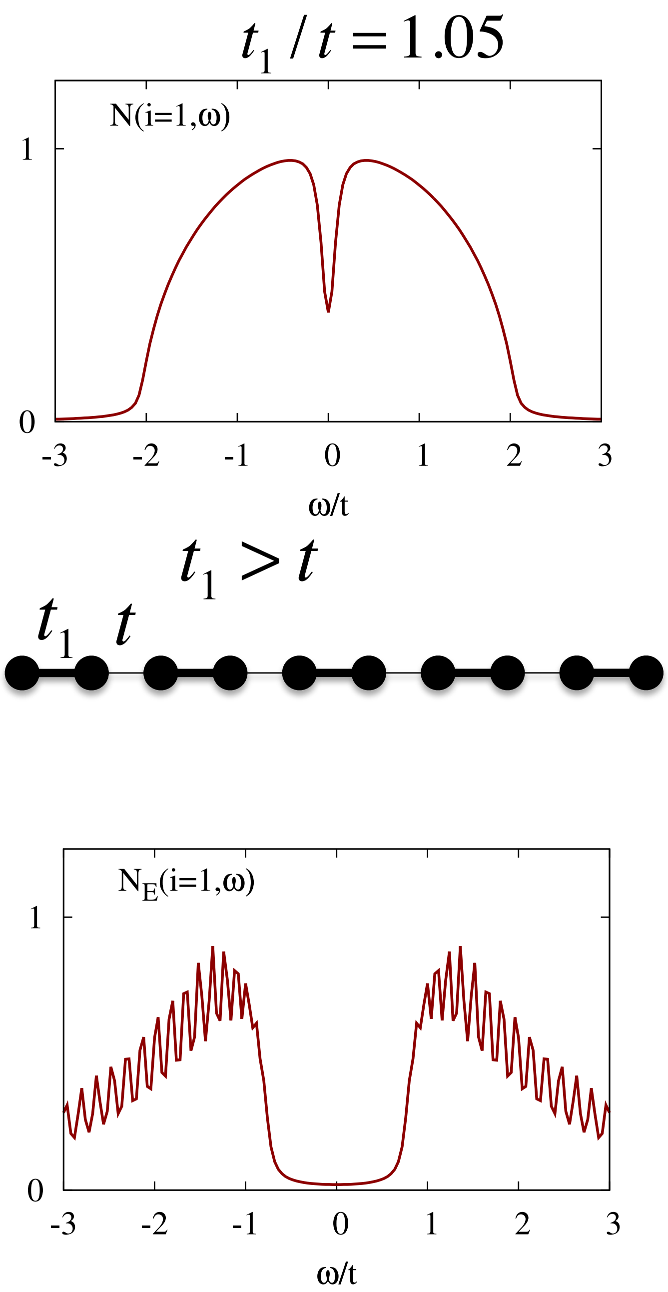
$$\hat{U}(\theta + d\theta, \theta) = e^{-d\theta \hat{H}_R}, \qquad \hat{H}_R = \int_0^\infty dx\, x\, \hat{h}(x)$$

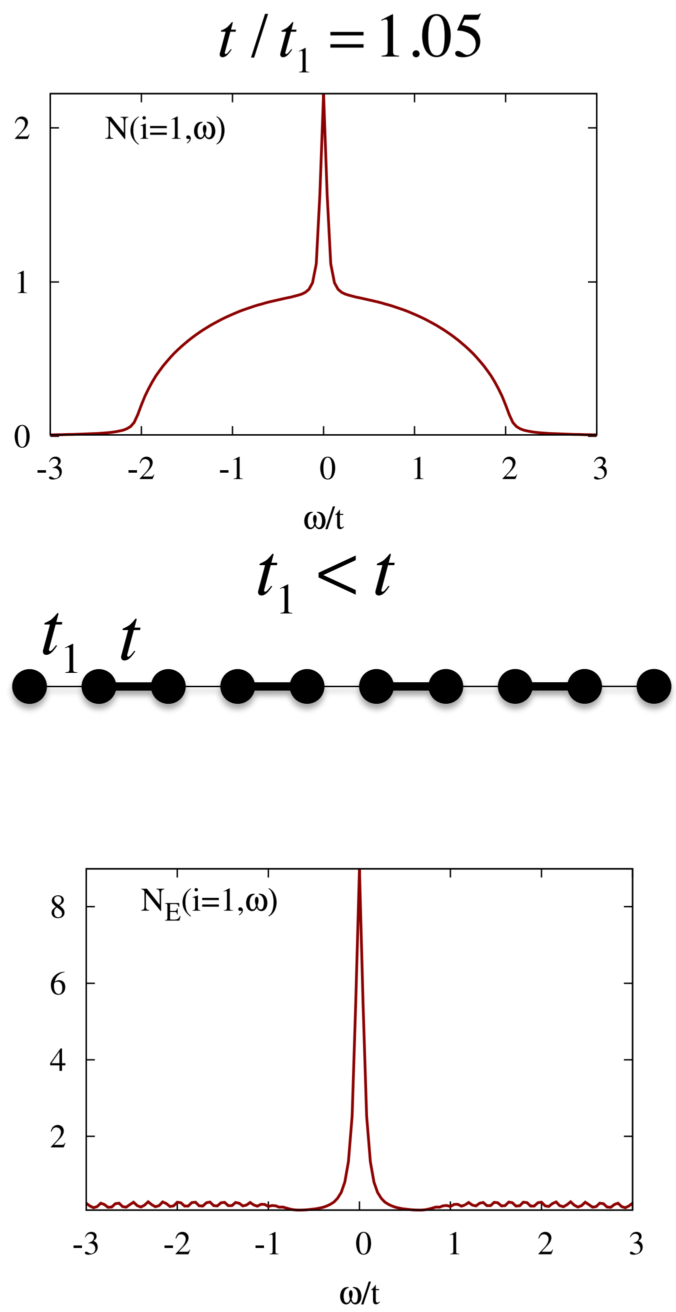
$$\langle \phi_A' | \hat{\rho}_A | \phi_A \rangle = \sum_{|\phi_B\rangle} \langle \phi_B \phi_A' | \psi_0 \rangle \langle \psi_0 | \phi_B \phi_A \rangle = \sum_{|\phi_B\rangle} \langle \phi_A' | e^{-\pi \hat{H}_R} | \phi_B \rangle \langle \phi_B | e^{-\pi \hat{H}_R} | \phi_A \rangle = \langle \phi_A' | e^{-2\pi \hat{H}_R} | \phi_A \rangle$$

$$\hat{\rho}_A = e^{-2\pi \hat{H}_R}$$

$$\hat{H}_E = 2\pi \hat{H}_R = 2\pi \int_0^\infty dx\, x\, \hat{h}(x)$$

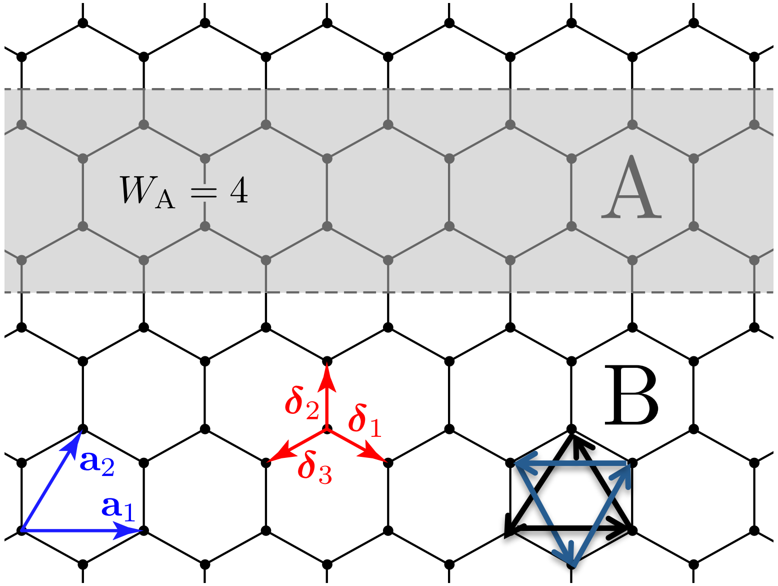
# Detecting topological states with entanglement spectrum.



# Entanglement spectrum for non-interacting topological insulators.

Partitioning $\mathcal{H}=\mathcal{H}_A\otimes\mathcal{H}_B$

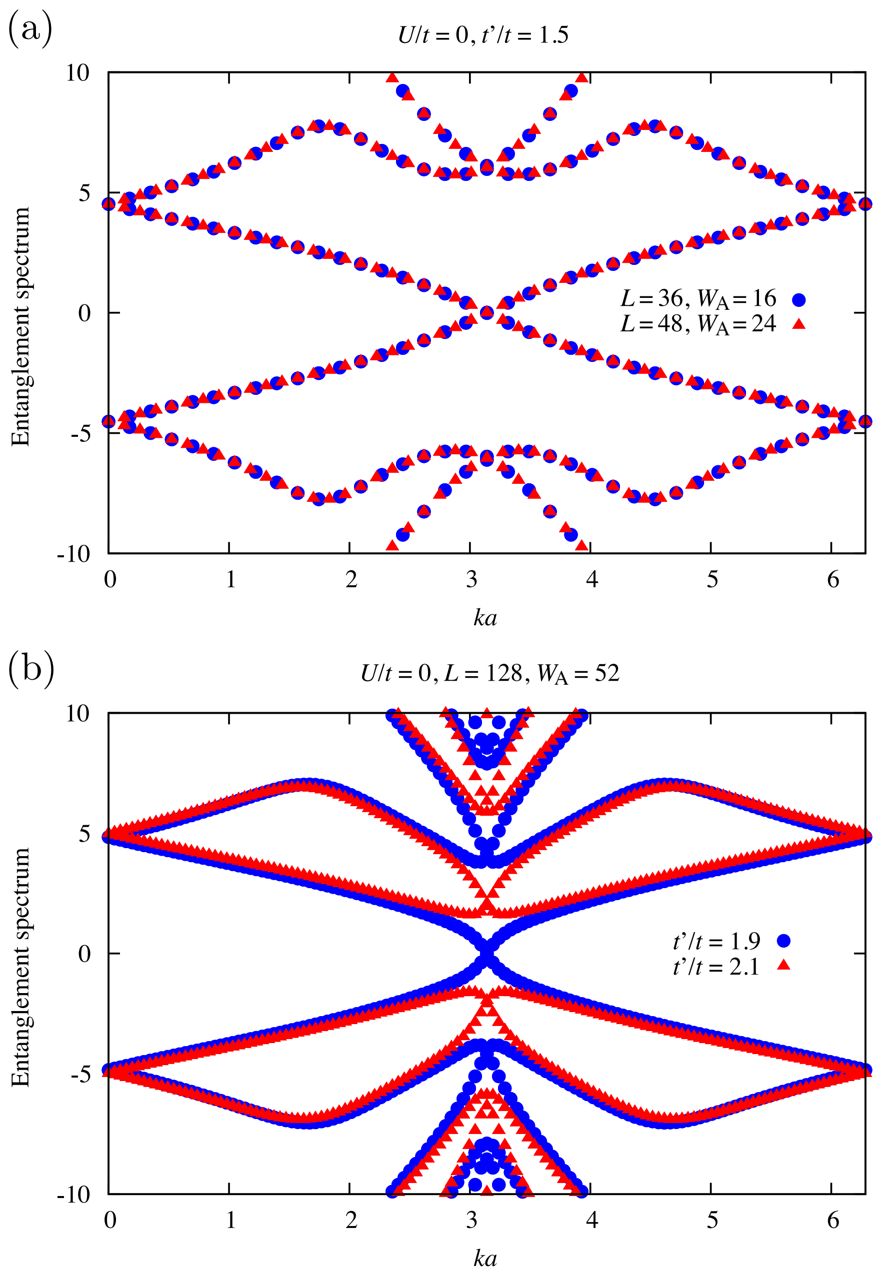



$$\hat{H}_{DKM} = \sum_{\mathbf{i},\mathbf{j}} \hat{c}_{\mathbf{i}}^{\dagger}\left(t_{\mathbf{i},\mathbf{j}} + i\lambda_{\mathbf{i},\mathbf{j}} \bullet \sigma\right)\hat{c}_{\mathbf{j}}^{\dagger}$$

$$t_{\boldsymbol{ij}} = \begin{cases} -t & \text{if } \boldsymbol{i}-\boldsymbol{j} = \pm\boldsymbol{\delta}_2, \pm\boldsymbol{\delta}_3 \\ -t' & \text{if } \boldsymbol{i}-\boldsymbol{j} = \pm\boldsymbol{\delta}_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{i} \xrightarrow{\hspace{2em}} \mathbf{j} = \lambda i\left(\hat{c}_{\mathbf{i}}^{\dagger}\sigma_z\hat{c}_{\mathbf{j}} - \hat{c}_{\mathbf{j}}^{\dagger}\sigma_z\hat{c}_{\mathbf{i}}\right)$$

Topological phase transition @ t′/t= 2



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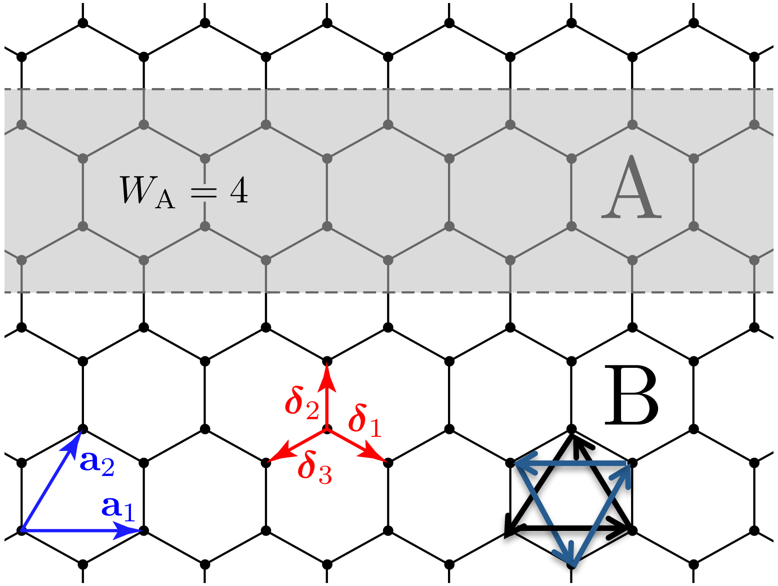


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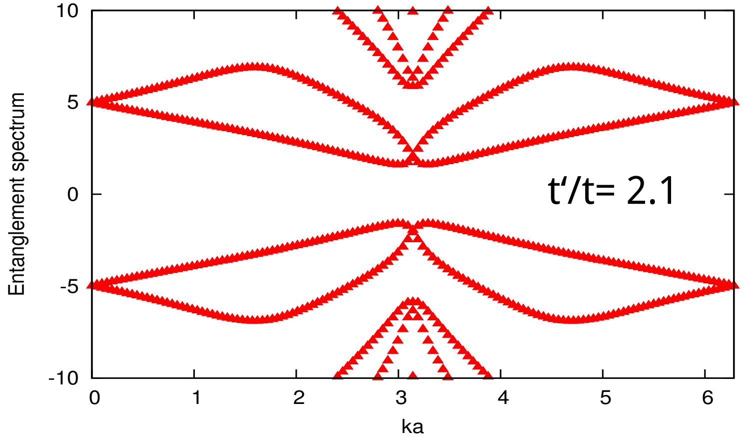
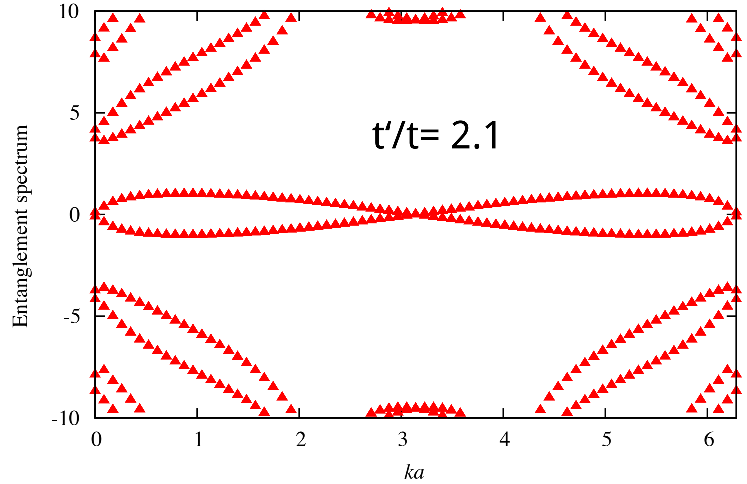
Topological phase transition @ t′/t= 2

Dimerization along the $\delta_1$ direction




Dimerization along the $\delta_2$ direction




See also
A, Chandran, V. Khemani, S. L. Sondhi Phys. Rev. Lett. 113, 060501 (2014)

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FFA, T. C. Lang, and F. Parisen Toldin Phys. Rev. B, 89, 125121 (2014)
FFA, PRB 91, 125146 (2015)

## Auxiliary field QMC. BSS: R. Blankenbecler, D. J. Scalapino, R. L. Sugar (1981)

$$\mathrm{Tr}\, e^{-\beta \hat{H}} \propto \int D\left\{\boldsymbol{\Phi}(\mathbf{i},\tau)\right\} e^{-S\left(\left\{\boldsymbol{\Phi}(\mathbf{i},\tau)\right\}\right)}$$

Trotter, Hubbard-Stratonovich

MC importance sampling

One body problem in external field

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Trotter, Hubbard-Stratonovich

MC importance sampling

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## Example

For $\hat{H} = \hat{H}_{KM} + \frac{1}{4}\sum_{\mathbf{i},\mathbf{j}} V_{\mathbf{i},\mathbf{j}} (\hat{c}^{\dagger}_{\mathbf{i}}\hat{c}_{\mathbf{i}} - 1)(\hat{c}^{\dagger}_{\mathbf{j}}\hat{c}_{\mathbf{j}} - 1), \quad \hat{c}^{\dagger}_{\mathbf{i}} = (\hat{c}^{\dagger}_{\mathbf{i},\uparrow}, \hat{c}^{\dagger}_{\mathbf{i},\downarrow})$

$$S\left(\left\{\Phi(i,\tau)\right\}\right) = \sum_{\mathbf{i},\mathbf{j},\tau} \Delta\tau\, \Phi(\mathbf{i},\tau) V^{-1}_{\mathbf{i},\mathbf{j}} \Phi(\mathbf{j},\tau) - \ln \mathrm{Tr}\left[\prod_{\tau=1}^{L_\tau} e^{-\Delta\tau \hat{H}_{KM}} e^{-\Delta\tau \sum_{\mathbf{i}} i\Phi(\mathbf{i},\tau)\left[\hat{c}^{\dagger}_{\mathbf{i}}\hat{c}_{\mathbf{i}} - 1\right]}\right]$$

- The action is real! → positive weights
  (U(1) spin symmetry, particle-hole symmetry, V positive definite)
- Scaling $\beta N^3$

M. V. Ulybyshev, P. V. Buividovich, M. I. Katsnelson, and M. I. Polikarpov. Phys. Rev. Lett., 111, 056801, (2013).
M. Hohenadler, F. Parisen Toldin, I. Herbut and F. Assaad, 90, 085146 (2014)

M. Hohenadler, T.C. Lang and FFA PRL 106, 2011
D. Zheng, C. Wu and G.-M. Zhang PRB 84, 2011

## Measuring observables.

$$\frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\hat{c}_x^{\dagger}\hat{c}_y\right]}{\mathrm{Tr}\, e^{-\beta\hat{H}}} = \int D\Phi\; P(\Phi)\; G_{x,y}\left(\Phi\right)$$

$$P(\Phi) = \frac{e^{-S(\Phi)}}{\int D\Phi\; e^{-S(\Phi)}}\,, \qquad G(\Phi) = (1+B_{L_\tau}\cdots B_1)^{-1}$$

Wicks theorem holds for a given field configuration →

Any equal time observable can be computed from G

A different way of writing Wick's theorem. T. Grover Phys. Rev. Lett., 111, 130402, (2013).

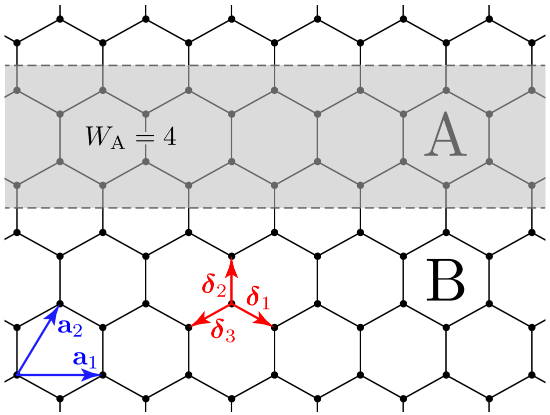
$$\hat{\rho} \equiv \frac{e^{-\beta\hat{H}}}{Z} = \int d\Phi\, P(\Phi)\hat{\rho}(\Phi), \qquad \hat{\rho}(\Phi) = \det\left[1 - G(\Phi)\right] e^{-\hat{c}^{\dagger}\ln\left[G^{-1}(\Phi)-1\right]\hat{c}}$$

$$\rightarrow \int d\Phi\, P(\Phi)\mathrm{Tr}\left[\hat{\rho}(\Phi)\hat{O}\right] = \left\langle \hat{O}\right\rangle$$

For all equal time observables.

## Renyi entanglement entropies

T. Grover Phys. Rev. Lett., 111, 130402, (2013).

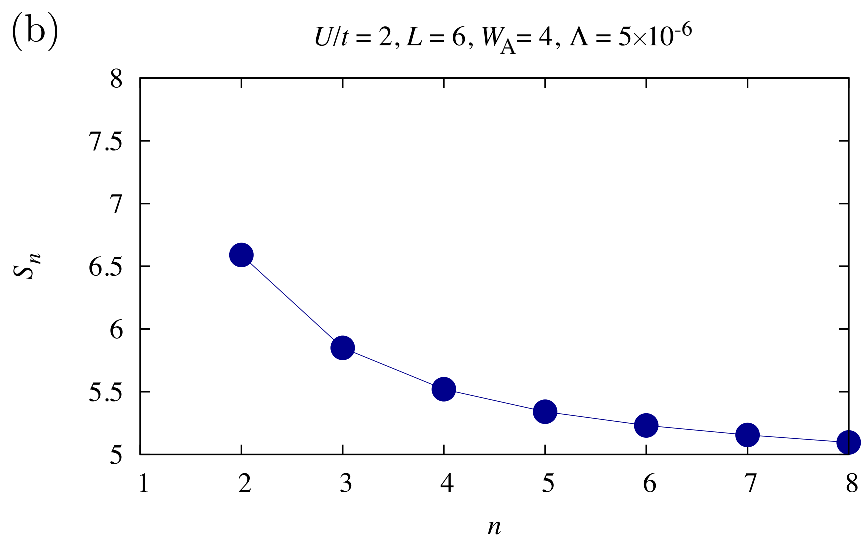



$$\hat{\rho}_A = \mathrm{Tr}_B \hat{\rho} \equiv \int d\Phi\, P(\Phi) \hat{\rho}_A(\Phi)$$

$$\mathrm{Tr}\hat{\rho}_A{}^n = \int \overbrace{d\Phi^1 \cdots d\Phi^n P(\Phi^1) \cdots P(\Phi^n)}^{\text{n-replicas}} \mathrm{Tr}\left[\hat{\rho}_A(\Phi^1) \cdots \hat{\rho}_A(\Phi^n)\right]$$

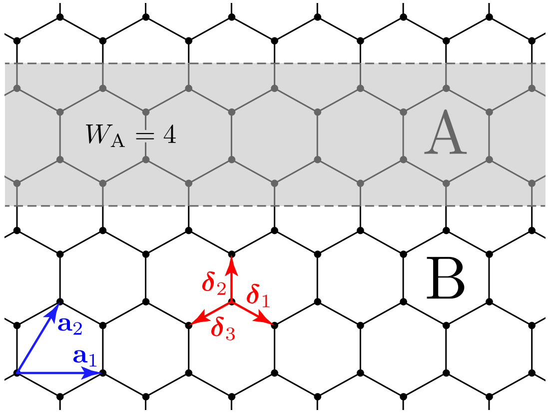
$$S_n = -\frac{1}{n-1} \ln \mathrm{Tr}\hat{\rho}_A{}^n$$

F. F. Assaad, T. C. Lang, and F. Parisen Toldin
Phys. Rev. B, 89, 125121, (2014)



## Entanglement Hamiltonian

T. Grover Phys. Rev. Lett., 111, 130402, (2013).




$$\hat{\rho}_A = e^{-\hat{H}_E} = \mathrm{Tr}_B \hat{\rho} \equiv \int d\Phi\, P(\Phi) \hat{\rho}_A(\Phi),$$

with $$\hat{\rho}_A(\Phi) = \overbrace{\det\left[1 - G_A(\Phi)\right]}^{\equiv e^{-\alpha(\Phi)}} e^{-\hat{c}^\dagger \overbrace{\ln\left[G_A^{-1}(\Phi) - 1\right]}^{\equiv h(\Phi)} \hat{c}}$$

Cumulant expansion: $$\hat{H}_E = \sum_{x,y} t_{x,y} \hat{c}_x^\dagger \hat{c}_y + \sum_{x,y,w,z} U_{x,y,w,z} \hat{c}_x^\dagger \hat{c}_y \hat{c}_w^\dagger \hat{c}_z + \cdots$$

$$t_{x,y} = \left\langle h(\Phi)_{x,y} \right\rangle - \left[ \left\langle \alpha(\Phi) h(\Phi)_{x,y} \right\rangle - \left\langle \alpha(\Phi) \right\rangle \left\langle h(\Phi)_{x,y} \right\rangle \right]$$

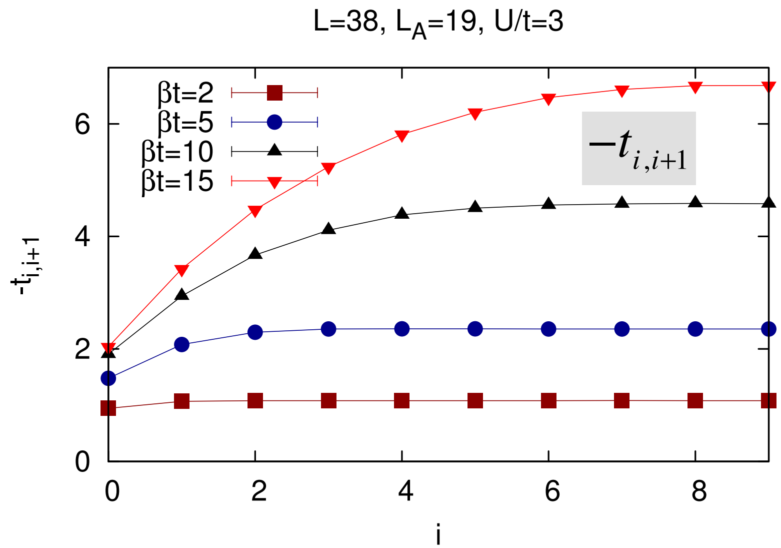
$$U_{x,y,w,z} = -\left[ \left\langle h(\Phi)_{x,y} h(\Phi)_{w,z} \right\rangle - \left\langle h(\Phi)_{x,y} \right\rangle \left\langle h(\Phi)_{w,z} \right\rangle \right]$$

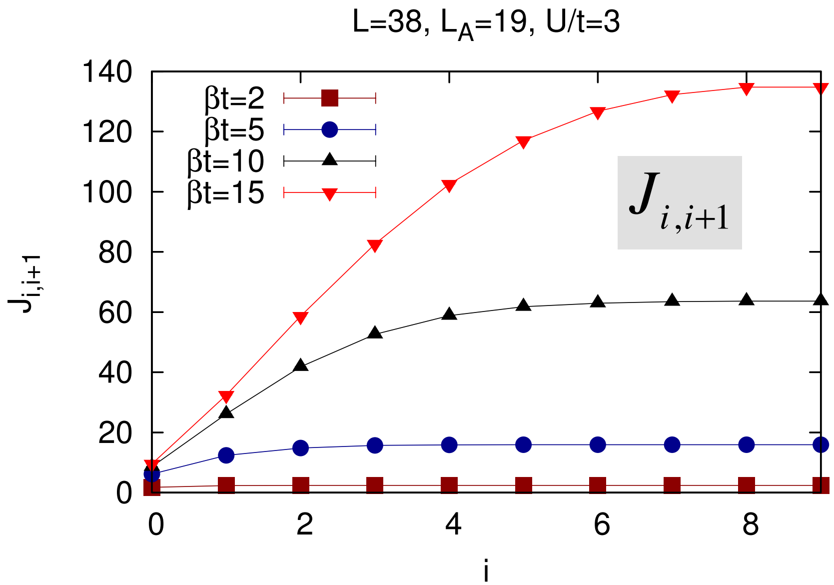
Note $$\left\langle h(\Phi)_{x,y} \right\rangle = \int d\Phi P(\Phi) h(\Phi)_{x,y}$$

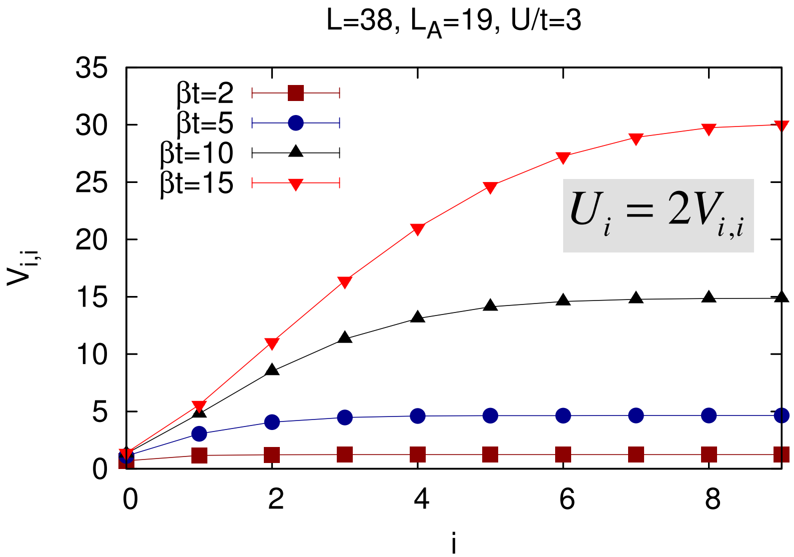
# 1D Hubbard model @ U/t = 3, <n>=1

$$H_E=\sum_{i,n,\sigma} -t_{i,i+n}\left(\hat{c}^\dagger_{i,\sigma}\hat{c}_{i+n,\sigma}+\hat{c}^\dagger_{i+n,\sigma}\hat{c}_{i,\sigma}\right)+\sum_{i,n}V_{i,i+n}\hat{n}_i\hat{n}_{i+n}+\sum_{i,n}\frac{-J_{i,i+n}}{4}\left(D^\dagger_{i,i+n}D_{i,i+n}+D_{i,i+n}D^\dagger_{i,i+n}\right)+\cdots$$

$$D^\dagger_{i,i+n}=\sum_{\sigma}\hat{c}^\dagger_{i,\sigma}\hat{c}_{i+n,\sigma}$$



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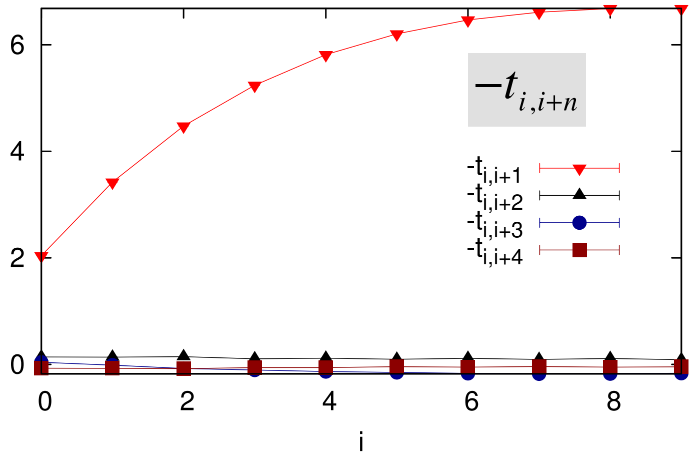
$$H_E = \sum_{i,n,\sigma} -t_{i,i+n}\left(\hat{c}^{\dagger}_{i,\sigma}\hat{c}_{i+n,\sigma} + \hat{c}^{\dagger}_{i+n,\sigma}\hat{c}_{i,\sigma}\right) + \sum_{i,n} V_{i,i+n}\hat{n}_i\hat{n}_{i+n} + \sum_{i,n}\frac{-J_{i,i+n}}{4}\left(D^{\dagger}_{i,i+n}D_{i,i+n} + D_{i,i+n}D^{\dagger}_{i,i+n}\right) + \cdots$$

$$D^{\dagger}_{i,i+n} = \sum_{\sigma}\hat{c}^{\dagger}_{i,\sigma}\hat{c}_{i+n,\sigma}$$

L=38, $L_A$=19, U/t=3,βt=15

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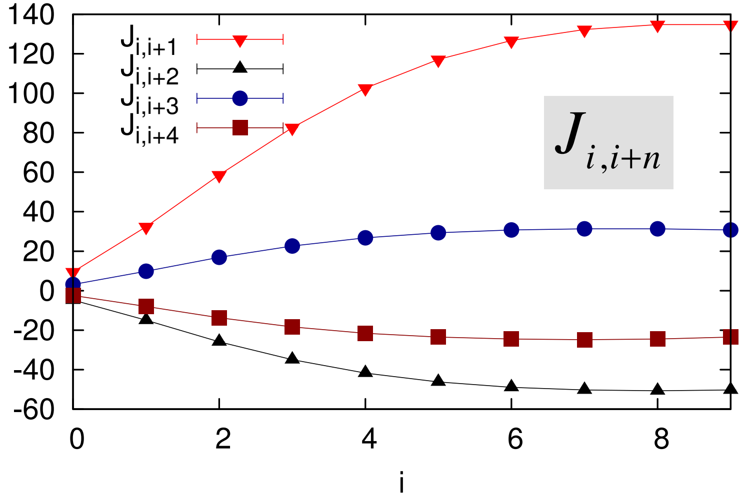


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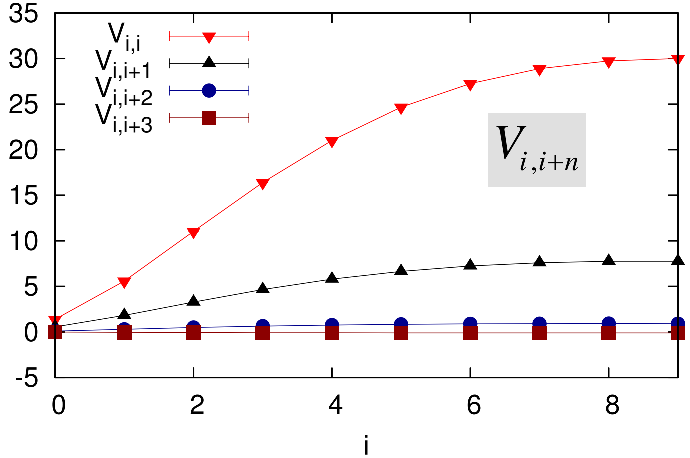


## Entanglement spectrum

F. F. Assaad, T. C. Lang, and F. Parisen Toldin   Phys. Rev. B, 89, 125121, (2014)

$$\left\langle a_x^\dagger(\tau) a_y \right\rangle = \frac{\mathrm{Tr}\left[ e^{-(\beta-\tau)H} a_x^\dagger e^{-\tau H} a_y \right]}{\mathrm{Tr}\left[ e^{-\beta H} \right]} \quad \rightarrow \quad \left\langle a_x^\dagger(\tau) a_x \right\rangle = \int d\omega \, \frac{e^{-\tau\omega}}{1+e^{-\beta\omega}} A(x,\omega)$$

$$A(x,\omega) = \frac{1}{Z} \sum_{n,m} \left( e^{-\beta E_n} + e^{-\beta E_m} \right) \left| \left\langle n \middle| a_x^\dagger \middle| m \right\rangle \right|^2 \delta\left( \omega - E_m + E_n \right)$$

$$\hat{\rho}_A = e^{-\hat{H}_E}$$

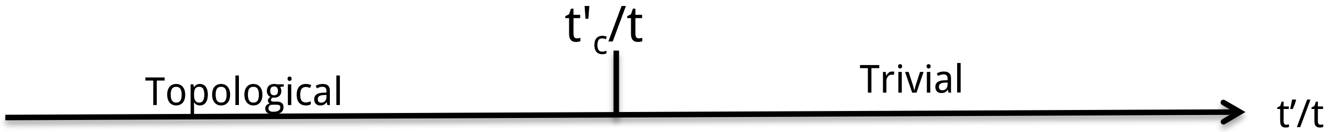
$$\left\langle a_x^\dagger(\tau_E) a_y \right\rangle_E = \frac{\mathrm{Tr}\left[ \hat{\rho}_A^{\,n-\tau_E} a_x^\dagger \hat{\rho}_A^{\,\tau_E} a_y \right]}{\mathrm{Tr}\left[ \hat{\rho}_A^{\,n} \right]} \quad \rightarrow \quad \left\langle a_x^\dagger(\tau_E) a_x \right\rangle_E = \int d\omega \, \frac{e^{-\tau_E \omega}}{1+e^{-n\omega}} A^E(x,\omega)$$

Note: n, $\tau_E$ are natural numbers → restricted to low energy sector of the entanglement spectrum.

# Entanglement spectrum

F. F. Assaad, T. C. Lang, and F. Parisen Toldin Phys. Rev. B, 89, 125121, (2014)

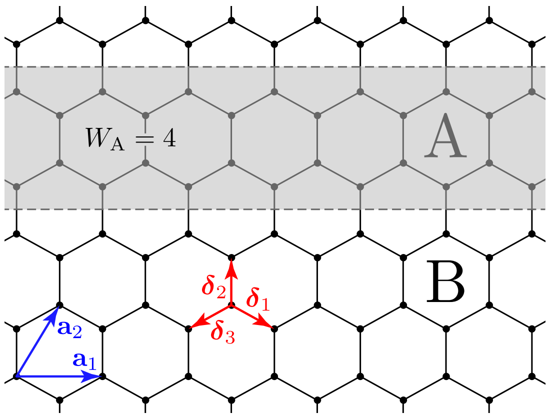
Example: Single particle entanglement spectral function for dimerized Kane-Mele Hubbard model.

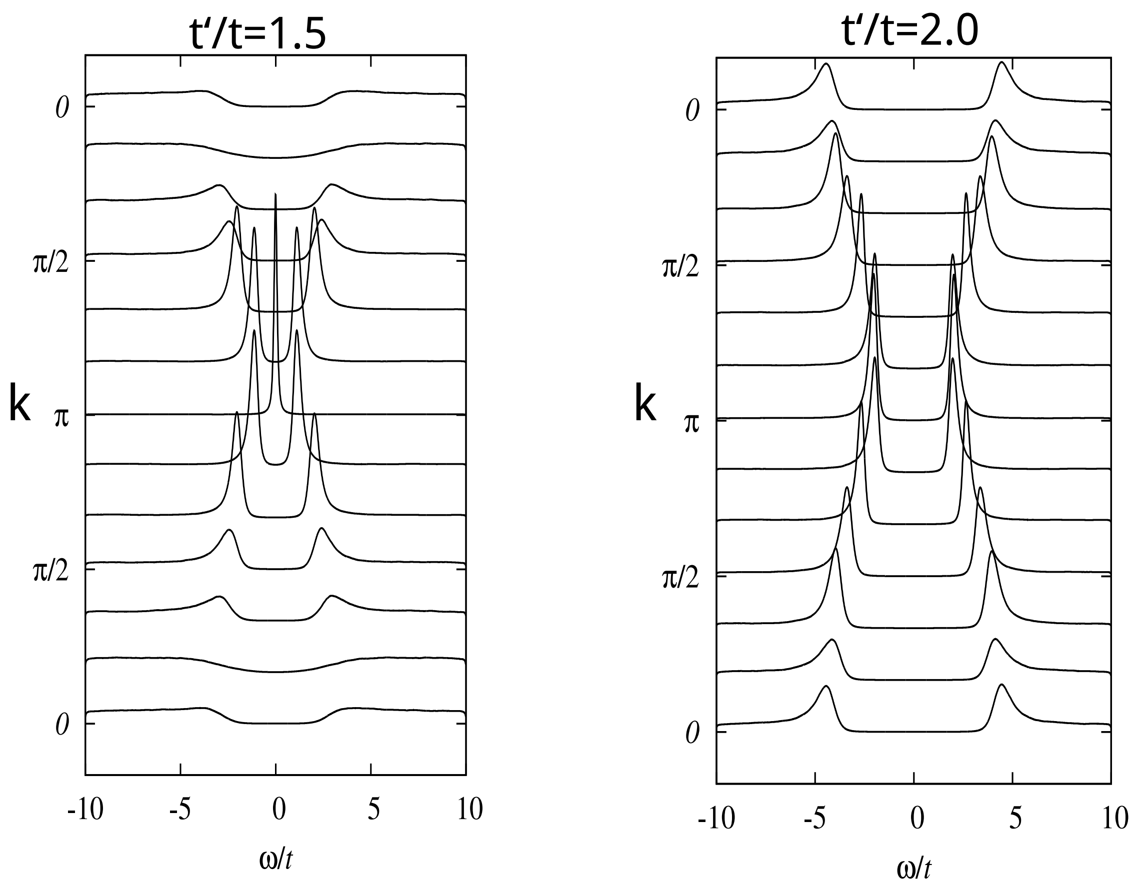



Dimerization along the $\delta_1$ direction

$$A^{E}_{1,1}(k,\omega) \ @ \ n = 8$$

U/t = 2, $\lambda$/t = 0.2




@U/t=2 : 1.95 < $t'_c$/t < 2

T. C. Lang, A. M. Essin, V. Gurarie, and S. Wessel,
Phys. Rev. B 87, 205101 (2013).

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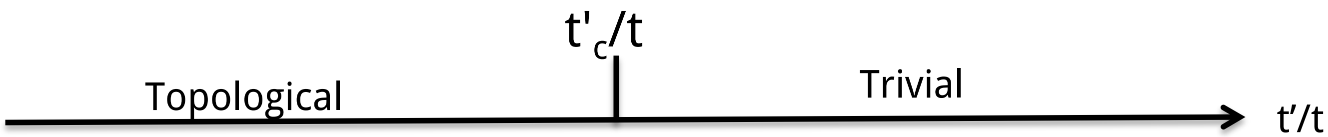
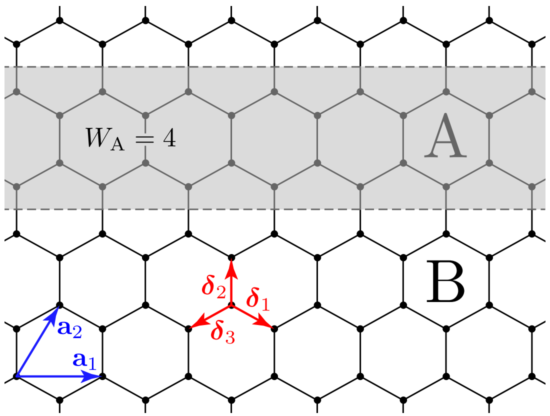



Dimerization along the $\delta_2$ direction

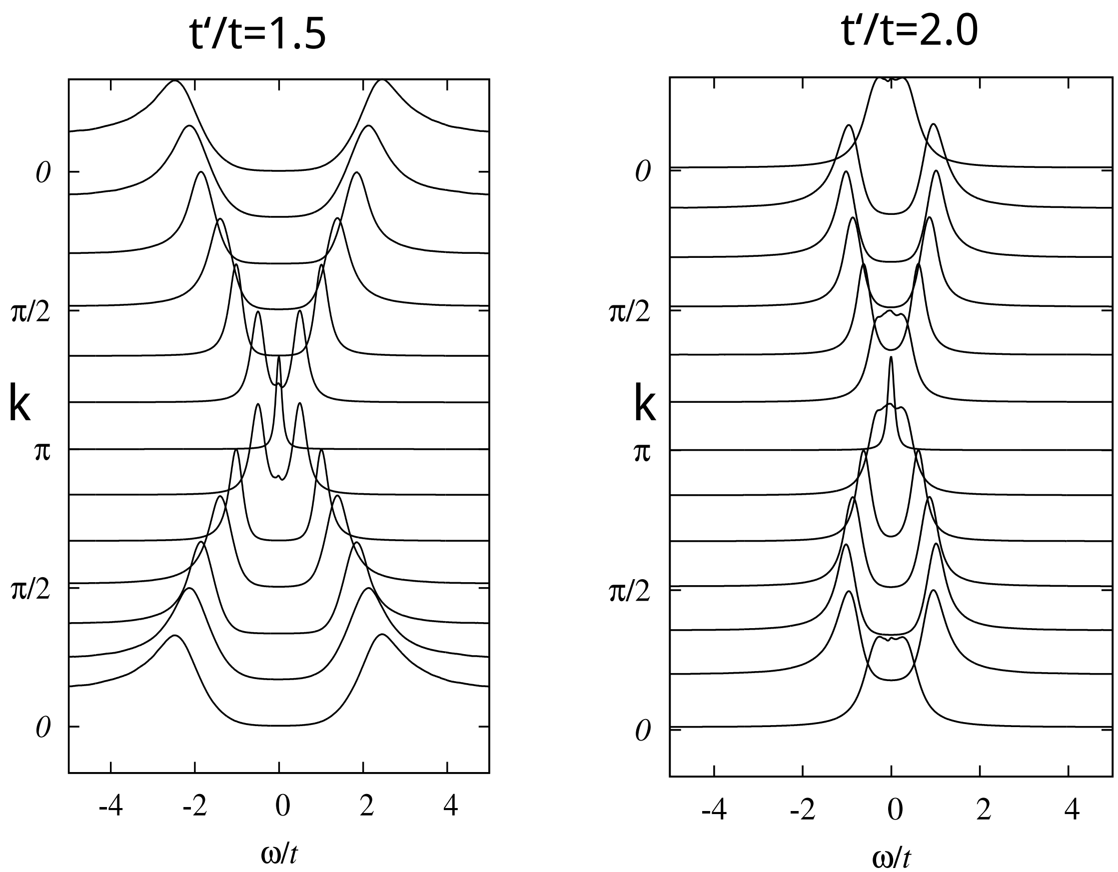
$U/t = 2,\ \lambda/t = 0.2$




@U/t=2 :  $1.95 < t'_c/t < 2$

T. C. Lang, A. M. Essin, V. Gurarie, and S. Wessel, Phys. Rev. B 87, 205101 (2013).

$A^E_{1,1}(k,\omega)$ @ $n = 8$



## Limitations of the Gaussian (or weak coupling) approach.

$$\hat{\rho}_A = \int d\Phi\, P(\Phi)\hat{\rho}_A(\Phi)$$

→ Fluctuations! e.g n=2

$$\mathrm{Tr}\hat{\rho}_A^{\,2} = \int d\Phi^1 d\Phi^2 \overbrace{P(\Phi^1)P(\Phi^2)}^{\text{Sample}}\ \overbrace{\mathrm{Tr}_A\left[\hat{\rho}_A(\Phi^1)\,\hat{\rho}_A(\Phi^2)\right]}^{\text{Measure}}$$

$$\frac{\left\langle \Delta\mathrm{Tr}\left[\hat{\rho}_A(\Phi^1)\,\hat{\rho}_A(\Phi^2)\right]\right\rangle}{\left\langle \mathrm{Tr}\left[\hat{\rho}_A(\Phi^1)\,\hat{\rho}_A(\Phi^2)\right]\right\rangle} \approx \frac{b/\sqrt{T_{CPU}}}{e^{-a\partial_A}} \ll 1 \qquad T_{CPU} \gg b^2\, e^{2a\partial_A}$$

☒

→ Entanglement spectrum

$$\left\langle a_x^\dagger(\tau_E)a_y\right\rangle_E = \frac{\mathrm{Tr}_A\left[\hat{\rho}_A^{\,n-\tau_E} a_x^\dagger \hat{\rho}_A^{\,\tau_E} a_y\right]}{\mathrm{Tr}_A\left[\hat{\rho}_A^{\,n}\right]}$$

☒

$$\mathrm{Tr}\hat{\rho}_A^{\,n} = \int d\Phi^1\cdots d\Phi^n\, \underbrace{P(\Phi^1)\cdots P(\Phi^n)\overbrace{\mathrm{Tr}_A\left[\hat{\rho}_A(\Phi^1)\cdots\hat{\rho}_A(\Phi^n)\right]}^{\text{Is positive}}}_{\text{Sample}}$$

→ Identical to replica approach FFA, PRB 91, 125146 (2015)

→ Entanglement Hamiltonian with cumulant expansion

☑

# Entanglement Hamiltonian and spectra from quantum Monte Carlo simulations of interacting fermions.

Fakher F. Assaad (KITP, June 23rd 2015 )

<u>Organization</u>

- Introduction: Entanglement Hamiltonian and spectra for free electrons.
- QMC
  - Weak coupling approaches → Gaussian representation of reduced density matrix
  - Strong coupling approaches → Replica
- Conclusions

F. Parisen Toldin

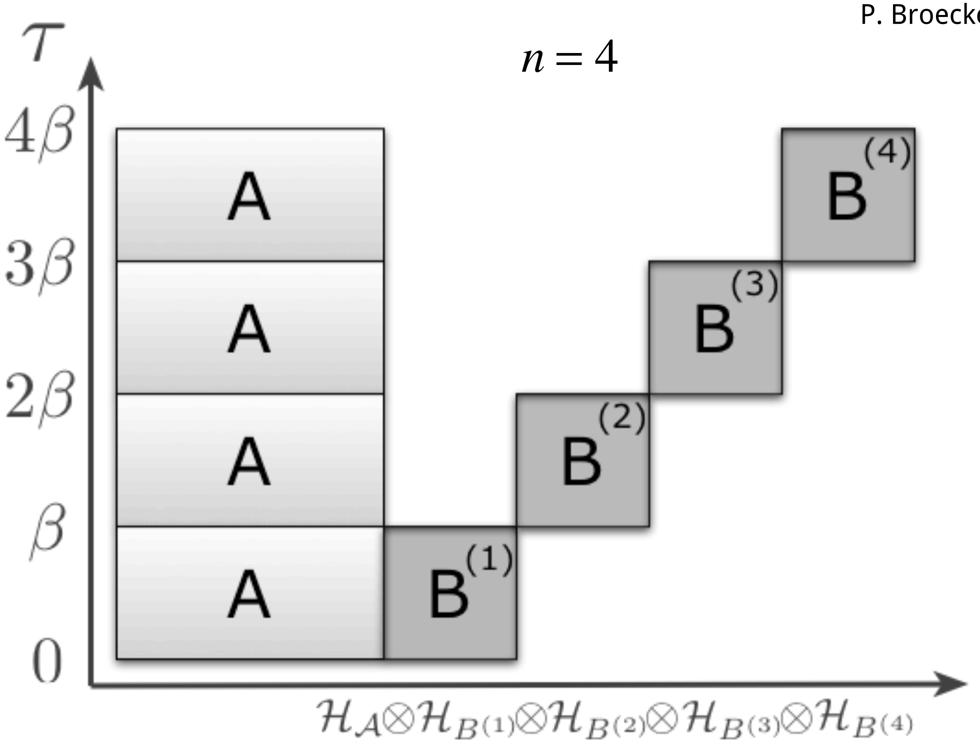
T. Lang

Julius-Maximilians-UNIVERSITÄT WÜRZBURG

FFA, T. C. Lang, and F. Parisen Toldin Phys. Rev. B, 89, 125121 (2014)
FFA, PRB 91, 125146 (2015)

# Entanglement spectrum for strongly  correlated electrons

M. B. Hastings, I. Gonzalez, A. B. Kallin, and R. G. Melko,  Phys. Rev. Lett, 157201, (2010).
P. Broecker and S. Trebst.  JSTAT, P08015, (2014).




Replica method: Hilbert space

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B^{(1)} \otimes \mathcal{H}_B^{(2)} \cdots \mathcal{H}_B^{(n)}$$

Time dependent Hamiltonian, $0 < \tau < n\beta$

$$\hat{H}(\tau) = \sum_{r=1}^{n} \Theta\left[\tau - (r-1)\beta\right] \Theta\left[r\beta - \tau\right] \hat{H}^{(r)}.$$

$\hat{H}^{(r)}$ Hamiltonian in Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B^{(r)}$

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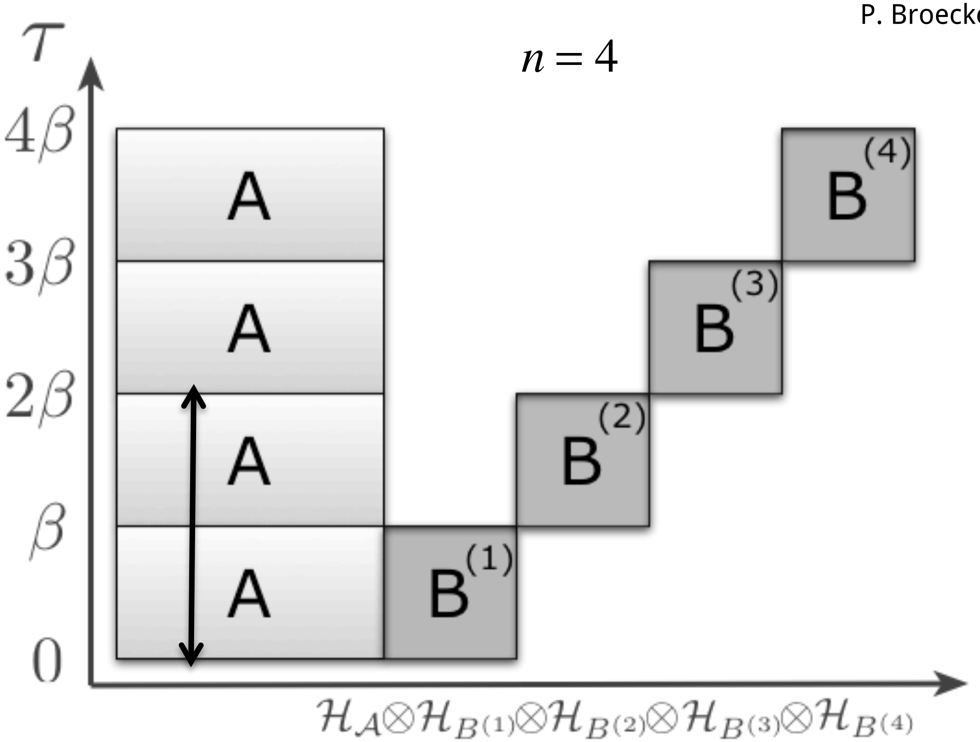
$n = 4$




Replica method: Hilbert space

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$\hat{H}^{(r)}$ Hamiltonian in Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B^{(r)}$

$$\frac{\mathrm{Tr}_{\mathcal{H}_{tot}}\left[\hat{U}(n\beta, \tau_E\beta)\hat{O}^\dagger \hat{U}(\tau_E\beta, 0)\hat{O}\right]}{\mathrm{Tr}_{\mathcal{H}_{tot}}\left[\hat{U}(n\beta, 0)\right]} = \frac{\mathrm{Tr}_{\mathcal{H}_A}\left[e^{-(n-\tau_E)\hat{H}_E}\hat{O}^\dagger e^{-\tau_E \hat{H}_E}\hat{O}\right]}{\mathrm{Tr}_{\mathcal{H}_A}\left[e^{-n\hat{H}_E}\right]}$$

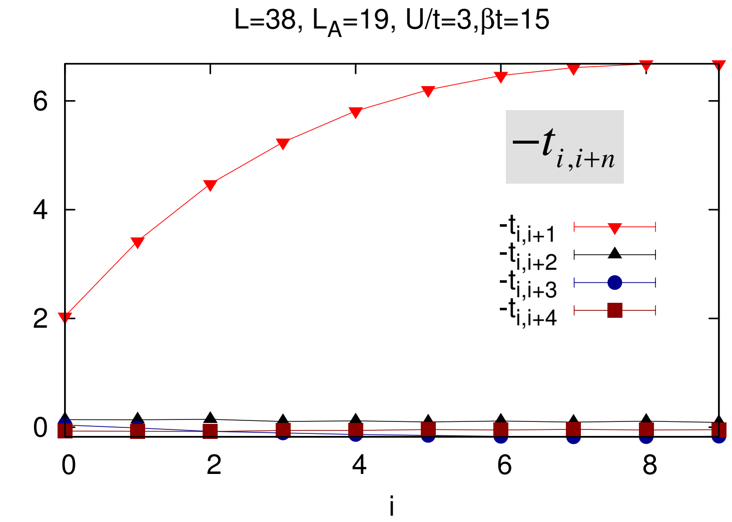
→ One can study equal time and dynamical properties of the entanglement Hamiltonian

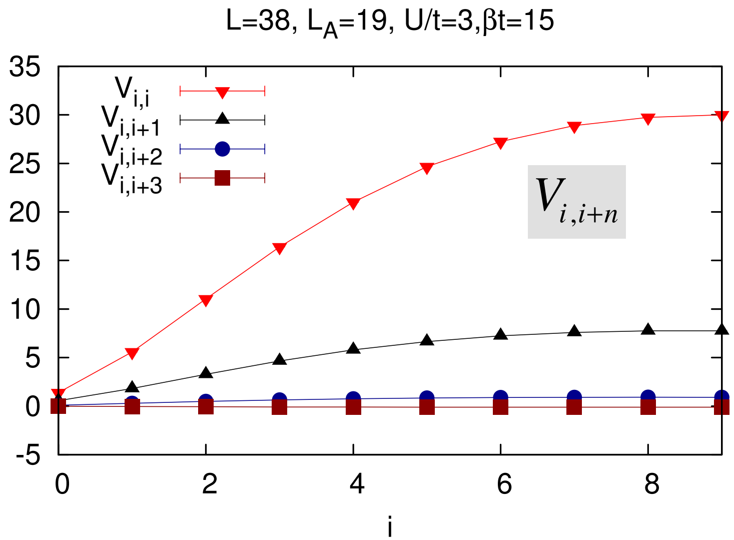
FFA, T. C. Lang, and F. Parisen Toldin Phys. Rev. B, 89, 125121 (2014), FFA, PRB 91, 125146 (2015).

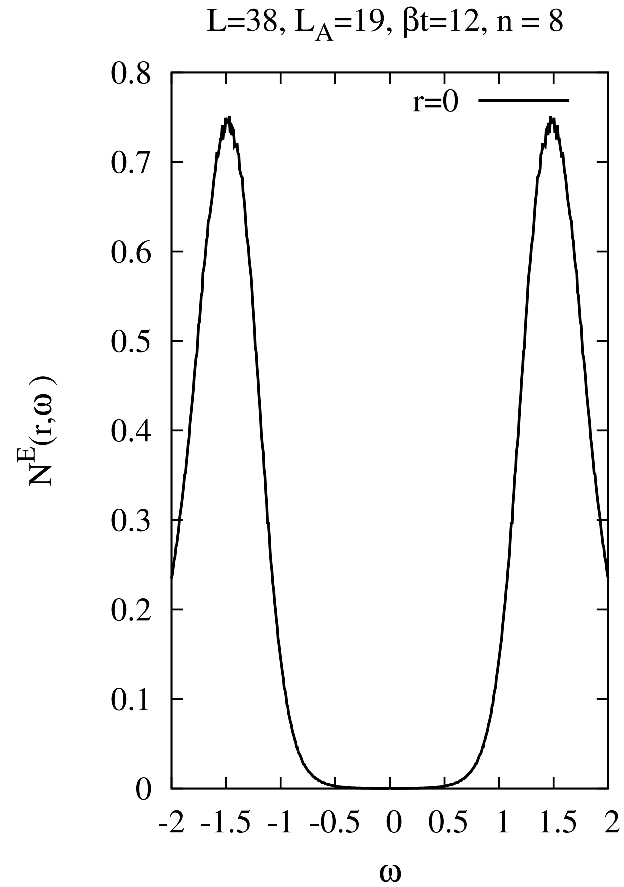
## One-dimensional Hubbard chain @ U/t=3, <n>=1

### Cumulant expansion

$$H_E = \sum_{i,n,\sigma} -t_{i,i+n}\left(\hat{c}^\dagger_{i,\sigma}\hat{c}_{i+n,\sigma} + \hat{c}^\dagger_{i+n,\sigma}\hat{c}_{i,\sigma}\right) + \sum_{i,n} V_{i,i+n}\hat{n}_i\hat{n}_{i+n} + \sum_{i,n}\frac{-J_{i,i+n}}{4}\left(D^\dagger_{i,i+n}D_{i,i+n} + D_{i,i+n}D^\dagger_{i,i+n}\right) + \cdots$$



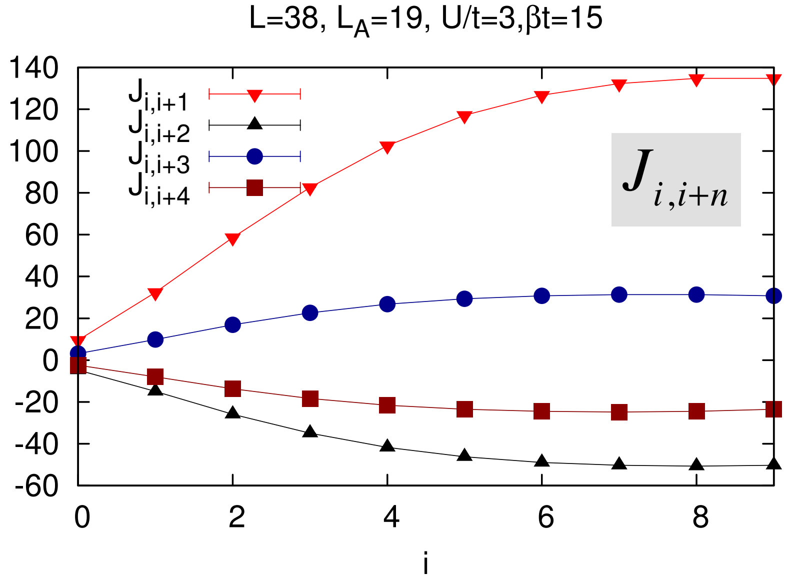



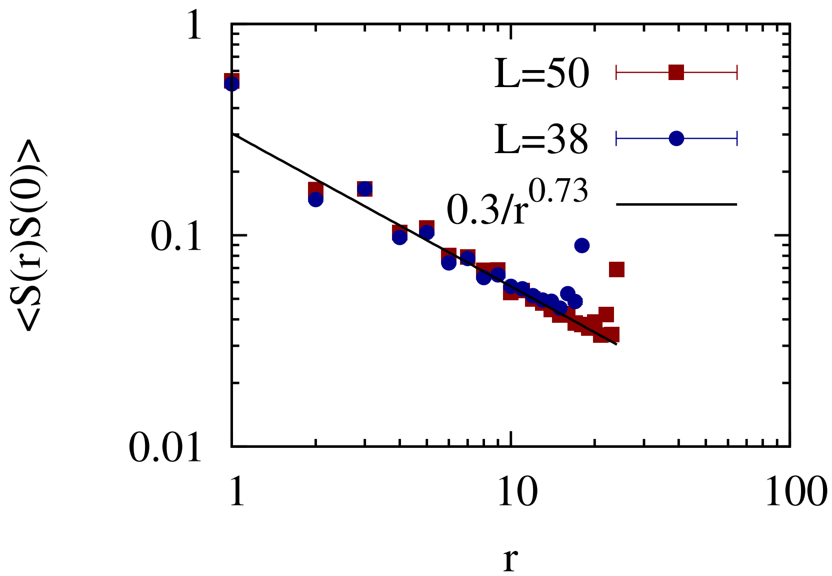

Single particle density of states of the entanglement Hamiltonian @ r=0. → Gap @ r > 0 grows.
→ Insulating state.

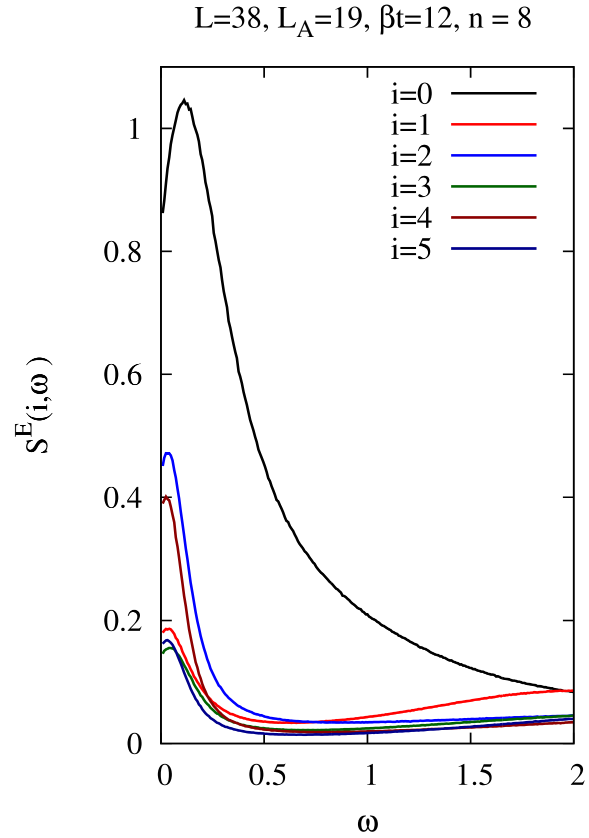
## One-dimensional Hubbard chain @ U/t=3, <n>=1

### Cumulant expansion

$$H_E = \sum_{i,n,\sigma} -t_{i,i+n}\left(\hat{c}^\dagger_{i,\sigma}\hat{c}_{i+n,\sigma} + \hat{c}^\dagger_{i+n,\sigma}\hat{c}_{i,\sigma}\right) + \sum_{i,n} V_{i,i+n}\hat{n}_i\hat{n}_{i+n} + \sum_{i,n}\frac{-J_{i,i+n}}{4}\left(D^\dagger_{i,i+n}D_{i,i+n} + D_{i,i+n}D^\dagger_{i,i+n}\right) + \cdots$$

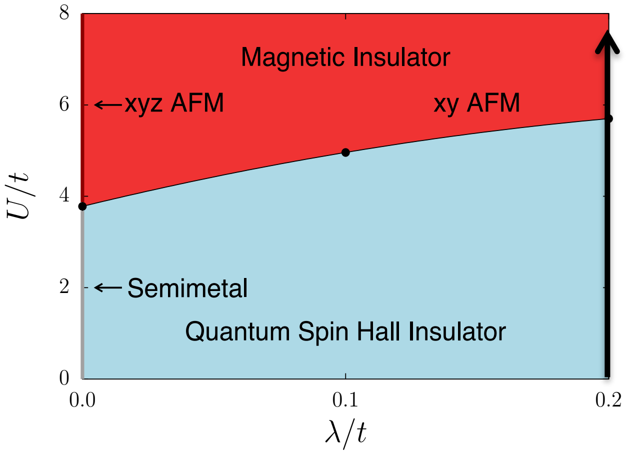


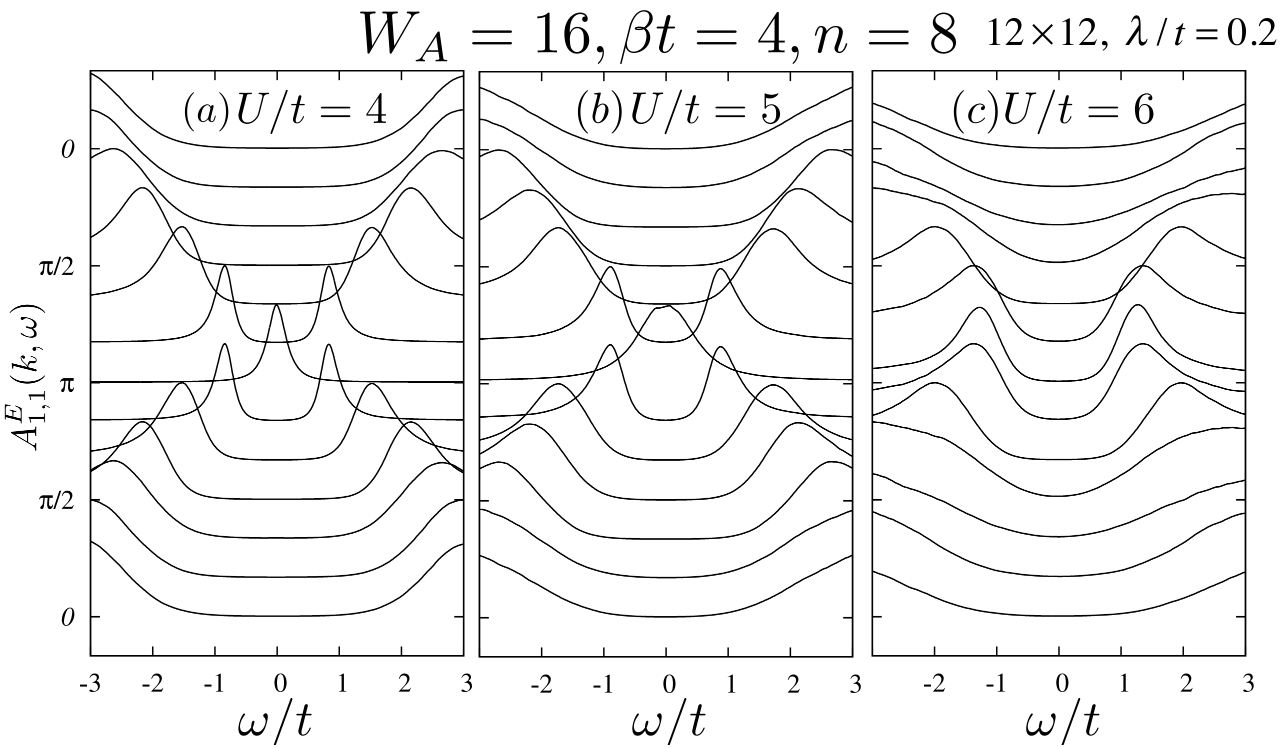



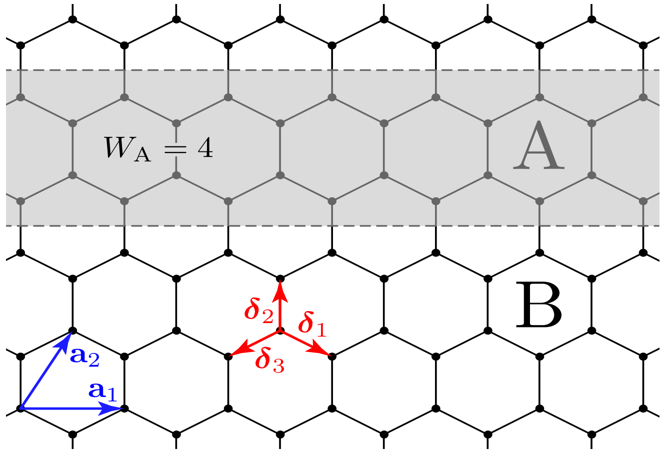




Local dynamical spin structure factor of the entanglement Hamiltonian → Gapless spin excitations

# Single-particle spectrum of the entanglement Hamiltonian




$$\frac{\mathrm{Tr}_{\mathcal{H}_A}\left[e^{-(n-\tau_E)\hat{H}_E}\hat{a}^{\dagger}_{k,m}e^{-\tau_E\hat{H}_E}\hat{a}_{k,m'}\right]}{\mathrm{Tr}_{\mathcal{H}_A}\left[e^{-n\hat{H}_E}\right]}$$

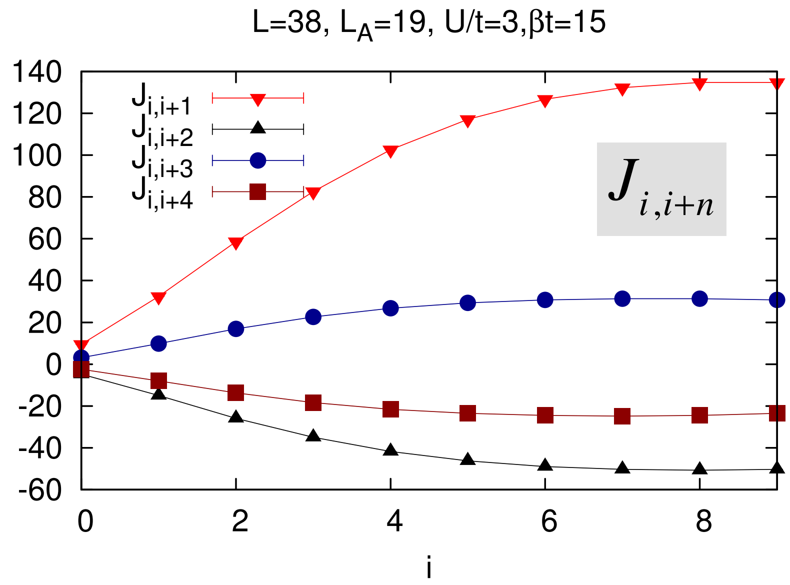
k: translation symmetry in $a_1$, m: orbital index across partition A

Wick rotation (Stochastic MaxEnt) to produce spectral function

## Summary

**Part I   Weak coupling methods.  Direct calculation of entanglement Hamiltonian (cumulant expansion)**

$$H_E = \sum_{i,n,\sigma} -t_{i,i+n}\left(\hat{c}^{\dagger}_{i,\sigma}\hat{c}_{i+n,\sigma} + \hat{c}^{\dagger}_{i+n,\sigma}\hat{c}_{i,\sigma}\right) + \sum_{i,n} V_{i,i+n}\hat{n}_i\hat{n}_{i+n} + \sum_{i,n}\frac{-J_{i,i+n}}{4}\left(D^{\dagger}_{i,i+n}D_{i,i+n} + D_{i,i+n}D^{\dagger}_{i,i+n}\right) + \cdots$$




**Part II   Method to compute entanglement spectrum of strongly correlated fermion systems with QMC.**