DYNAMICS OF 1D QUANTUM SYSTEMS WITH TENSOR NETWORKS

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H. Kim, J. I. Cirac, M. Hastings, D. Huse

arXiv:1410.4186

Max Planck Institut
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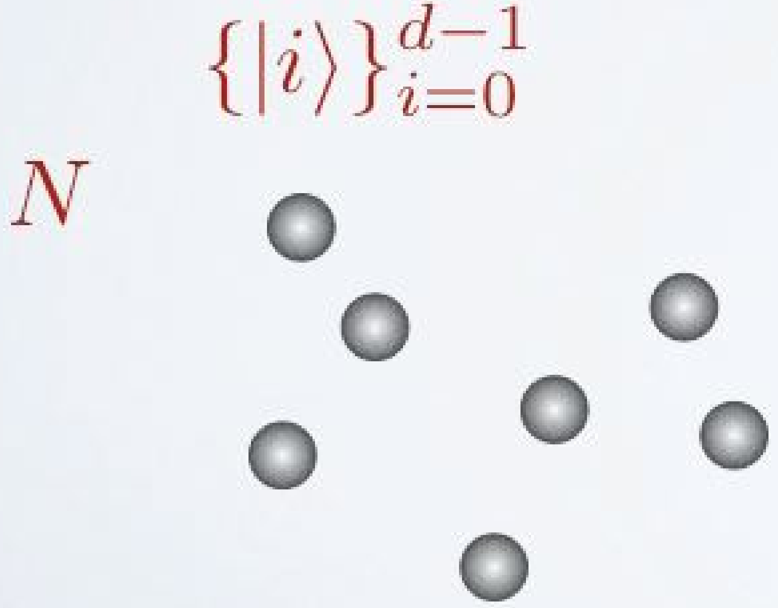
(Garching)

KITP 5.5.2015

WHAT ARE TNS?

- TNS = Tensor Network States

Context: quantum many body systems



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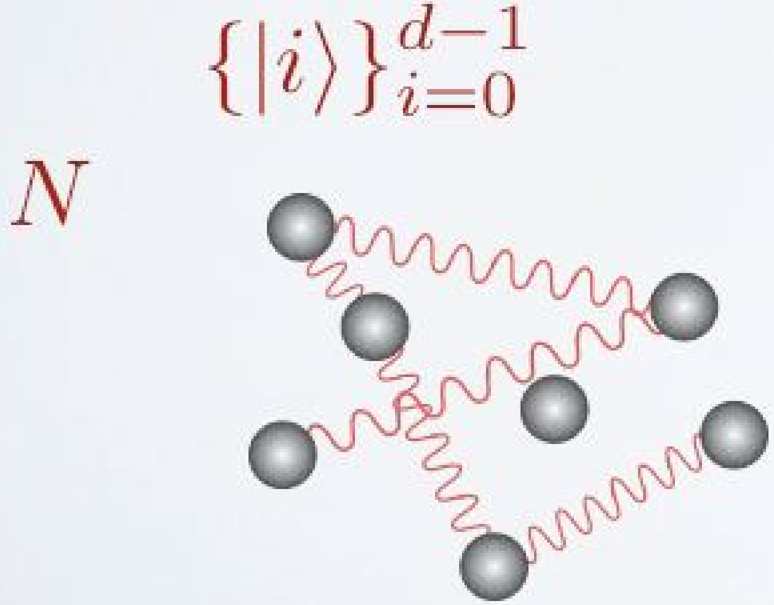


interacting with each other

Goal: describe equilibrium states

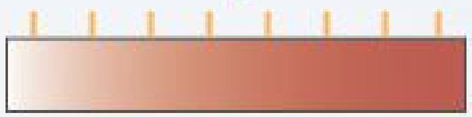
ground, thermal states

WHAT ARE TNS?

- TNS = Tensor Network States

A general state of the N-body Hilbert space has exponentially many coefficients

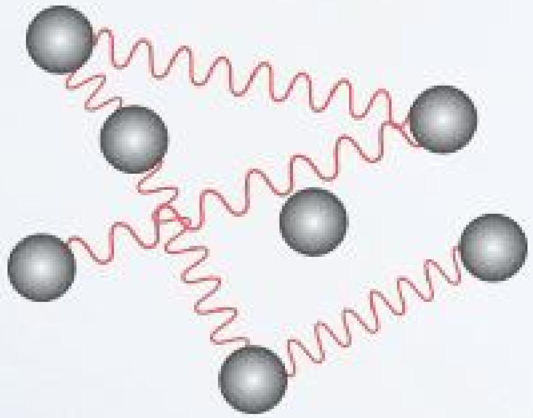
$$|\Psi\rangle = \sum_{i_j} \boxed{c_{i_1 \dots i_N}} |i_1 \dots i_N\rangle$$



N-legged tensor

d^N

N



WHY SHOULD TNS BE USEFUL?

Which properties characterize physically interesting states?

finite range
gapped
Hamiltonians

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states with
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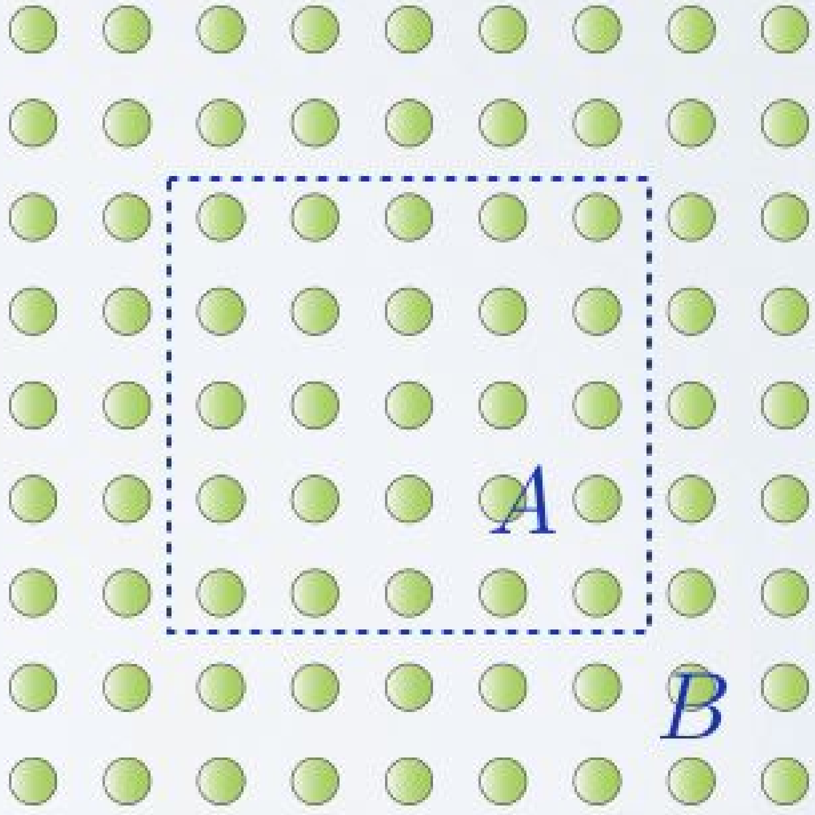
Area law



WHY SHOULD TNS BE USEFUL?

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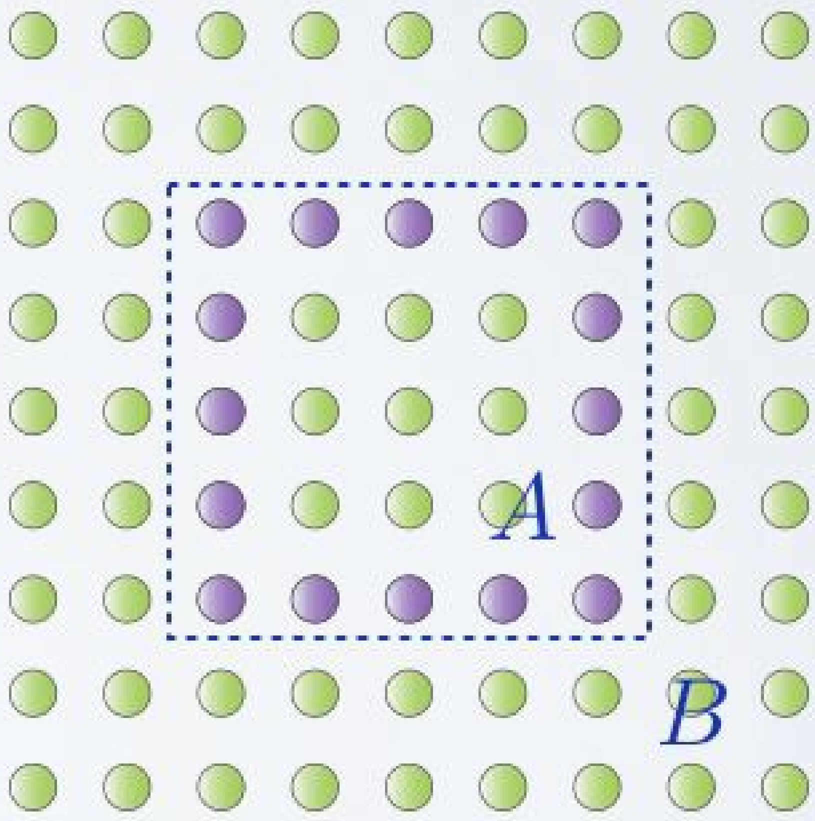
Area law

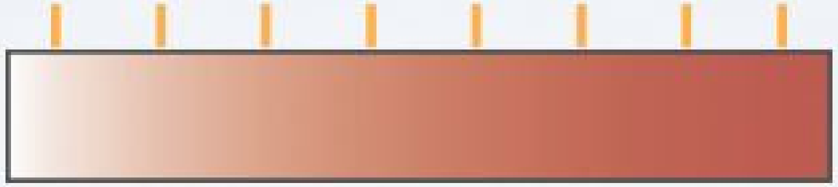
$S_{A\max} \propto |\delta A|$



TNS parametrize the
structure of entanglement

1D SYSTEMS: MPS

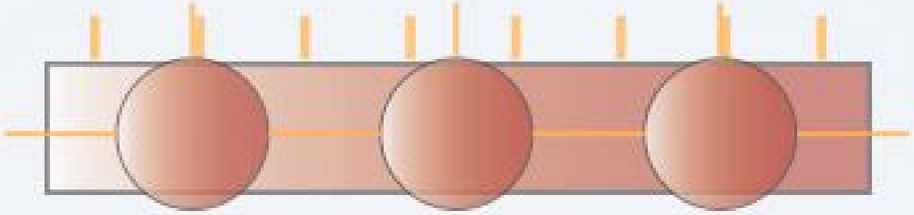
- MPS = Matrix Product States



$$|\Psi\rangle = \sum_{i_1 \ldots i_N} c_{i_1 \ldots i_N} |i_1 \ldots i_N\rangle$$

1D SYSTEMS: MPS

- MPS = Matrix Product States



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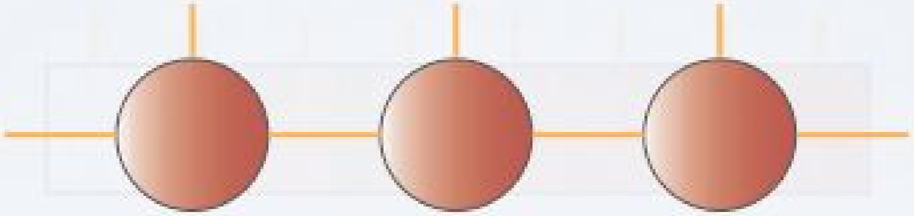

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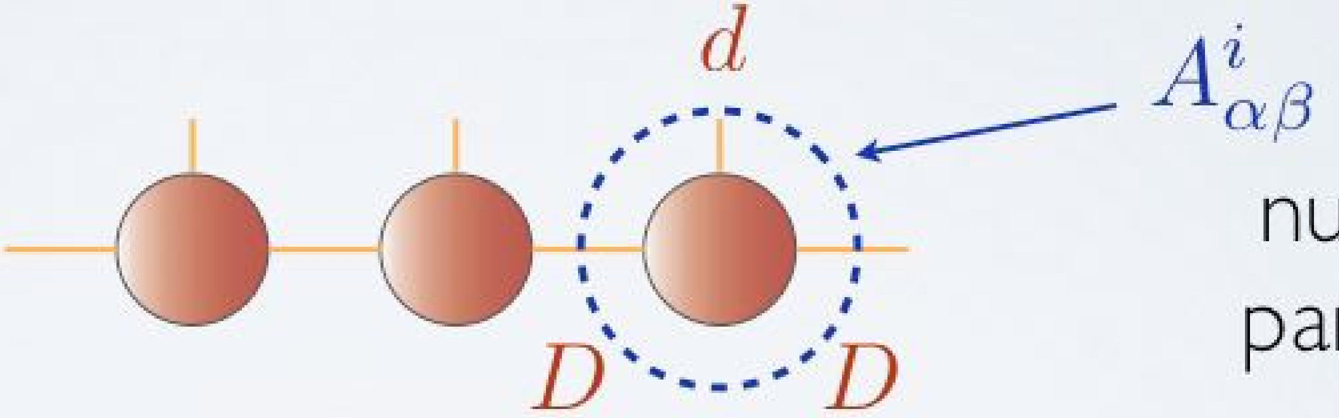
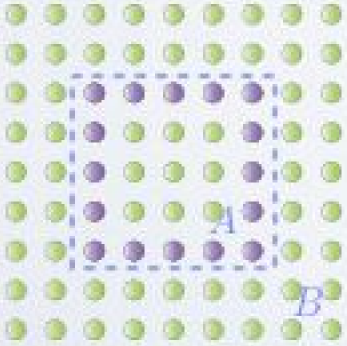


number of parameters

$$NdD^2$$

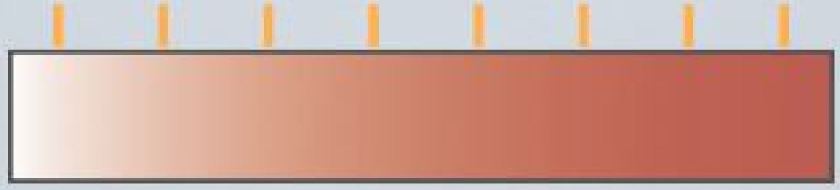
$$|\Psi\rangle = \sum_{i_1 \dots i_N} \mathrm{tr}(A_1^{i_1} A_2^{i_2} \dots A_N^{i_N}) |i_1 \dots i_N\rangle$$



Area law by construction

MIXED STATES / OPERATORS

- MPO = Matrix Product Operator



MIXED STATES / OPERATORS

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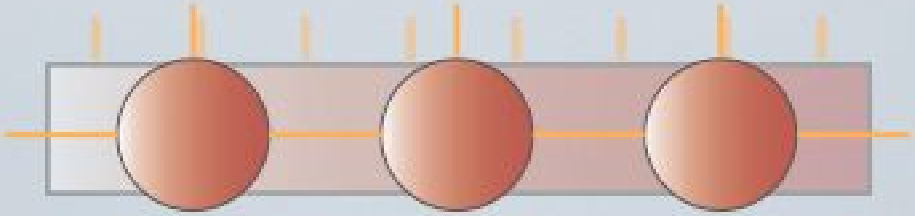

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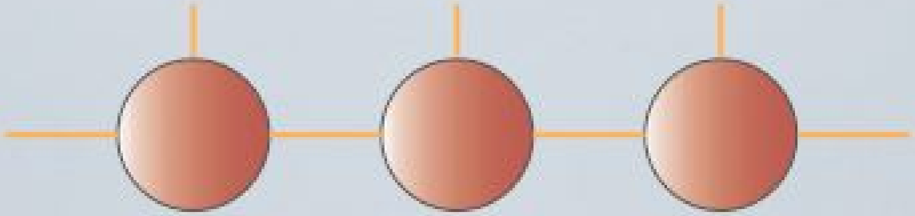

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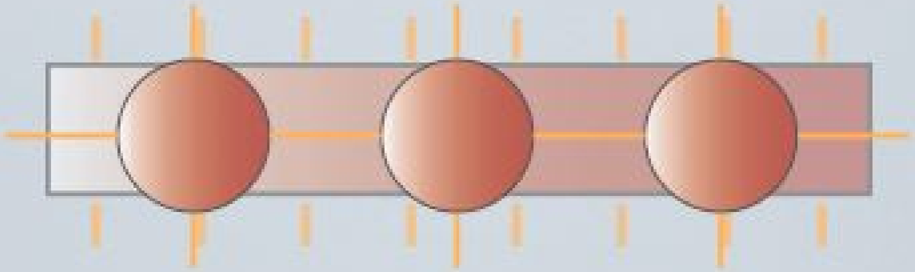
MIXED STATES / OPERATORS

- MPO = Matrix Product Operator

Same kind of ansatz for operators



$$\hat{M} = \sum_{i_1, j_1 \ldots i_N, j_N} \mathrm{tr}(M_1^{i_1 j_1} M_2^{i_2 j_2} \ldots M_N^{i_N j_N}) |i_1 \ldots i_N\rangle\langle j_1 \ldots j_N|$$

Verstraete et al., PRL 2004

Pirvu et al., NJP 2010

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Same kind of
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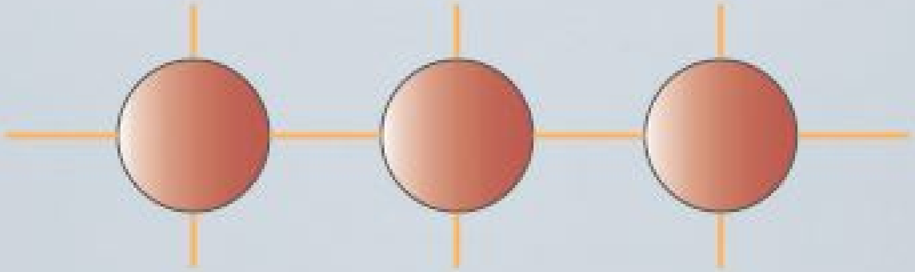

Bounded
operator space
entanglement
entropy

$$\hat{M} = \sum_{i_1,j_1 \ldots i_N,j_N} \mathrm{tr}(M_1^{i_1 j_1} M_2^{i_2 j_2} \ldots M_N^{i_N j_N}) |i_1 \ldots i_N\rangle\langle j_1 \ldots j_N|$$

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SOME MPS PROPERTIES

good approximation of ground states

- gapped finite range Hamiltonian ⇒ area law (ground state)

  Verstraete, Cirac, PRB 2006

  Hastings J. Stat. Phys 2007

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good approximation of ground states

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Verstraete, Cirac, PRB 2006

Hastings J. Stat. Phys 2007

extremely successful for GS, low energy

White, PRL 1992

little entangled

Verstraete, Porras, Cirac, PRL 2004

Schollwöck, RMP 2005, Ann. Phys. 2011

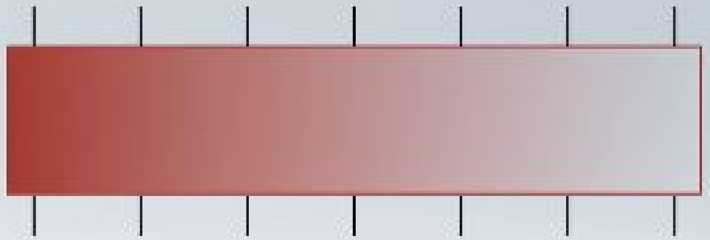
time evolution can be simulated too

Vidal, PRL 2003, PRL 2007
White, Feiguin, PRL 2004
Daley et al., 2004
Haegeman et al., 2011

DYNAMICS & MPS

Approximate action of local operators on MPS

time evolution



Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

DYNAMICS & MPS

Approximate action of local operators on MPS

time evolution

$$U(t) \to [U(\delta)]^M$$

Vidal, PRL 2003, 2004

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DYNAMICS & MPS

Approximate action of local operators on MPS

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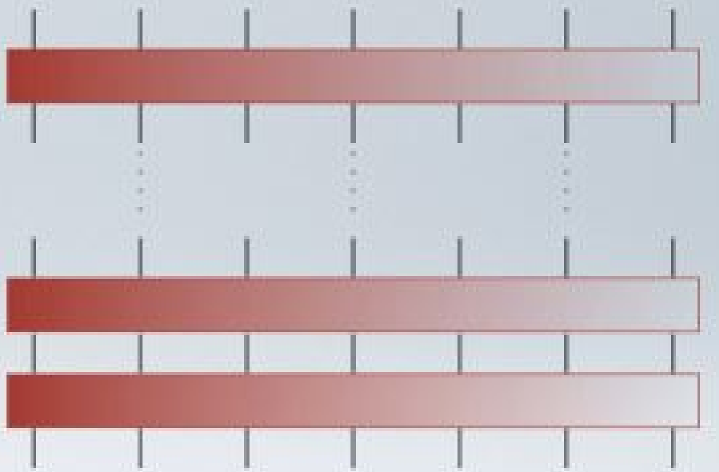
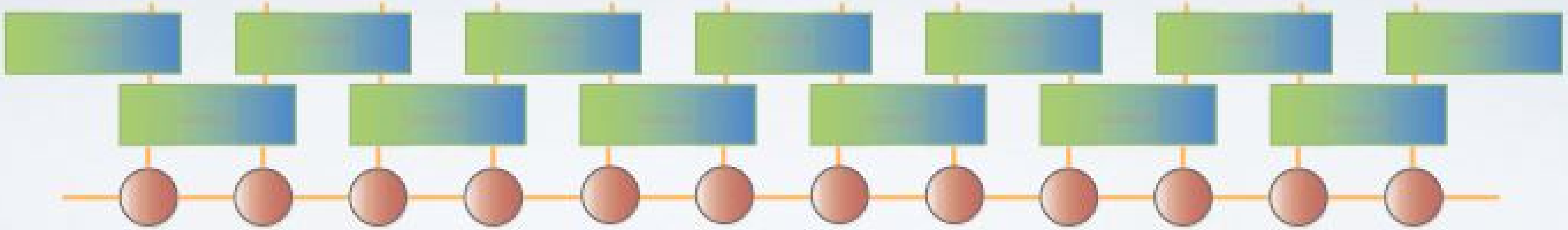


$$U(t) \rightarrow [U(\delta)]^M$$

as local terms!



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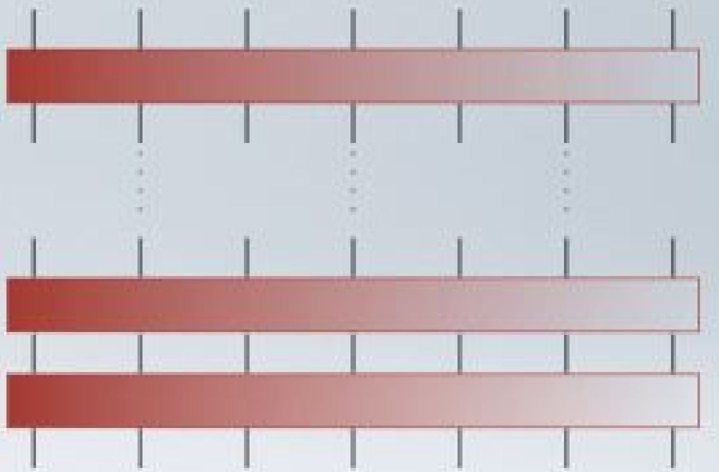

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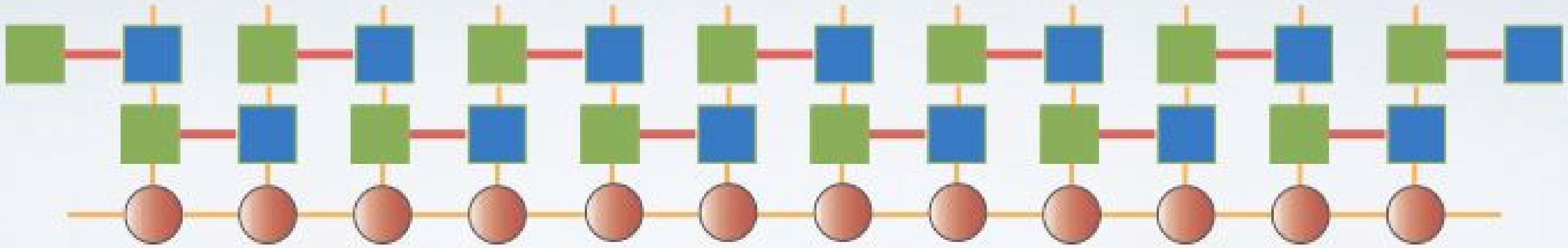

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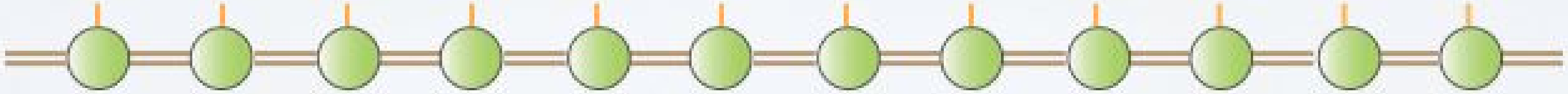

TEBD
t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

DYNAMICS & MPS

Entropy of evolved state may grow linearly

Osborne, PRL 2006
Schuch et al., NJP 2008

DYNAMICS & MPS

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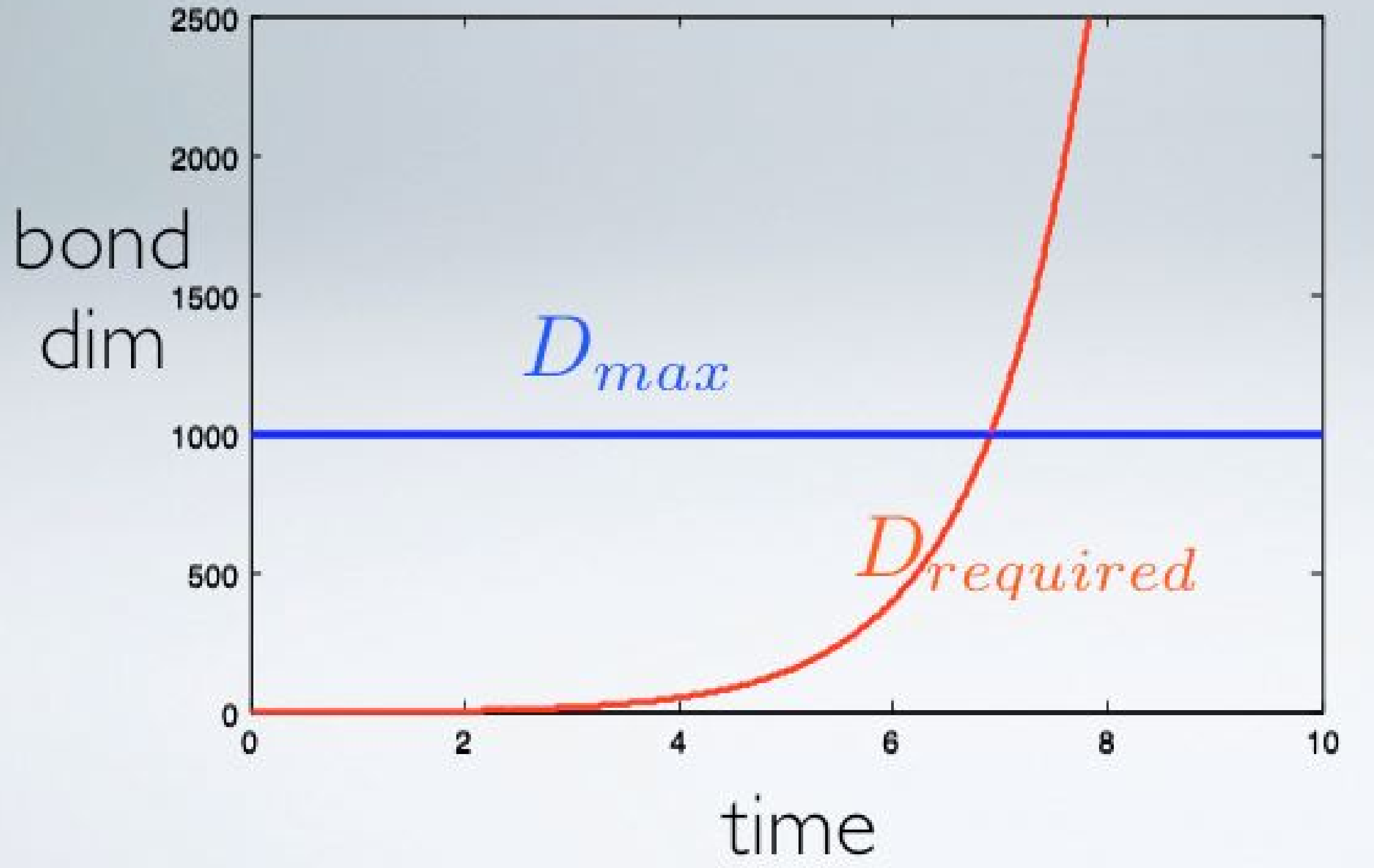


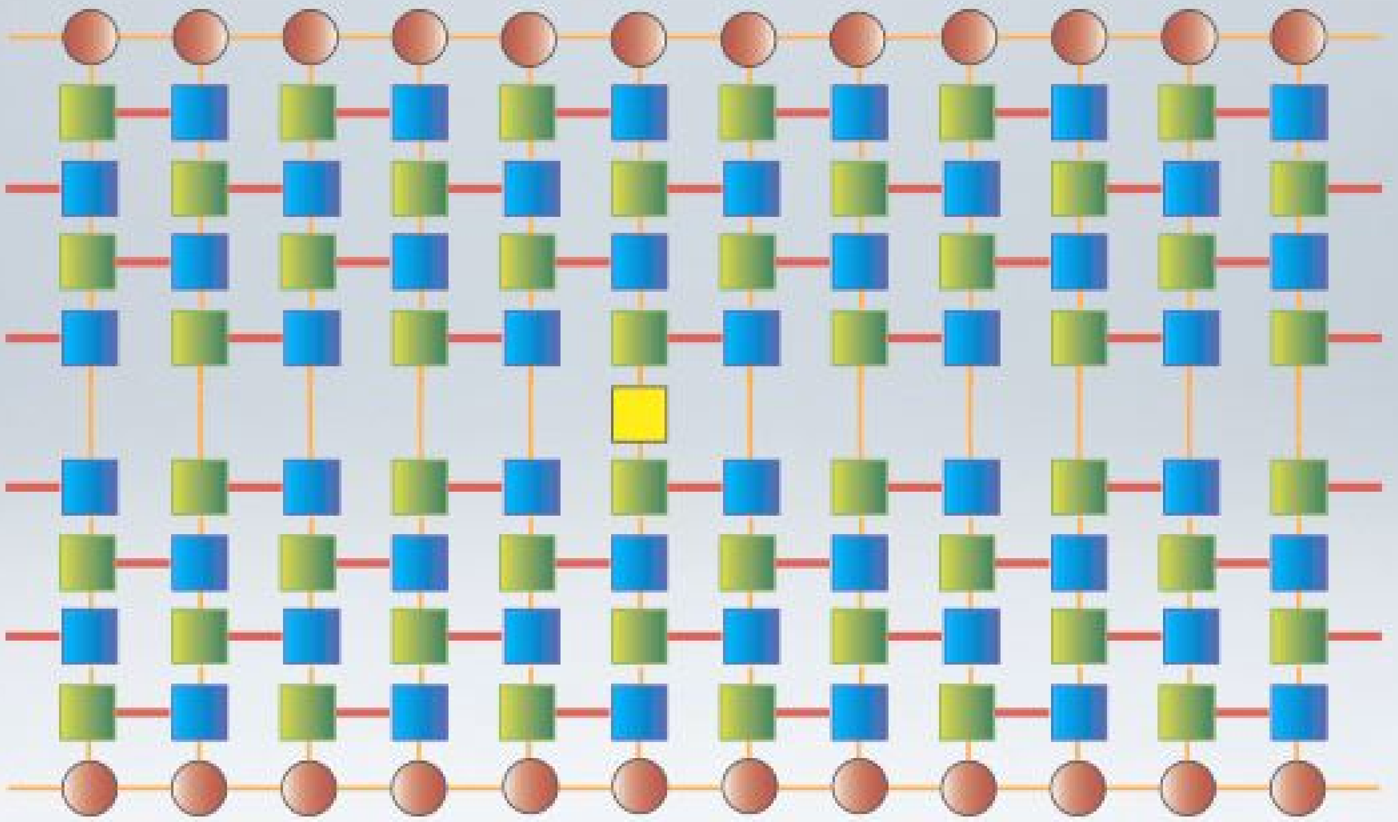
Osborne, PRL 2006
Schuch et al., NJP 2008

required bond for fixed precision

$$D \sim e^{\alpha t}$$

ALTERNATIVELY

time dependent
observables as TN

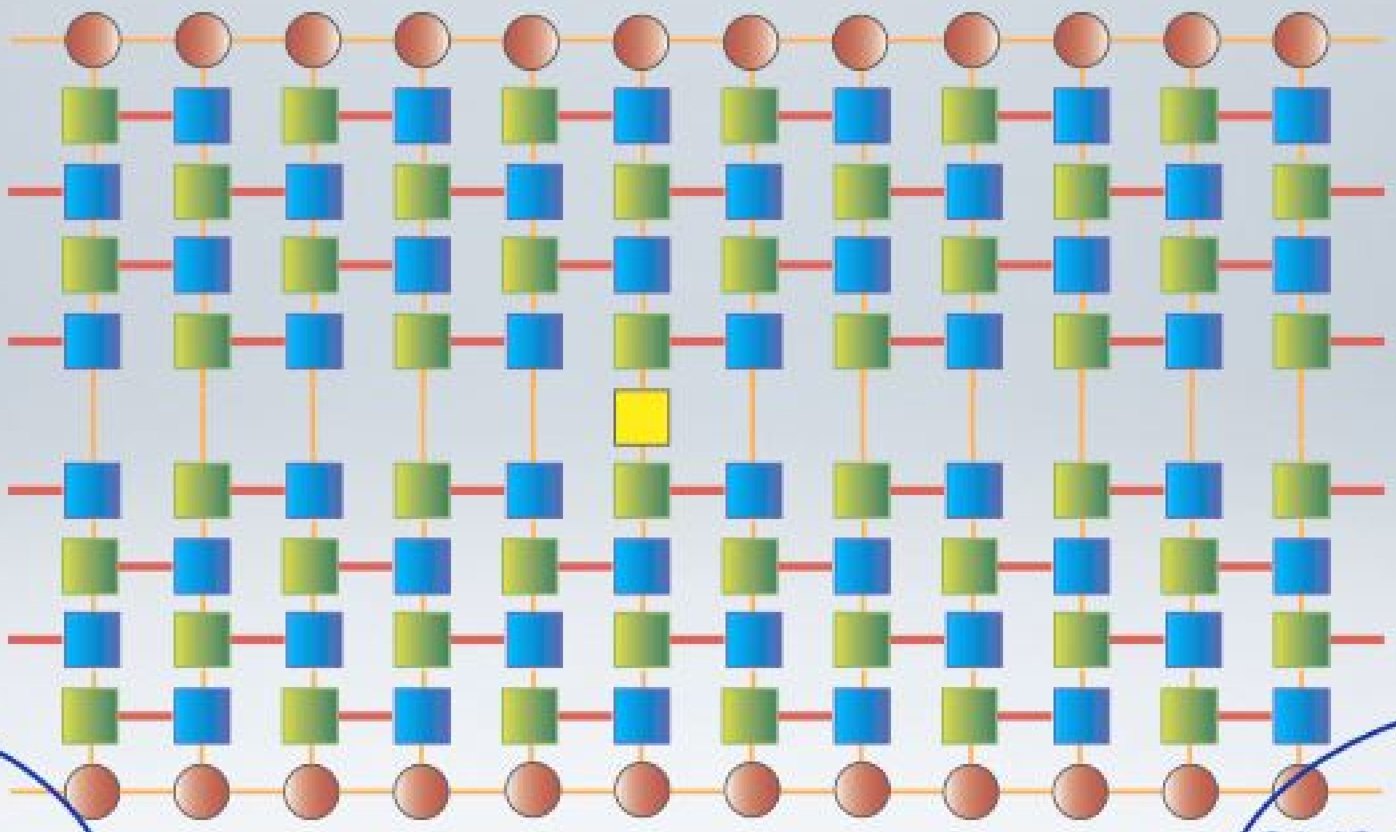


MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

ALTERNATIVELY

time dependent observables as TN

problem is contracting the network



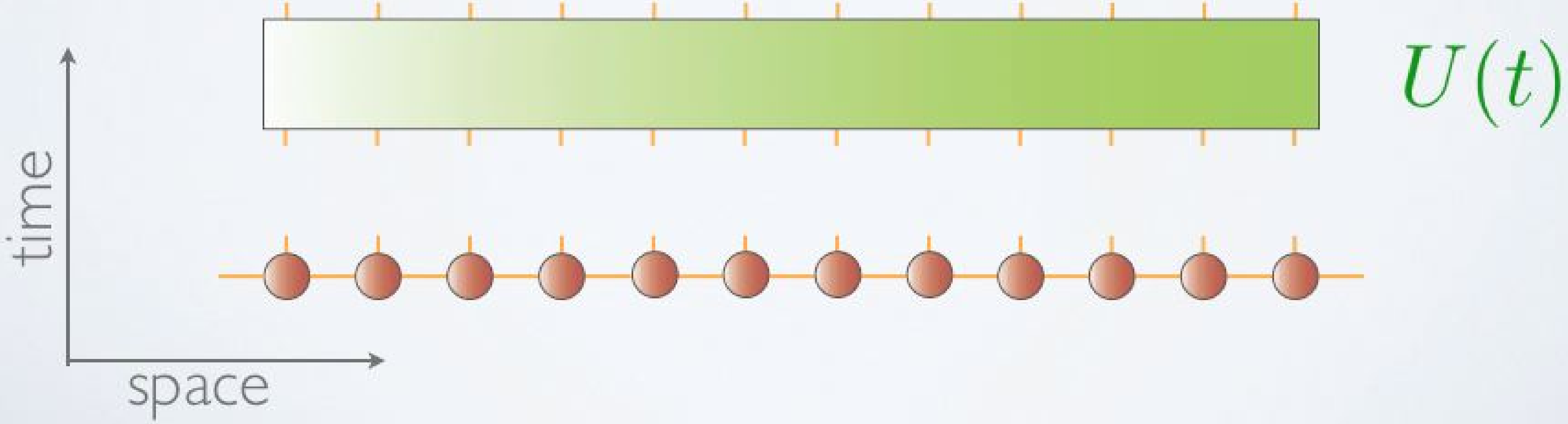
TN describe observables, not states

exact contraction not possible
#P complete

MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

OBSERVABLE AS TN

$\sim e^{-iHt}$



OBSERVABLE AS TN

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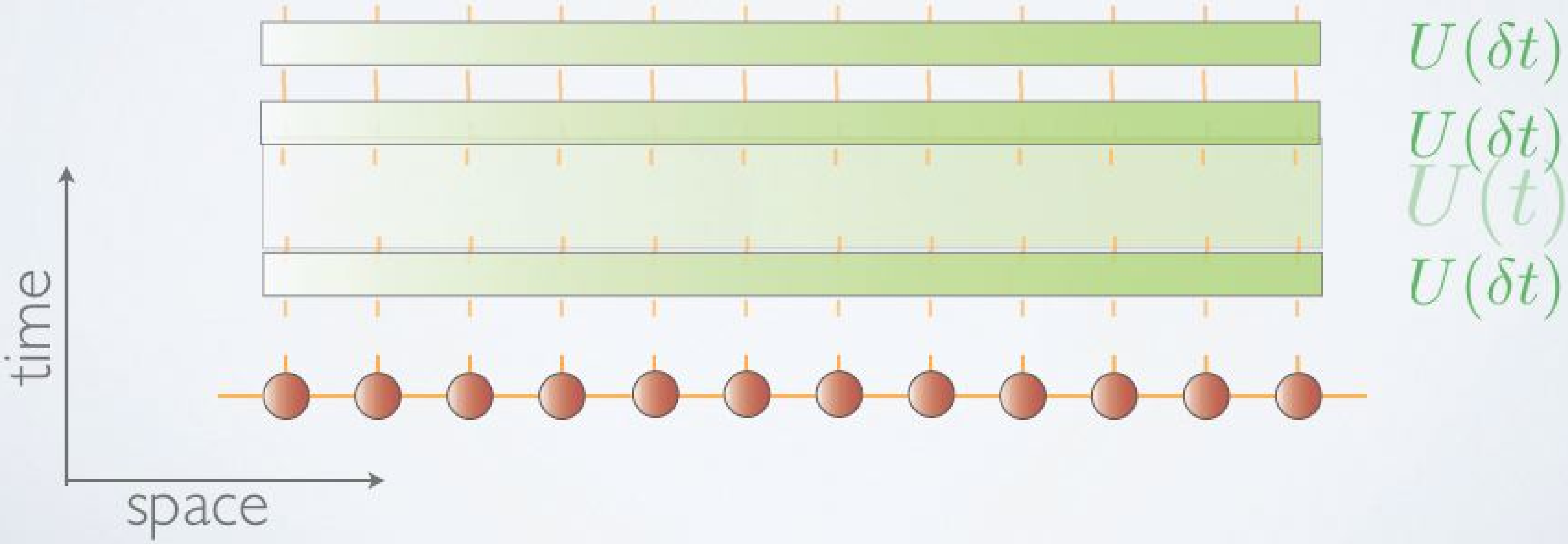

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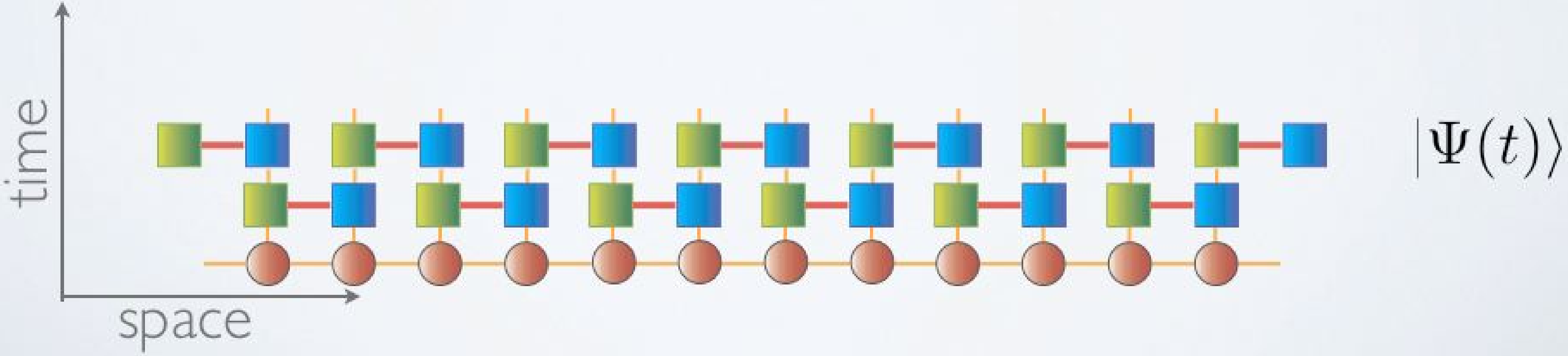
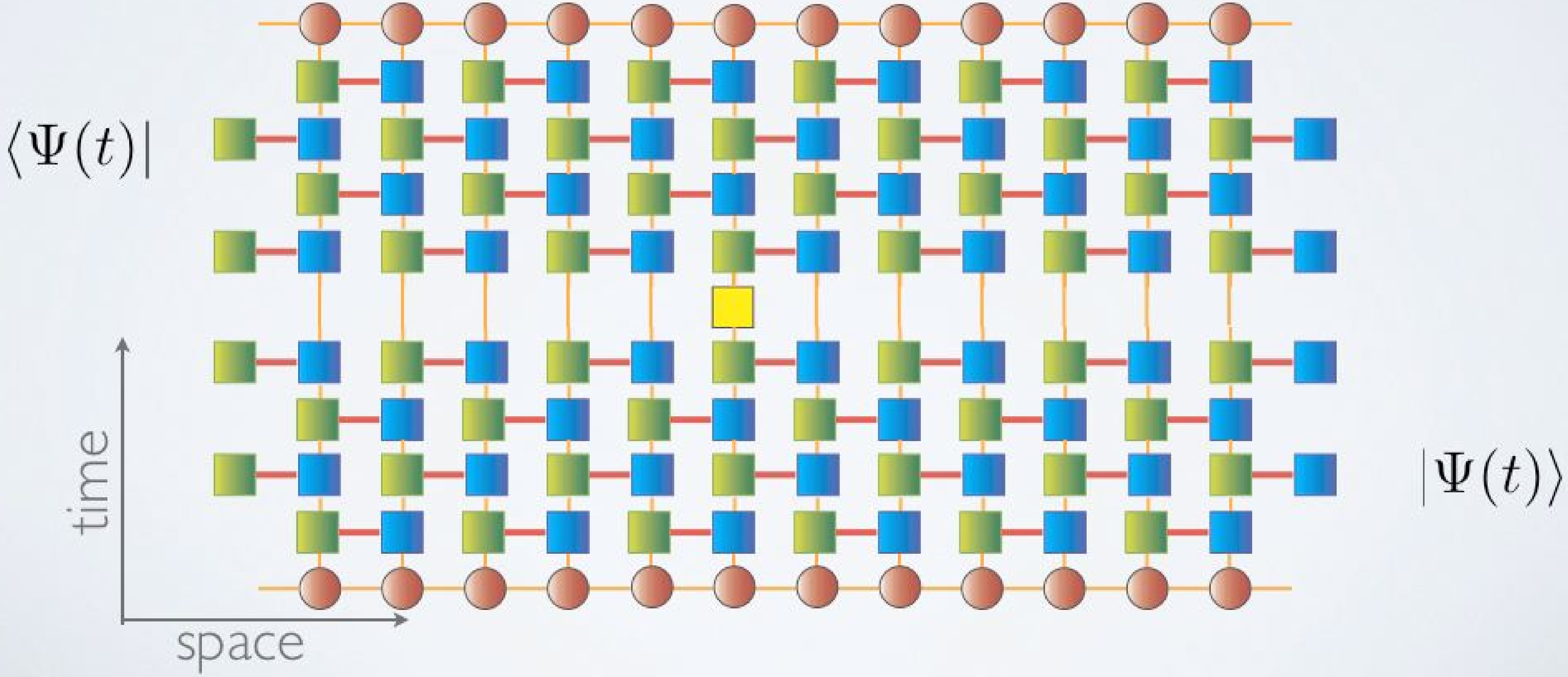


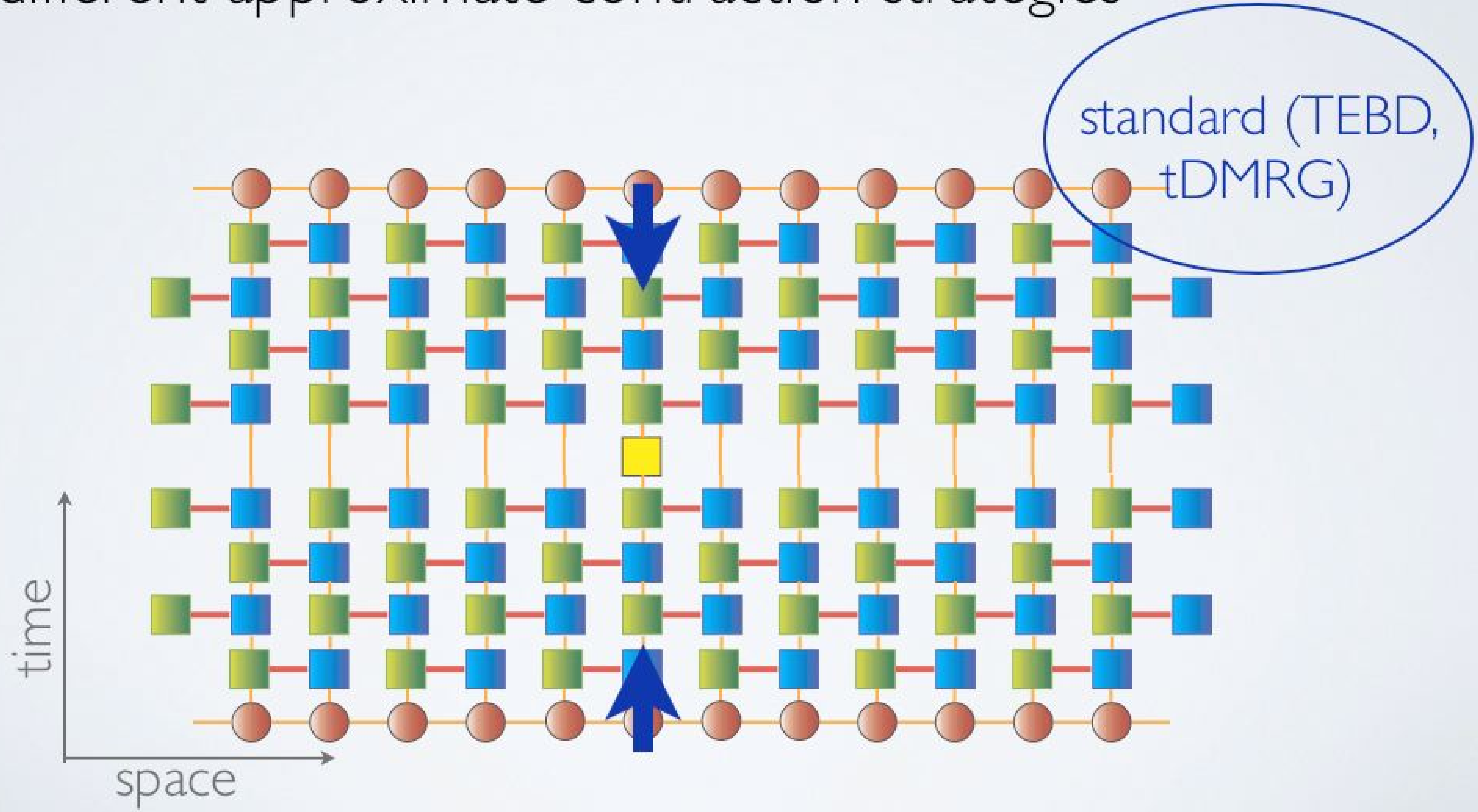
OBSERVABLE AS TN

$$\langle \Psi(t) | O | \Psi(t) \rangle$$



OBSERVABLE AS TN

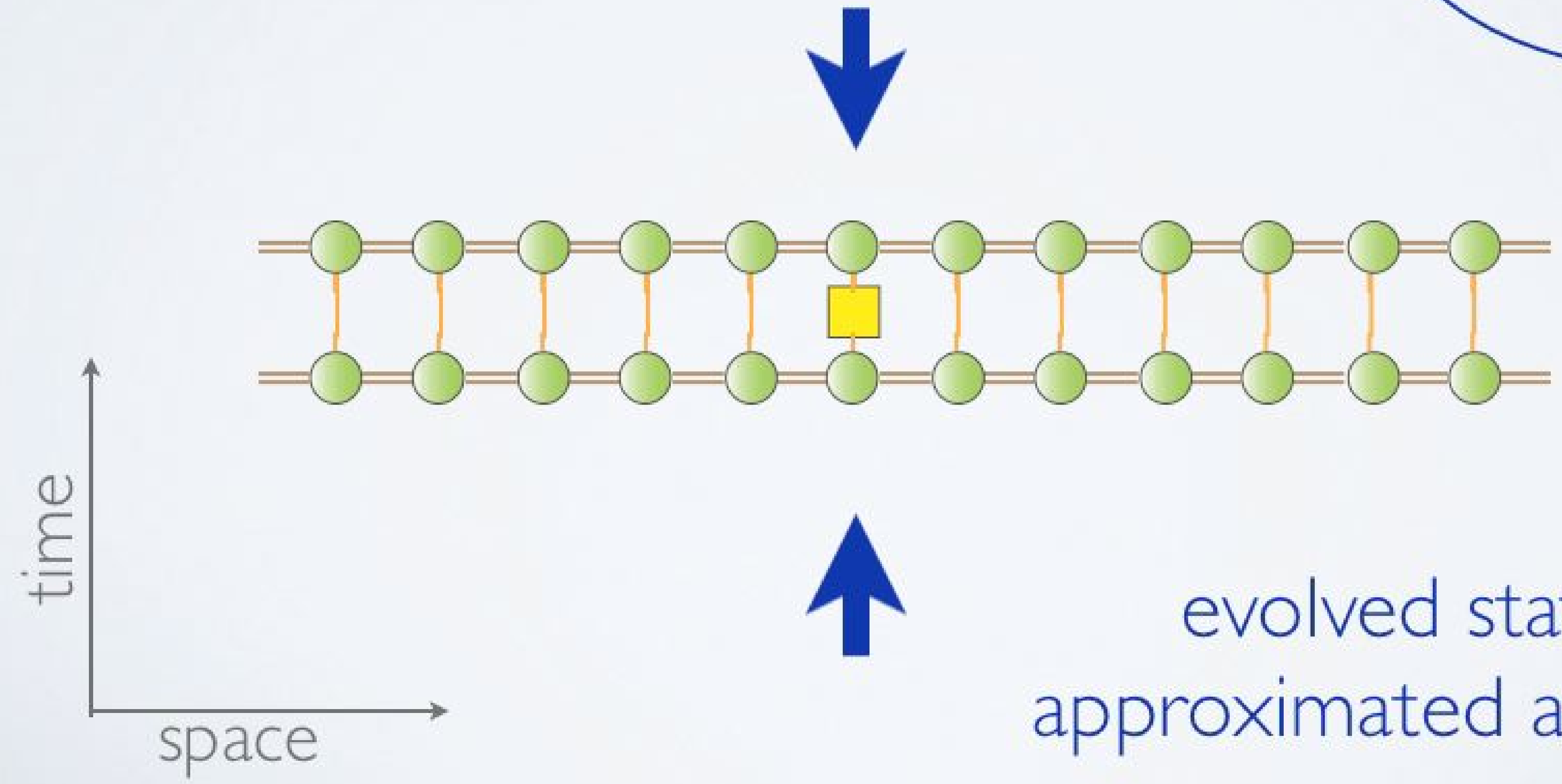
different approximate contraction strategies



OBSERVABLE AS TN

different approximate contraction strategies

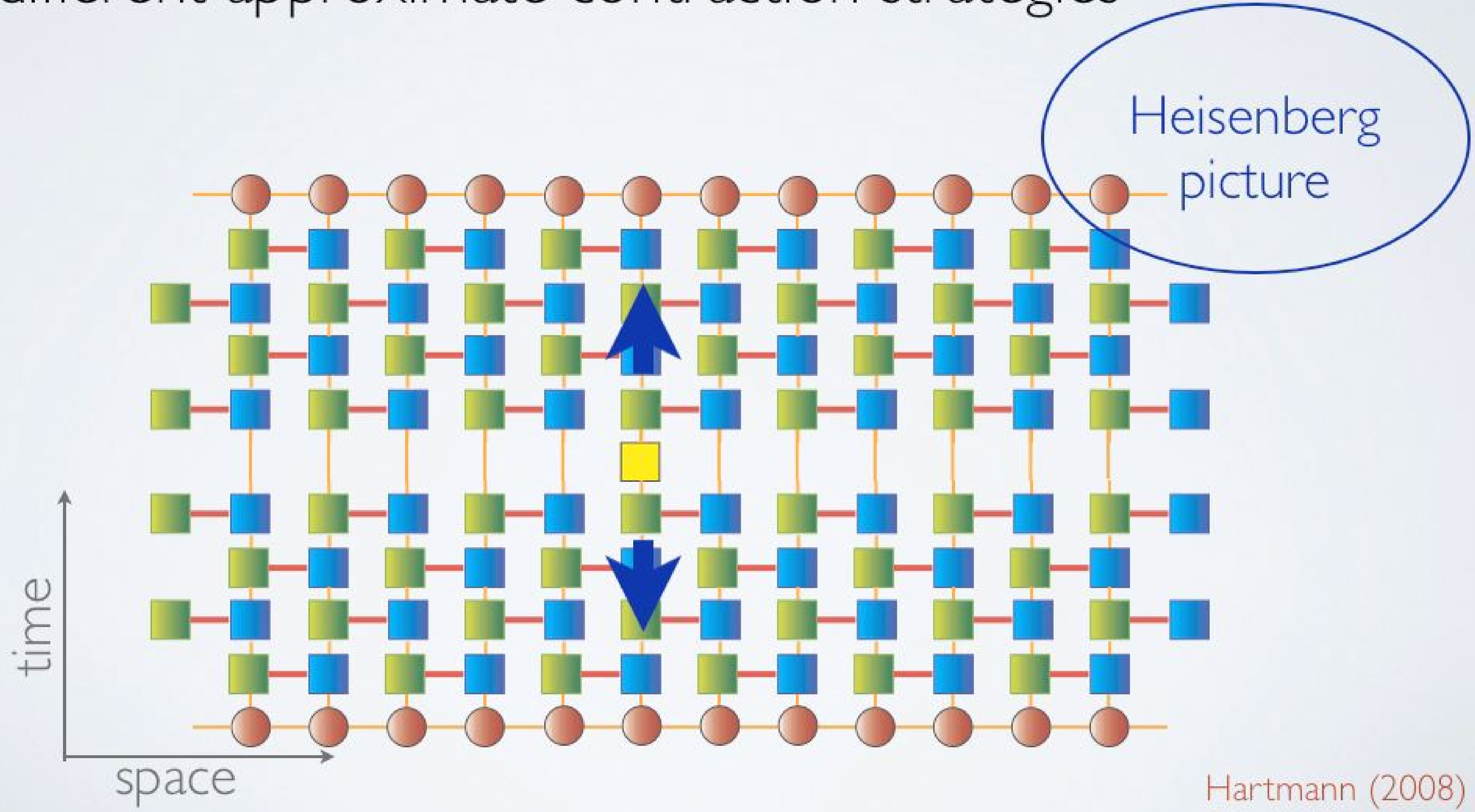
standard (TEBD, tDMRG)



evolved state approximated as MPS

OBSERVABLE AS TN

different approximate contraction strategies



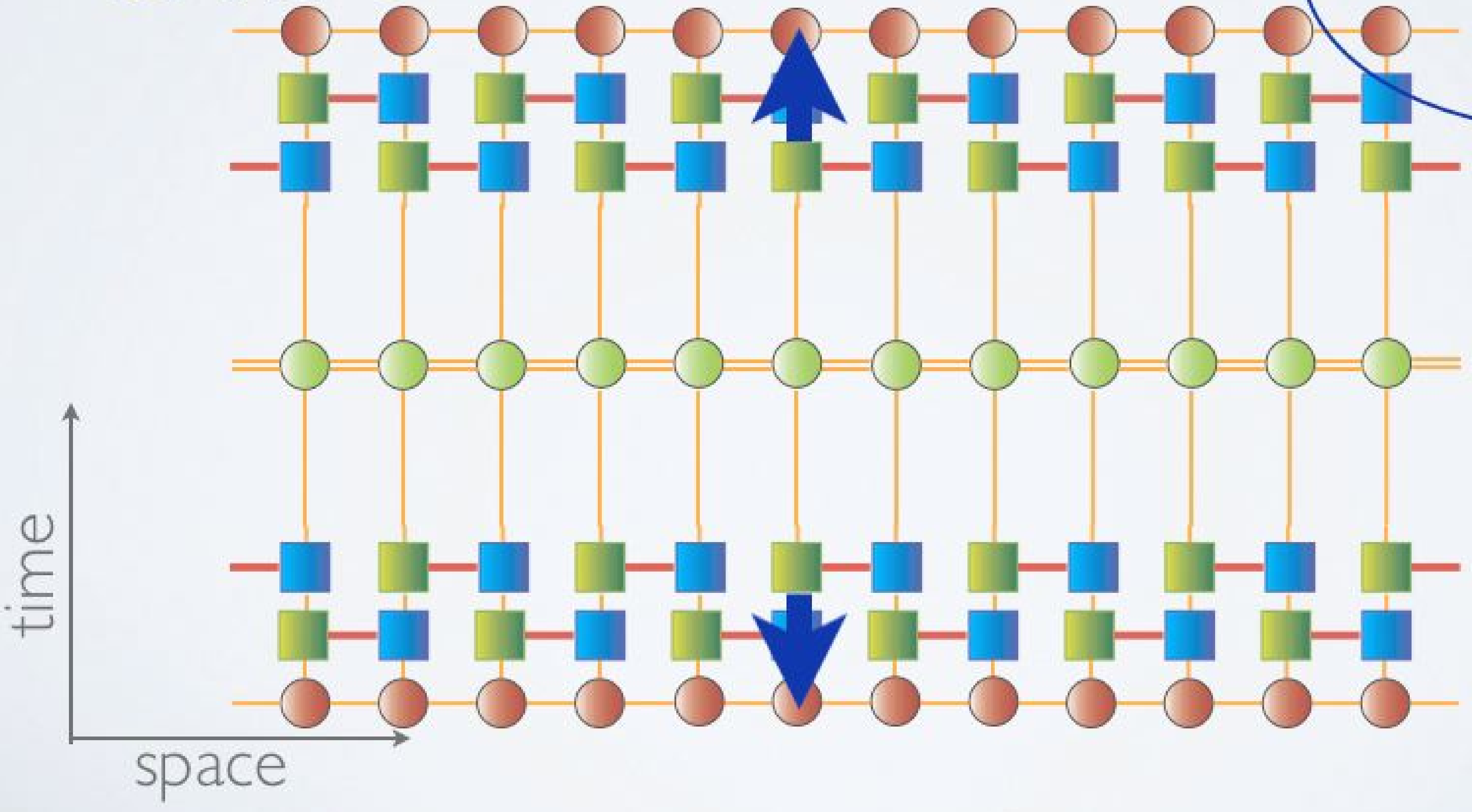
Hartmann (2008)

OBSERVABLE AS TN

different approximate contraction strategies

evolved operator as MPO

Heisenberg picture



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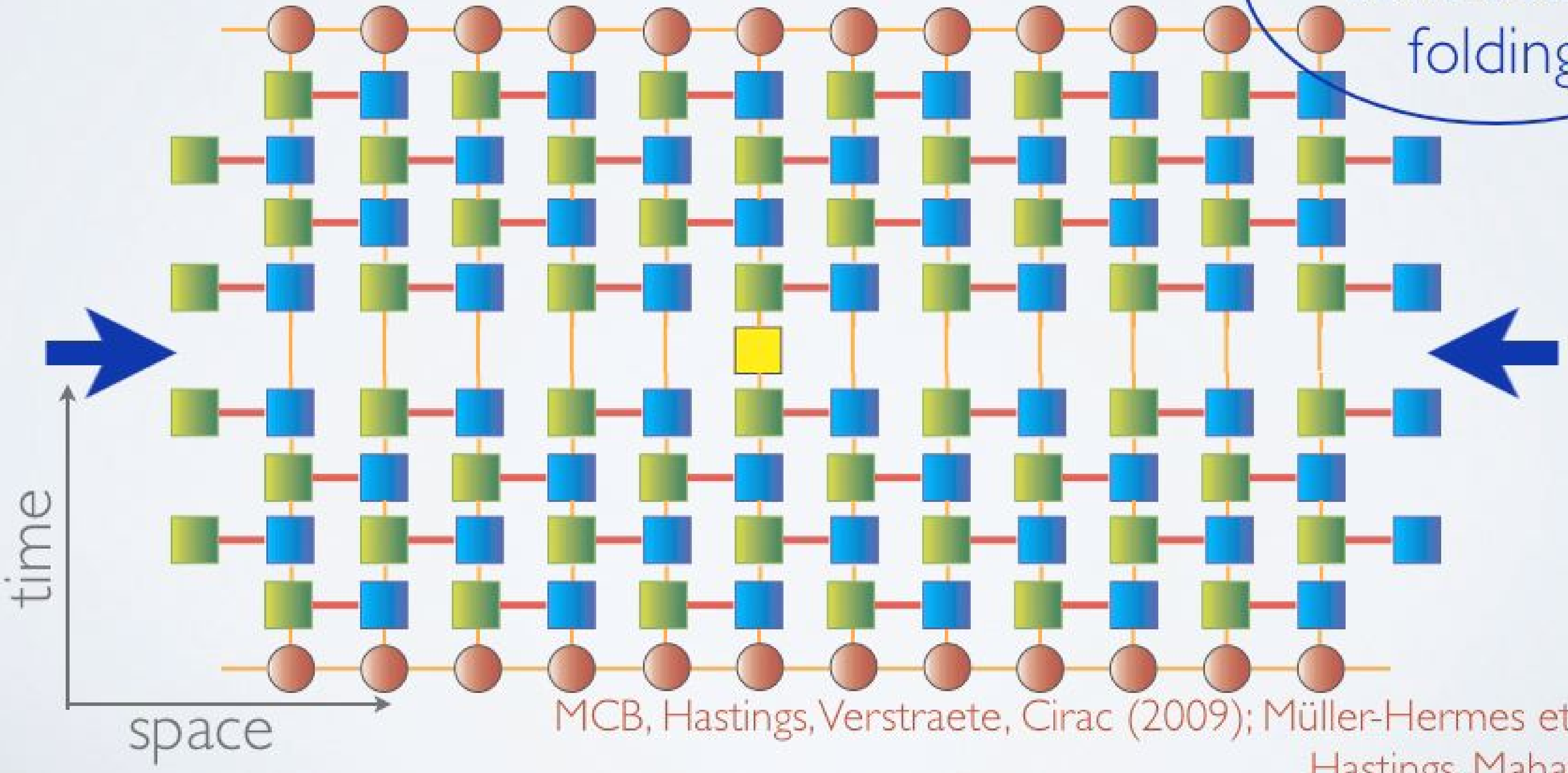
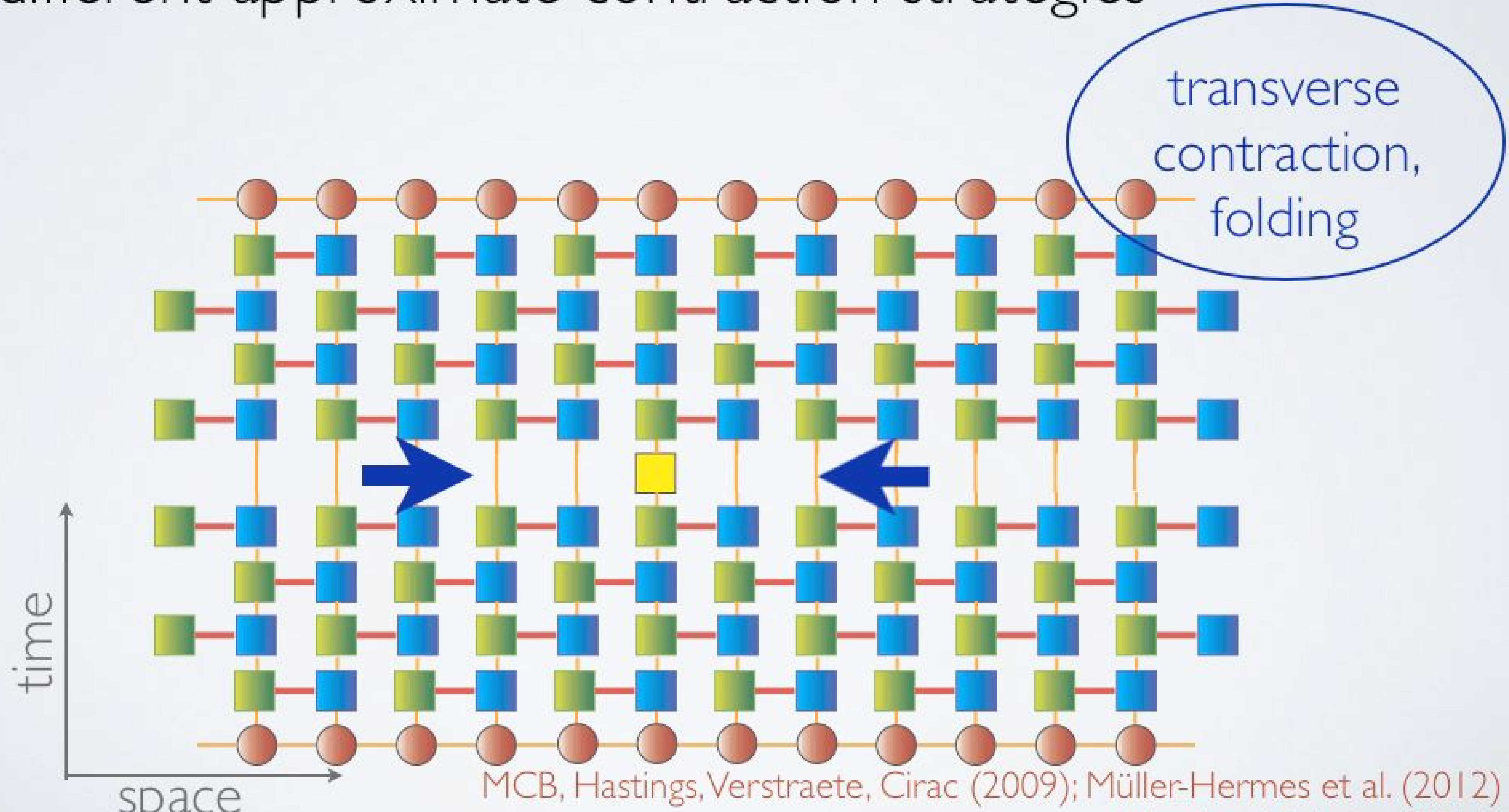


transverse contraction, folding

MCB, Hastings, Verstraete, Cirac (2009); Müller-Hermes et al. (2012)
Hastings, Mahajan (2014)

OBSERVABLE AS TN

different approximate contraction strategies



MCB, Hastings, Verstraete, Cirac (2009); Müller-Hermes et al. (2012)
Hastings, Mahajan (2014)

APPLICATION TO THERMALIZATION

Global quench scenario: closed quantum system initialized out of equilibrium

APPLICATION TO THERMALIZATION

thermalization of infinite quantum spin chain

MCB, Cirac, Hastings, PRL 2011

APPLICATION TO THERMALIZATION

thermalization of infinite quantum spin chain

fix non-integrable Hamiltonian

$$H = -\sum_i \left(\sigma_z^i \sigma_z^{i+1} + g\sigma_x^i + h\sigma_z^i\right)$$

varying initial state

MCB, Cirac, Hastings, PRL 2011

APPLICATION TO THERMALIZATION

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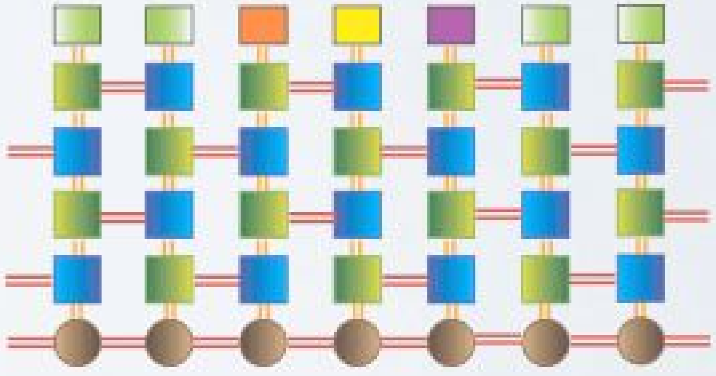
$$H = -\sum_i \left(\sigma_z^i \sigma_z^{i+1} + g\sigma_x^i + h\sigma_z^i\right)$$

varying initial state 

compute ρ_N for small number of sites



MCB, Cirac, Hastings, PRL 2011

APPLICATION TO THERMALIZATION

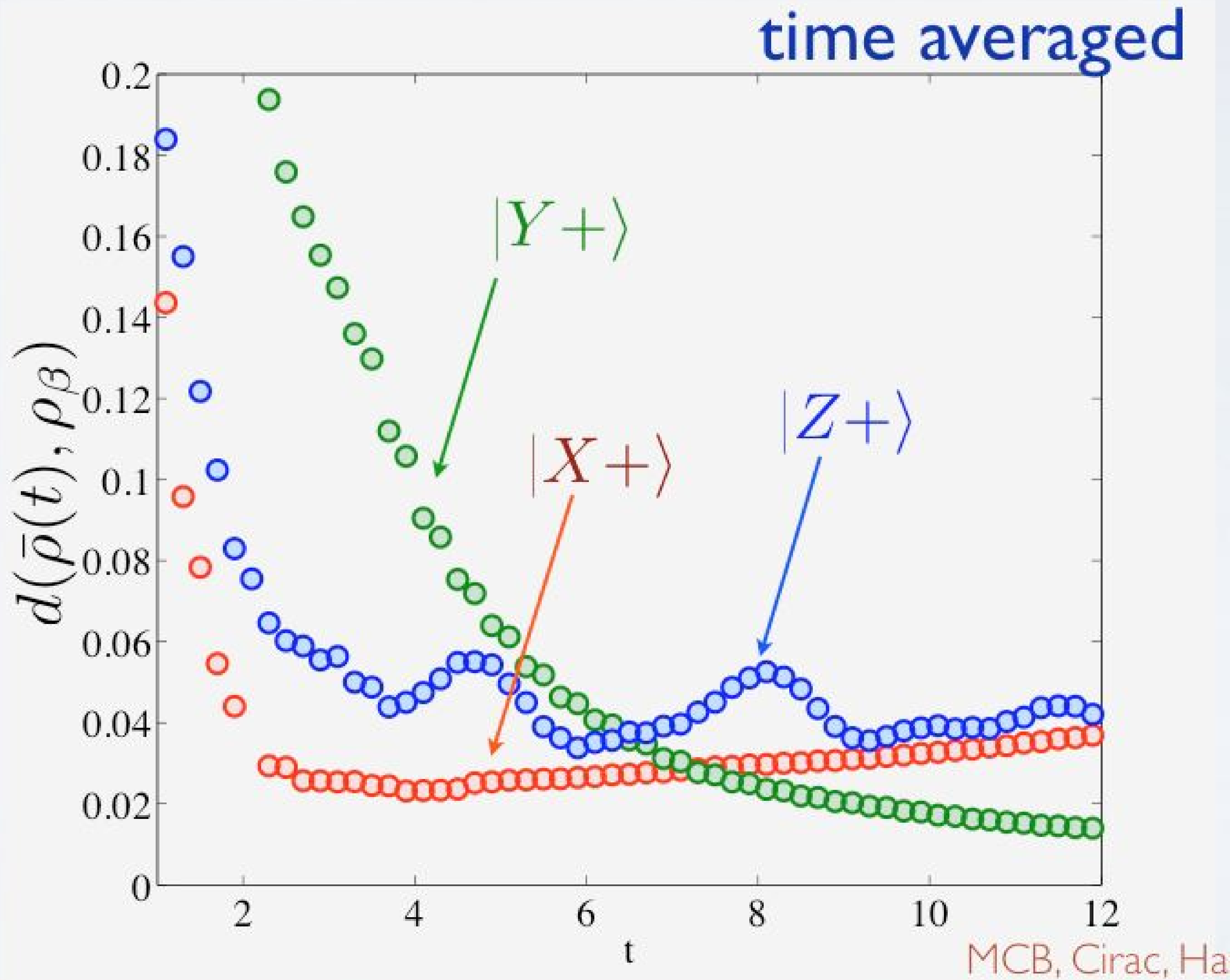
We observed different regimes of thermalization for the same Hamiltonian parameters

non-integrable regime

strong instantaneous state relaxes $|Y+\rangle$

MCB, Cirac, Hastings, PRL 2011

APPLICATION TO THERMALIZATION



MCB, Cirac, Hastings, PRL 2011
Hastings, Mahajan (2014)

different perspective

What are the slowest evolving (local) operators?

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

different perspective

What are the slowest evolving (local) operators?

$$\frac{dA(t)}{dt} = \boxed{i[H, A(t)]}$$

numerical study using ED and TNS

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

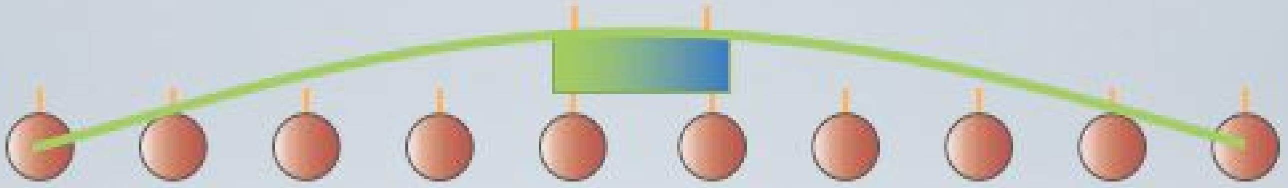
Scenario

1D non-integrable spin chain

$$H = \sum_i \left(\sigma_z^i \sigma_z^{i+1} + g\sigma_x^i + h\sigma_z^i \right)$$

only local conserved quantity is energy density

operator acting on M central sites



slow operator: inhomogeneity of energy density

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

Goal: minimizing $\|[H, A_M]\|$

$$\left\| \frac{dA_M}{dt} \right\| = \|[A_M, H]\|$$

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

Goal: minimizing $\|[H, A_M]\|$

$$\|A_M(t) - A_M(0)\| \leq \chi(M)t$$

$$\|A\|_2 = \sqrt{\mathrm{tr}(A^\dagger A)}$$ Frobenius norm

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

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$$\lambda_M = \min_{A_M} \frac{\|[A_M, H]\|_2^2}{\|A_M\|_2^2}$$

physical meaning

$$\rho \sim I + \epsilon A_M$$ high T state

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

Look for the A_M with smallest $\|[H, A_M]\|_2$

$$H = \sum_i \left(\sigma_z^i \sigma_z^{i+1} + g\sigma_x^i + h\sigma_z^i\right)$$

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

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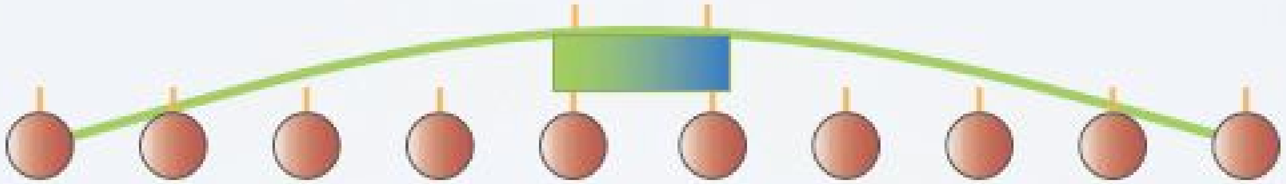
$$H = \sum_i \left(\sigma_z^i \sigma_z^{i+1} + g\sigma_x^i + h\sigma_z^i\right)$$

energy density fluctuation ⇒ diffusive mode

$$E_M = \sum_{n=-M/2}^{M/2} c_n h_{n,n+1}$$



$$c_n \sim \cos\frac{\pi n}{M}$$

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

Results

there are operators slower than energy diffusion for a non-integrable system

I. exhaustive numerical search

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

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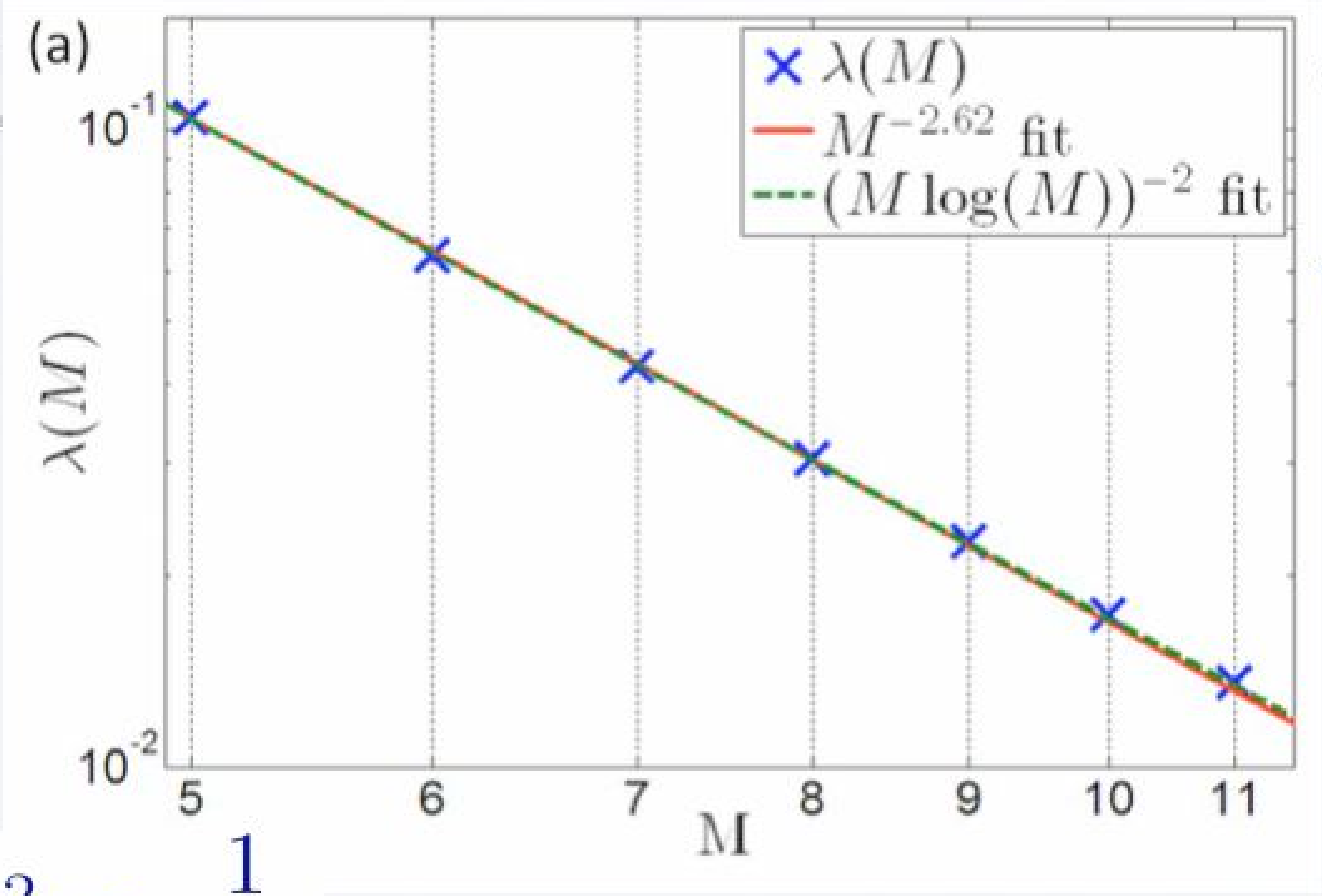


nitian operator

(Pauli)

$$\|[H,O_M]\|_2^2 \leq \frac{1}{M^\alpha}$$

$$\alpha > 2$$

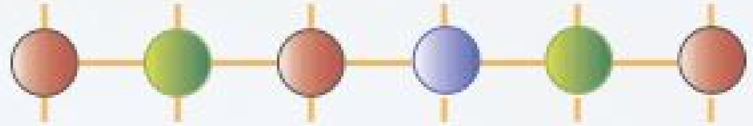
Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

Results

there are operators slower than energy diffusion for a non-integrable system

I. exhaustive numerical search

II. MPO ansatz

$O_M =$ 

variational search over MPO with bond dimension D

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

Results

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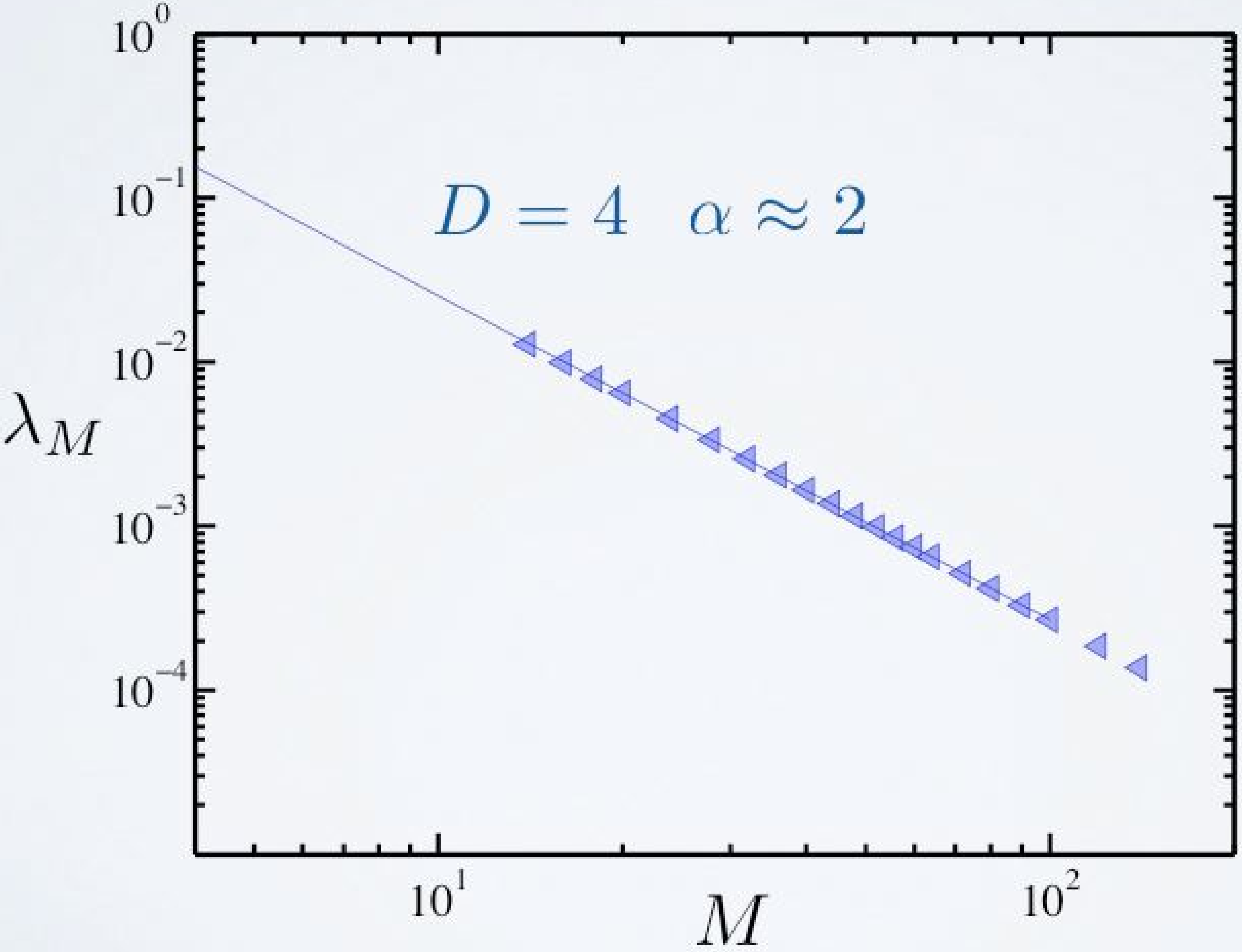

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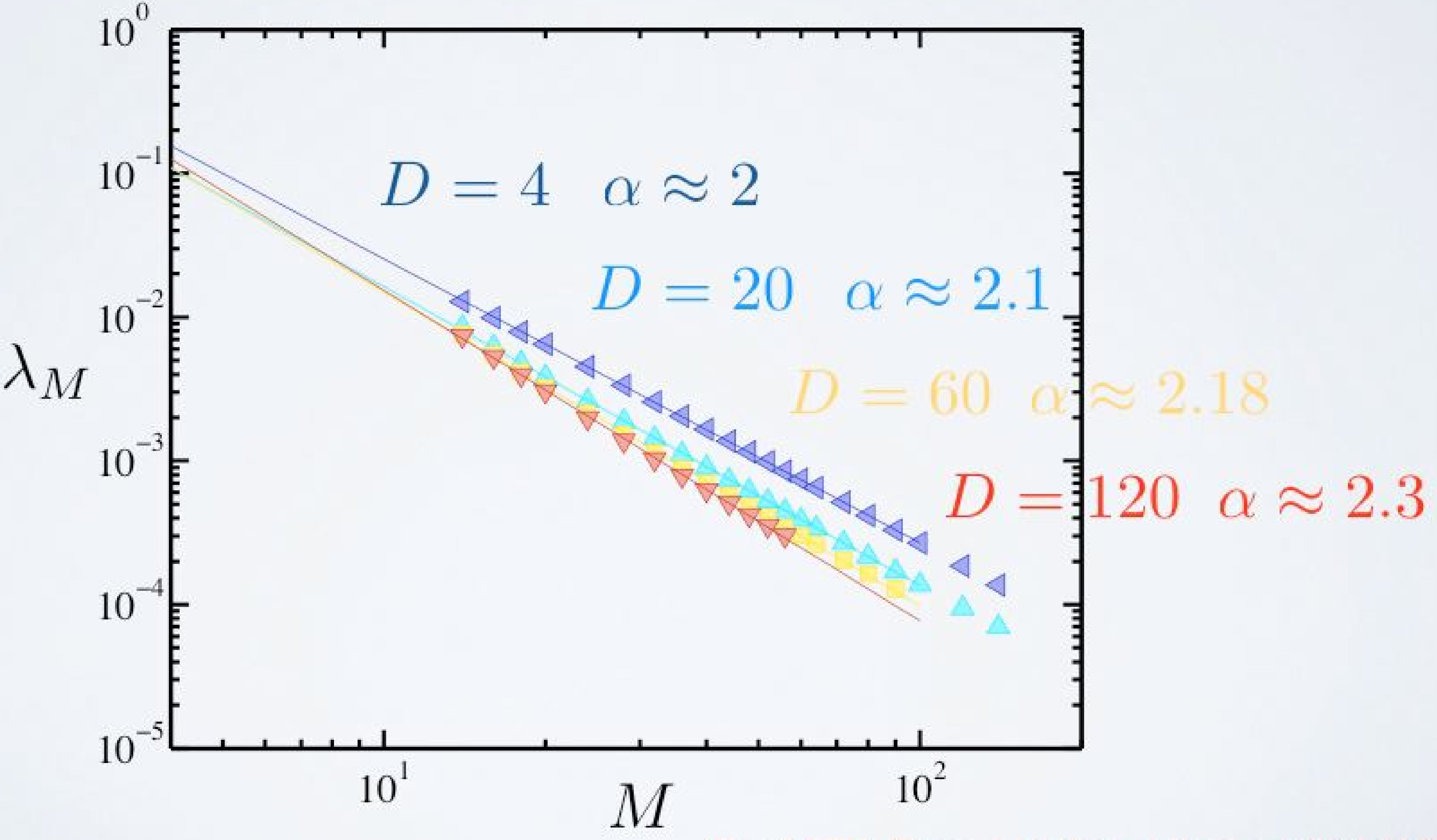
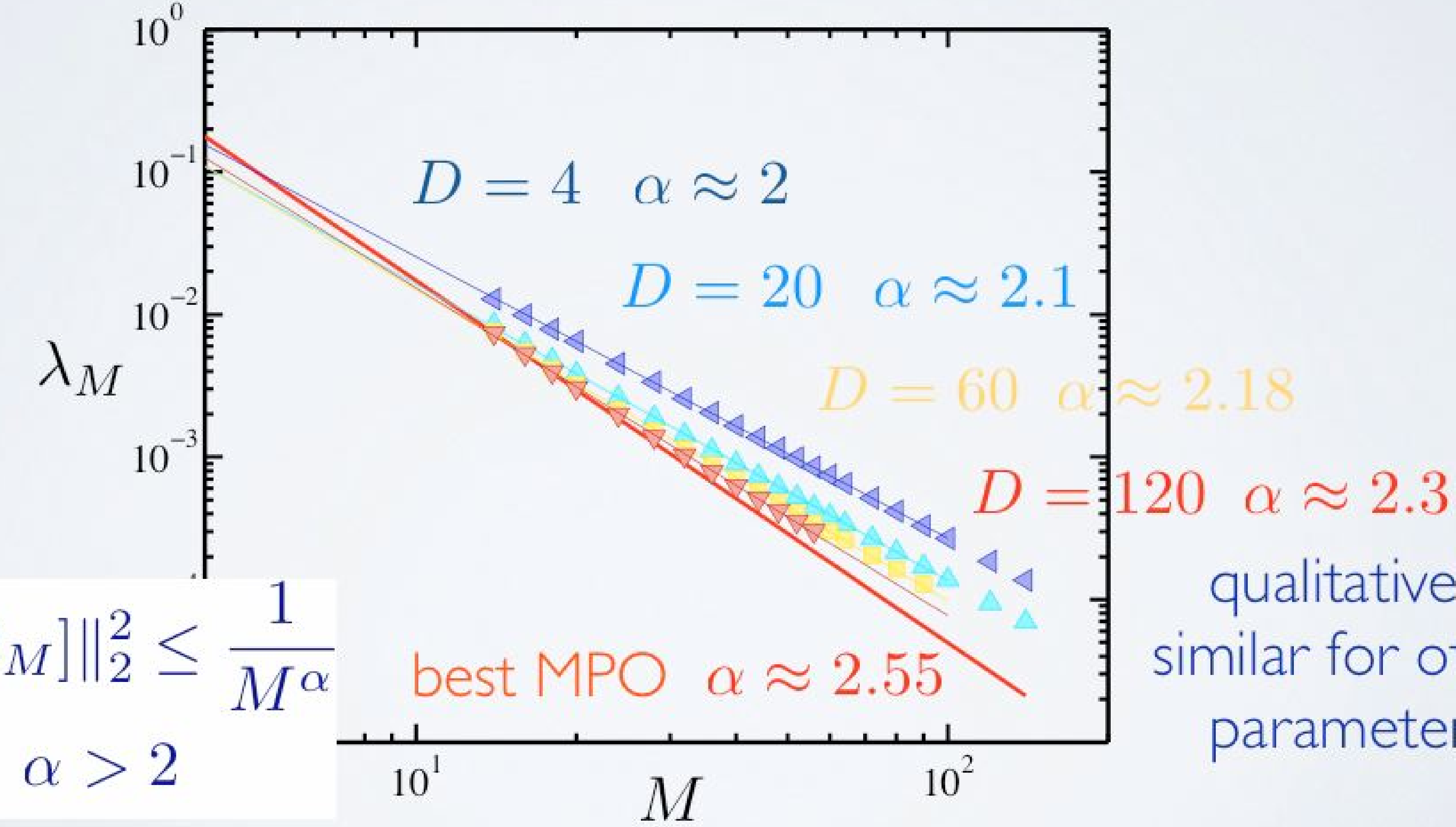
there are MPO operators slower than energy diffusion for a non-integrable system



Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

Results

there are MPO operators slower than energy diffusion for a non-integrable system



$$\|[H, O_M]\|_2^2 \leq \frac{1}{M^\alpha}$$

$$\alpha > 2$$

qualitatively similar for other parameters

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

Results

there are operators slower than energy diffusion for a non-integrable system

$$H = \sum_i \left(\sigma_z^i \sigma_z^{i+1} + g\sigma_x^i + h\sigma_z^i \right)$$

look at system with no conserved local energy density: Floquet

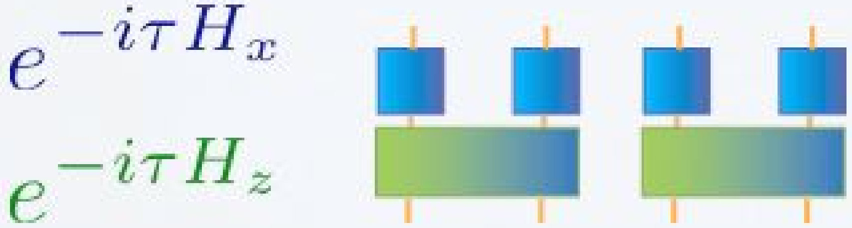
Results

there are operators slower than energy diffusion for a non-integrable system

$$H = \sum_i \left(\sigma_z^i \sigma_z^{i+1} + \boxed{g\sigma_x^i} + h\sigma_z^i\right)$$

look at system with no conserved local energy density: Floquet

$$U = \boxed{e^{-i\tau H_x}} e^{-i\tau H_z}$$



Strategy: minimizing

$$\lambda_M = \min_{A_M} \frac{\|[A_M, U]\|_2^2}{\|A_M\|_2^2}$$

similar physical meaning

$$\rho \sim I + \epsilon A_M$$

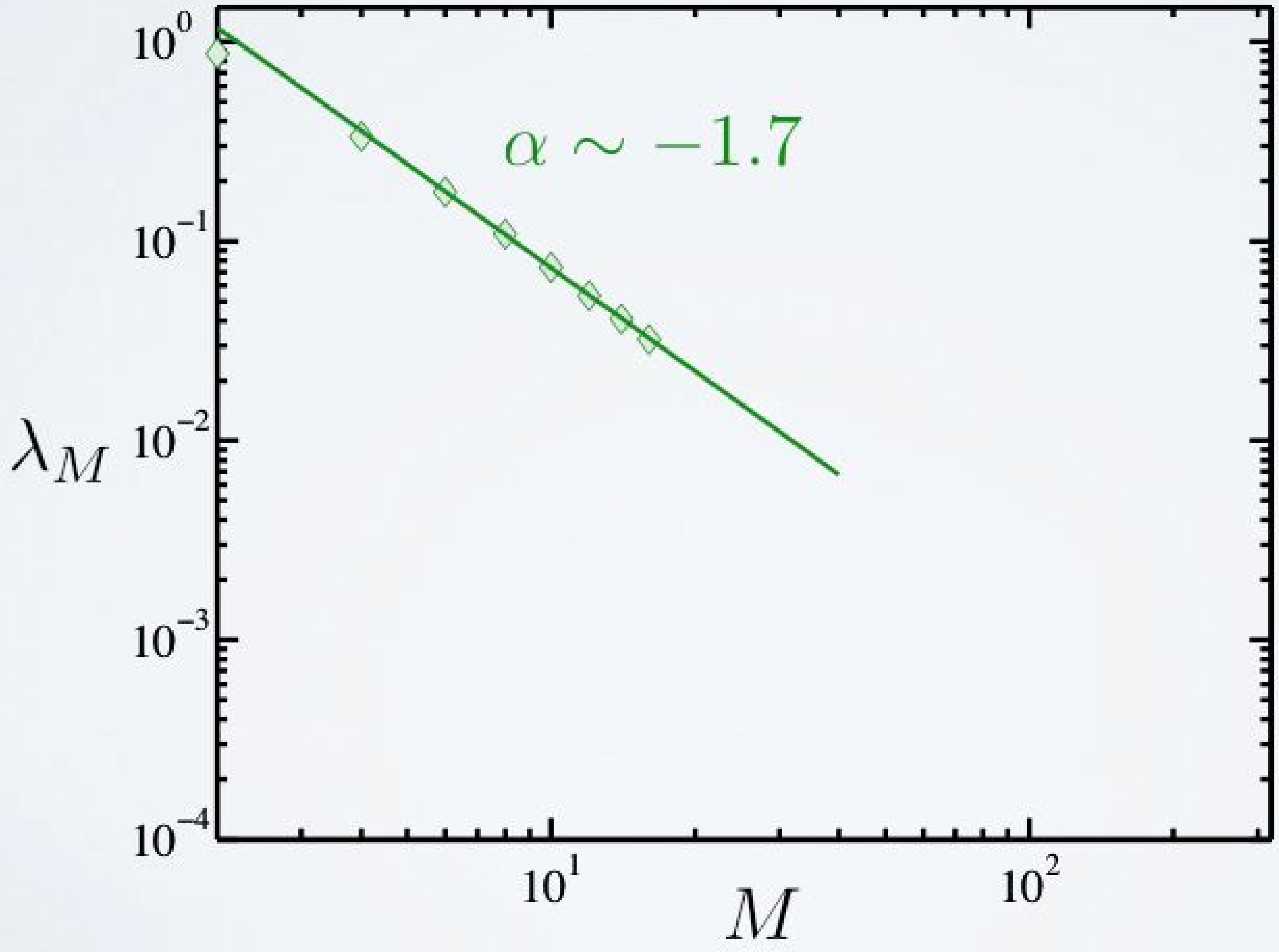
bounds also thermalization time (step number)

$$\left|\langle A_M^{(N)}\rangle - \langle A_M\rangle_\beta\right| \geq 1 - N\sqrt{\lambda_M}$$

$$N_{th} \geq \frac{1}{\sqrt{\lambda_M}}$$

Results

I. exhaustive numerical search for $M<12$



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$$A_M = \sum_{n=-M}^{M} c_n U^n O_0 U^{-n}$$

filtered operator

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variational parameters

single site traceless operator

minimization requires computing $\mathrm{tr}\left(A_M U A_M U^\dagger\right)$

Results

I. exhaustive numerical search for $M<12$

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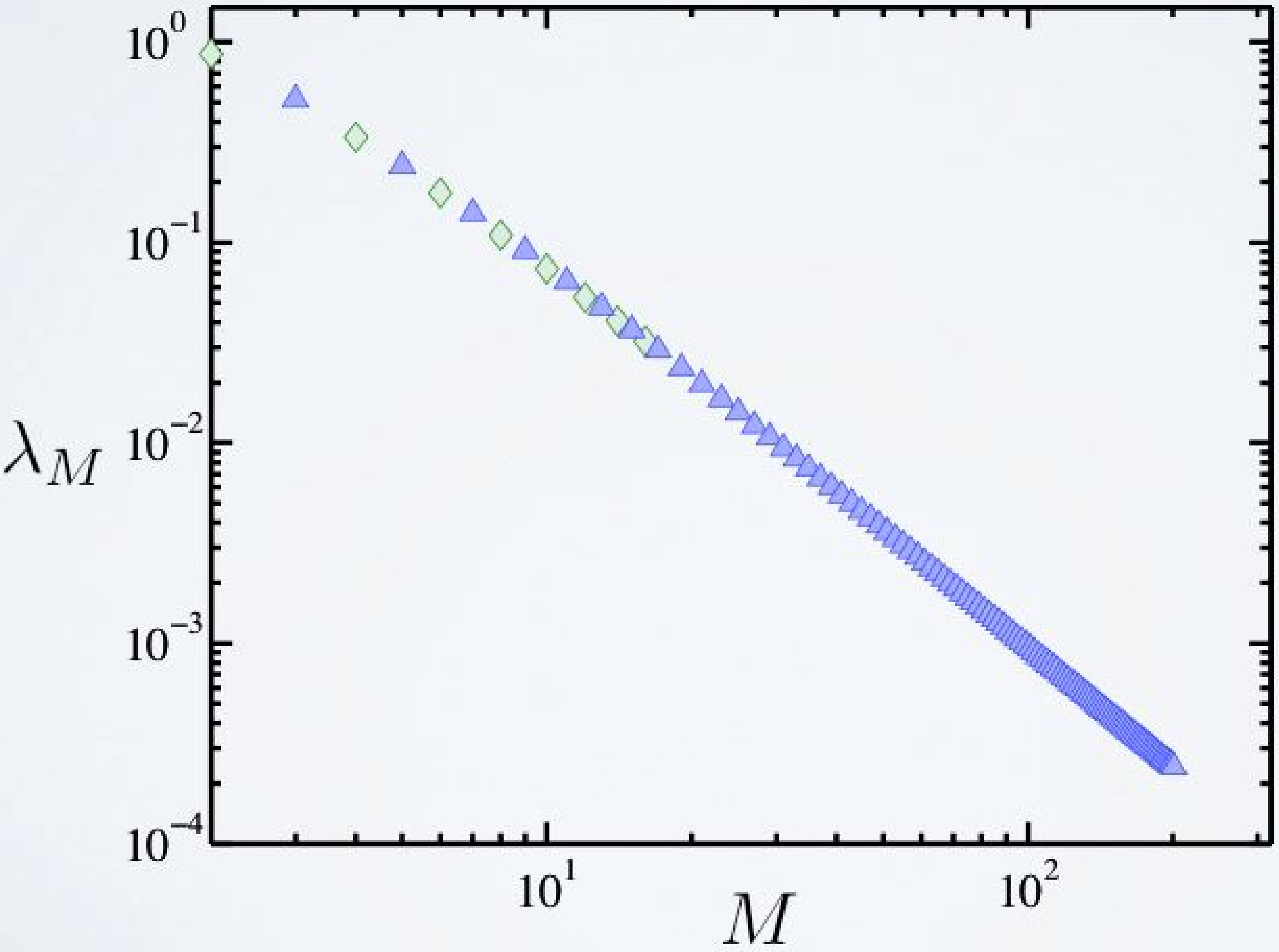

Results

I. exhaustive numerical search for ***M<12***

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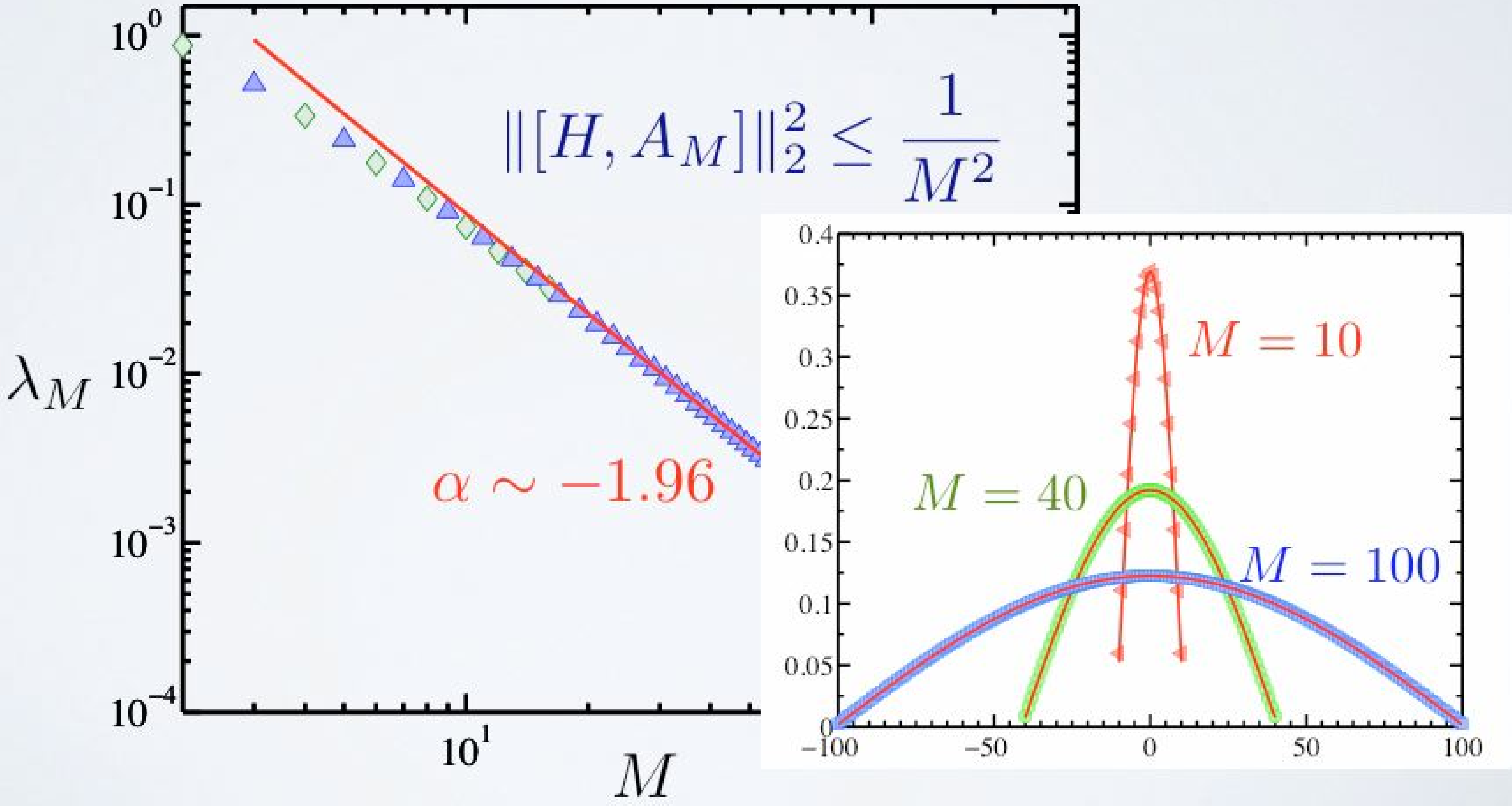
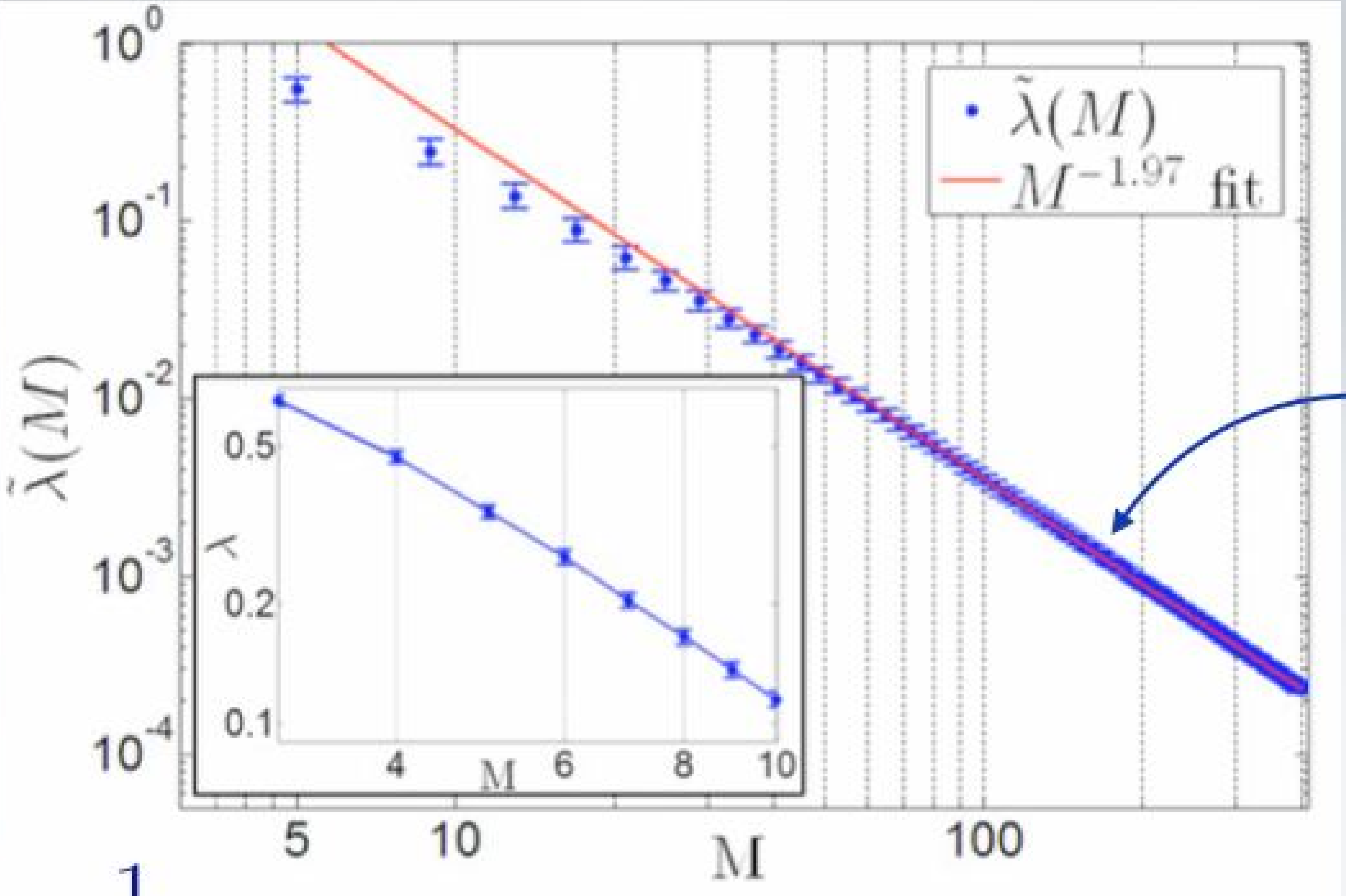


Results

more generic system without conserved local energy: random quantum circuit

only locality of evolution

but operators with even smaller commutator exist



explicit construction

$$\|[H, A_M]\|_2^2 \leq \frac{1}{M^2}$$

Kim, MCB, Cirac, Hastings, Huse, arXiv:1410.4186

MANY BODY LOCALIZATION

Anderson localization: single particle states localized due to disorder

environment destroys localization

Interactions and disorder more interesting scenario

weak interactions ⇒ MBL phase

Basko, Aleiner, Altshuler, Ann. Phys. 2006
Gornyi, Mirlin, Polyakov, PRL 2005

MANY BODY LOCALIZATION

Existing numerical studies use exact diagonalization to access full spectrum/very long times

limited to small systems

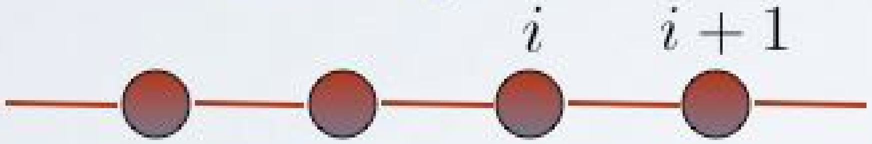
TN techniques to reach larger system sizes

long time evolution of mixed states

collaboration in progress with N. Yao, M. Lukin

TNS/QI & MBL

$$H = \sum \left(S_x^{[i]} S_x^{[i+1]} + S_y^{[i]} S_y^{[i+1]} + J S_z^{[i]} S_z^{[i+1]} + h_i S_i^z \right)$$



Oganesyan, Huse, PRB 2007
Pal, Huse, PRB 2010
Luitz et al., 2014

at $J=0$ non-interacting XY localized for $h>0$

at $J=1$ shows MBL for $h\sim3\text{-}3.5$

TNS/QI & MBL

(Q)INFORMATION AND MBL

(Q)INFORMATION AND MBL

known scenario: global quenches for pure states

single particle localization saturation of S
logarithmic growth of S

Bardarsson, Pollmann, Moore, PRL 2012

(Q)INFORMATION AND MBL

known scenario: global quenches for pure states

single particle localization saturation of S

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Bardarsson, Pollmann, Moore, PRL 2012

$$I(A:B) = S(A) + S(B) - S(AB)$$

measures correlations between subsystems

(Q)INFORMATION AND MBL

$$\rho_\Phi \propto Id^{\otimes \frac{L-L_c}{2}} \otimes |\Phi\rangle\langle\Phi| \otimes Id^{\otimes \frac{L-L_c}{2}}$$

$|0\rangle$

can be used to encode a qubit

(Q)INFORMATION AND MBL

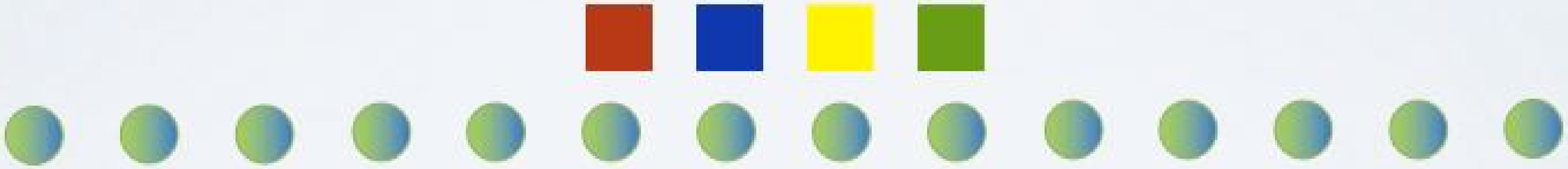
$$\rho_\Phi \propto Id^{\otimes \frac{L-L_c}{2}} \otimes |\Phi\rangle\langle\Phi| \otimes Id^{\otimes \frac{L-L_c}{2}}$$

$|1\rangle$

can be used to encode a qubit

$$I(\text{central } L_c \text{ sites} : \text{edges})$$

RECOVERY FIDELITY



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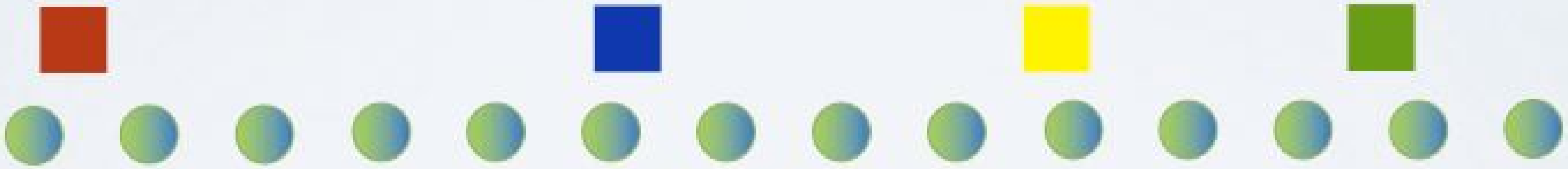

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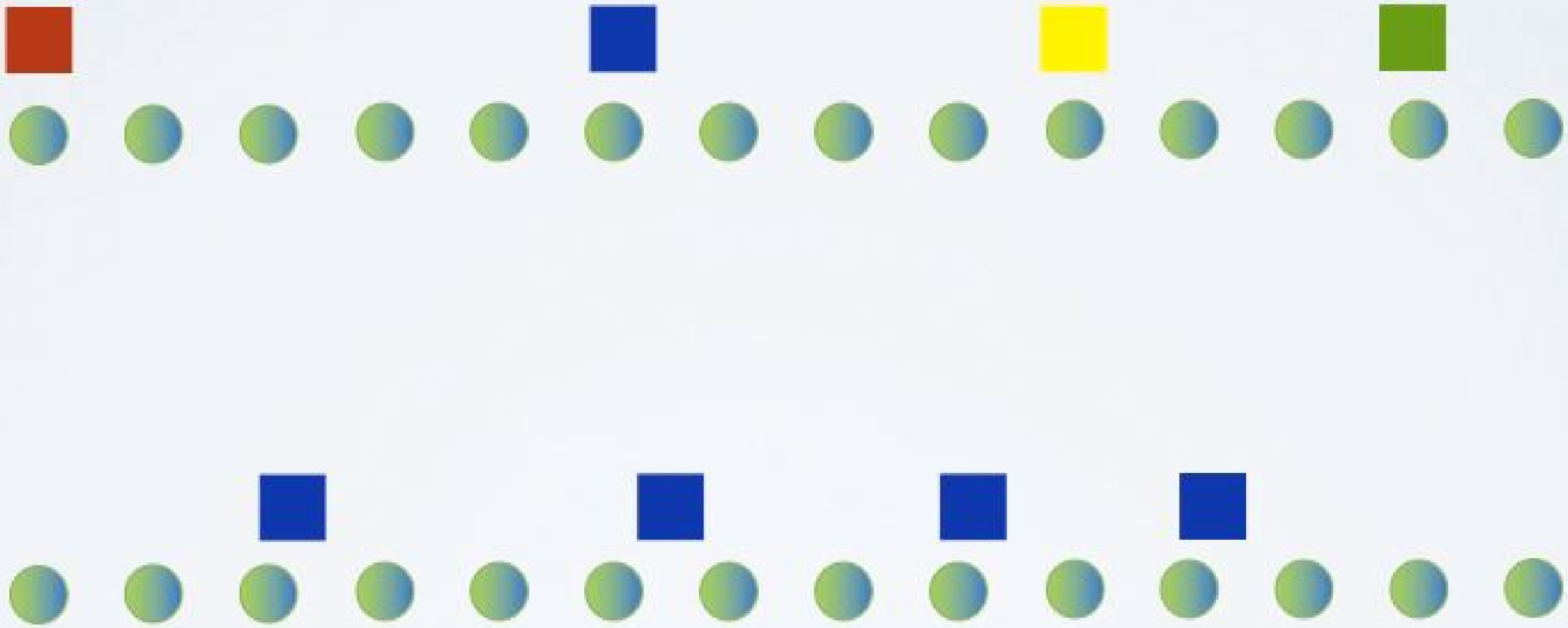

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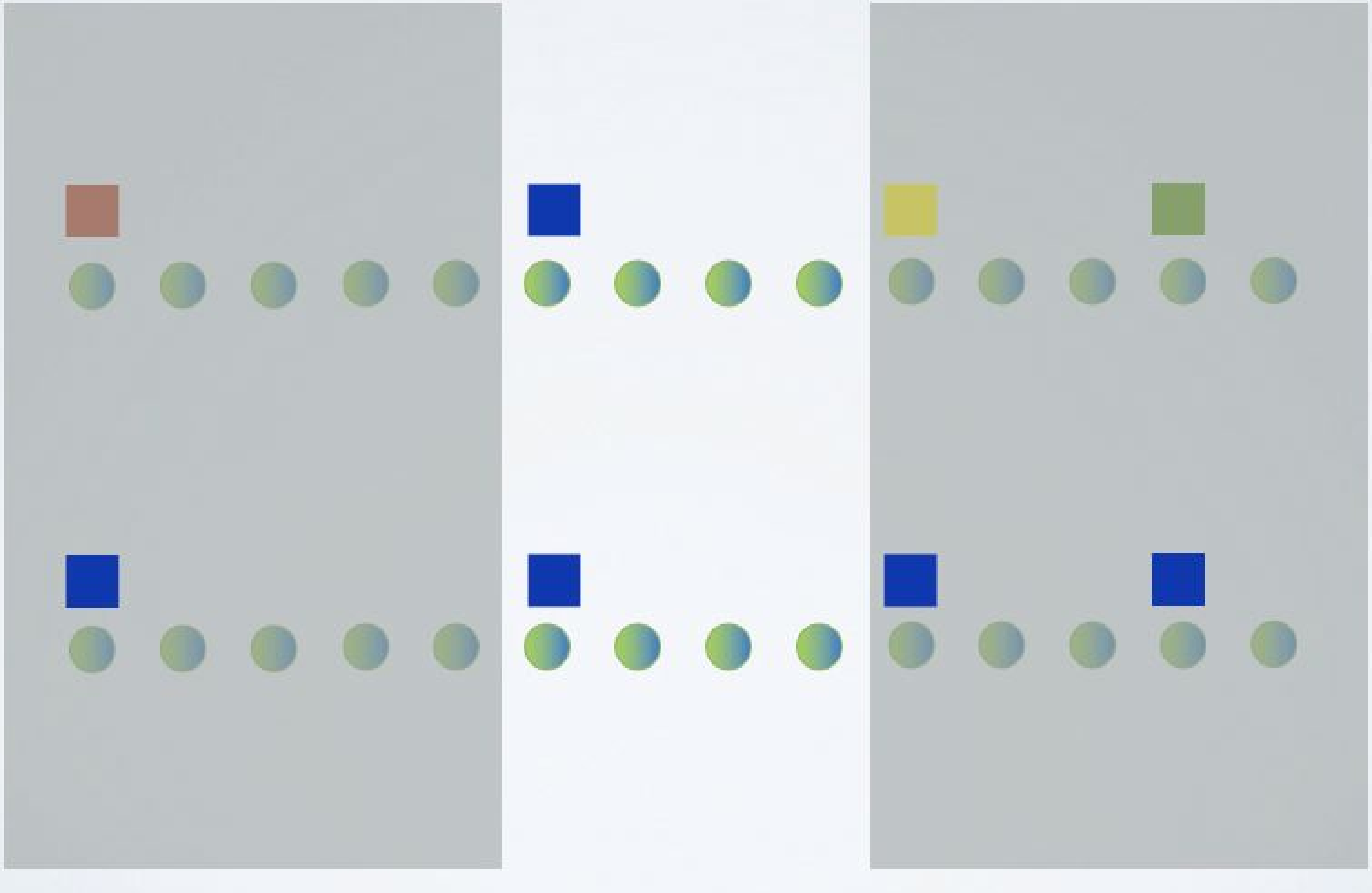
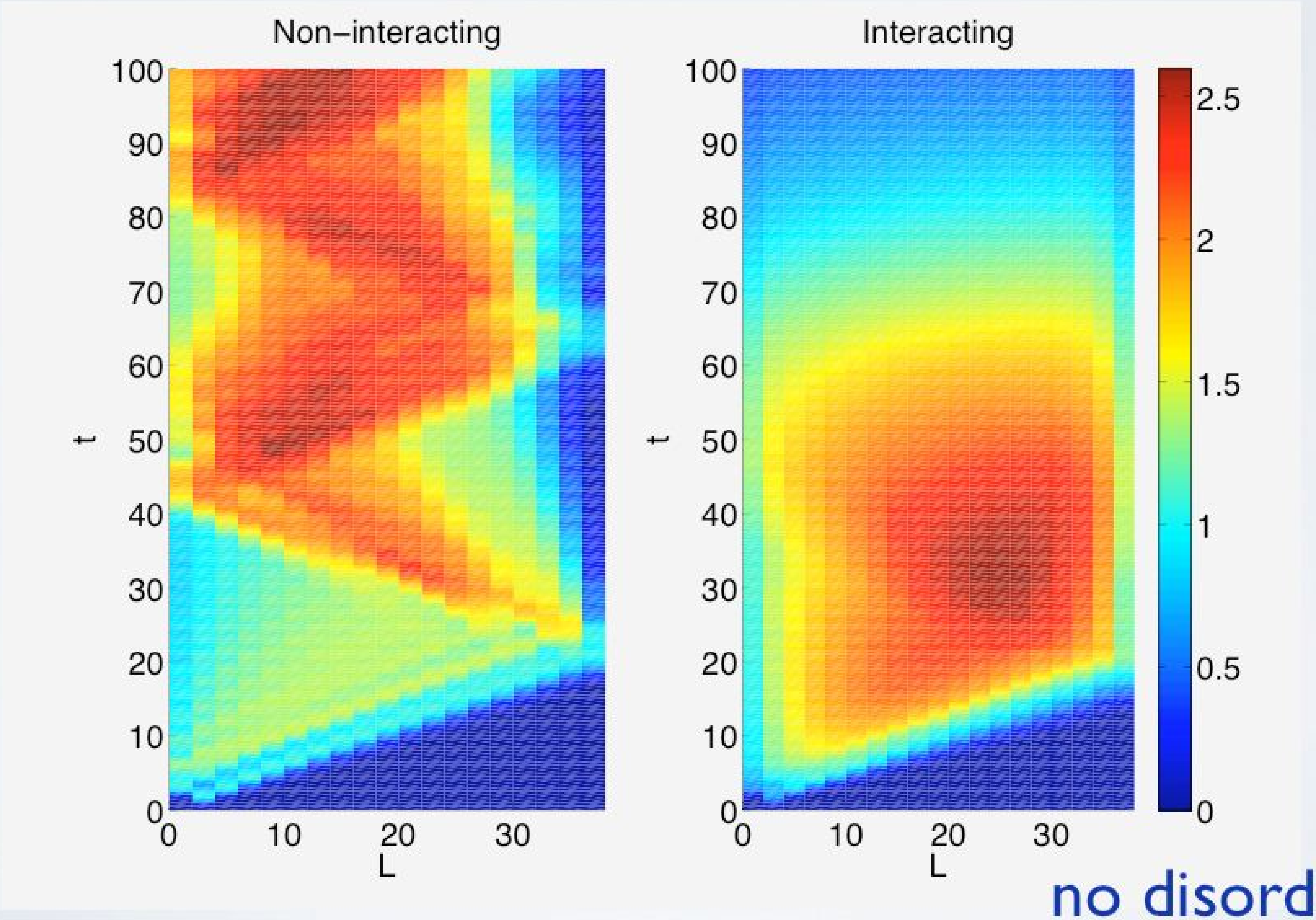
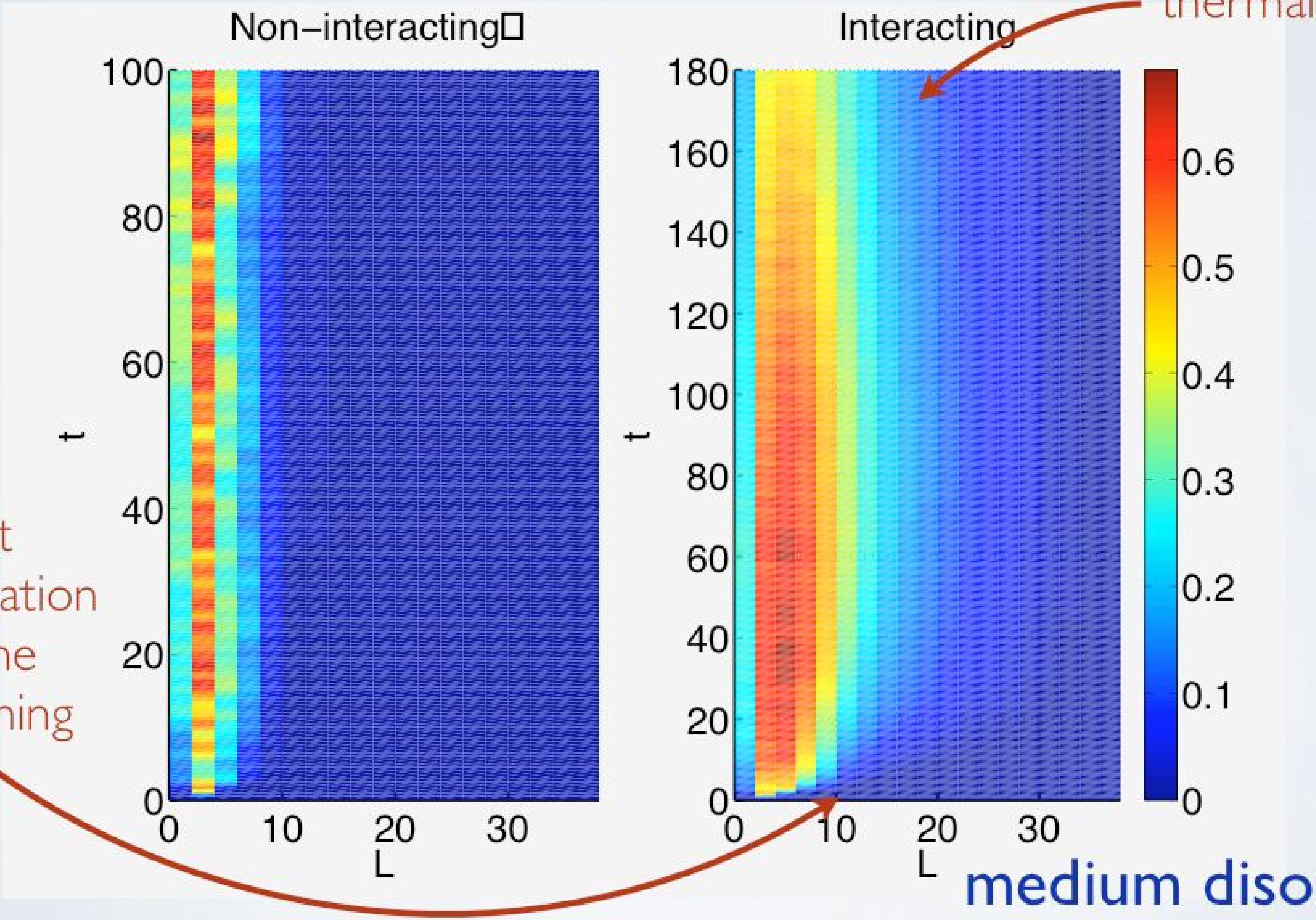


MUTUAL INFORMATION



no disorder

MUTUAL INFORMATION



thermalization

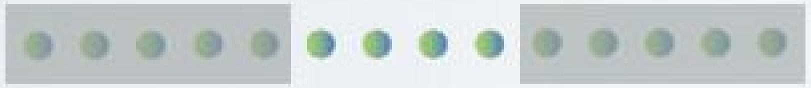
fast propagation at the beginning

medium disorder

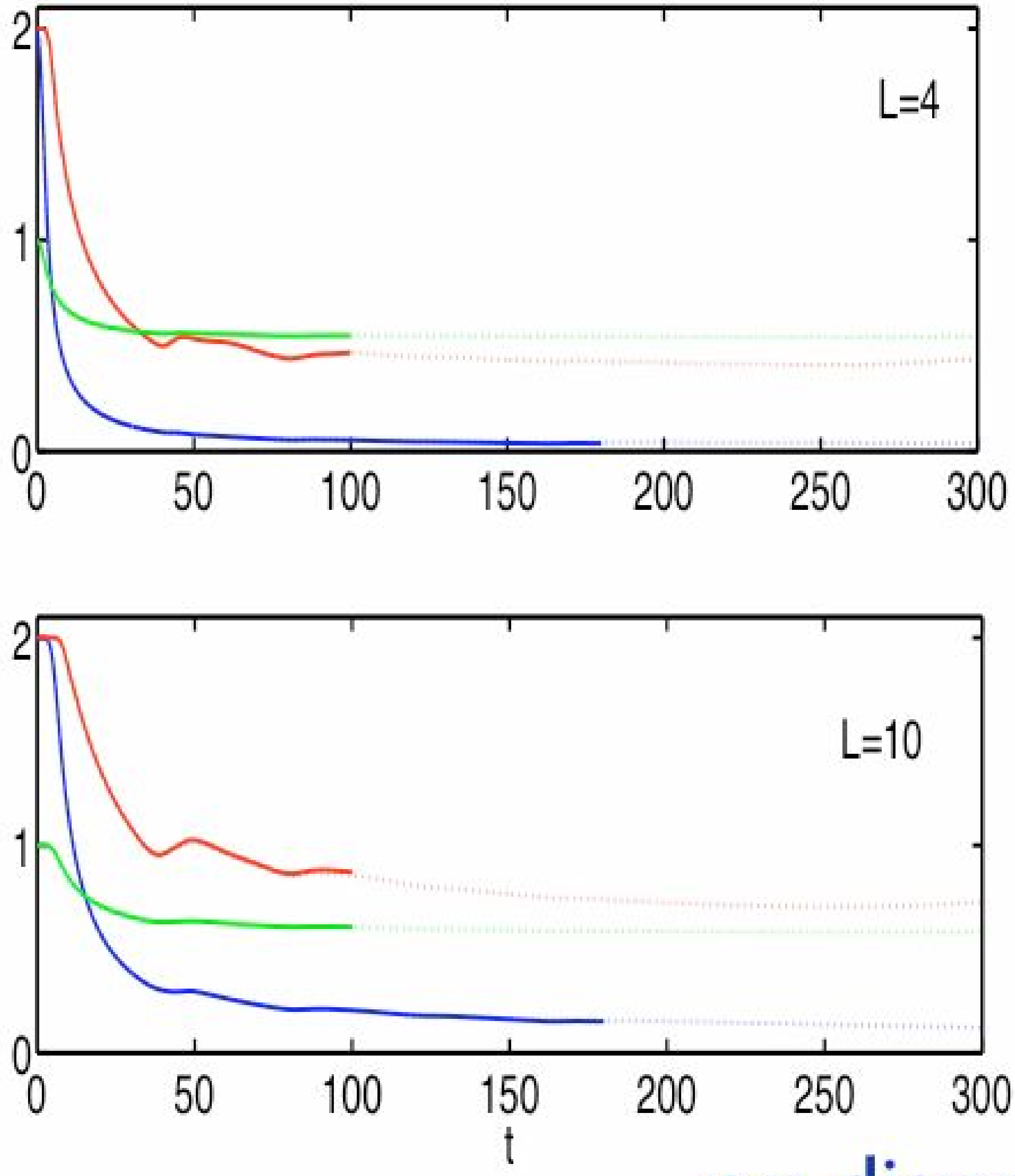
RECOVERY FIDELITY

non interacting

look at
middle 4 sites



look at middle
10 sites



no disorder

RECOVERY FIDELITY

look at
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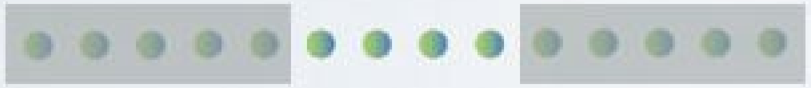

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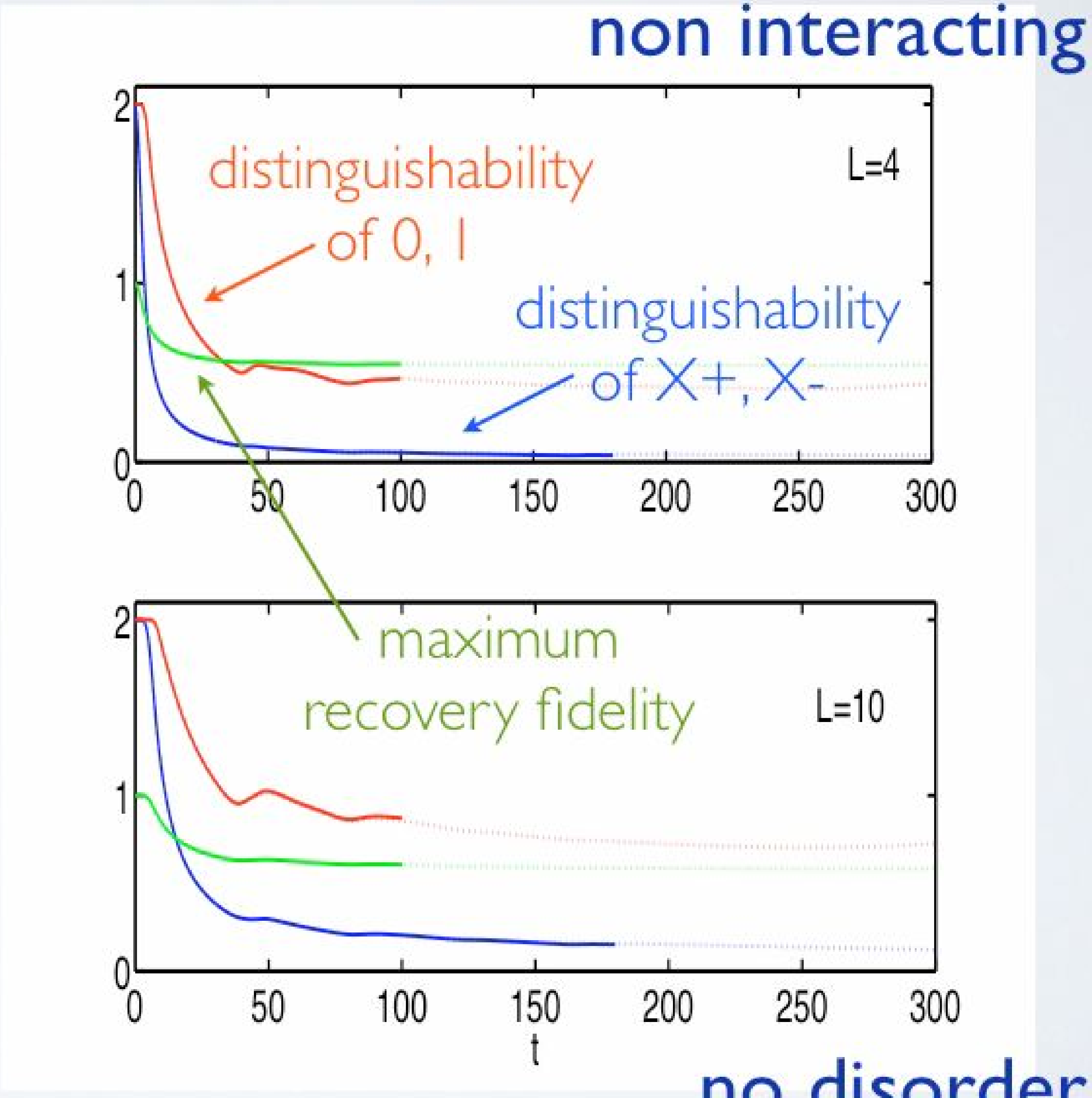


non interacting

distinguishability
of 0, 1

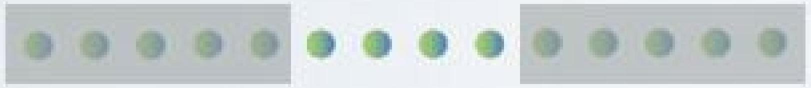
distinguishability
of X+, X-

maximum
recovery fidelity

no disorder

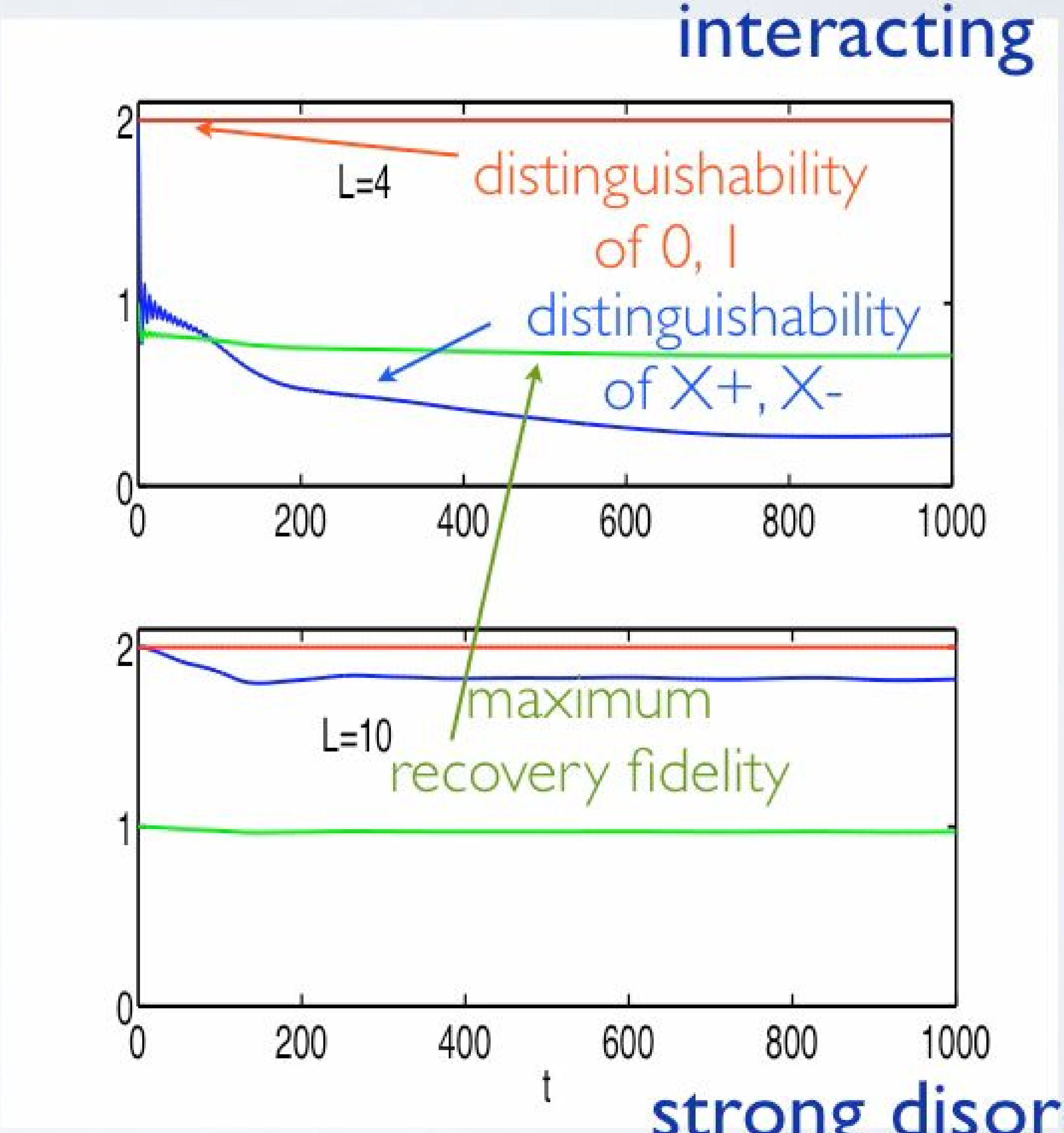
RECOVERY FIDELITY

look at
middle 4 sites



different behavior
Z/X

look at middle
10 sites



interacting

distinguishability
of 0, 1

distinguishability
of X+, X-

maximum
recovery fidelity

strong disorder

CONCLUSIONS

Versatile TNS tools: can be used for out-of-equilibrium time evolution

approximations involved

state/operators

more general TN contraction

entanglement in TN

Applications to non-equilibrium

THANKS!

MPQ

Max Planck Institut
of Quantum Optics
(Garching)

siqs