Entanglement entropy and gauge fields

H. C., M.Huerta and A. Rosabal (2012)
H. C., M. Huerta (2014)
H. C., M. Huerta, in preparation

Puzzling results →

- Lattice calculations with extended lattice (Buividovich-Polikarpov, Donnelly, 2008)
- Negative contact term (Kabat, 1995)
- Mismatch on the log. coefficient (Dowker, 2010)

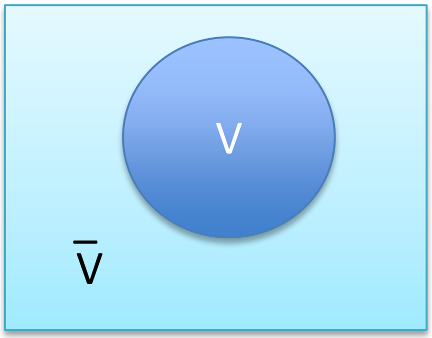
↓

Are gauge fields special regarding entanglement entropy?

Outline

- Entanglement entropy and other measures of information
- Lattice gauge fields and gauge invariant operator algebra: constraints
- The entropy for local operator algebras with center
- Ambiguities: Local algebras vs. regions. The entropy
- The continuum limit: mutual information and relative entropy
- Examples: scalar with center, Maxwell field in 2+1, topological model
- Final Comments

Entanglement entropy and other information measures



$\rho = |0\rangle\langle 0|$ Density matrix

Spatial set V → Partition of the global Hilbert space

$\mathcal{H}_V \otimes \mathcal{H}_{\bar{V}}$ → $\rho_V = \mathrm{tr}_{\mathcal{H}_{\bar{V}}}(\rho)$ → $\mathrm{tr}(\rho \mathcal{O}_V) = \mathrm{tr}(\rho_V \mathcal{O}_V)$

$$S(V) = -\mathrm{tr}(\rho_V \log \rho_V)$$

Other measures…

EE has UV divergencies local on the boundary → Subtracted boundary terms

↑

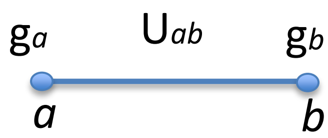
$S(\rho^1_A \,|\, \rho^0_A) = \mathrm{tr}(\rho^1_A \log \rho^1_A - \rho^1_A \log \rho^0_A)$ Relative entropy

$I(A,B) = S(A) + S(B) - S(A \cup B)$ Mutual information

$(S(\rho_{A\cup B} \,|\, \rho_A \otimes \rho_B))$

Lattice gauge theories and operator algebra

- G : U_l , g_i



Oriented link $l = (a,b)$

$$U_{\bar{l}} = U_{(ba)} = U^{-1}_{(ab)} = U^{-1}_{l}$$

$$U'_{(ab)} = g_a U_{(ab)} g_b^{-1}.$$ → Gauge transformation law

- Wave functionals $|\Psi\rangle \equiv \Psi[U]$, where $U = \{U_{(ab)}\}$ → Assignation of group elememts to all links

  Subspace of physical states $\mathcal{H} \subset \mathcal{V}$ → $\Psi[U] = \Psi[U^g]$

- Algebra of physical operators $\mathcal{B}(\mathcal{H})$
  - Link algebra (non gauge invariant)

→ $(\hat{L}^l_g \Psi)[U_1, ..., U_N] = \Psi[U_1, ..., gU_l, ..., U_n]$ $\quad \hat{L}^l_g = \hat{L}^{\bar{l}}_{g^{-1}},\ \hat{L}^l_{g_1}\hat{L}^l_{g_2} = \hat{L}^l_{g_1 g_2}$

Completing the algebra

→ $(\hat{U}^r_l \Psi)[U] = U^r_l \Psi[U]$

Analogous to the momentum and coordinate op.

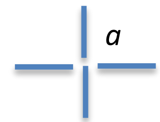
$\hat{L}$ and $\hat{U}$ → Two commuting algebras (for abelian groups) which do not commute to each other. Together they are a generating set for the algebra in the unphysical space v.

Gauge invariant algebra

$\hat{T}_{g_a}$: Operator induced by the gauge transformation by an element g on the vertex *a*

$$(\hat{T}_{g_a}\Psi)[U] = \Psi[U^{g_a}]$$

$$\hat{T}_{g_a} = \prod_{b} \hat{L}_g^{(ab)}$$



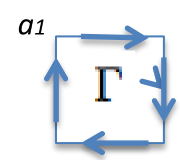
- $\hat{L}_g^l$ gauge invariant
- $\hat{U}_{(a,b)}$ are not gauge invariant

$$\left(\hat{T}_{g_a}\hat{U}^r_{(ab)}\Psi\right)[U] = g^r U^r_{(ab)}\Psi[U] \neq \left(\hat{U}^r_{(ab)}\Psi\right)[U]$$

Gauge inv. version of the coordinate operators?

→ Wilson Loops

$$\hat{W}^r_\Gamma = \hat{U}^r_{(a_1a_2)}\hat{U}^r_{(a_2a_3)}\ldots\hat{U}^r_{(a_ka_1)}$$



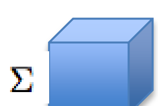
Oriented closed path

Generating set of operators

$\{\hat{L}_g^l , \hat{W}_\Gamma^r\}$

$$(\hat{T}_{g_a}\Psi)[U] = \Psi[U] \longrightarrow \hat{T}_{g_a} = \prod_{b} \hat{L}_g^{(ab)} \equiv 1$$

$$\prod_{\Sigma} W^r_\Gamma = 1$$



Constraint eqs.

For continuous groups

Parametrizations: $U_l = e^{iaA_l}$, g = $e^{i\phi}$ and $\hat{L}_l \leftrightarrow e^{i\lambda \hat{E}_l}$ $\hat{W}_\Gamma = e^{ie\sum_\Gamma a\hat{A}_l}$

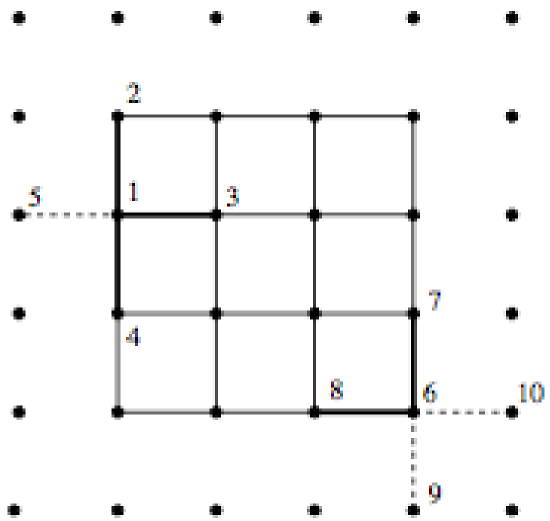
$[\hat{E}_l, \hat{A}_{l'}] = -i\delta_{l,l'}$, $[\hat{E}_l, \hat{U}_{l'}] = \hat{U}_l \delta_{l,l'}$ → Coordinate operator and its conjugated momentum

- Gauss law $\hat{T}_{g_a} = \prod_b \hat{L}_g^{(ab)} \equiv 1 \rightarrow \sum_b \hat{E}_{(ab)} = 0$ The total electric flux through (d-2) dim. closed surfaces
- Magnetic flux conservation $\prod W_\Gamma = 1$ The magnetic flux through bidimensional closed surfaces

Example: Two dimensional lattice

The constraint: Some operators will commute with all other

Figure 1: The product of three link operators on the square $\hat{L}_g^{(12)}\hat{L}_g^{(13)}\hat{L}_g^{(14)}$ is equal to a link operator outside the square, $\hat{L}_g^{(51)}$, and hence it commutes with the rest of the operators on the square. The same occurs for the product $\hat{L}_g^{(67)}\hat{L}_g^{(68)} = \hat{L}_g^{(96)}\hat{L}_g^{(10\,6)}$ on the corner.



Local algebras, constraints and center

Scalar field case:

Operators for a lattice site *a* → $\phi(a)$ $\pi(a)$

$$[\phi(a),\pi(b)] = i\delta_{a,b}\,, \qquad [\phi(a),\phi(b)] = 0\,, \qquad [\pi(a),\pi(b)] = 0$$

- For a lattice region V

$\mathcal{A}_V$ → Generated by $\phi(a)\,\pi(a)$ within V (including all polynomials)

Given a set of operators $\mathcal{G}$ and its commutant $\mathcal{G}'$

$\mathcal{A}_V = \{\phi(a),\pi(a), a\,\mathrm{in}\,V\}''$ with $\mathcal{G}''$ double commutant

"Von Neumann double commutant theorem"

$\mathcal{A}_V \cap (\mathcal{A}_V)' = 1$ Interpretation of $\mathcal{A}_V$ as a factor in a tensor product

$\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{V'}$ ⟷ Two algebras $\mathcal{A}_V$ and $\mathcal{A}_{V'}$

$$O_V \otimes 1_{V'} \text{ and } 1_V \otimes O_{V'}$$

Operator algebras with center

General situation: $\mathcal{A}_V \cap (\mathcal{A}_V)' = \mathcal{Z}_V$ Non trivial center

$$\begin{pmatrix} (\lambda^1) & 0 & \dots & 0 \\ 0 & (\lambda^2) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & (\lambda^m) \end{pmatrix} \in \mathcal{Z}_V$$

Choosing a basis that simoultaneously diagonalize all the commuting operators in the center

$$\mathcal{A}\mathcal{A}' \equiv (\mathcal{A} \cup \mathcal{A}')'' = \begin{pmatrix} \mathcal{A}_1 \otimes \mathcal{A}'_1 & 0 & \dots & 0 \\ 0 & \mathcal{A}_2 \otimes \mathcal{A}'_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \mathcal{A}_m \otimes \mathcal{A}'_m \end{pmatrix}$$

The algebra generated by $\mathcal{A}$ and its commutant

Block diagonal form

Effective superselection rule in the algebra.
Rduced state: the unique state that belongs to the algebra and reproduces expectation values

$$\rho_\mathcal{A} \in \mathcal{A} \text{ and } \mathrm{tr}(\rho_\mathcal{A}\mathcal{O}) = \mathrm{tr}(\rho\mathcal{O}),$$

$$\rho_{\mathcal{A}\mathcal{A}'} = \begin{pmatrix} p_1\rho_{\mathcal{A}_1\mathcal{A}'_1} & 0 & \dots & 0 \\ 0 & p_2\rho_{\mathcal{A}_2\mathcal{A}'_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & p_m\rho_{\mathcal{A}_m\mathcal{A}'_m} \end{pmatrix}$$

$\sum_k p_k = 1$

Partial trace on each block →

$$\rho_\mathcal{A} = \begin{pmatrix} p_1\rho_{\mathcal{A}_1} & 0 & \dots & 0 \\ 0 & p_2\rho_{\mathcal{A}_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & p_m\rho_{\mathcal{A}_m} \end{pmatrix}$$

Entropy and more…

$$S(V) = -\mathrm{tr}(\rho_A \log \rho_A) = H(\{p_k\}) + \sum_k p_k S(\rho_{A_k})$$

$$H(\{p_k\}) = -\sum_k p_k \log(p_k)$$

Average of quantum contributions

Shannon entropy of a classical probability distribution

$$S(\mathcal{A}) = S(\mathcal{A}') \qquad \text{(if global state is pure)}$$

Relative entropy

$$S(\rho_A^1|\rho_A^0) = \mathrm{tr}(\rho_A^1 \log \rho_A^1 - \rho_A^1 \log \rho_A^0)$$

$$S(\rho_A^1|\rho_A^0) = \sum_k p_k^1 \log(p_k^1/p_k^0) + \sum_k p_k^1 S(\rho_{A_k}^1|\rho_{A_k}^0)$$

$$S(\rho_A^1|\rho_A^0) \leq S(\rho_B^1|\rho_B^0) \qquad \text{for } \mathcal{A} \subseteq \mathcal{B}$$

Increasing with inclusion of algebras

Mutual Information

$$I(\mathcal{A},\mathcal{B}) = S(\rho_{\mathcal{AB}}|\rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}) = S(\mathcal{A}) + S(\mathcal{B}) - S(\mathcal{AB})$$

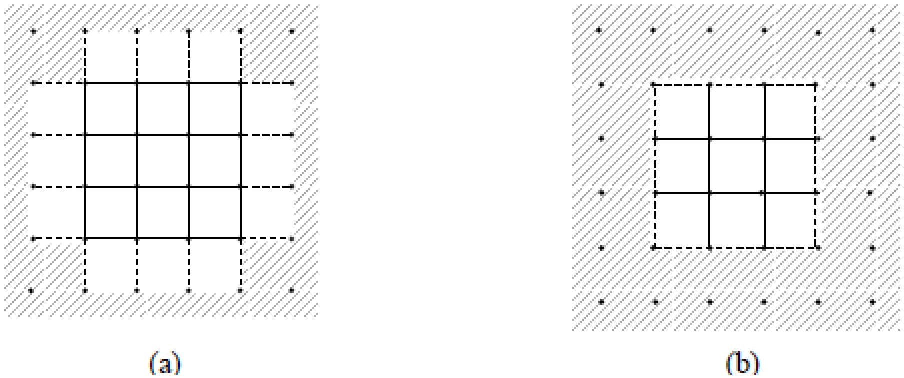
$$I(\mathcal{A},\mathcal{B}) = \sum_{k_A,k_B} p_{k_A,k_B} \log(p_{k_A,k_B}/(p_{k_A}p_{k_B})) + \sum_{k_A,k_B} p_{k_A,k_B} S(\rho_{k_A,k_B}|\rho_{k_A} \otimes \rho_{k_B})$$

$$I(\mathcal{A},\mathcal{B}) \leq \mathcal{I}(\mathcal{A},\mathcal{C}) \qquad \mathcal{B} \subseteq \mathcal{C}$$

Ambiguities in the assignment of algebras to regions

- Two geometric choices with non trivial center

Electric center



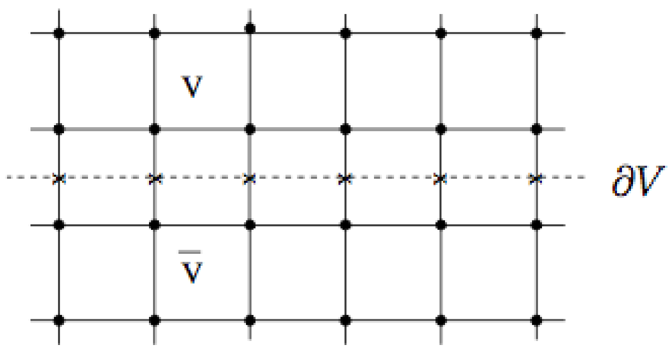
Magnetic center

(a) The algebra of the square with an electric center choice. The center is formed by the link operators shown with dashed lines. The commutant is represented by the shaded region, having the same center. (b) The square with a magnetic center choice. The center is formed by a single loop at the boundary in this two-dimensional example.

- The extended lattice construction

P. V. Buividovich and M. I. Polikarpov, "Entanglement entropy in gauge theories and the holographic principle for electric strings," Phys. Lett. B **670**, 141 (2008) [arXiv:0806.3376 [hep-th]].

W. Donnelly, "Decomposition of entanglement entropy in lattice gauge theory," Phys. Rev. D **85**, 085004 (2012) [arXiv:1109.0036 [hep-th]].



$l_\partial \rightarrow l_{\partial V}$ and $l_{\partial \bar{V}} \quad \Longrightarrow \quad \mathcal{L} \rightarrow \mathcal{L}'$

$\mathcal{H} \rightarrow \mathcal{H}' \Longrightarrow \mathcal{H}'_V \otimes \mathcal{H}'_{\bar{V}}$

Gauge transformations act independently on each side of the boundary

$$\Psi'(U_{l_1}, \ldots, U_{l_{\partial V}}, U_{l_{\partial \bar{V}}}, \ldots, U_{l_N}) \equiv \Psi(U_{l_1}, \ldots, U_{l_{\partial V}}.U_{l_{\partial \bar{V}}}, \ldots, U_{l_N})$$

$$\rho_V = \mathrm{tr}_{\mathcal{H}_{\bar{V}}}(\Psi \Psi^\dagger)$$

Equivalent to the electric center choice

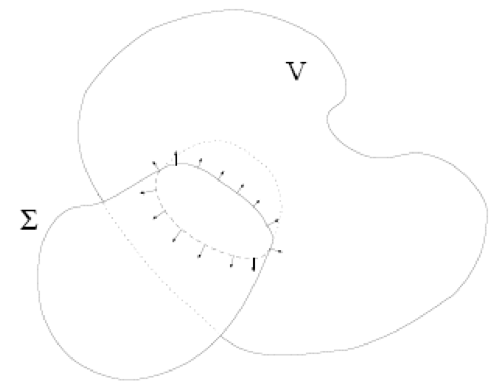
Recent construction by Ghosh, Soni, Trivedi (2015): take the full Hilbert space of non gauge invariant functionals, but a gauge invariant state. The resulting entropy also coincides witht he electric center choice. Generalization for non-abelian theories of electric center choice in this paper and in L.Y.Hung, Y. Wan (2015).

Trivial center choices: bona-fide entanglement entropy

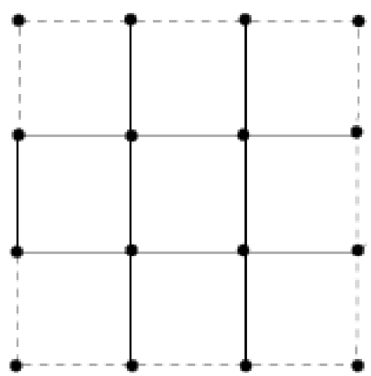
Constraints →

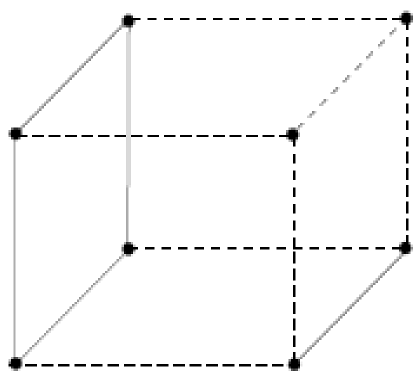
- The set ∂V^- of links on the boundary whose link op. do not belong to the algebra must be a maximal tree
- Balanced value of magnetic and electric degrees of freedom

- Electric constraints: The set ∂V^- must be connected and the set of links belonging to the algebra should not contain all the links attached to a single point in the boundary



- Magnetic constraints: the set ∂V^- must not contain any close path, otherwise its Wilson loop operator belongs to the center





Many choices
All break lattice symmetries

Maximal tree as a gauge fixing (or the triumphant return of the gauge dependence)

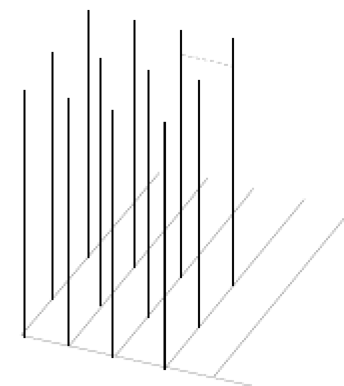
- Lattice gauge fixing → Fixed link variables on a maximal tree T

Physical variables are attached to the links complementary to the maximal tree $U_l \in \bar{T}$

Choosing a subset $V \subseteq \bar{T}$ → $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{V'}$

→ In general

problems with localization → No entanglement entropy of local regions



→ Maximal tree on the surface The trivial center case

- any link on $\bar{T} \cap \partial V$ necessarily closes a loop on ∂V and does not go far away.

If we extend the maximal tree on ∂V to all the space (what can always be done), closing the tree with a new link l inside of V, the corresponding loop cannot pass through the boundary of V.

↓

This prescription cuts the degrees of freedom in two: the ones inside and the ones outside V

How ambiguous is the entropy?

- If we change the prescription at the boundary how much does the entropy change?

1- Free scalar field in the lattice and a non trivial center

On a subset of the boundary we consider only the fields but not its conjugated momentum → Center $\hat{\phi}_a$ with $a \in A \subseteq \partial V$

$$p(\{\phi\}_A) = \sqrt{\det(M_A/(2\pi))}\; e^{-\frac{1}{2}\phi_a M_A^{ab}\phi_b}$$

$$X_{ab}^A \equiv \langle \hat{\phi}_a \hat{\phi}_b \rangle_A = \int (\Pi_{a\in A} d\phi_a)\, p(\{\phi\}_A)\, \phi_a \phi_b = (M_A^{-1})_{ab}$$

$$H = -\int (\Pi_{a\in A} d\phi_a)\, p(\{\phi\}_A) \log(p(\{\phi\}_A)) = \mathrm{tr}\left(\frac{1}{2} - \log\left(\frac{M}{2\pi}\right)\right)$$

→ For continuous gauge groups in the lattice, H is ill defined and depending on the interpretation S can be infinite or even negative -> Kabat contact term (H.C., M. Huerta, A. Rosabal 2012). Recent work by Donnelly, Wall (2014) and K. Huang (2014) suggests this is the case.

→ The entropy strongly depends on the size of the center (ambiguity proportional to area and degree of freedom in the center)

→ Mutual information and relative entropy unambiguos classical term

$$-\int (\Pi_{a\in AB} d\phi_a)\, p_{AB}(\{\phi\}) \log(p_{AB}(\{\phi\})/(p_A(\{\phi\})p_B(\{\phi\})))$$

How ambiguous is the entropy?

Topological Z_2 model

Z_2 : variables zl = $\{1, -1\}$

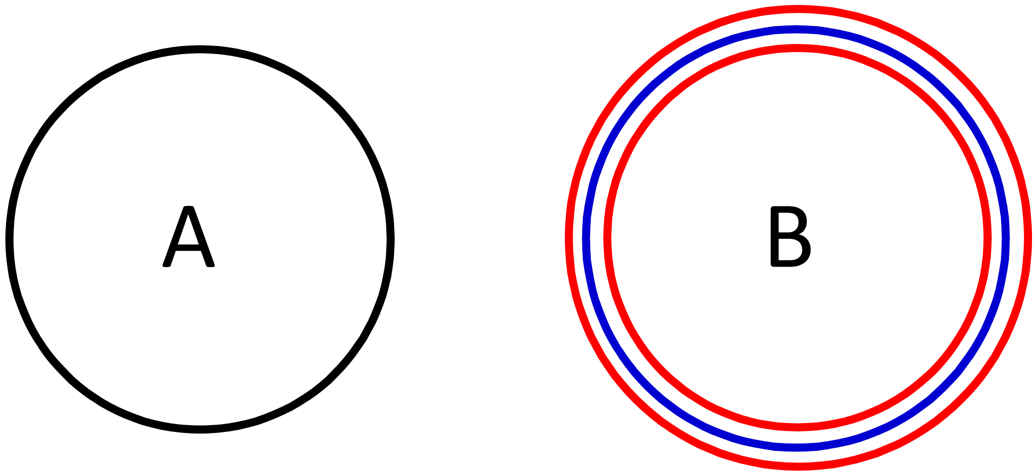
$$\Psi[z_l] = K \sum_{\Gamma} \Pi_{l\in\Gamma}\ z_l$$ Topological wave function

$$\Psi[z_l] = K \sum_{\Gamma} W_{\Gamma}\ \Psi_0[z_l]$$ → Gauge invariant

For different center choices

$$S_E(V) = (L_V - n_\partial)\log(2)$$
$$S_M(V) = 0 \qquad S_{\rm ent}(V) = 0$$

How ambiguous are mutual information and relative entropy under change of algebra choice? The continuum limit



The blue and red prescriptions satisfy

$I(A,B)<I(A,B')<I(A,B'')$

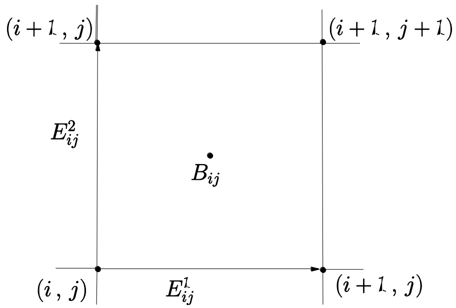
In the continuum they have to have the same limit

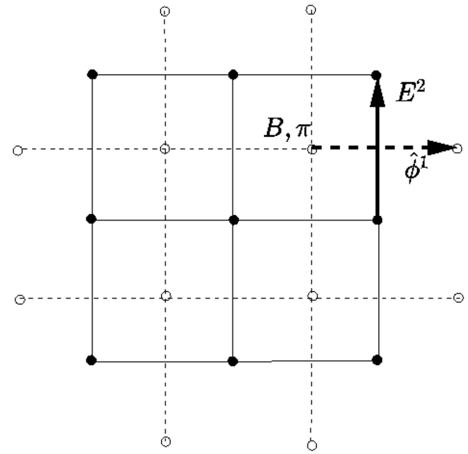
Mutual information and relative entropy are not ambiguous in the continuum limit.
They have mild ambiguities (as functions of regions) in the lattice.
Classical mutual information of the center goes to zero in the continuum limit.

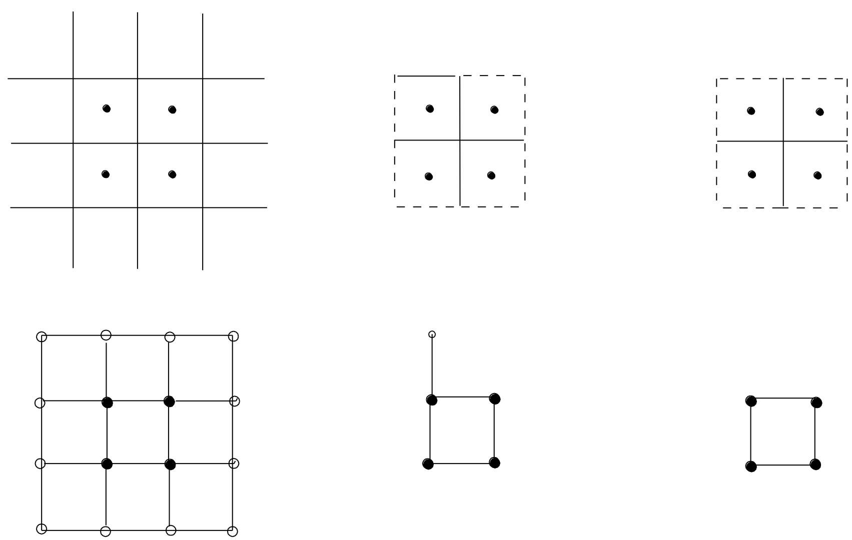
Example: Maxwell field in 2+1.

Dual to the gradient of a massless scalar field

$$\partial_\mu \phi = \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}$$







Entropy for free fields and Gaussian states

$$[q_i, p_j] = iC_{ij}$$

$$\begin{aligned} \langle p_i, p_j \rangle &= P_{i,j} \\ \langle q_i, q_j \rangle &= X_{i,j} \\ \langle q_i, p_j \rangle &= \frac{i}{2} C_{ij} \end{aligned}$$

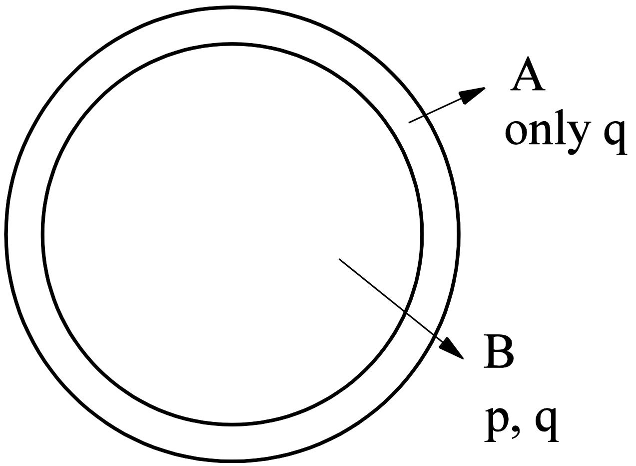
For the electromagnetic field these are E and B variables

If the «region» (set of variables) V=AUB, and A is the center here taken as the variables q in A, we have for the quantum entropy (independent of the sector in the center!)

$$S_Q(V) = \mathrm{tr}\left((\Theta + 1/2)\log(\Theta + 1/2) - (\Theta - 1/2)\log(\Theta - 1/2)\right)$$

$$\Theta = \sqrt{\tilde{X}\tilde{P}}$$

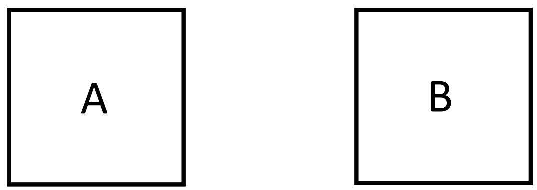
$$\tilde{X} = (X_V^{-1}|_B)^{-1} = (C_{VB}^T X_V^{-1} C_{VB})^{-1}, \qquad \tilde{P} = P_B\,.$$



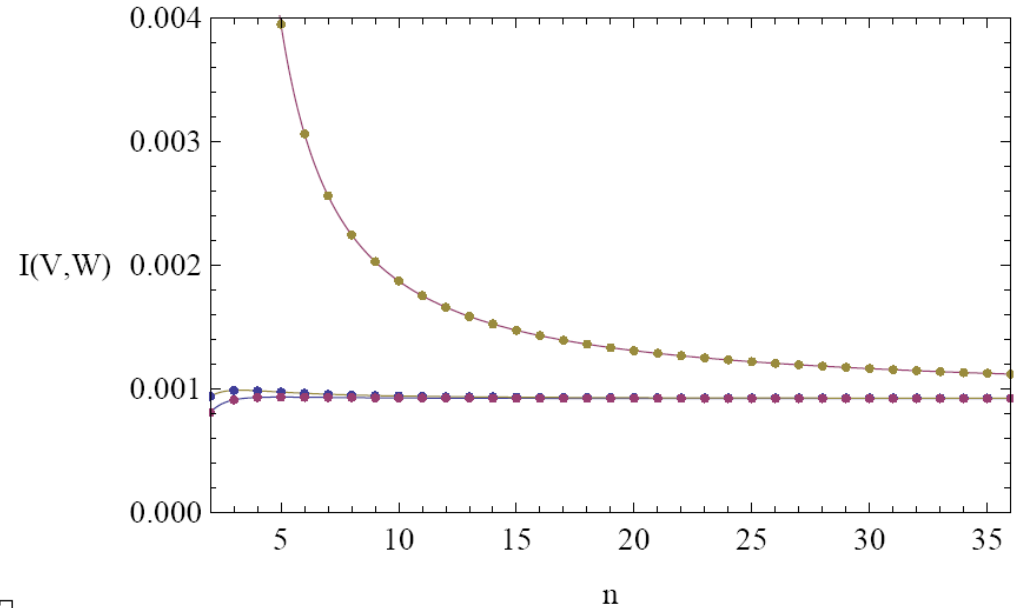
While the classical entropy of the center is

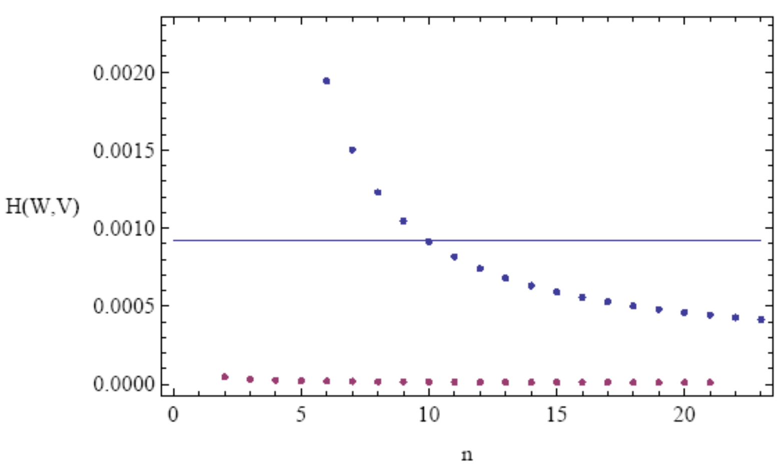
$$H(A) = \frac{1}{2}\mathrm{tr}\left(1 + \log\left(2\pi X_A\right)\right)$$

The continuum limit. Mutual information for electric center, magnetic center, and no center have the same limits



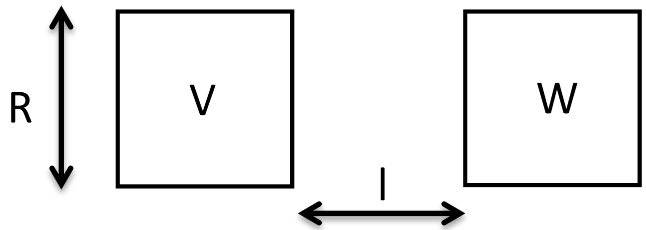
$$I_0^E = 0.000923$$
$$I_0^T = 0.000924$$
$$I_0^M = 0.000924$$





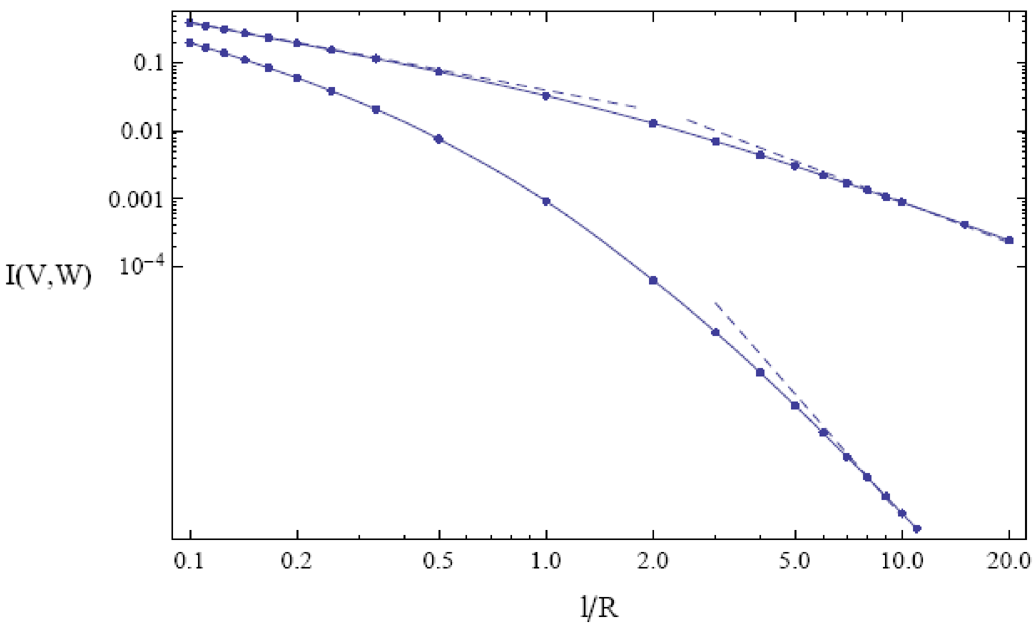
Classical mutual information goes to zero

Mutual information: Maxwell versus massless scalar field



$$I_{\text{gauge}}(V,W) \leq I_{\text{scalar}}(V,W)$$

The gauge model is a subalgebra of the scalar model.



$$I_{\text{scalar}} \sim a_s \left(\frac{l}{R}\right)^{-2}, \qquad \frac{l}{R} \gg 1$$

$$I_{\text{gauge}} \sim a_g \left(\frac{l}{R}\right)^{-6}, \qquad \frac{l}{R} \gg 1$$

For short distances both models have the same limit of mutual information as can be computed by dimensional reduction

$$I(V,W) \sim k\frac{R}{l} + \ldots \qquad \frac{l}{R} \ll 1$$

Logarithmic term

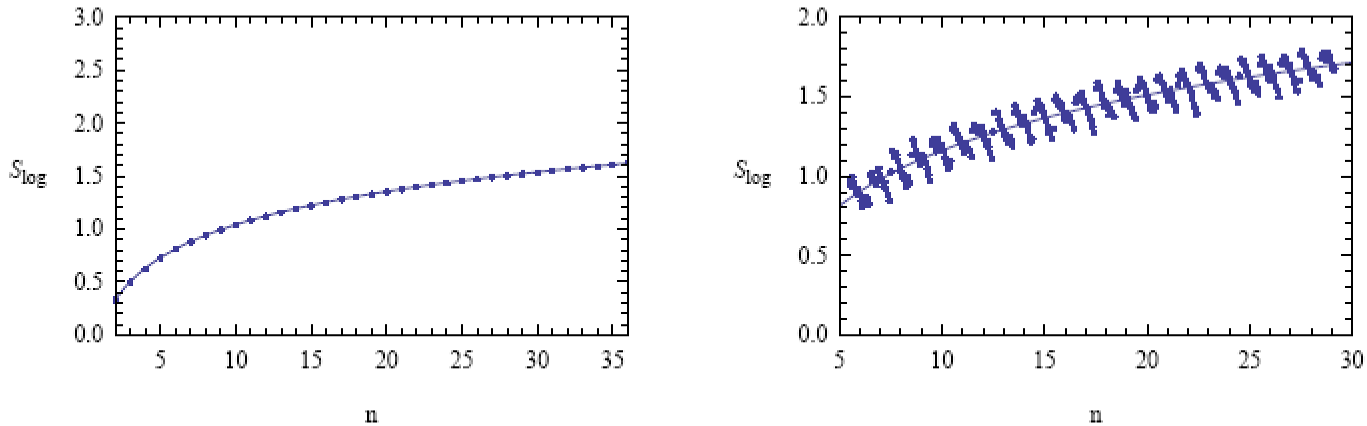
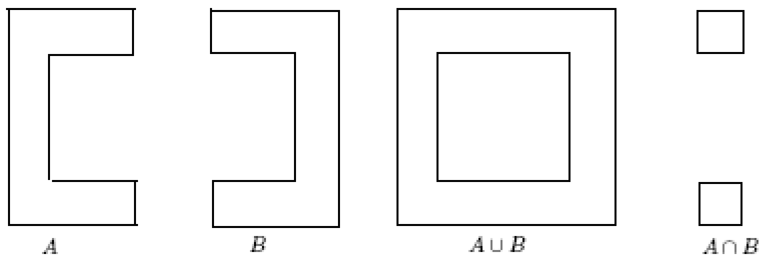


Figure 10: Left panel: Entropy of a square of size n where the linear term in a fit $S = c_1 n + c_0 + s_{\log} \log(n)$ has been subtracted. The logarithmic term is shown with a solid line. Right panel: Entropy of circles of radius n in the square lattice, computed with $1/10$ steps for the radius. The linear term has been subtracted. Both figures are for a trivial center.

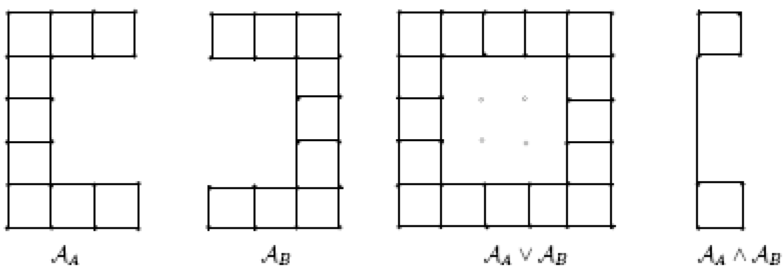
$$c_{\log} = \frac{N_c}{2} + c^s_{\log} = \frac{N_c}{2} - \sum_v s(\theta_v)$$

Topological logarithmic coeficient is positive and proportional to number of conected components, but not represented in ultraviolet mutual information! (related to divergences of F-charge of F theorem for Maxwell, Agon, Headrick, Jafferis, Kasco (2013) and Klebanov, Pufu, Sachdev, safdi (2012))

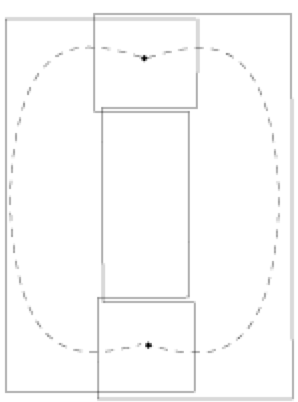
Strong subadditivity



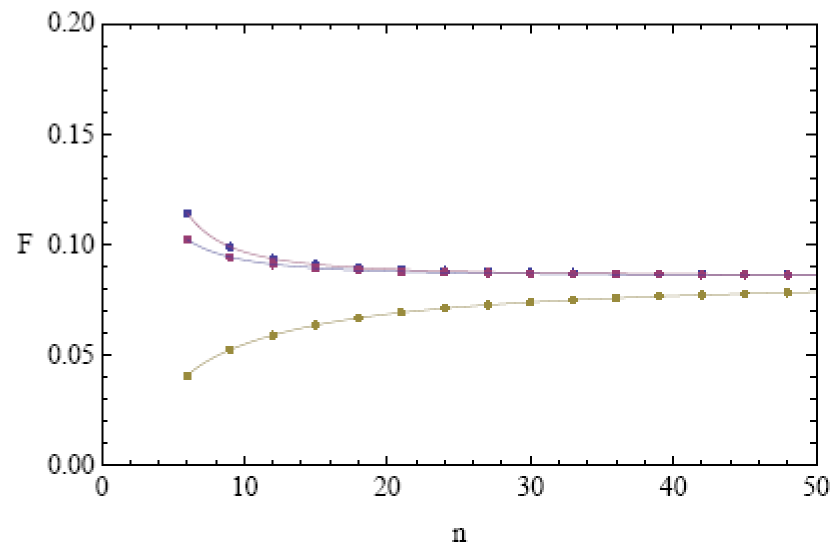
$$S(A) + S(B) - S(A \cup B) - S(A \cap B) = -\frac{1}{2}\log(R/\epsilon) + \text{const} < 0$$



$$F(\mathcal{A}_A, \mathcal{A}_B) = S(\mathcal{A}_A) + S(\mathcal{A}_B) - S(\mathcal{A}_A \vee \mathcal{A}_B) - S(\mathcal{A}_A \wedge \mathcal{A}_B) \geq 0$$



Somewhat pathological: The algebra of the intersection of regions is not the intersection of algebras. Problem does not appear with charges



Conclusions, comments

In the lattice:

a) Entropy well defined for gauge invariant operator algebras (with or without center) if center does not contain continuous variables. (Otherwise only relative entropy quantities well defined even in the lattice: lattice regularization not enough in this case)

b) Local entropy not an entanglement entropy when there is a center.

c) Ambiguities in assignation of algebras to regions. («region» concept badly defined in the lattice in terms of physical content of model!)

d) Local algebras without center and entanglement entropy can be defined, but in many different ways. These can be correlated with gauge fixings, what can also be thought as a «gauge dependence of entanglement entropy».

e) Entropy can vary widely with choice of algebra (milder dependence for relative entropy).

f) Extended lattice construction a special algebra choice (hence, this construction does not introduce external elements)

In the QFT

a) Relative entropy quantities well defined as functions of regions (independent of choice of algebra). Entropy has the same ambiguities and divergences as for other fields.
Are gauge fields special regarding entanglement entropy?: NO! (barring pathologies for free fields)

b) Kabat contact term produced by classical center. But classical entropy cancel in relative entropy quantities in continuous limit.

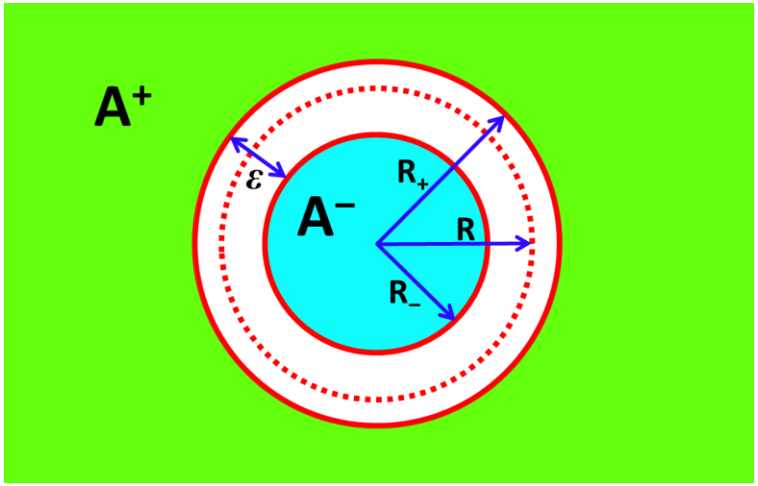
c) Dowker coefficient of log term of a sphere in d=3+1 (equal to 16/45) found to be correct by simulations in radial lattices for vector spherical harmonics decomposition. Does not coincide with anomaly coefficient 31/45! Suggested by Donnelly, Wall (2014) and K. Huang (2014) the electric center provides missing part. But this is not universal. Charges (even very massive) probably restore anomaly in mutual information at small distances -> similar phenomena is expected for topological theories in 2+1.

In quantum gravity

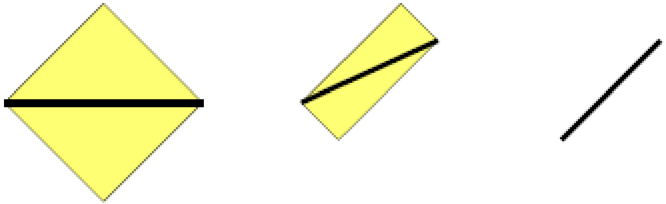
Can quantum gravity make unambiguous the negative contact term and use it to produce a finite entropy, retaining a meaning as entropy?

Mutual information as a geometric regulator
for entropy. For small enough ε contribution of charged
particles should change log term.

For topological theories mutual information is zero
until ε crosses the gap scale and the topological constant
term can appear (this is negative and it has to be
supported by an area term that also should show up after
the gap scale).



Similar phenomena also expected for universal area term in massive theories:
Different contribution for free and interacting fields no matter how small the interaction is.
This difference should also be seen for small ε in mutual information, when UV fix point
scales are tested.



Relative entropy between two states (eg. Thermal and
vacuum) in the limit of null surfaces very different
for interacting and free cases: it is zero in the interacting case
no matter how small the coupling constant is.

Srednicki lattice in 3+1

Vector spherical harmonics

$$\bar{Y}^r_{lm} = Y_{lm}(\theta,\phi)\hat{r}\,,$$
$$\bar{Y}^e_{lm} = \frac{r\bar{\nabla}Y_{lm}}{\sqrt{l(l+1)}} \quad l>0$$
$$\bar{Y}^m_{lm} = \frac{\bar{r}\times\bar{\nabla}Y_{lm}}{\sqrt{l(l+1)}} \quad l>0$$

$$\bar{E} = E^r_{lm}(r)\bar{Y}^r_{lm}(\theta,\phi) + E^e_{lm}(r)\bar{Y}^e_{lm}(\theta,\phi) + E^m_{lm}(r)\bar{Y}^m_{lm}(\theta,\phi)$$

$$\nabla . E = 0 \quad \longrightarrow \quad \frac{\partial E^r}{\partial r} + \frac{2}{r}E^r = \frac{\sqrt{l(l+1)}}{r}E^e$$

Constraint equations: keep only radial and magnetic variables

$$[B_i(x), E_j(x')] = i\epsilon_{ijl}\partial_l\delta^3(x-x')$$
$$[E^r_{lm}(r), B^m_{l'm'}(r')] = -[E^m_{lm}(r), B^r_{l'm'}(r')] = \frac{\sqrt{l(l+1)}}{r^3}\delta(r-r')\delta_{l,l'}\delta_{m,m'}$$

Commutation relations

$$\tilde{E}^m = rE^m \,,\quad \tilde{B}^m = rB^m\,,$$
$$\tilde{E}^r = \frac{r^2}{\sqrt{l(l+1)}}E^r \,,\quad \tilde{B}^r = \frac{r^2}{\sqrt{l(l+1)}}B^r$$

Redefinition of variables

$$\left[\tilde{E}^r_{lm}(r), \tilde{B}^m_{l'm'}(r')\right] = \delta(r-r')\delta_{l,l'}\delta_{m,m'}$$

Canonically commuting variables

$$H_{lm} = \int dr\left[(\tilde{B}^m)^2 + \left(\frac{d\tilde{E}^r}{dr}\right)^2 + \frac{l(l+1)}{r^2}(\tilde{E}^r)^2\right] + (\tilde{E}\leftrightarrow\tilde{B})$$

radial Hamiltonian

Discretize, evaluate correlation fucntions and entropy

Some more comments

When relative entropy is decomposed as

$$S(\rho_V|\rho_V^0) = \mathrm{tr}\left(\rho_V \log \rho_V - \rho_V \log \rho_V^0\right) \;\; = \Delta\langle K\rangle - \Delta S$$

$$\boxed{\Delta S \leq \Delta\langle K\rangle}$$ Bekenstein bound

Again the change in entropy for the classical center is ambiguous for continuous variables. Ambiguities are terms local on the boundary that are exactly equal on both sides. Good definition using difference of mutual information.

2- Topological Z_2 model

Z_2 : variables $z_l = \{1, -1\}$

$$\Psi[z_l] = K \sum_{\Gamma} \Pi_{l\in\Gamma}\ z_l$$ Topological wave function

$$\Psi[z_l] = K \sum_{\Gamma} W_{\Gamma}\ \Psi_0[z_l]$$ → Gauge invariant

$$W_{\Gamma}\Psi[z_l] = K \sum_{\Gamma'} W_{\Gamma}W_{\Gamma'}\ \Psi_0[z_l] = K \sum_{\Gamma'} W_{\Gamma'}\ \Psi_0[z_l] = \Psi[z_l]$$ Since W op. form a group

Link operators Pauli matrices σ_x^l

$$W_{\Gamma}\sigma_x^l = \sigma_x^l W_{\Gamma} \text{ if } l \notin \Gamma, \text{ and } W_{\Gamma}\sigma_x^l = -\sigma_x^l W_{\Gamma} \text{ if } l \in \Gamma$$

Any expectation value is equivalent to the expectation value of link operators

$$\langle\Psi|\sigma_{l_1}\ldots\sigma_{l_k}|\Psi\rangle = \langle\Psi|\sigma_{l_1}\ldots\sigma_{l_k}W_{\Gamma}|\Psi\rangle = -\langle\Psi|W_{\Gamma}\sigma_{l_1}\ldots\sigma_{l_k}|\Psi\rangle = 0$$

Or 1 if the product of link operators is a constraint

There are no correlations in this model except the ones introduced by the constraint equations

↕

Topological model

Topological Z_2 model

Electric Center

N independent link operators σ^l_{x} with eigenvalues ± 1

↓

2^N Sectors labeled by $\lambda = \{\pm 1, \ldots, \pm 1\}$
with the same probability for each sector $p_\lambda = 2^{-N}$

(all expectation values of prod. of link operators are equal to zero)

- Classical contribution $H = N \log(2)$ ⟶ N = $L - n_\partial$

(There is one independent constraint for each boundary)

- Quantum contribution

Quantum entropy of the algebra once the eigenvalues of the operators in the center are fixed to $\lambda = \{\pm 1, \ldots, \pm 1\}$ → $\mathcal{A}_\lambda$ restricted to the sector λ

Operators $\mathcal{O}_V^\lambda = P_\lambda \mathcal{O}_V P_\lambda$ with $P_\lambda = \frac{1}{2}(1 \pm \sigma^l_x)$

$p_\lambda \mathrm{tr}(\rho_V^\lambda \mathcal{O}_V^\lambda) = \langle \Psi | \mathcal{O}_V^\lambda | \Psi \rangle$

$\mathrm{tr}(\rho_V^\lambda W_\Gamma^\lambda) = 1$ ⟶ ρ_V^λ pure state (diagonal with only one 1) ⟶ No contribution

For different center choices

$$S_E(V) = (L_V - n_\partial)\log(2)$$
$$S_M(V) = 0 \qquad S_{\mathrm{ent}}(V) = 0$$