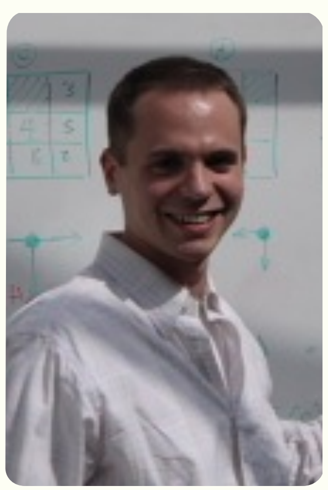
PARTICLE PARTIONED ENTANGLEMENT IN QUANTUM FLUIDS

Measuring Rényi entropies in the spatial continuum



Chris Herdman
UVM /
U Waterloo

Stephen Inglis
U Waterloo /
LMU

P.N. Roy
U Waterloo

Roger Melko
U Waterloo

Phys. Rev. B, 89, 140501 (2014)
Phys. Rev. E, 90, 013308 (2014)
arXiv:1412.6529 (2015)

KITP 2015

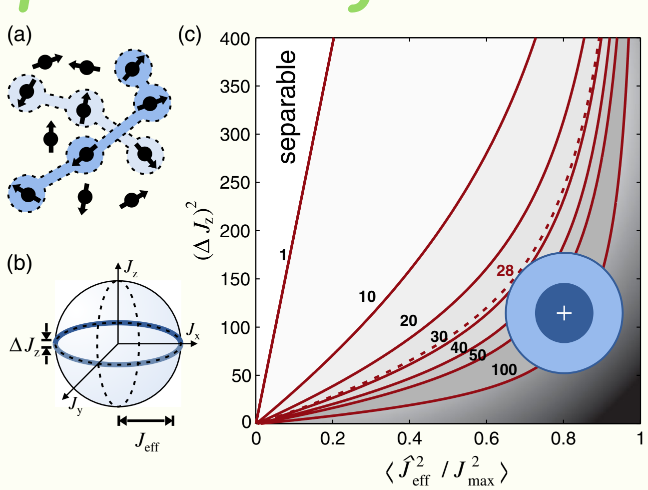
Adrian Del Maestro
University of Vermont

Entanglement in quantum liquids and gases

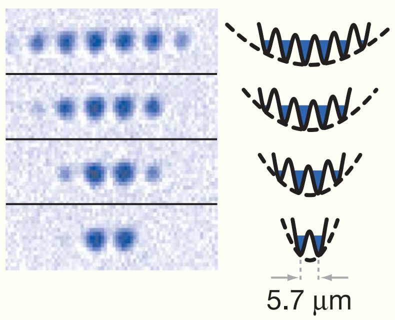
Much theoretical work has focused on systems with discrete Hilbert spaces: qubits, insulating lattice models, …

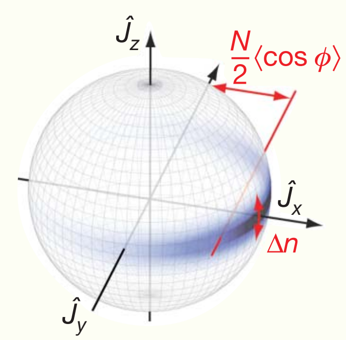
Experiments employ the quantum and positional states of ultra-cold atomic gasses and BECs

observation and manipulation of Dicke states



B. Lücke, *et.al.*, PRL 112, 155304 (2014)



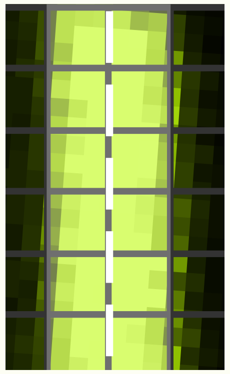


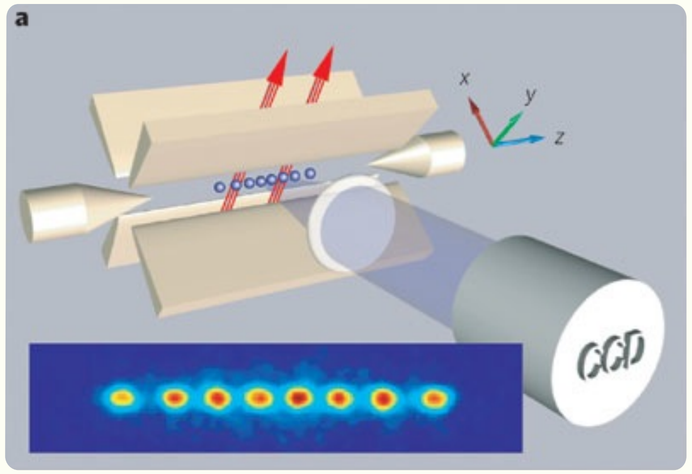
ultra high-precision quantum interferometry

.Estève, *et al.*,
Nature 455, 1216 (2008)

Rényi entropy in lattice gases

R. Islam, *et.al.*, (2015)





multiparticle entanglement of trapped ions

T. Monz, *et.al.*, PRL 102, 040501 (2009)

boson sampling

C. Shen, *et al.*, PRL 112, 050504 (2014)

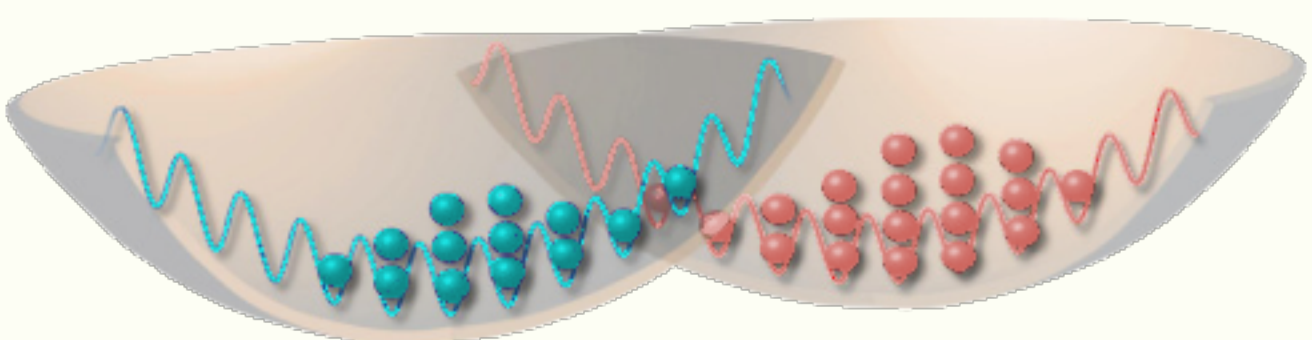
Describing quantum liquids and gases

governed by the general many-body Hamiltonian

$$H = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m_i} \nabla_i^2 + U_i \right) + \sum_{i<j} V_{ij},$$

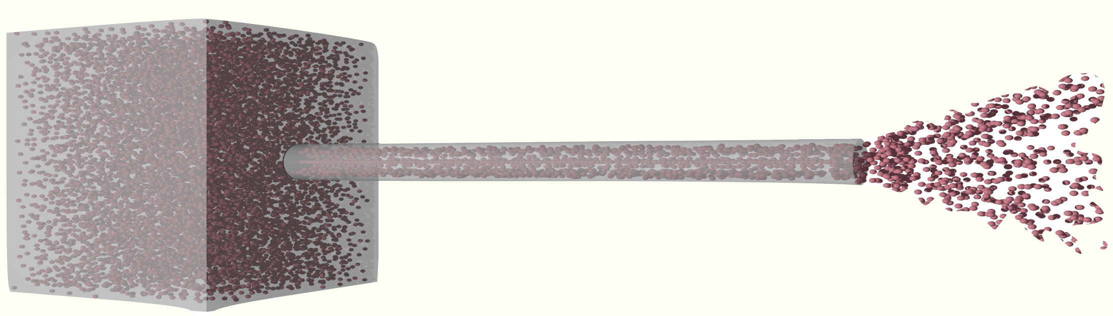
external potential

interaction potential



trapped ions with a periodic lattice potential

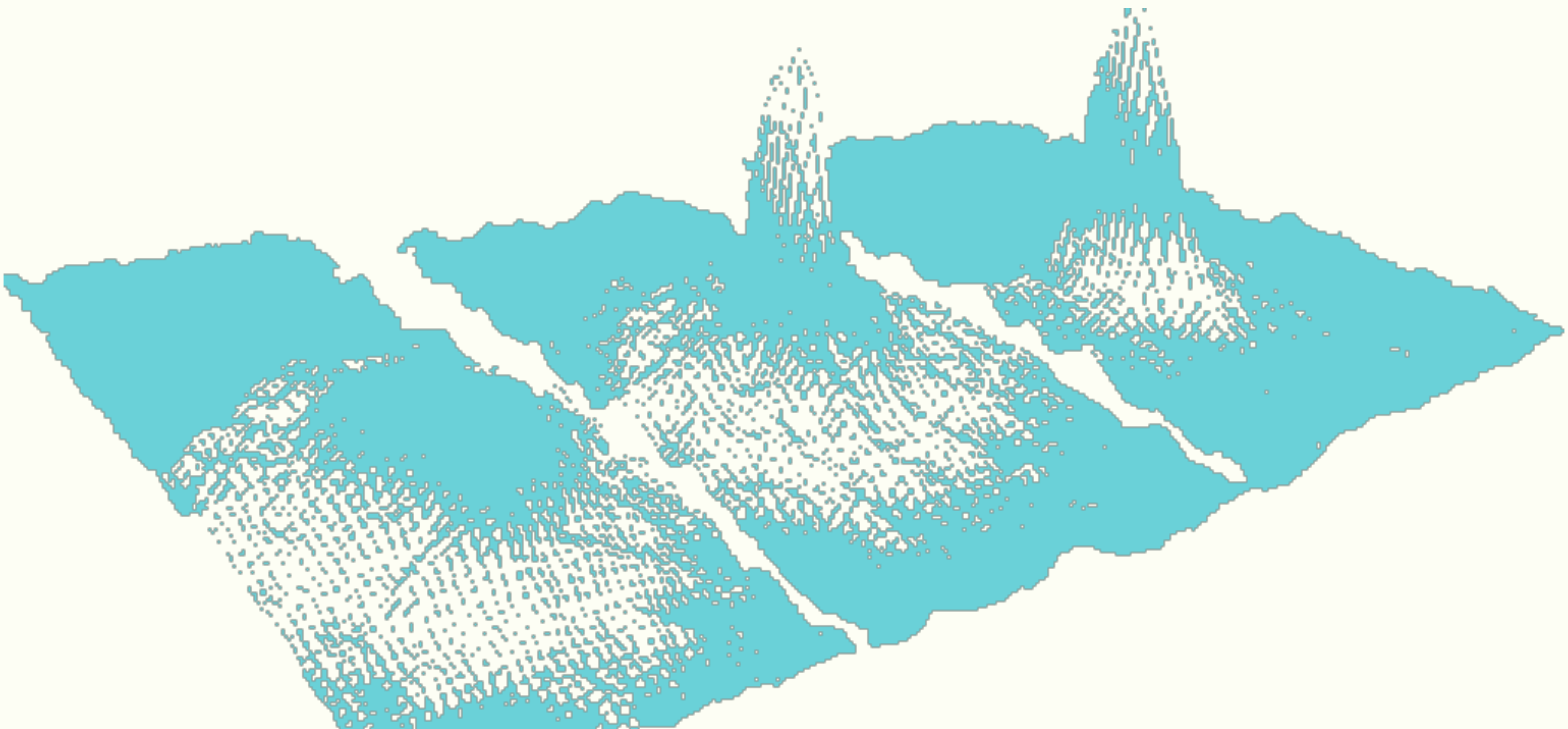
J. Wernsdorfer et al. PRA, 81, 043620 (2010)



quantum nanofluids of helium-4

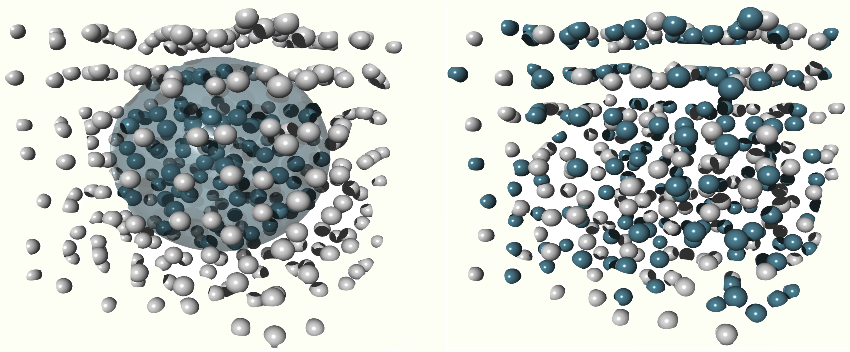
B. Kulchytskyy et al. PRB, 88, 064512 (2013)

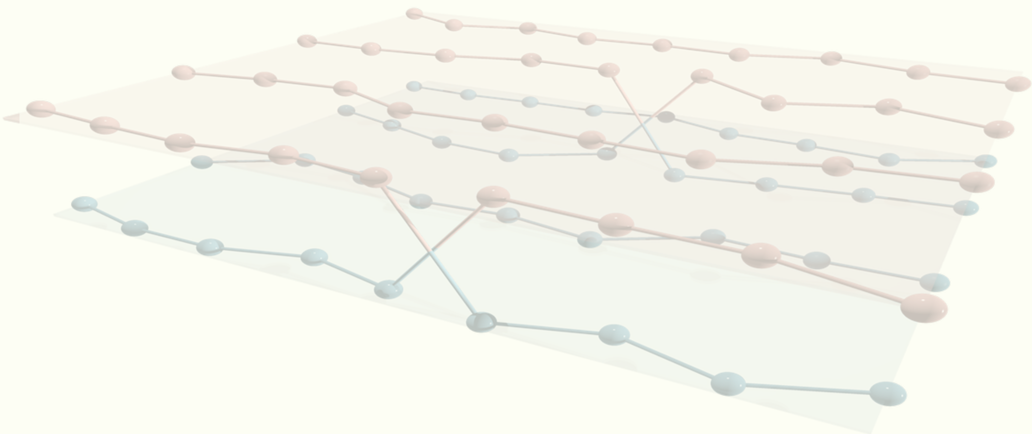
Can we quantify, optimize & employ the entanglement in quantum fluids?



Quantifying Entanglement

bipartite Rényi entropies in the spatial continuum



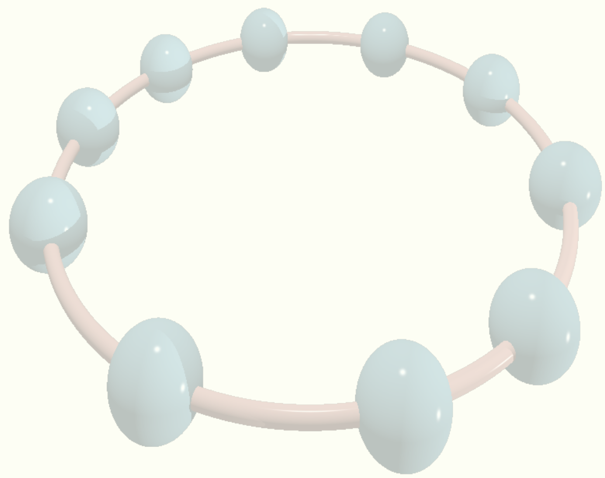


Algorithmic Development

measurement and benchmarking using path integral quantum Monte Carlo

Applications in 1d

interacting bosons and the connection between entanglement and condensate fraction



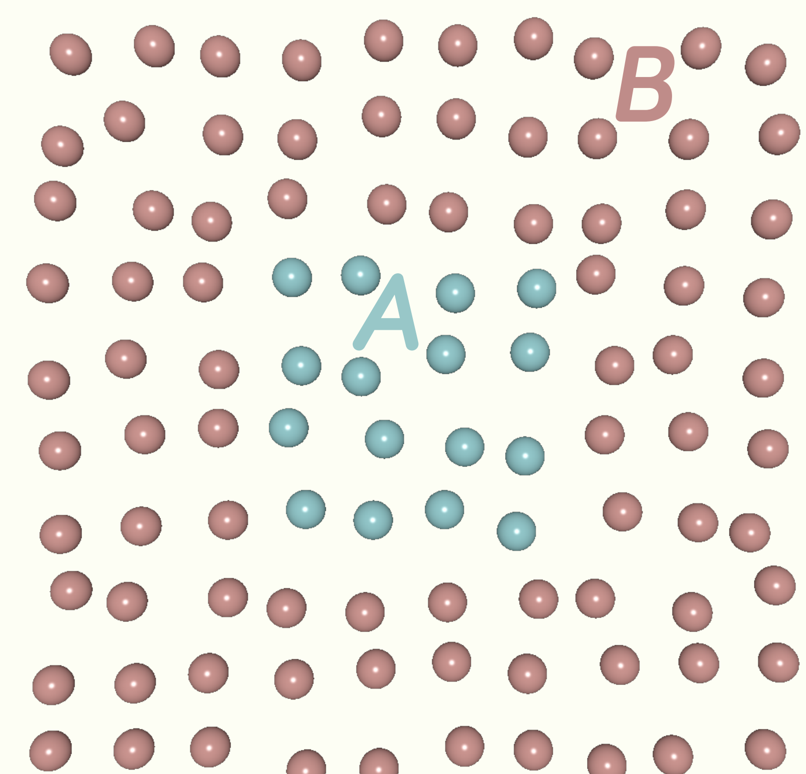
Quantifying entanglement: a prescription

1. *Prepare a system in the spatial continuum*
2. *Bipartition into two subsystems: A & B*
3. *Compute the reduced matrix of region A by tracing over all degrees of freedom in region B*
4. *Measure the entanglement entropy*

$$|\Psi\rangle \stackrel{?}{=} \begin{cases} |\varphi\rangle_A \otimes |\chi\rangle_B \\ \sum_a |\phi_a\rangle_A \otimes |\chi\rangle_B \end{cases}$$

$$\rho \equiv |\Psi\rangle\langle\Psi| \rightarrow \rho_A = \operatorname*{Tr}_B \rho$$

$$S(\rho_A) = -\mathrm{Tr}\rho_A \log \rho_A$$



Rényi Entropies

$$\boxed{S_\alpha[\rho_A] = \frac{1}{1-\alpha} \log \mathrm{Tr}\, \rho_A^\alpha}$$

Different bipartitions of itinerant particles

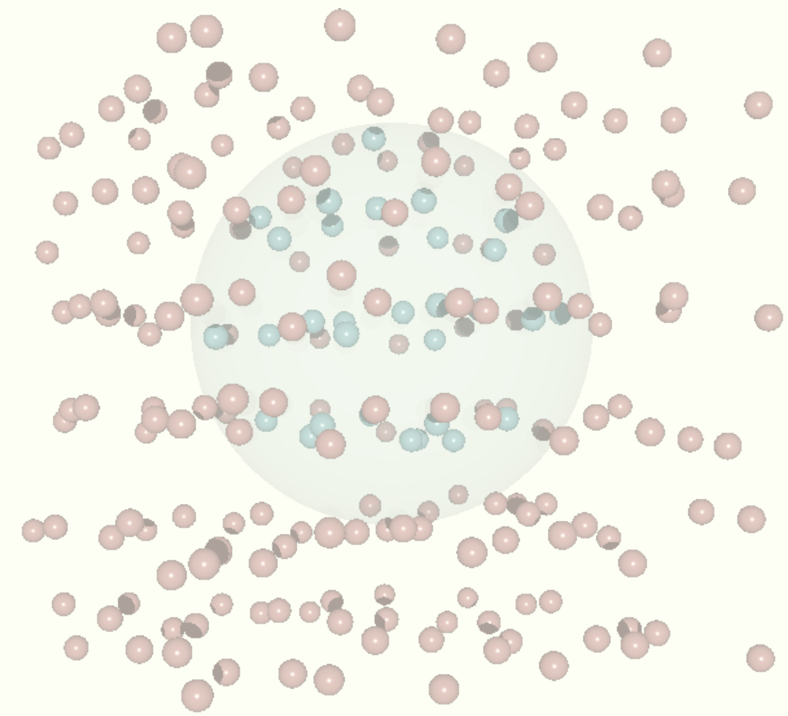
for identical particles in the spatial continuum, various ways to partition ground state

Spatial Bipartition

Constructed from the Fock space of single-particle modes

$$|\Psi\rangle = \sum_{\boldsymbol{n}_A, \boldsymbol{n}_B} c_{\boldsymbol{n}_A \boldsymbol{n}_B} \left|\boldsymbol{n}_A\right\rangle \otimes \left|\boldsymbol{n}_B\right\rangle$$

$$\rho_A \to S(A)$$



Particle Bipartition

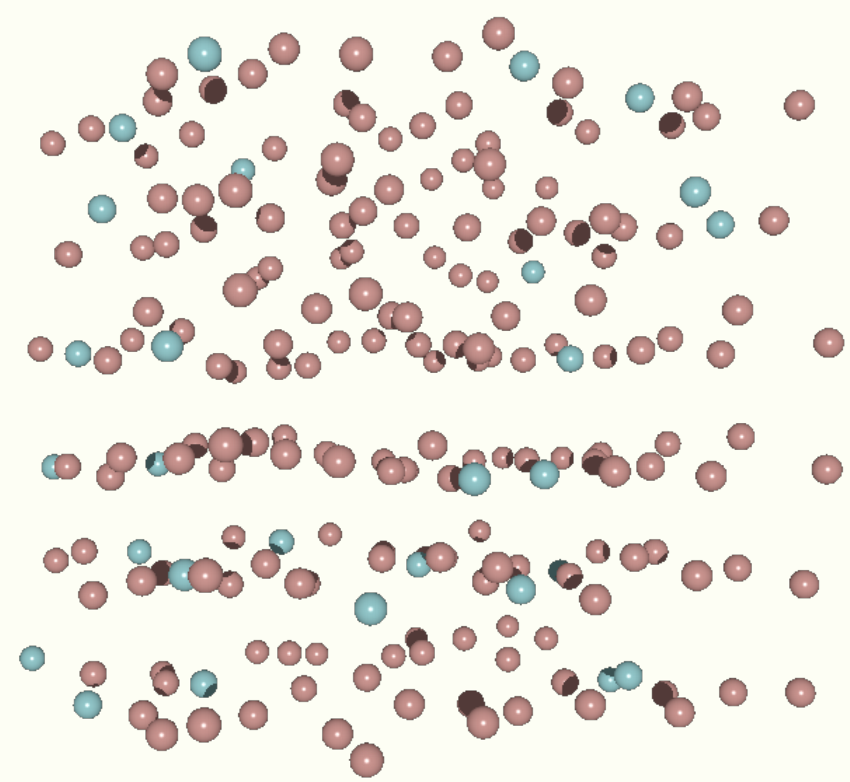
Artificially label a subset of n particles

$$|\Psi\rangle = |\boldsymbol{r}_1 \cdots \boldsymbol{r}_{\boldsymbol{N}}\rangle$$

$$\rho_n = \int d\boldsymbol{r}_{n+1} \cdots d\boldsymbol{r}_N \langle\Psi|\rho|\Psi\rangle$$

n-body density matrix

$$\rho_n \to S(n)$$



Example: entanglement in the free Bose gas

$$|\mathrm{BEC}\rangle \equiv \frac{1}{\sqrt{N!}}\left(\phi_0^\dagger\right)^N |\mathbf{0}\rangle$$

Spatial Bipartition

entanglement is non-zero and is generated via number fluctuations

$$S_2(A) \sim \frac{1}{2}\log \ell_A$$

Particle Bipartition

Ground state is already in product-form in first quantization

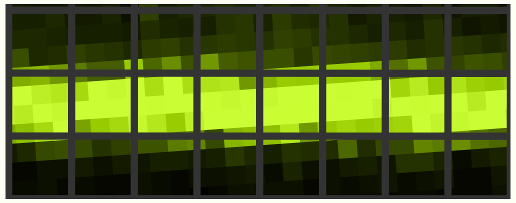
$$S_2(n) = 0$$

C. Simon, PRA 66, 052323 (2002)
W. Ding and K. Yang, PRA 80, 012329 (2009)

How do interactions change this picture?

"toy" quantum fluid: 1d Bose-Hubbard model

$$H_{\mathrm{BH}} = \sum_j \left[-t \left(b_j^\dagger b_{j+1} + \mathrm{h.c.} \right) + \frac{U}{2} n_j \left(n_j - 1 \right) - \mu_j n_j \right]$$



R. Islam et al. (2015)

3 types of candidate ground states

$$|\mathrm{BEC}\rangle \equiv \frac{1}{\sqrt{N!}} \left(\phi_0^\dagger \right)^N |\mathbf{0}\rangle$$

$$|\mathrm{Mott}\rangle \equiv \prod_j b_j^\dagger |\mathbf{0}\rangle$$

$$|\mathrm{Cat}\rangle \equiv \sum_j \frac{1}{\sqrt{L}\sqrt{N!}} \left(b_j^\dagger \right)^N |\mathbf{0}\rangle$$

State	Particle Entanglement	Spatial Entanglement
BEC	0	$1/2 \log L$
Mott	$L \log 2$	0
Cat	$\log L$	$\log 2$

O. Zozulya, M. Haque, and K. Schoutens, PRA 78, 042326 (2008)

Can any of this entanglement be put to use?

Or is it all just fluffy bunnies?

J. Dunningham, A. Rau, and K. Burnett, Science 307, 872 (2005)

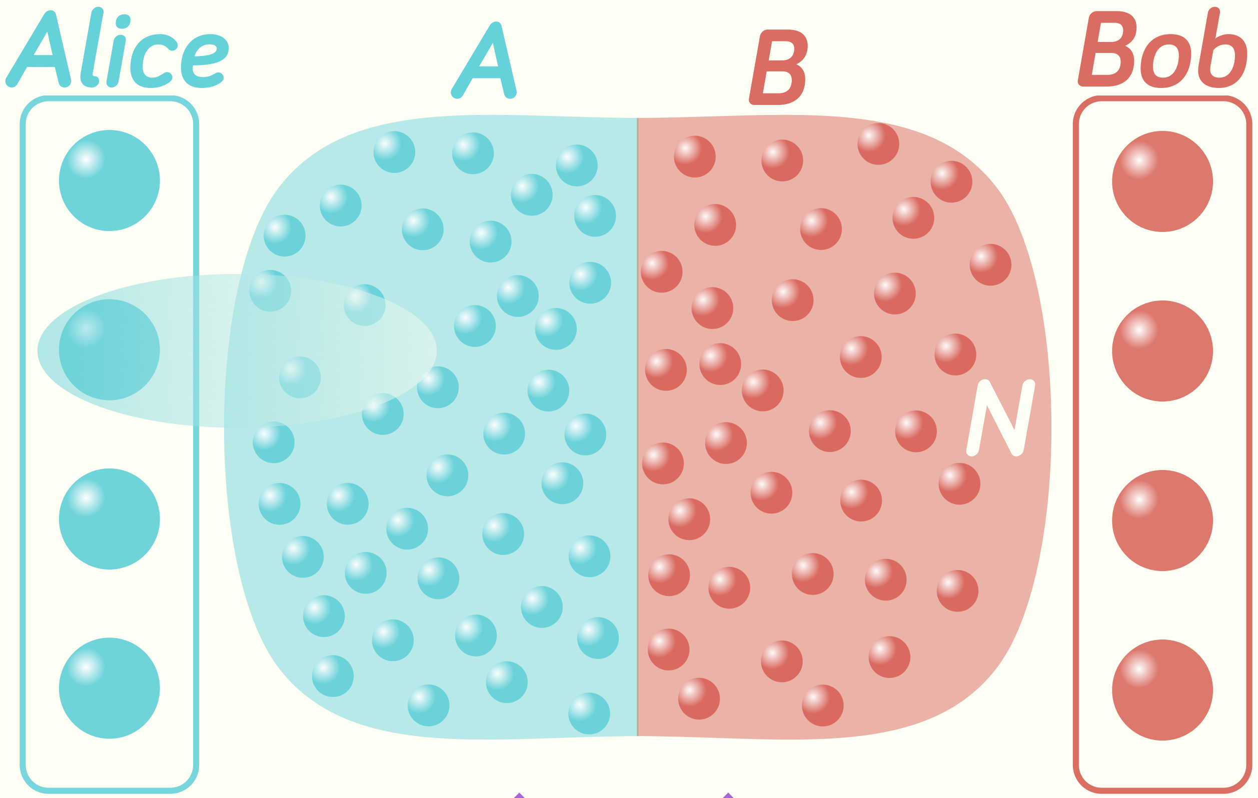
Using entanglement as a resource requires ability to perform local physical operations on subsystems

Particle Entanglement

inaccessible due to the indistinguishability of particles

Spatial Entanglement

particle number conservation prohibits swapping to conventional register



$$\mathrm{SWAP}\left\{\left(|0_{\mathrm{reg}}\rangle + e^{i\phi}|1_{\mathrm{reg}}\rangle\right) \otimes \left(|0_A\rangle \otimes |1_B\rangle\right)\right\} = |0_{\mathrm{reg}}\rangle \otimes \left(\cancel{|0_A\rangle} + \mathrm{e}^{i\phi}|1_A\rangle\right) \otimes |1_B\rangle$$

The Entanglement of Particles

Get around these difficulties by combining the two measures. H. M. Wiseman and J. A. Vaccaro, PRL 91, 097902 (2003)

$$E_p\left(A\right) \equiv \sum_n P_n S\left(\rho_{A,n}\right)$$

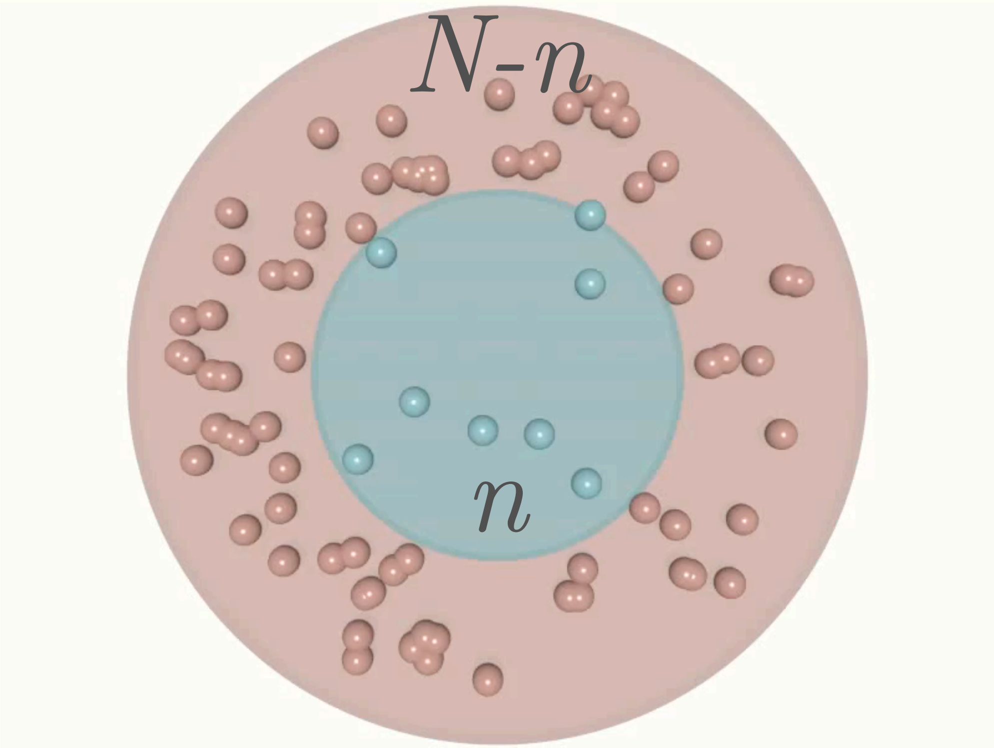
$$\rho_{A,n} \equiv \frac{1}{P_n} \hat{P}_n \rho_A \hat{P}_n$$

probability

projection operator

$$E_p(A) < S(A)$$

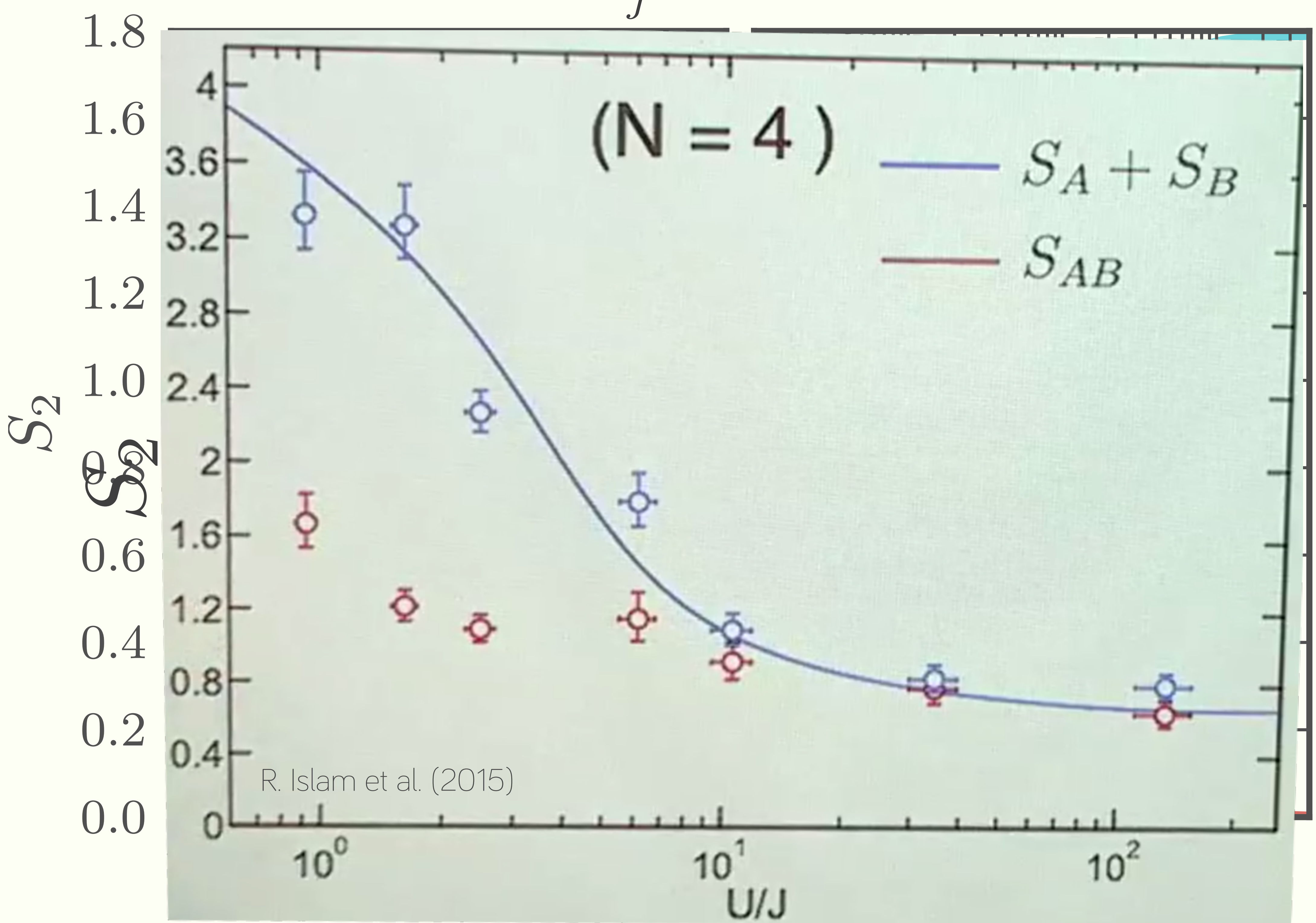
$$E_p(A) > 0 \Rightarrow S(n) > 0$$



E_P is the maximal amount of entanglement that can be produced between quantum registers by local operations.

Back to the Bose-Hubbard model

$$H_{\mathrm{BH}} = \sum_j \left[-t \left(b_j^\dagger b_{j+1} + \mathrm{h.c.} \right) + \frac{U}{2} n_j \left(n_j - 1 \right) \right]$$

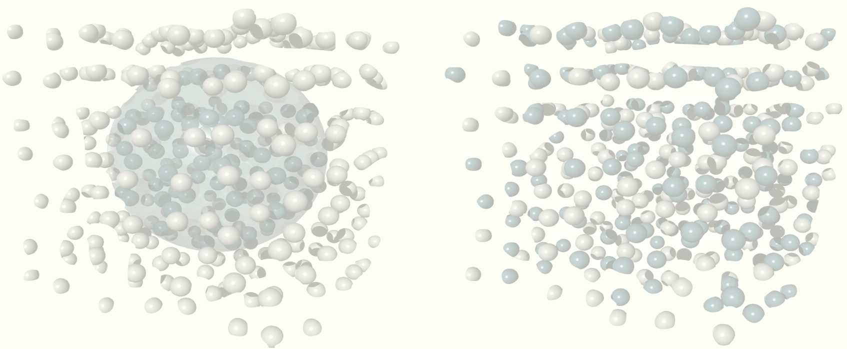


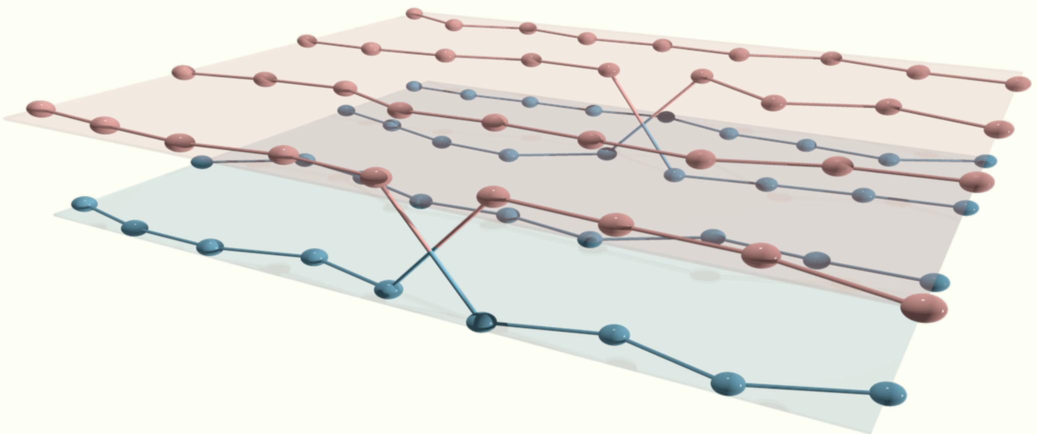
peaked at transition

C. Herdman *et al.* PRE, 90, 013308 (2014)

Quantifying Entanglement

bipartite Rényi entropies in the spatial continuum



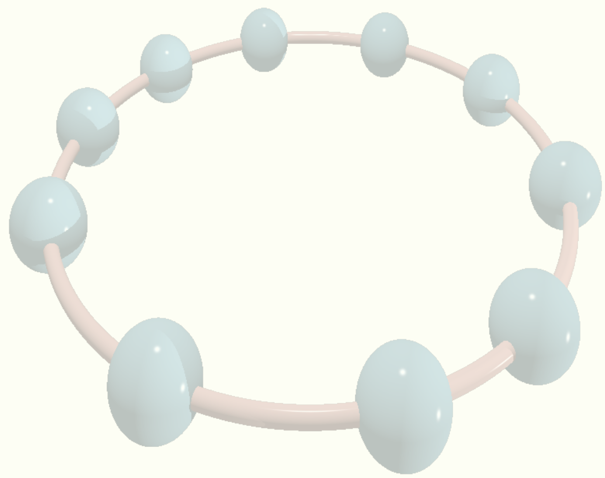


Algorithmic Development

measurement and benchmarking using path integral quantum Monte Carlo

Applications in 1d

interacting bosons and the connection between entanglement and condensate fraction



Path integral ground state quantum Monte Carlo

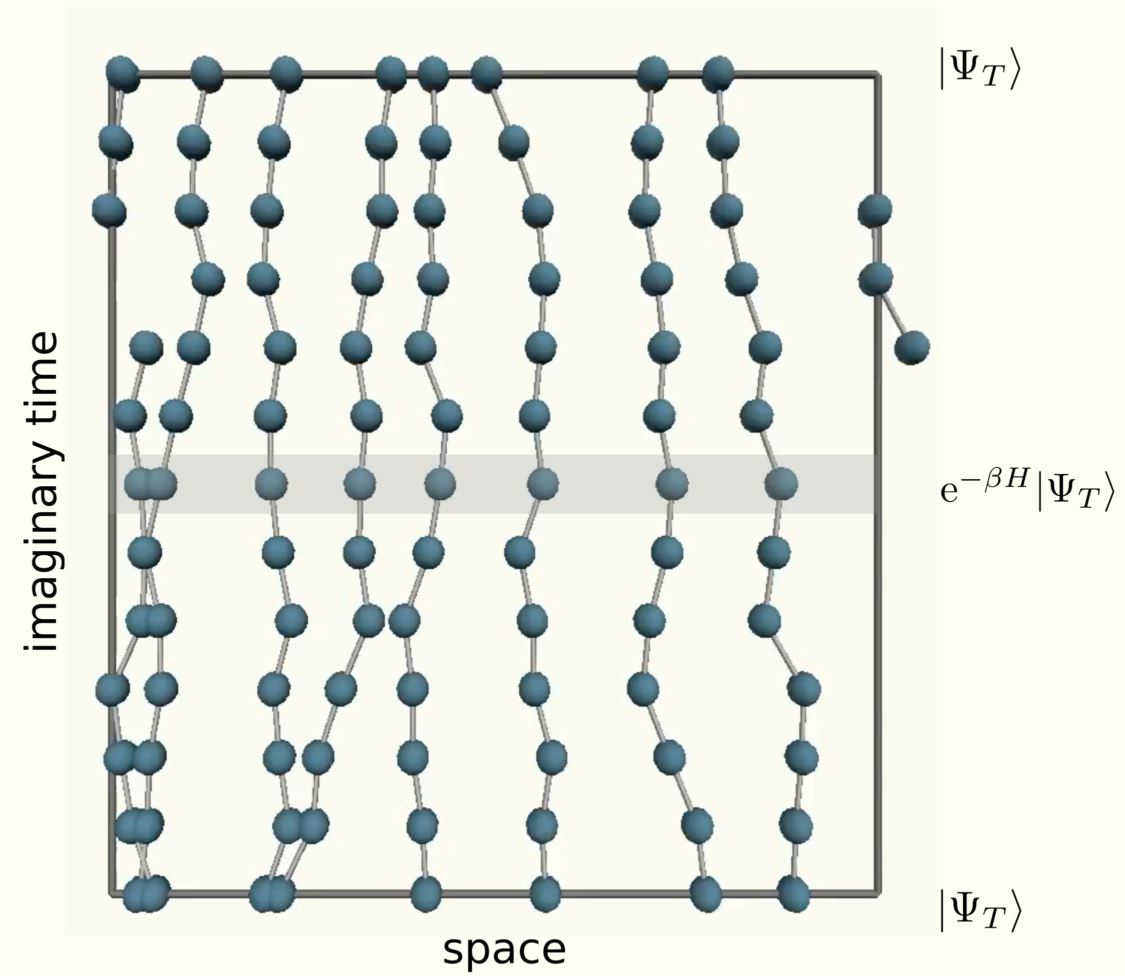
Description

$$H = \sum_{i=1}^{N}\left(-\frac{\hbar^2}{2m_i}\nabla_i^2 + U_i\right) + \sum_{i<j} V_{ij},$$

Projecting

trial wave function onto ground state

$$\left|\Psi\right\rangle = \lim_{\beta\to\infty} \mathrm{e}^{-\beta H}\left|\Psi_T\right\rangle$$



Configurations

discrete imaginary time worldlines constructed from products of short time propagator

$$\rho_\tau(\boldsymbol{R},\boldsymbol{R}') = \langle\boldsymbol{R}|\mathrm{e}^{-\tau H}|\boldsymbol{R}'\rangle$$

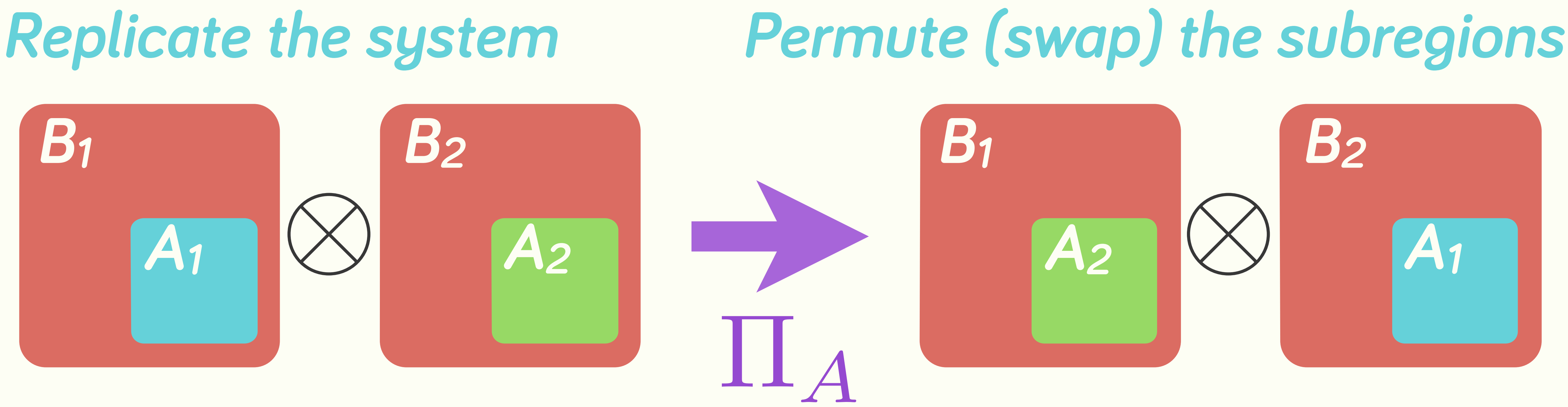
Observables

exact method for computing ground state expectation values

$$\langle\hat{\mathcal{O}}\rangle = \lim_{\beta\to\infty} \frac{\left\langle\Psi_\mathrm{T}\middle|e^{-\beta H}\hat{\mathcal{O}}e^{-\beta H}\middle|\Psi_\mathrm{T}\right\rangle}{\left\langle\Psi_\mathrm{T}\middle|e^{-2\beta H}\middle|\Psi_\mathrm{T}\right\rangle}$$

D. M. Ceperley, RMP 67, 279 (1995)
A. Sarsa, *et. al.*, J. Chem. Phys. 113, 1366 (2000)

Computing Rényi entropies in Monte Carlo



Technology imported from QFT to QMC

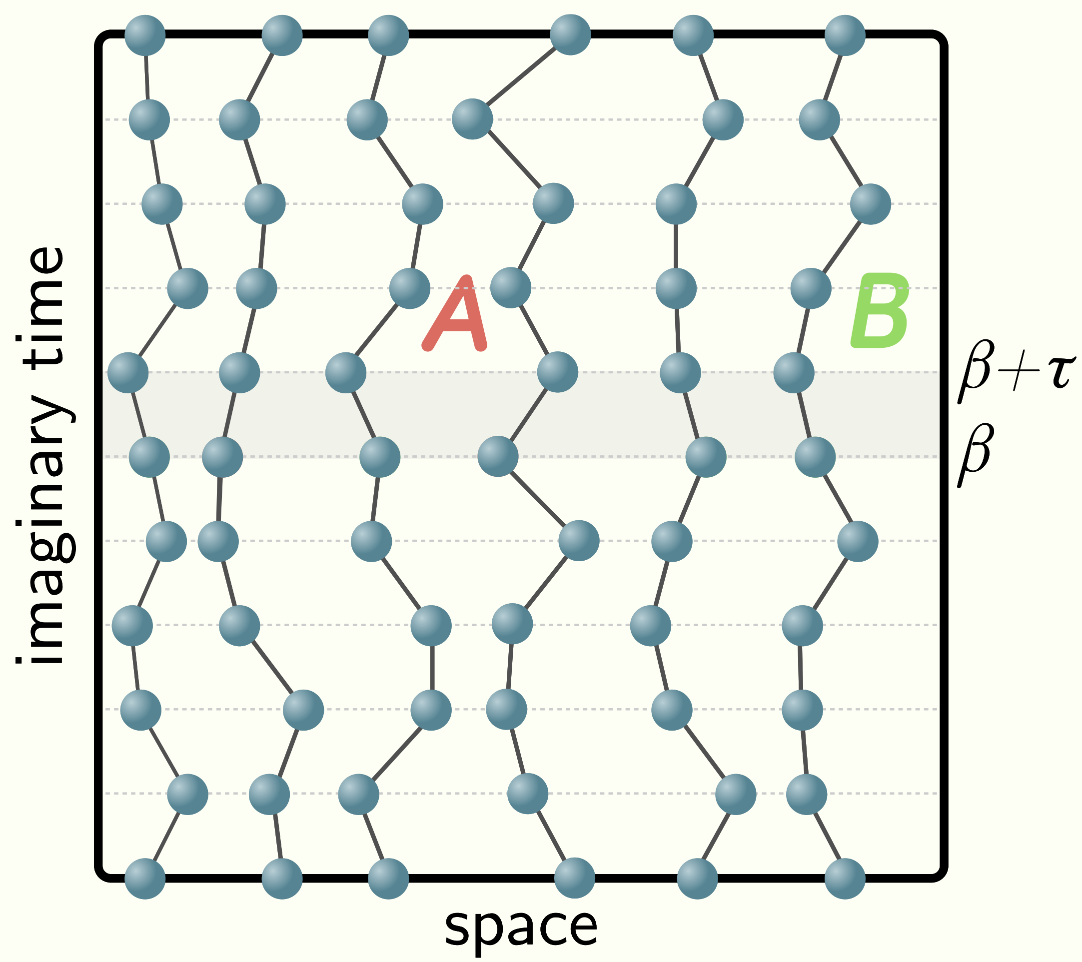
C. Holzhey, F. Larsen, and F. Wilczek, Nuclear Physics B 424, 443 (1994).
P. Calabrese and J. Cardy, J. Stat. Mech.: Theor. Exp. 2004, P06002 (2004)
M. B. Hastings, I. González, A. B. Kallin, and R. G. Melko, PRL 104, 157201 (2010)
R. Melko, A. Kallin, and M. Hastings, PRB 82, 100409 (2010)

For α = 2 replicas, expectation value of the permutation operator is a measure of the 2nd Rényi entropy.

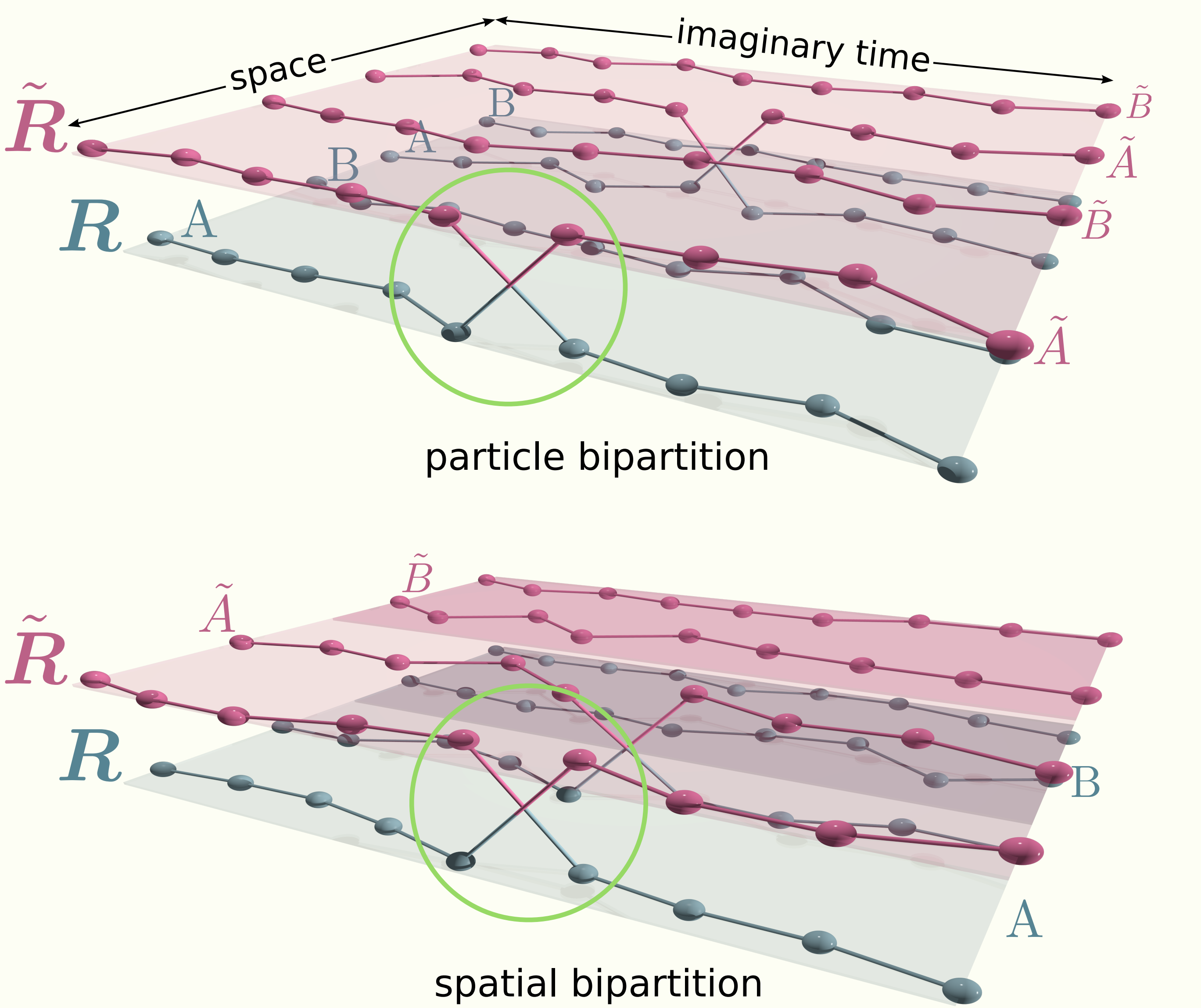
$$\boxed{S_2 = -\log\langle\Pi_A\rangle}$$

Porting to the path integral representation

Break continuous space paths at the center time slice β



The bipartitions only exist at this time slice.
Broken links are in A.



$$\langle \Pi_2^A \rangle \sim \left\langle \rho_\tau^A \left(\boldsymbol{R}^\beta \otimes \tilde{\boldsymbol{R}}^\beta ; \Pi_2^A \left[\boldsymbol{R}^{\beta+\tau} \otimes \tilde{\boldsymbol{R}}^{\beta+\tau} \right] \right) \right\rangle$$

C. Herdman *et al.* Phys. Rev. B, 89, 140501 (2014)
C. Herdman *et al.* Phys. Rev. E, 90, 013308 (2014)

Benchmarking on a non-trivial model

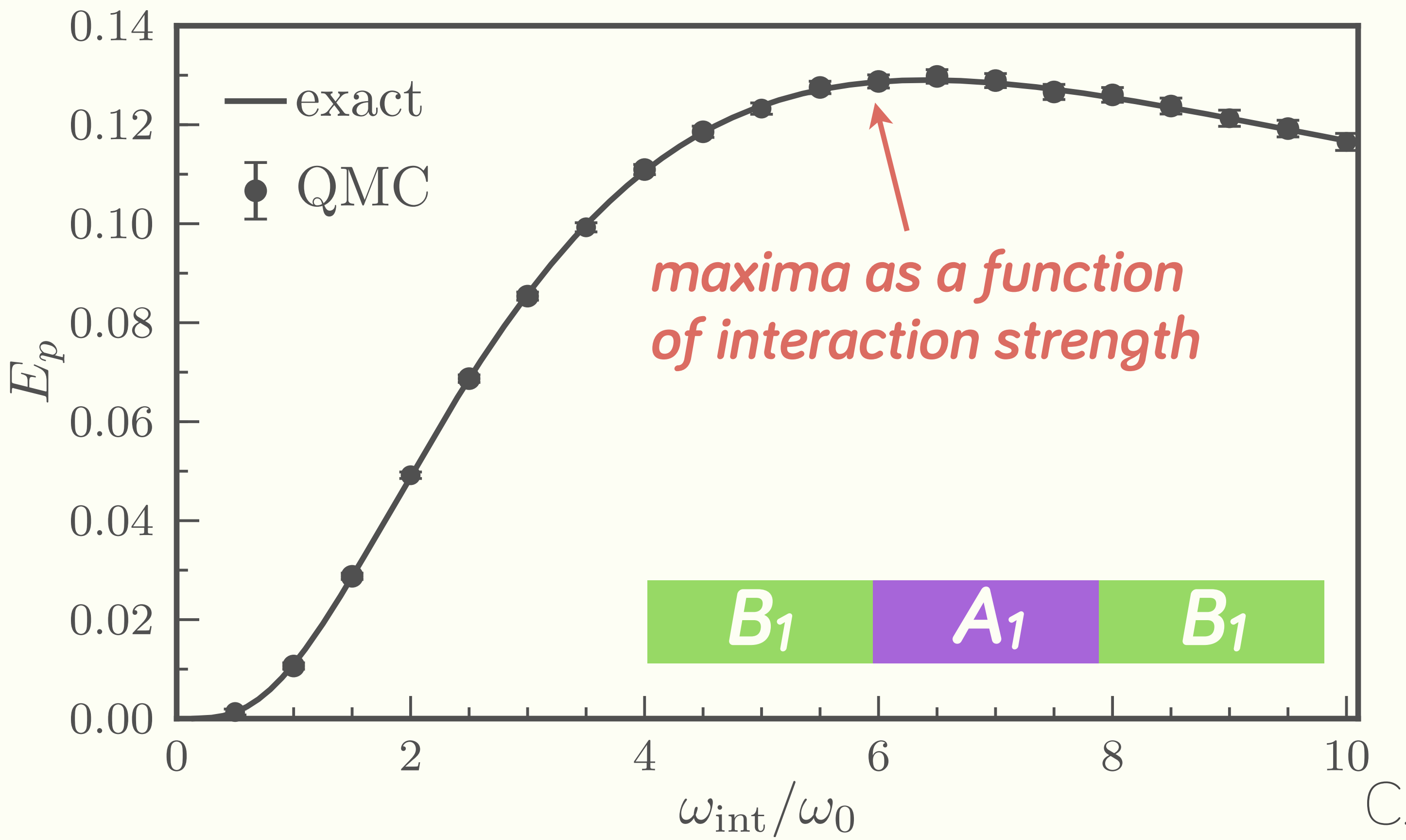
N-Harmonium in 1d

harmonically interacting and confined bosons

$$H = \sum_{i=1}^{N}\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx_i^2} + \frac{1}{2}m\omega_0^2 x_i^2 + \frac{1}{2}m\omega_{\text{int}}^2 \sum_{j>i}(x_i - x_j)^2\right]$$

exact solution can be computed using Wigner quasi-distributions for bosons or fermions C. L. Benavides-Riveros, I. V. Toranzo, and J. S. Dehesa, JPB 47 195503 (2014)

QMC Results: Entanglement of Particles



The useful entanglement is zero for non-interacting particles and peaks at some value of the interaction strength

C. Herdman *et al.*, Phys. Rev. E, 90, 013308 (2014)

Quantifying Entanglement

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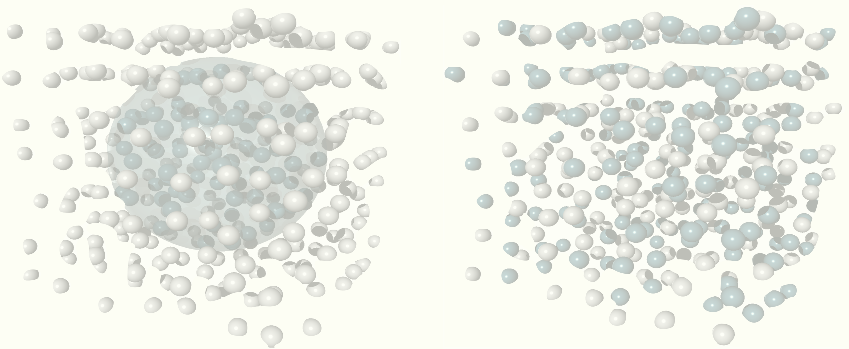
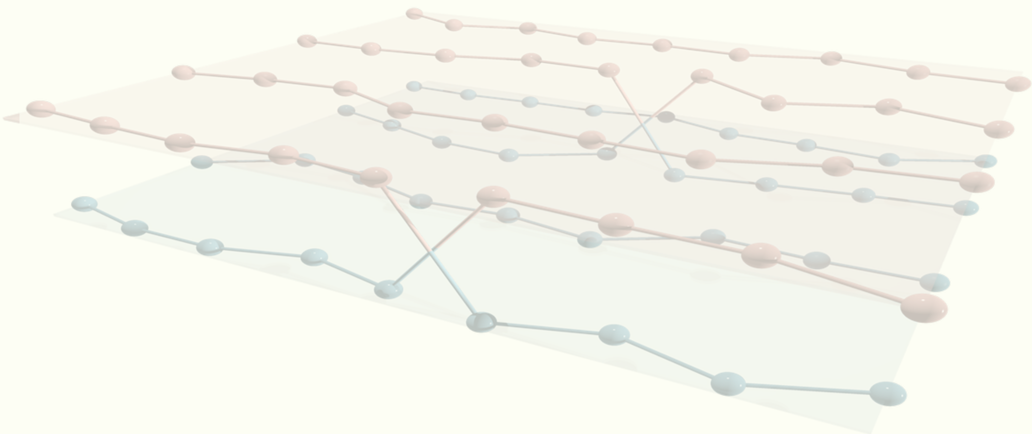


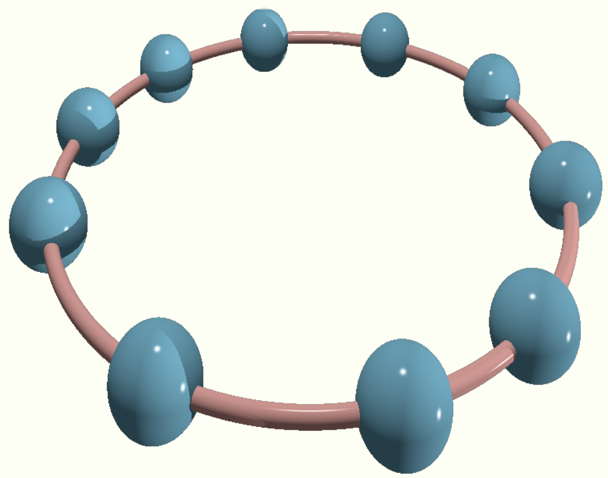


Algorithmic Development

measurement and benchmarking using path integral quantum Monte Carlo

Applications to 1d bosons

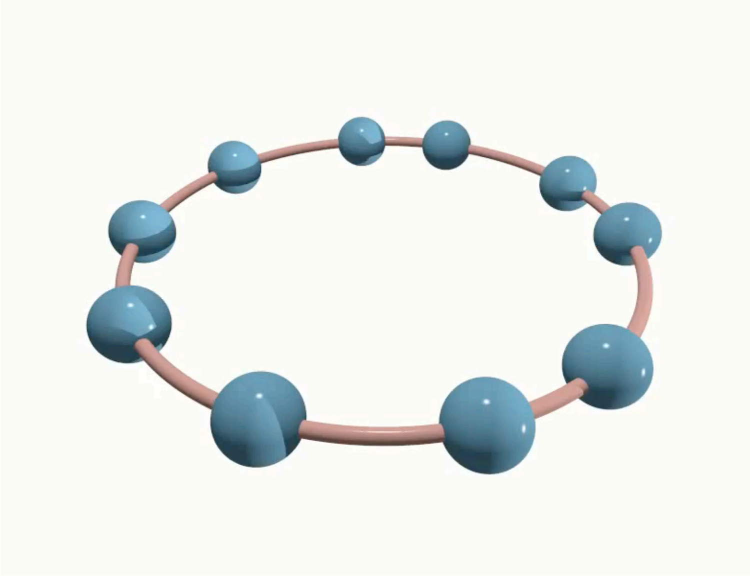
interactions and the connection between entanglement and condensate fraction



Moving towards a physically realizable system

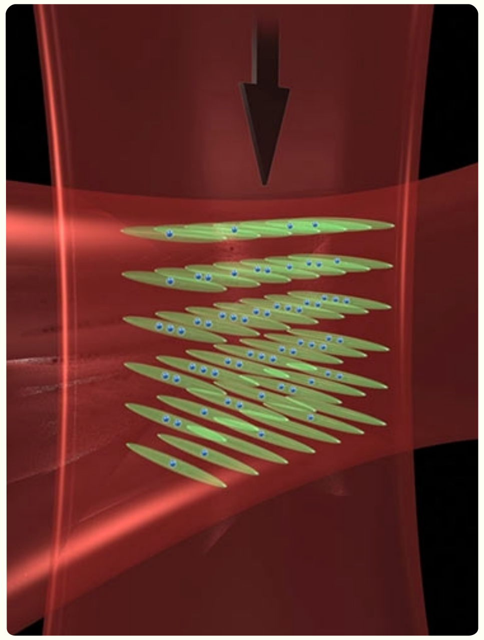
one dimensional short-range interacting bosons

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx_i^2} + \frac{2c}{\sqrt{2\pi\sigma^2}} \sum_{j>i} \mathrm{e}^{-|x_i - x_j|^2/2\sigma^2} \right]$$



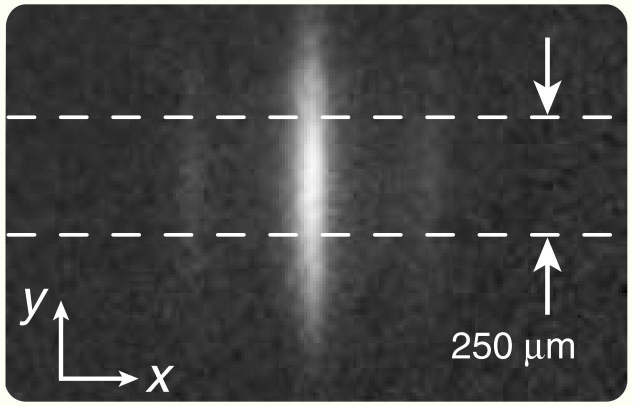
as σ→0 & σ/c → const. we recover the Lieb-Liniger model of delta-function interacting bosons.

E. H. Lieb and W. Liniger, PR 130, 1605 (1963)



c →∞: Tonks-Girardeau gas

B. Paredes, *et al.*, Nature 429, 277 (2004)
T. Kinoshita, *et al.*, Science 305, 1125 (2004)

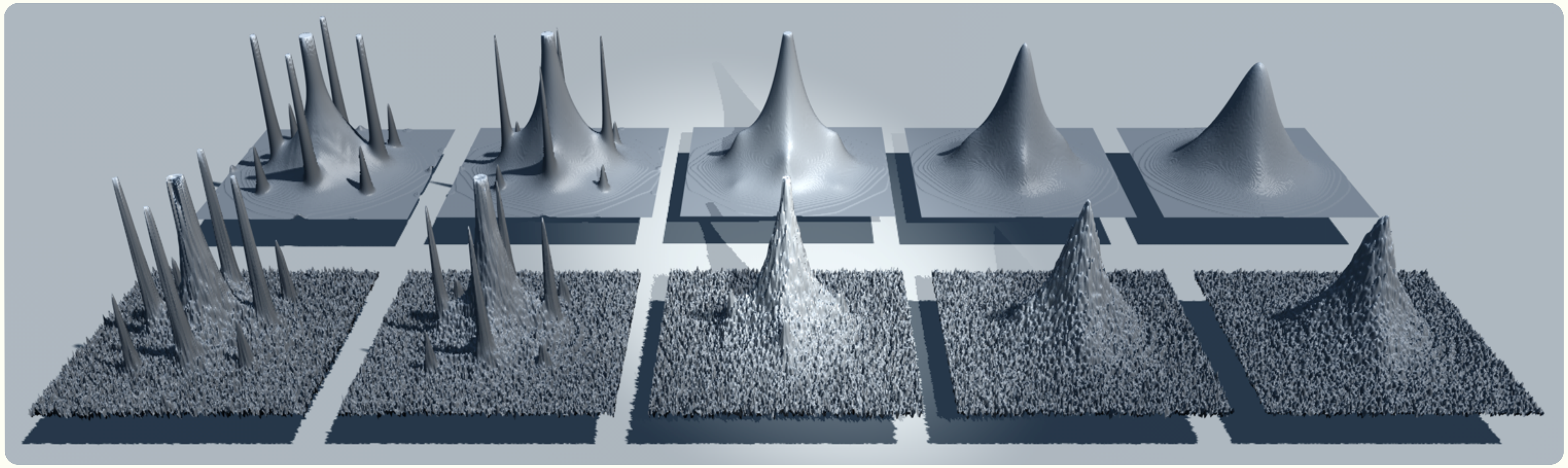


Single particle entanglement is related to the condensate fraction!

the fractional population of particles in the zero-momentum state

$$n_0 = \lim_{|x-x'|\to\infty} \rho_1(x,x') \sim |\Psi_0(x)|^2$$

Easily accessible in experiments and simulations!



QMC

experiment

S. Trotzky, *et al.*, Nat. Phys. 6, 998 (2010)

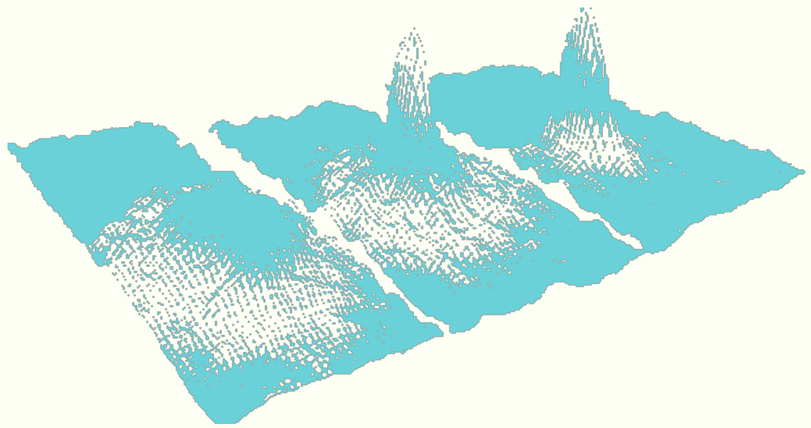
Single particle entanglement is related to the condensate fraction!

$$S_2(n=1) = -\log \mathrm{Tr}\, \rho_1^2$$

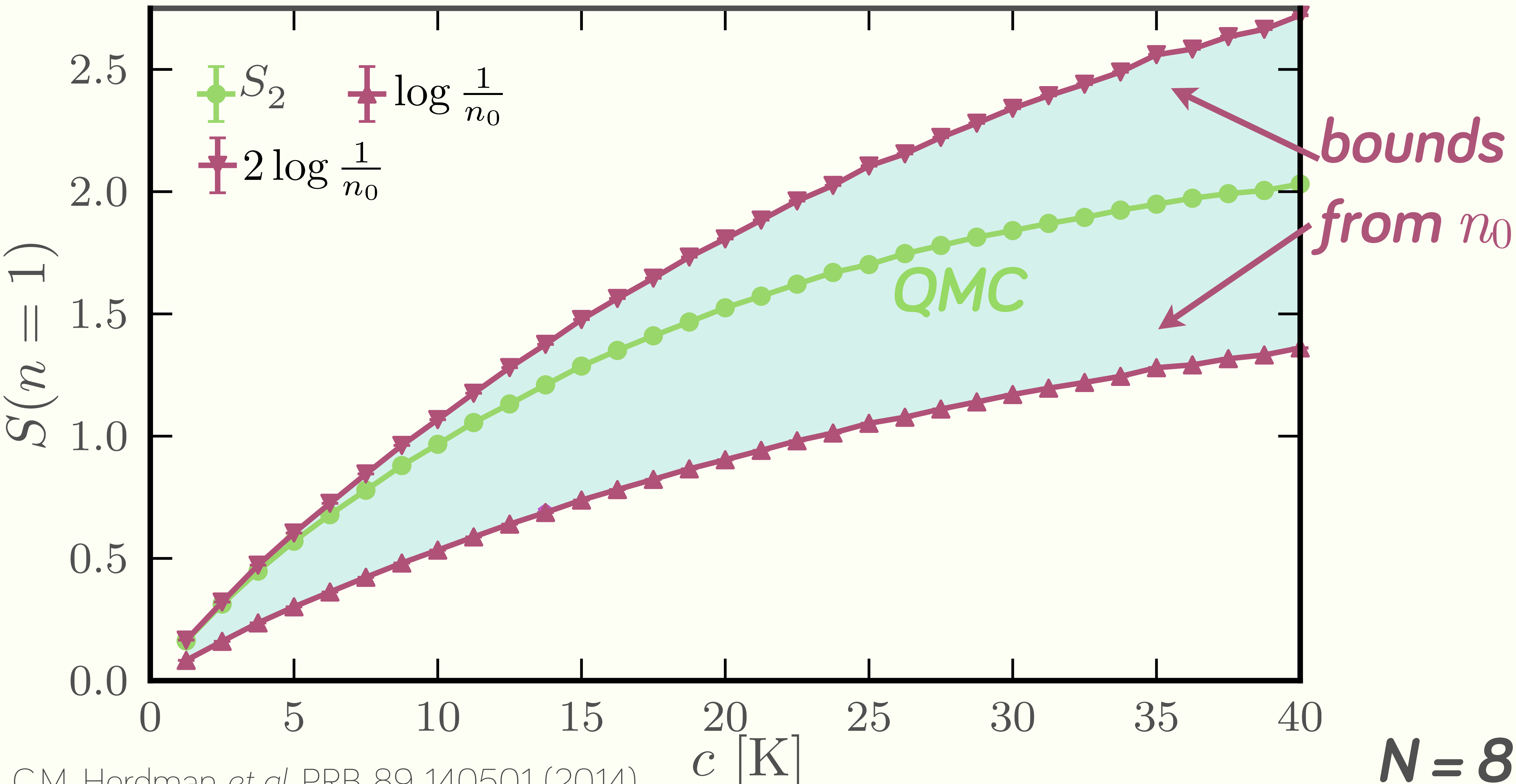
- n_0 is the largest eigenvalue of the one-body density matrix
- determines the single copy entropy: $S_\infty = -\log n_0$
- determines the "max-entropy": $2S_\infty = -2\log n_0$

$$\boxed{\log \frac{1}{n_0} \leq S_2(n=1) \leq 2\log \frac{1}{n_0}}$$

Bounding entanglement with the condensate fraction



$$\log \frac{1}{n_0} \leq S_2(n=1) \leq 2\log \frac{1}{n_0}$$



C.M. Herdman *et al.* PRB, 89, 140501 (2014)

Finite size scaling and universality

Universal "area"-like law for particle entanglement

A canonical scaling function for particle entanglement entropy

O. Zozulya, M. Haque, and K. Schoutens, PRA 78, 042326 (2008)

$$S(n) = an \log N + b$$

$$H_{\mathrm{TLL}} = \frac{\hbar v}{2\pi} \int dx \left[K \left(\partial_x \phi\right)^2 + \frac{1}{K} \left(\partial_x \theta - \rho_0\right)^2 \right]$$

S.-I. Tomonaga, Prog. Theo. Phys. 5, 544 (1951)
J.M. Luttinger, J. Math. Phys. 4, 1154 (1963)
F.D.M. Haldane, PRL 47, 1840 (1981)

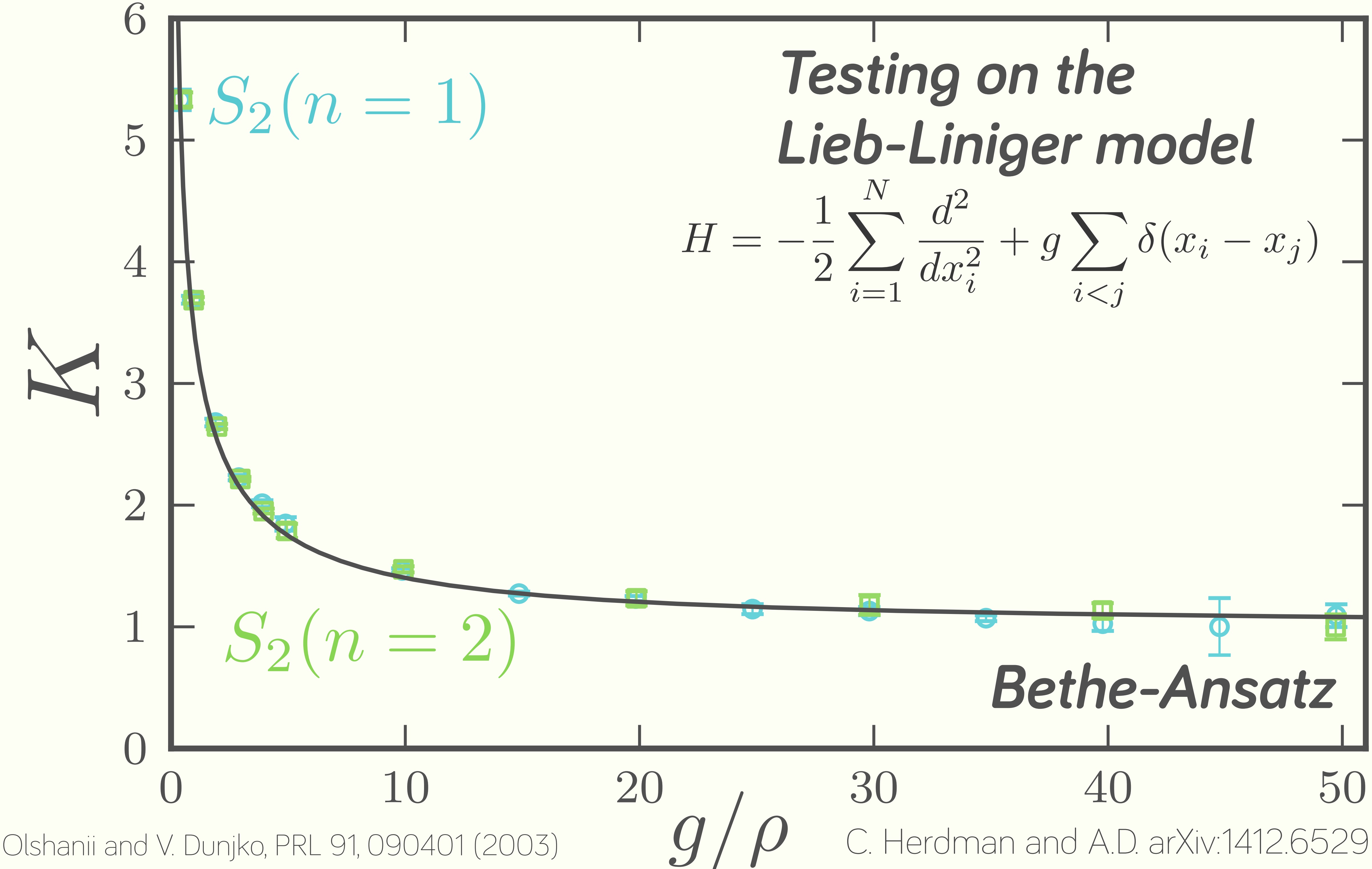
Luttinger Liquid bosonic one-body density matrix:

$$\rho_1(x, x') = \langle \Psi^\dagger(x) \Psi(x') \rangle \sim \frac{1}{|x - x'|^{1/2K}}$$

One-Particle Entanglement $S_2(n=1) = -\log \mathrm{Tr}\, \rho_1^2$

Bosonic Luttinger liquid scaling

$$S_2(n=1) \simeq \boxed{\frac{1}{K}\log N} - \log\left[1 - \frac{1}{K}\left(\frac{N}{2}\right)^{1/K-1}\right] + \log\frac{K-1}{2^{1/K}K}$$



M. Olshanii and V. Dunjko, PRL 91, 090401 (2003)

C. Herdman and A.D. arXiv:1412.6529 (2015)

Open questions & what's next

$$\boxed{S_2\left(n\right) = \frac{n}{K}\log N + \text{const.} + \mathcal{O}\left(\frac{1}{N^{1-1/K}}\right)}$$

- have only numerically confirmed $n > 1$ scaling
- what about other Rényi entropies?
- $n = 1$ pre-factor for a Fermionic Luttinger liquid?
- higher dimensions? ab initio simulations?
- relation to fluctuation entanglement?
- entanglement of particles in more realistic systems
- corrections to scaling for spatial bipartitions in the continuum?

We can quantify entanglement in ultracold Bose gases!

Experimental measurement & optimization

Bound entanglement via the condensate fraction and learn how to optimize the functional entanglement that can be transferred to a register for quantum information processing.

Applications to low dimensional field theories

Scaling of the one-particle entanglement is related to the Luttinger parameter of the effective field theory.

http://delmaestro.org/adrian

http://code.delmaestro.org

@agdelma

Computing resources and partners in research

Vermont Complex Systems Center

Calcul Québec

compute • calcul
CANADA

VACC
Vermont Advanced Computing Core
University of Vermont

XSEDE
Extreme Science and Engineering
Discovery Environment