Scaling of Entanglement in Scale Invariant 2+1 Dimensional Quantum Field Theories

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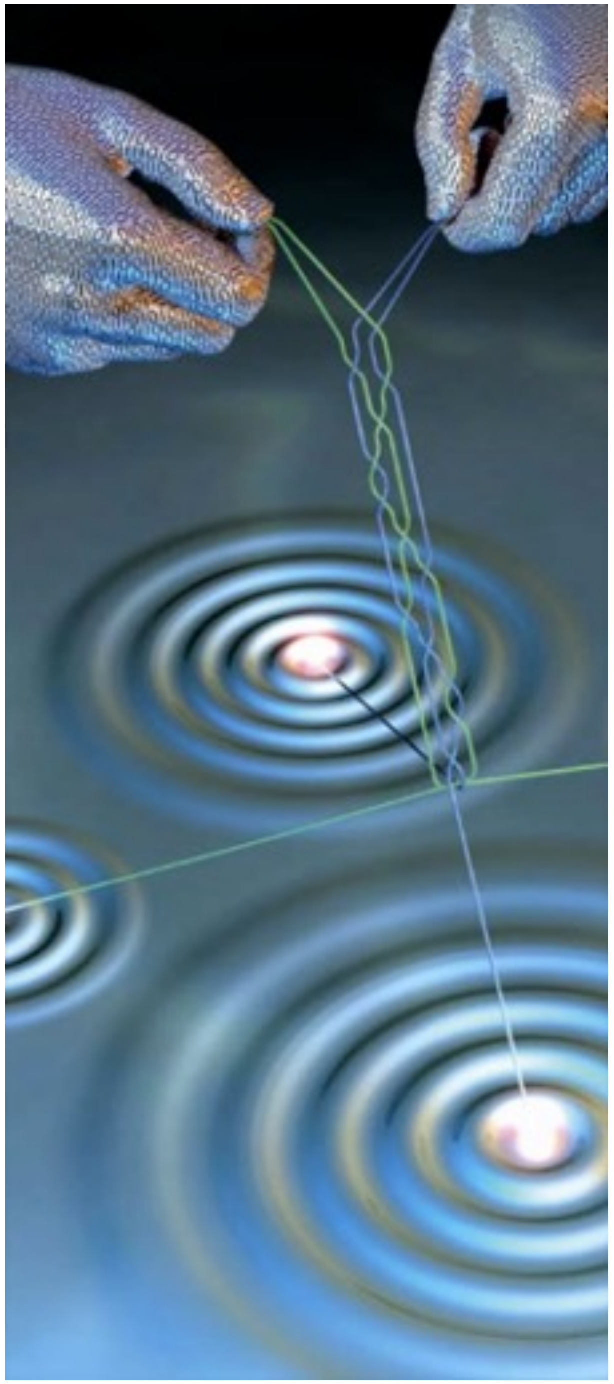
Entanglement

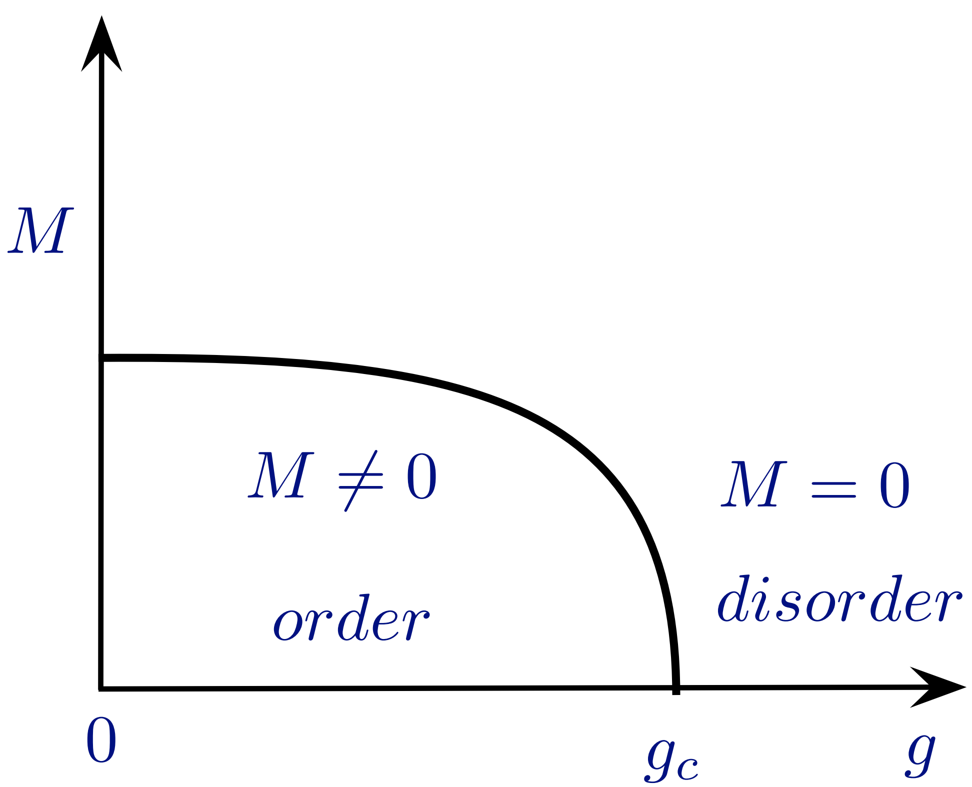


Quantum Phase Transitions and Quantum Criticality

- Phase transitions at T=0 as a function of some coupling constant
- Typical example: Ising model in a transverse field

$$H = -\sum_n \sigma_3(n)\sigma_3(n+1) - g\sum_n \sigma_1(n)$$

$\sigma_1(n)$ and $\sigma_3(n)$ are 2 x 2 Pauli matrices at each site n



- Global $\mathbb{Z}_2$ symmetry and spontaneous symmetry breaking
- Order parameter $\langle M \rangle \neq 0$
- Quantum critical point at g_c
- Scale Invariance
- Energy gap $\Delta \sim |g-g_c|^{\nu z}$
- Correlation length $\xi \sim |g-g_c|^{-\nu}$
- Ising model: z=1 (effectively Lorentz invariant)

Quantum Criticality and QFT

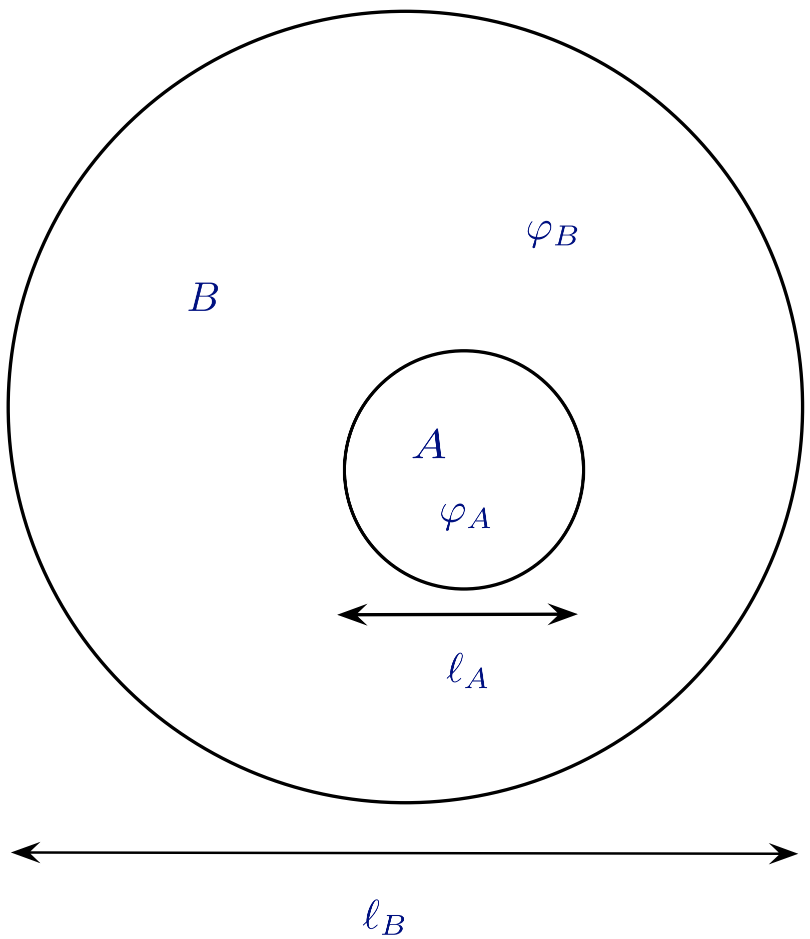
- This description of a **quantum critical point** is similar to a **classical critical point** at a thermal phase transition
- In both problems it describes the behavior of **local fluctuating operators**: their correlations and scaling
- This is not an accident. For $z=1$, d-dimensional quantum criticality $\Rightarrow$ d+1-dimensional classical criticality
- Quantum Mechanics dictates the dynamics: dynamic critical exponent z
- The scaling behavior at and near a quantum critical point is described by (and defines) a local Quantum Field Theory
- Question: is there a more intrinsically quantum mechanical signature of a quantum critical point, i.e. without a classical analog?
- Natural candidate: **Quantum Entanglement**, measured by the **von Neumann Entanglement Entropy**

Quantum Entanglement in Condensed Matter

The application of the concept of entanglement in Condensed Matter (and in QFT) raises some questions

- It measures the global entanglement between regions A and B and hence very non-local quantum correlations
- We would like to determine its behavior, e.g. universal properties, dependence on the geometry of the regions, etc, in ordered and disordered phases, in topological phases and at quantum criticality
- In particular: how is the scaling of the entanglement entropy related to the scaling properties of local observables?

Density Matrix and Entanglement Entropy



- Pure state in A ∪ B: $\Psi[\varphi_A,\varphi_B]$
- Density Matrix

$$\langle \varphi_A, \varphi_B | \rho_{A\cup B} | \varphi'_A, \varphi'_B \rangle = \Psi[\varphi_A, \varphi_B] \Psi^*[\varphi'_A, \varphi'_B]$$

Observing only A: mixed state with a reduced density matrix ρ_A

$$\langle \varphi_A | \rho_{A\cup B} | \varphi'_A \rangle = \mathrm{tr}_B \rho_{A\cup B}$$

von Neumann Entanglement Entropy:

$$S_{vN} = -\mathrm{tr}\,(\rho_A \log \rho_A)$$

Scaling of Entanglement

- We will consider regions A and B such that

$$\ell_B \gg \ell_A \gg \xi \gg a$$

ξ is the correlation length (if finite) and a is the short-distance cutoff

- In a massive phase in d space dimensions ξ is finite and the entanglement entropy obeys the **Area Law**

$$S_{vN} = \alpha \left(\frac{\ell_A}{a} \right)^{d-1} + \ldots \qquad \text{(Srednicki, 1993)}$$

- The Area Law is obeyed in all local theories, massive or massless, with d>1. This law is due to short range correlations encoded in the ground state wave functional
- The prefactor of the Area Law is **not universal** as it depends on the short-distance cutoff a
- Are there **universal subleading contributions** to the entanglement entropy?

Scaling of Entanglement in Topological Phases

- Topological phases are fully gapped states with trivial local properties but are globally non-trivial.
- Their ground states span a finite-dimensional Hilbert space whose dimension depends on the topology of the space
- The effective field theories are **topological quantum field theories** (TQFT)
- The prototype topological phases are the Laughlin states of **fractional quantum Hall fluids** (and their generalizations)
- The effective field theory is a **Chern-Simons gauge theory**
- For a topologically trivial region A, the entanglement entropy has the scaling $S_{vN}=\alpha\ (l/a)-\gamma+\ldots$ (Kitaev and Preskill (2006), Levin and Wen (2006)). The coefficient γ is expressed in terms of a topological invariant $\mathcal{D}$, the effective quantum dimension: $\gamma= \ln \mathcal{D}$
- For topologically non-trivial regions (i.e cylinder sections of a 2-torus) the coefficient γ depends also of the linear combination of ground state of the surface and on the topological data of the TQFT (Dong, Fradkin, Leigh, Nowling, 2008)

Scaling of Entanglement in 1+1 Dimensions

- The case where the scaling of entanglement is fully understood is in d=1 space dimensions
- At quantum criticality these systems are described by relativistic conformal field theories with central charge c
- The entanglement entropy of a 1+1 dimensional CFT obeys the logarithmic scaling law (Holzhey, Larsen and Wilczek(1994): Cardy and Calabrese (2004): Vidal, Latorre, Rico and Kitaev (2003))

$$S_{vN} = \frac{c}{3} \log \left(\frac{\ell_A}{a} \right) + \dots$$

- This scaling law is also found at infinite disorder fixed points of random spin chains (Refael and Moore, 2004)
- Subdominant finite terms yield universal contributions to the mutual information for two intervals and depend on the full operator content of the CFT

Scaling of Entanglement in General Dimension

- For general *odd* space dimension d the entanglement entropy has an expansion of the form $S_{vN} = \alpha\, (l/a_0)^{d-1} + \alpha'\, (l/a_0)^{d-3} + \ldots + s \ln (l/a_0) + \ldots$

- This expression is verified both in free field theories (Casini and Huerta, 2008) and in the AdS/CFT correspondence using the Ryu-Takayanagi ansatz (2006)

- The first term is the Area Law. The coefficients α, α', etc. are non-universal

- The coefficient s of the logarithmic term is universal

- In d=3 (i.e. 3+1 dimensions) CFT, the coefficient s is expressed in terms of integrals of the entangling surface and of the central charges a and c

Scaling of Entanglement in 2+1 Dimensions

- Much less is known about the universal terms in the scaling of entanglement in odd space-time dimensions at quantum criticality

- For simply-connected regions with the shape of a **disk** in d=2 dimensions the entanglement entropy has the form of the F theorem: $S_{vN}=\alpha\, (l/a)-F+\ldots$, where the constant term *F* is finite and universal (Klebanov, Pufu, Safdi (2011); Casini, Huerta and Myers (2011))

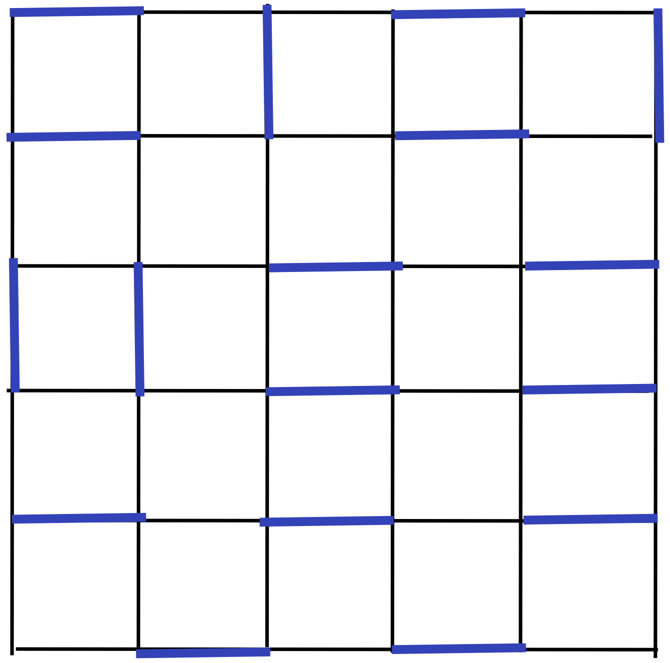
- Here we will discuss the current evidence for universal finite terms in several d=2 quantum critical systems of interest

- We will focus on the case in which the d=2 surface is a 2-torus and the observed regions are cylindrical sections of the torus.

Quantum Dimer Models

- Quantum dimer models are simple models of 2d frustrated (or large N) antiferromagnets (Kivelson and Rokhsar, 1988)
- The space of states $|C\rangle$ are the dimer coverings of the lattice, each dimer being regarded as a spin singlet (**valence bond**) of a pair of nearby spins
- The ground states have the RVB form

$$|\Psi_0\rangle = \sum_{\{\mathcal{C}\}} e^{-\beta E[\mathcal{C}]} |\mathcal{C}\rangle$$



- For small enough β, on bipartite lattices (square, honeycomb, etc) these states represent quantum critical points
- On non-bipartite lattices (also for small enough β) they represent $\mathbb{Z}_2$ topological states (deconfined $\mathbb{Z}_2$ gauge theories or Kitaev's Toric Code states)
- For large β they represent ordered (VBS) states

The Quantum Lifshitz Model

- The Quantum Lifshitz Model is the effective continuum theory of the critical Quantum Dimer Model (Ardonne, Fendley, Fradkin, 2004)
- It is a free field theory in two space dimensions for a scalar field ϕ, a compactified boson representing the coarse-grained configurations of dimers (as dual "heights")
- It has dynamic critical exponent z=2
    - Hamiltonian: $$\mathcal{H} = \frac{\Pi^2}{2} + \frac{\kappa^2}{2} \left(\nabla^2 \phi\right)^2, \qquad [\phi(\mathbf{x}), \Pi(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})$$
- Ground State Wave Functional: $$\Psi_0[\phi(\mathbf{x})] = \text{const} \times e^{-\frac{\kappa}{2}\int d^2x \, \left(\nabla\phi(\mathbf{x})\right)^2}$$
    - The ground state wave functional is conformally invariant

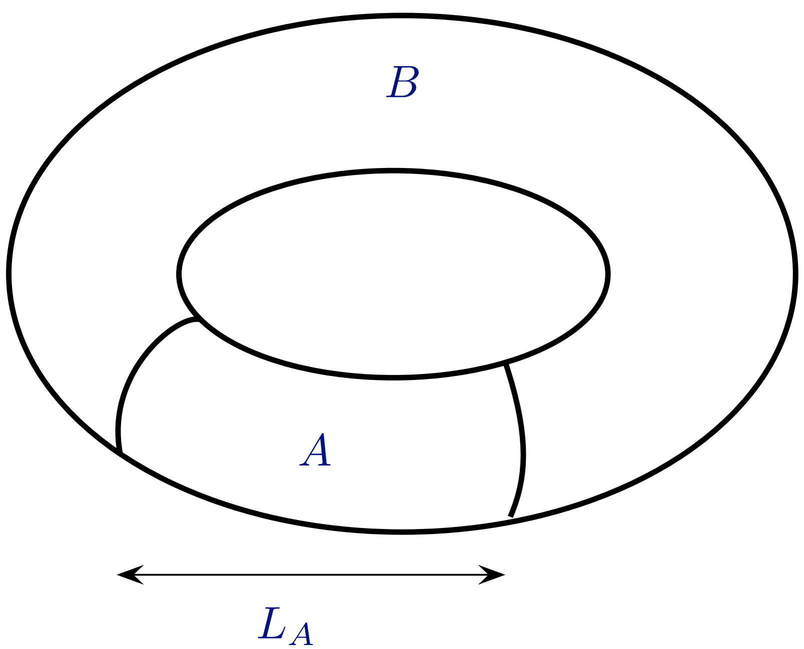
Entanglement Scaling in the QLM

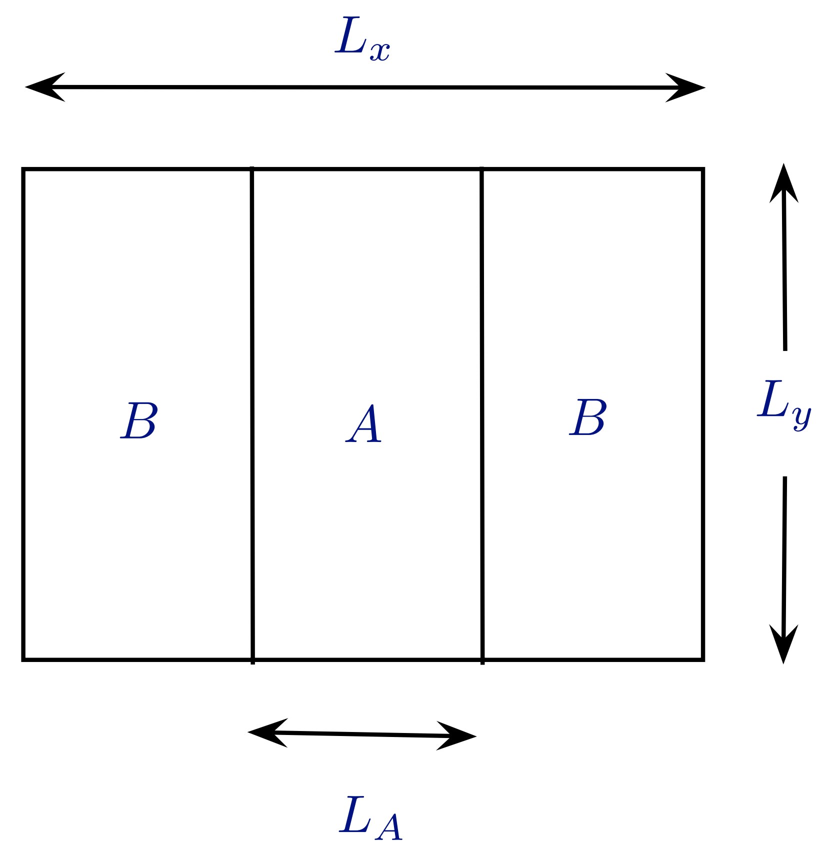
- On **simply connected region** A with a smooth boundary the entanglement entropy obeys the scaling **$S_{vN}=\alpha$ (l/a)+γ+…**, where γ is universal (Fradkin and Moore, 2006)
- If the boundary has corners (cusps), or ends at the edge of the system, S_{vN} has **universal logarithmic terms** (Zaletel, Bardarson and Moore, 2011)
- The Rényi entanglement entropy S_n is computed in terms of ratios of 2D classical partition functions

$$\mathrm{tr}\rho_A^n = \frac{Z_n}{Z^n}$$

Z_n: partition function of n copies glued at the boundary or the observed region A
Z: partition function (norm of the wave function) for one copy

- On a 2 torus the entanglement entropy of a cylindrical section has a **universal finite term which depends on the aspect ratio of the cylinder** (Hsu, Mulligan, Fradkin, Kim (2008); Hsu and Fradkin (2010); Oshikawa (2010); Stéphan, Furukawa, Misguich and Pasquier (2009, 2012))
- For a 2 torus with aspect ratio L_y/L_x and for a cylindrical region A with aspect ratio $u = L_A/L_x$, the entanglement entropy is $S_{vN}=2\alpha\ (L_y/a)+\beta\ J(u)$ (Stéphan et al)





$$J(u) = \log\left(\frac{\lambda}{2}\frac{\eta(\tau)^2}{\theta_3(\lambda\tau)\theta_3(\tau/\lambda)}\frac{\theta_3(\lambda u\tau)\theta_3(\lambda(1-u)\tau)}{\eta(2u\tau)\eta(2(1-u)\tau)}\right)$$

$\theta_3(z)$: Jacobi theta-function

$\eta(z)$: Dedekind eta-function

For the quantum dimer model at the RK point, $\lambda=2$

$$\tau = i\frac{L_y}{L_x}$$

Scaling of Entanglement in the 2+1 dimensional Ising Model in a Transverse Field

- Inglis and Melko (2013) studied numerically the second Rényi entropy S_2 at the quantum critical point of the 2D TFIM on a 2 torus
- This system is in the universality class of the ϕ^4 Wilson-Fisher fixed point in 2+1 dimensions
- Two different scaling functions:
- $S_2 = 2\alpha\, L_y + \beta \log(\sin(\pi L_x/L))$
- $S_2 = 2\alpha\, L_y + \beta\, J(u) + \text{const}$

The QLM scaling works surprisingly well even though is not relativistic!

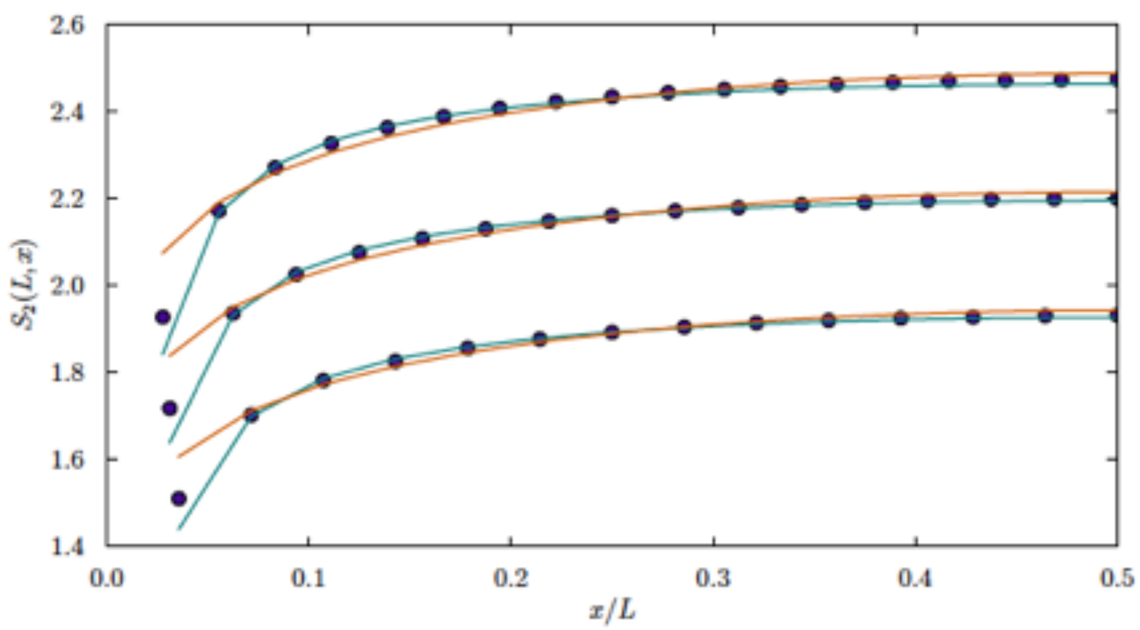


Figure 7. The entanglement entropy of the $L = 28, 32, 36$ systems using two cylinders (figure 1), along with fits to (orange) equation (27) and (teal) equation (28). Notice the lack of any even–odd effect in the entanglement as a function of cut length.

Holographic Entanglement on the 2 torus

Chen, Cho, Faulkner and Fradkin (2015) used the **Ryu-Takayanagi ansatz** to obtain the finite term in the **von Neumann entanglement entropy for a cylindrical section of a 2 torus**

$$S_{vN} = \frac{\mathcal{A}}{4G_N}$$

$\mathcal{A}$ is the area of the minimal surface ending on the boundary where the QFT lives

On toroidal geometries the minimal surface falls on the AdS_4 soliton geometry (Horowitz and Myers)

- The EE is sensitive to which cycle of the torus contracts since the cut is along y
- In the thin torus limit, $L_y \ll L_x$, the EE saturates
- This is natural for fermions with anti-periodic boundary conditions
- In the opposite, **thin slice**, limit we find universal scaling
- The von Neumann EE has the scaling form $S_{vN}=\alpha\, L_y+ \beta\, j(u)$, where j(u) has two different forms depending on whether the aspect ratio of the torus is $L_y/L_x>1$ or $L_y/L_x<1$

Parametric for of j(u) for $L_y/L_x>1$

$$u(\chi) = \frac{3\chi^{1/3}(1-\chi)^{1/2}}{2\pi}\int_0^1 \frac{d\zeta\zeta^2}{(1-\chi\zeta^3)}\frac{1}{\sqrt{P(\chi,\zeta)}} \qquad j(\chi) = \chi^{-1/3}\left(\int_0^1 \frac{d\zeta}{\zeta^2}\left(\frac{1}{\sqrt{P(\chi,\zeta)}}-1\right)-1\right)$$

$$P(\chi,\zeta) = 1-\chi\zeta^3-(1-\chi)\zeta^4$$

Parametric form for j(u) for $L_y/L_x<1$

$$j(\chi) = \chi^{-1/3}\left(\int_0^1 \frac{d\zeta}{\zeta^2}\left(\frac{\sqrt{1-\chi\zeta^3}}{\sqrt{P(\chi,\zeta)}}-1\right)-1\right)$$

$$\frac{L_x}{L_y}u = \frac{3}{2\pi}\chi^{1/3}(1-\chi)^{1/2}\int_0^1 \frac{d\zeta\zeta^2}{\sqrt{1-\chi\zeta^3}}\frac{1}{\sqrt{P(\chi,\zeta)}}$$

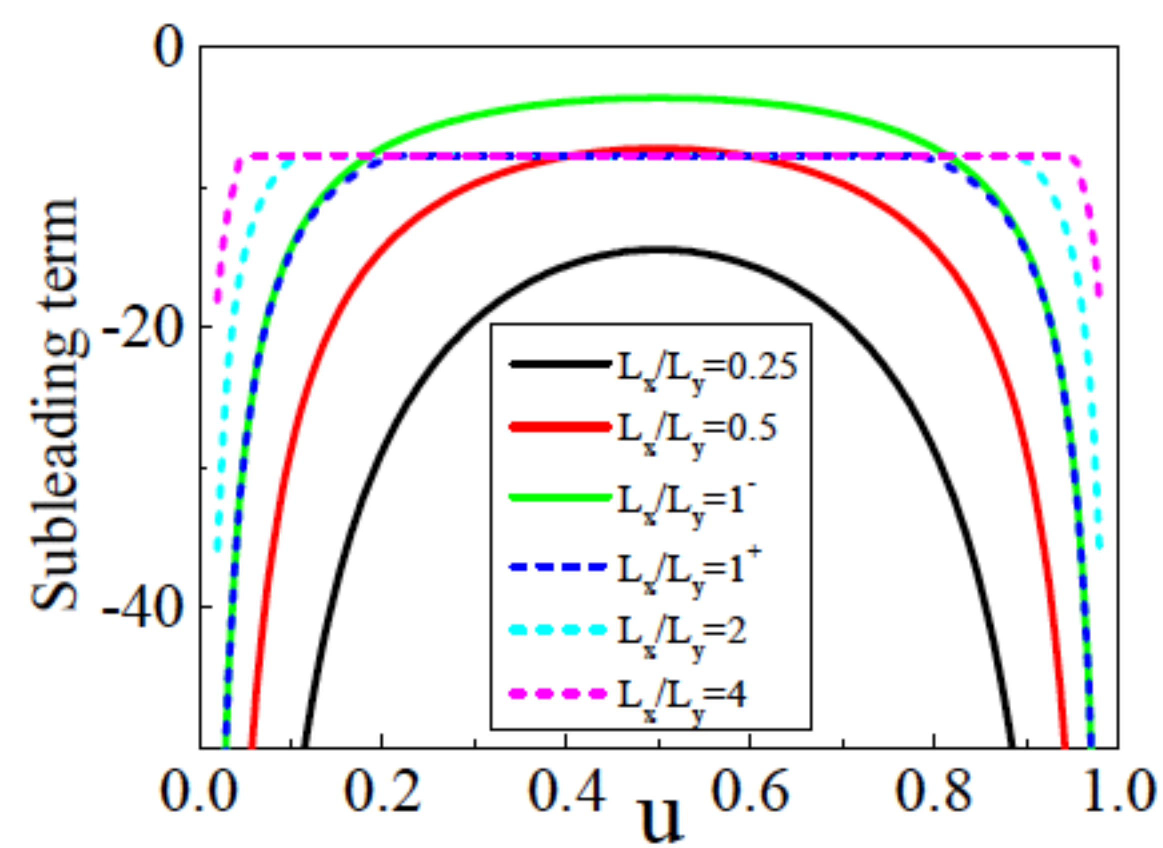


FIG. 2. The subleading term for the minimal surface for various values of L_x/L_y. The solid curves are for $j(u)$ when $L_x \leq L_y$ and the dashed curves are for $\tilde{j}(u)$ when $L_x > L_y$.

Tests of scaling in 2d free field fermionic models

- We computed the entanglement entropy for cylindrical regions of a 2 torus for free massless Dirac fermions in 2+1 dimensions and also for free massless "Lifshitz" (QBT) fermions, in both cases with anti-period boundary conditions
- Dirac fermions in 2+1 dimensions are a stable phase and hence an UV fixed point
- "Lifshitz" fermions in 2+1 dimensions have z=2 and are are at the marginal dimension (asymptotically free)
- We used a lattice version of these models and used standard results for the reduced density matrix and the entanglement entropy (Peschel, 2001)
- In the Dirac case we accounted for doublers

Scaling of entanglement for free massless Dirac fermions in 2+1 dimensions

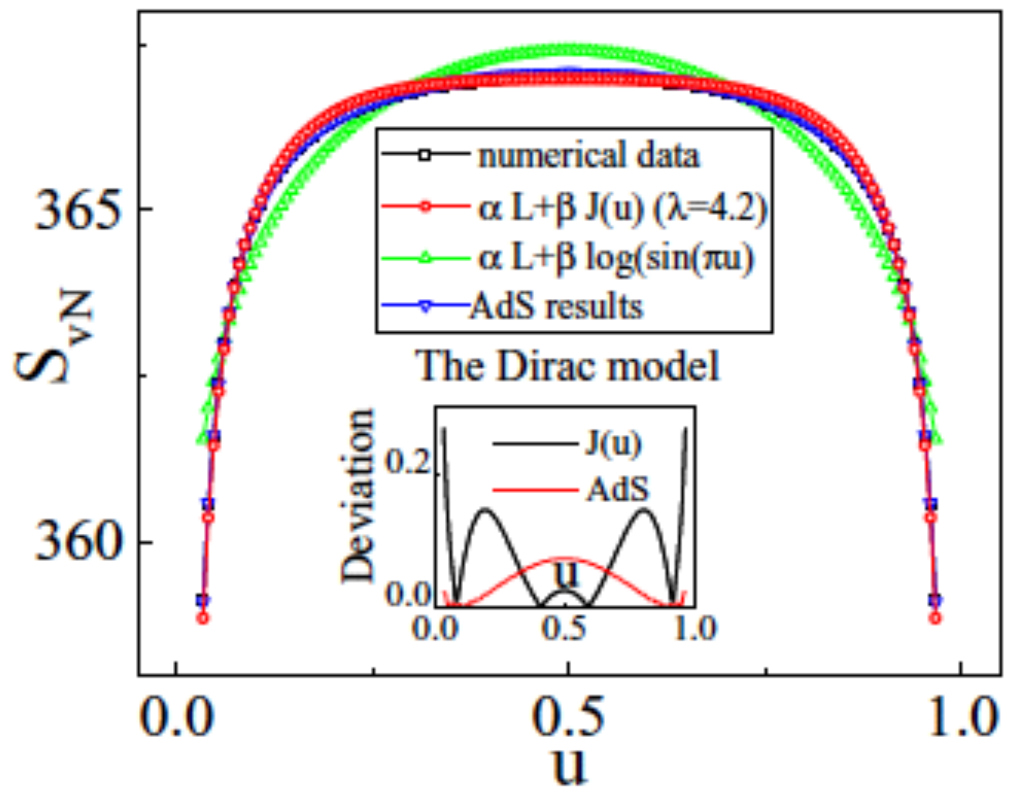


Figure 7. S_{vN} for the Dirac model as a function of u with $L = L_x = L_y = 300$. The red curve is the fitting function with the form $S_{\mathrm{vN}} = \alpha L + \beta J(u)$. The numerical data is in black curve. The blue curve is the holographic entropy. (The black and blue curves are hard to see in the figure since they are almost overlapping with the blue curve.) The inset is the absolute deviation for $S_{\mathrm{vN}} = \alpha L + \beta J(u)$ (black curve) and the holographic entropy (red curve) with the numerical data. In both cases, the deviation is less than 1% for the whole region, but the holographic result appears to be the most accurate. The green curve is the fitting function with the form $S_{\mathrm{vN}} = \alpha L + \beta \log(\sin(\pi u))$.

Scaling of entanglement for free massless "Lifshitz" fermions in 2+1 dimensions

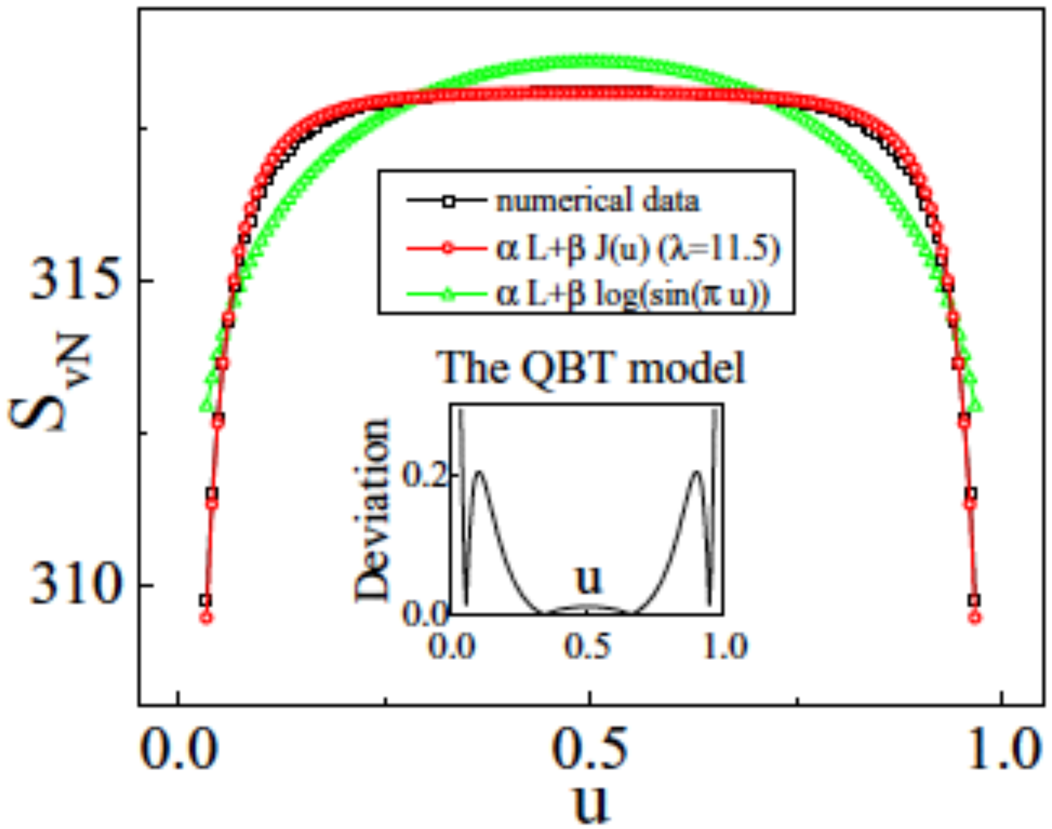


Figure 8. $S_{\rm vN}$ for the QBT model as a function of u. The bipartition geometry is the same as the Dirac model. The inset is the absolute deviation for the fitting function $S_{\rm vN} = \alpha L + \beta J(u)$ with numerical data. The deviation is less than 1% for the whole region of u.

Conclusions

- In 2+1 dimensions we expect to find universal finite corrections to the entanglement entropy
- The properties of these finite terms, e.g. how they relate to the scaling properties of local observables (dimensions, etc), is poorly understood
- On cylindrical regions the universal finite terms are given in terms of scaling functions of the aspect ratios of the cylinders and of the 2 torus
- We presented expressions for the entanglement entropy derived from holography on toroidal geometries which fit remarkably well even free massless Dirac fermions (on anti-periodic boundary conditions)
- The finite term derived by Stéphan et al for the QLM fits remarkably well even for Dirac fermions which is a relativistic theory (although the holographic results is better)
- How well does the holographic result fit the 2D Ising model in a transverse field? (Roger will tell us?)