Generalized F-Theorem and ϵ-Expansion

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Entanglement in Strongly-Correlated Quantum Matter

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Based mainly on

- IK, S. Pufu, B. Safdi, arXiv:1105.4598
- IK, T. Nishioka, S. Pufu, B. Safdi, arXiv:1207.3360
- S. Giombi, IK, arXiv:1409.1937
- L. Fei, S. Giombi, IK, G. Tarnopolsky, arXiv:1502.07271
- L. Fei, S. Giombi, IK, G. Tarnopolsky,

  work in progress

The c-theorem

- A deep problem in QFT is how to define a `good’ measure of the number of degrees of freedom which decreases along RG flows and is stationary at fixed points.
- In two dimensions this problem was beautifully solved by Alexander Zamolodchikov who, using two-point functions of the stress-energy tensor, found the c-function which satisfies these properties.

- At RG fixed points the c-function coincides with the Virasoro central charge, which is also the Weyl anomaly
$$\langle T^a_a \rangle = -\frac{c}{12} R$$

- Determines the thermal free energy.
- Determines the EE of a segment of size r Holzhey, Larsen, Wilczek

$$S(r) = \frac{c}{3} \log(r/\epsilon) + c_0$$

- $c_{IR} < c_{UV}$ follows from boost invariance and SSA Casini, Huerta

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

- The central charge can also be found using the 2-d CFT on the sphere of radius R:

$$F = -\log Z = -c/3 \log R$$

The a-theorem

- In d=4 there are two Weyl anomaly coefficients

$$\langle T^a_a \rangle = -\frac{a}{16\pi^2}\left(R^2_{abcd} - 4R^2_{ab} + R^2\right) + \frac{c}{16\pi^2} C^2_{abcd}$$

- One of them, called a is proportional to the 4-d Euler density. It can be extracted from the Euclidean path integral on the 4-d sphere:

F=-log Z = a log R

- Cardy conjectured that the a-coefficient decreases along any RG flow.
- A proof was provided a few years ago. Komargodski, Schwimmer

The F-theorem

- How do we extend these successes to odd dimensions where there are no anomalies? This is interesting, especially in d=3 where there are many CFTs, some of them describing critical points in statistical mechanics and condensed matter physics.
- The free energy on the 3-sphere $F = -\ln|Z_{S^3}|$
- In a CFT, F is a well-defined, regulator independent quantity (there are no Weyl invariant counter terms).
- F-theorem: $F_{IR} < F_{UV}$ Jafferis, IK, Pufu, Safdi

The Entanglement Connection

- Remarkably, -F is the universal long range entanglement entropy across a circle of radius R in any 2+1 dimensional CFT. Casini, Huerta, Myers

$$S(R) = \alpha \frac{2\pi R}{\epsilon} - F$$

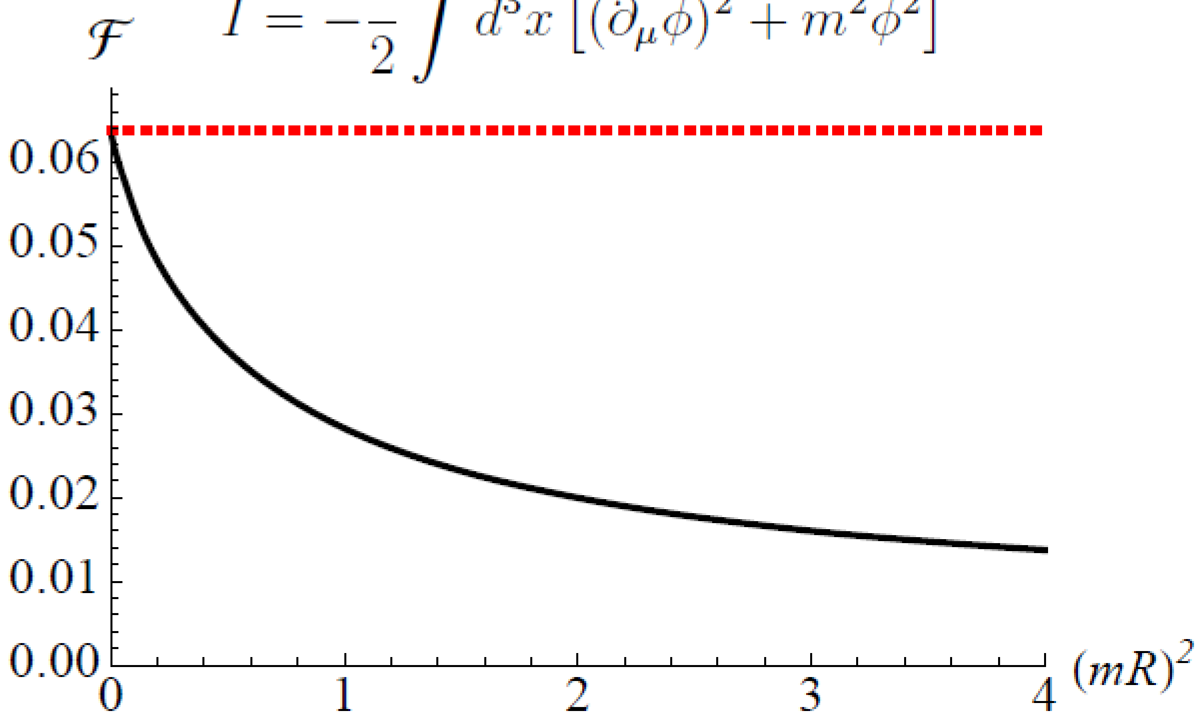
- Using the language of EE, the F-theorem was formulated and its proof was found. Myers, Sinha; Casini, Huerta
- The c-function used in the proof is the Renormalized Entanglement Entropy (REE). Liu, Mezei

$$\mathcal{F}(R) = -S(R) + RS'(R)$$

Non-Stationarity

- For some RG flows the REE is non-stationary, e.g. for the massive free scalar field. IK, Nishioka, Pufu, Safdi

$$I = -\frac{1}{2}\int d^3x\,\left[(\partial_\mu\phi)^2 + m^2\phi^2\right]$$



$$\partial_{(mR)^2}\mathcal{F} \text{ is negative and nonzero at } (mR)^2 = 0$$

Calculating F

- The simplest CFT’s involve free conformal scalar and fermion fields. Adding mass terms makes such a theory flow to a theory with no massless degrees of freedom in the IR where F=0.
- For consistency with the F-theorem, the F-values for free massless fields should be positive.

Conformal Scalar on S^d

- In any dimension

$$F_S = -\log|Z_S| = \frac{1}{2}\log\det\left[\mu_0^{-2}\mathcal{O}_S\right] \qquad \mathcal{O}_S \equiv -\nabla^2 + \frac{d-2}{4(d-1)}R$$

- The eigenvalues and degeneracies are

$$\lambda_n = \left(n + \frac{d-1}{2}\right)^2 - \frac{1}{4} \qquad n \geq 0 \qquad m_n = \frac{(2n+d-1)(n+d-2)!}{(d-1)!n!}$$

$$F_S = \frac{1}{2}\sum_{n=0}^{\infty} m_n \left[-2\log(\mu_0 a) + \log\left(n + \frac{d}{2}\right) + \log\left(n - 1 + \frac{d}{2}\right)\right]$$

- Using zeta-function regularization in d=3,

$$F_B = -\frac{1}{2}\frac{d}{ds}\left[2\zeta(s-2, 1/2) + \frac{1}{2}\zeta(s, 1/2)\right]\Bigg|_{s=0} = \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx .0638$$

- Furthermore, it is possible to derive an integral representation valid in continuous dimension d:

$$
\begin{aligned}
F_s &= \frac{1}{2}\log\det\left(-\nabla^2 + \frac{1}{4}d(d-2)\right) \\
&= -\frac{1}{\sin(\frac{\pi d}{2})\Gamma\left(1+d\right)}\int_0^1 du\, u \sin\pi u\, \Gamma\left(\frac{d}{2}+u\right)\Gamma\left(\frac{d}{2}-u\right)
\end{aligned}
$$

- Near even d, it has simple poles whose coefficients are the a-anomalies.
- For example, in $d = 4 - \epsilon$

$$F_s = \frac{1}{90\epsilon} + \dots$$

Slightly Relevant Operators

- Perturb a CFT by a relevant operator of dimension $\Delta = d - \epsilon$

$$S = S_0 + \lambda_0 \int d^d x \sqrt{G} O(x)$$

- The path integral on a sphere is

$$\log \left| \frac{Z(\lambda_0)}{Z(0)} \right| = \sum_{n=1}^{\infty} \frac{(-\lambda_0)^n}{n!} \int d^d x_1 \sqrt{G} \cdots \int d^d x_n \sqrt{G} \langle O(x_1) \cdots O(x_n) \rangle_0$$

- The 1-pt function vanishes.

- The 2- and 3-pt function are determined by conformal invariance in terms of the chordal distance

$$\langle O(x)O(y)\rangle_0 = \frac{1}{s(x,y)^{2(d-\epsilon)}}\,,$$

$$\langle O(x)O(y)O(z)\rangle_0 = \frac{C}{s(x,y)^{d-\epsilon}s(y,z)^{d-\epsilon}s(z,x)^{d-\epsilon}}$$

- The change in the free energy is

$$\delta F(\lambda_0) \equiv F(\lambda_0) - F(0) = -\frac{\lambda_0^2}{2}I_2 + \frac{\lambda_0^3}{6}I_3 + \mathcal{O}(\lambda_0^4)$$

$$I_2 = \int d^d x\sqrt{G}\int d^d y\sqrt{G}\,\langle O(x)O(y)\rangle_0 = \frac{(2a)^{2\epsilon}\pi^{d+1/2}}{2^{d-1}}\frac{\Gamma\left(-\frac{d}{2}+\epsilon\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma(\epsilon)}\,,$$

$$I_3 = \int d^d x\sqrt{G}\int d^d y\sqrt{G}\int d^d z\sqrt{G}\,\langle O(x)O(y)O(z)\rangle_0 = \frac{8\pi^{3(d+1)/2}a^{3\epsilon}}{\Gamma(d)}\frac{\Gamma\left(-\frac{d}{2}+\frac{3\epsilon}{2}\right)}{\Gamma\left(\frac{1+\epsilon}{2}\right)^3}C$$

- The beta function for the dimensionless coupling is

$$g = \lambda\mu^{-\epsilon}$$

$$\beta(g) = \mu\frac{dg}{d\mu} = -\epsilon g + \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}Cg^2 + \mathcal{O}(g^3)$$

- Integrating the RG equation

$$\lambda_0(2a)^\epsilon = g + \frac{C\pi^{d/2}}{\epsilon\Gamma\left(\frac{d}{2}\right)}g^2 + \mathcal{O}(g^3)$$

$$\delta F(g) = (-1)^{\frac{d+1}{2}}\frac{2\pi^{d+1}}{d!}\left[-\frac{1}{2}\epsilon g^2 + \frac{1}{3}\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}Cg^3 + \mathcal{O}(g^4)\right]$$

$$\frac{dF}{dg} = (-1)^{\frac{d+1}{2}}\frac{2\pi^{d+1}}{d!}\beta(g) + \mathcal{O}(g^2)$$

- This “F-function” is stationary at the fixed points.

- There exists a robust IR fixed point at

$$g^* = \frac{\Gamma\left(\frac{d}{2}\right)\epsilon}{\pi^{d/2}C} + \mathcal{O}(\epsilon^2)$$

- The 3-sphere free energy decreases

$$\delta F(g^*)|_{d=3} = -\frac{\pi^2\epsilon^3}{72C^2}$$

- A similar calculation for d=1 provided initial evidence for the g-theorem conjectured by Affleck and Ludwig.
- For a general odd dimension, what decreases along RG flow is IK, Pufu, Safdi

$$\tilde{F} = (-1)^{\frac{d+1}{2}} F = (-1)^{\frac{d-1}{2}} \log Z_{S^d}$$

Double-Trace Flows

- If we perturb a large N CFT by a relevant double-trace operator, it flows to another fixed point in the IR

$$Z = \int D\phi \exp\left(-S_0 - \frac{\lambda_0}{2}\int d^dx\sqrt{G}\Phi^2\right)$$

- If in the UV the dimension of Φ is Δ, in the IR it is d- Δ
- F can be calculated using the Hubbard-Stratonovich method

$$\frac{Z}{Z_0} = \frac{1}{\int D\sigma \exp(\frac{1}{2\lambda_0}\int d^dx\sqrt{G}\sigma^2)}\int D\sigma \left\langle \exp\left[\int d^dx\sqrt{G}\left(\frac{1}{2\lambda_0}\sigma^2 + \sigma\Phi\right)\right]\right\rangle_0$$

- The change in F between IR and UV is of order 1 and is computable Gubser, IK; Diaz, Dorn

$$\delta F_\Delta = \frac{1}{2}\sum_{l=0}^{\infty} M_d(l)\log\left(\frac{\Gamma(l+\Delta)}{\Gamma(d+l-\Delta)}\right) \qquad M_d(l) = \frac{\Gamma(l+d-1)(2l+d-1)}{l!\Gamma(d)}$$

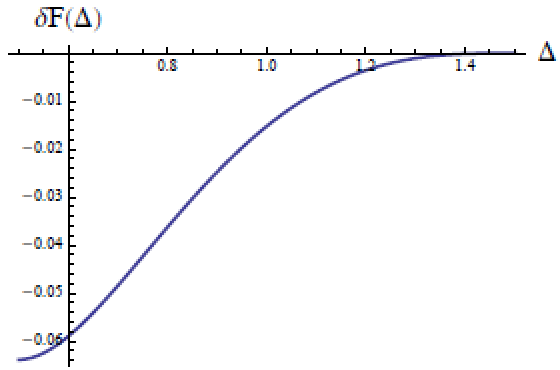
- In all dimensions d

$$\begin{aligned}\delta F_\Delta &= \Gamma(-d)\int_0^{\Delta-\frac{d}{2}} du\, u\left[\frac{\Gamma\left(\frac{d}{2}-u\right)}{\Gamma\left(1-u-\frac{d}{2}\right)} - \frac{\Gamma\left(\frac{d}{2}+u\right)}{\Gamma\left(1+u-\frac{d}{2}\right)}\right] \\ &= -\frac{1}{\sin(\frac{\pi d}{2})\Gamma\left(1+d\right)}\int_0^{\Delta-\frac{d}{2}} du\, u\sin\pi u\,\Gamma\left(\frac{d}{2}+u\right)\Gamma\left(\frac{d}{2}-u\right)\end{aligned}$$

- For d=3

$$\delta F_\Delta = -\frac{\pi}{6}\int_\Delta^{3/2} dx(x-1)(x-\frac{3}{2})(x-2)\cot(\pi x)$$

- The change in free energy is negative, in support of the F-theorem



- The particular case Δ=1 corresponds to the critical O(N) model

$$\delta F_{\Delta=1} = -\frac{\zeta(3)}{8\pi^2} \approx -.0152$$

O(N) Model

- The critical O(N) model is obtained via a double-trace perturbation of the theory of N free real scalars

$$S[\vec{\Phi}] = \frac{1}{2}\int d^3x \left[\partial\vec{\Phi}\cdot\partial\vec{\Phi} + m^2\vec{\Phi}^2 + \frac{\lambda}{2N}\left(\vec{\Phi}\cdot\vec{\Phi}\right)^2\right]$$

- Using our free field and double-trace results

$$F_{\text{crit}} = \frac{N}{16}\left(2\log(2) - 3\frac{\zeta(3)}{\pi^2}\right) - \frac{\zeta(3)}{8\pi^2} + O(1/N)$$

- A further relevant perturbation takes it to the Goldstone phase where

$$F_{\text{Gold}} = \frac{N-1}{16}\left(2\log(2) - 3\frac{\zeta(3)}{\pi^2}\right)$$

- The flow from the critical to the Goldstone phase provided a counter-example to the proposal that the thermal free energy decreases along RG flow. Sachdev
- Yet, there is no contradiction with the F-theorem since for large N

$$F_{\text{Goldstone}} - F_{\text{crit}} = -\frac{1}{16}\left(2\log(2) - 5\frac{\zeta(3)}{\pi^2}\right) \approx -0.0486$$

Interacting O(N) models in d>4?

- The scalar model with $\frac{\lambda}{4}(\phi^i\phi^i)^2$ interaction is IR trivial in d>4, but it has unitary large N UV fixed points for 4<d<6 (*Parisi '75*). Can be seen by doing large N expansion in the Hubbard-Stratonovich approach

$$S = \int d^d x \left(\frac{1}{2}(\partial \phi^i)^2 + \frac{1}{2}\sigma\phi^i\phi^i - \frac{\sigma^2}{4\lambda} \right)$$

- For d>4, the quadratic term for σ can be dropped in the UV, and one finds a large N CFT where the dimension of the singlet scalar operator $\sigma \sim \phi^i\phi^i$ is 2+O(1/N), as before. This is above the unitarity bound for d<6.

Interacting O(N) model in 4<d<6

- Is there an alternate description of this interacting scalar CFT as a more conventional *IR fixed point* of some other theory?
- In the large N approach, we see that $\Delta=2+O(1/N)$ approaches the free field value as d->6. This suggests to look for a theory with N+1 scalars near d=6.
- Proposal: work in d=6-ϵ and look for IR fixed points in the cubic O(N) symmetric theory Fei, Giombi, IK

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2} g_1 \sigma (\phi^i)^2 + \frac{1}{6} g_2 \sigma^3$$

- Interactions are relevant in d<6. The theory is free in the UV, and can have non-trivial IR fixed points.

Perturbative fixed points in d=6-ϵ

- The one-loop beta functions in d=6-ϵ

$$\beta_1 = -\frac{\epsilon g_1}{2} + \frac{(N-8)g_1^3 - 12 g_1^2 g_2 + g_1 g_2^2}{12(4\pi)^3}$$
$$\beta_2 = -\frac{\epsilon g_2}{2} + \frac{-4N g_1^3 + N g_1^2 g_2 - 3 g_2^3}{4(4\pi)^3}$$

- At large N, one finds a unitary, IR stable fixed point at *real* couplings

$$g_1^* = \sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1 + \frac{22}{N} + \dots\right), \qquad g_2^* = 6\sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1 + \frac{162}{N} + \dots\right)$$

Perturbative fixed points in d=6-ϵ

- The conformal dimensions of operators at the IR fixed point can be computed to any order in the 1/N expansion

$$\Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi = 2 - \frac{\epsilon}{2} + \frac{1}{(4\pi)^3}\frac{(g_1^*)^2}{6}$$
$$= 2 - \frac{\epsilon}{2} + \frac{\epsilon}{N} + \frac{44\epsilon}{N^2} + \frac{1936\epsilon}{N^3} + \ldots$$

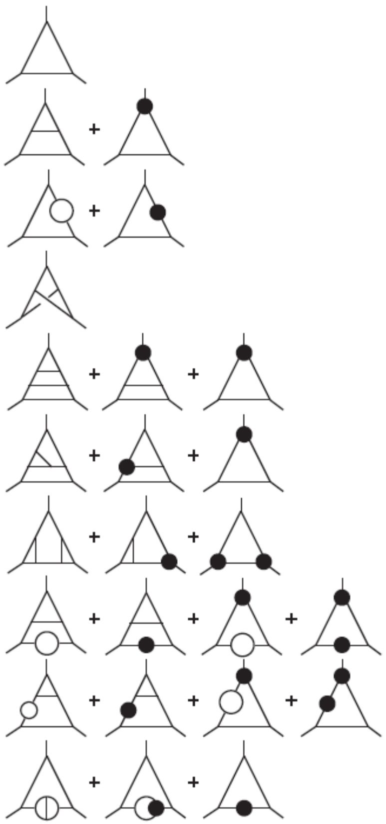
$$\Delta_\sigma = \frac{d}{2} - 1 + \gamma_\sigma = 2 - \frac{\epsilon}{2} + \frac{1}{(4\pi)^3}\frac{N(g_1^*)^2 + (g_2^*)^2}{12}$$
$$= 2 + \frac{40\epsilon}{N} + \frac{6800\epsilon}{N^2} + \ldots$$

- These match the known large N results of A.N. Vasiliev et al. expanded in d=6-ϵ
- Dimensions of some composite operators also checked to agree with known large N results *Lang, Ruhl; Petkou*
- This is a non-trivial check that the IR fixed points of the cubic theory in d=6-ϵ indeed provide an alternative ``UV-complete" description of the UV fixed points of the quartic scalar field theory in 4<d<6.

Three loop calculations

- We have checked this agreement at higher orders in ϵ.
- Three-loop calculation of the beta functions and scaling dimensions *Fei, Giombi, IK, Tarnopolsky*

$$
\begin{aligned}
\beta_1(g_1,g_2) = &-\frac{\epsilon}{2}g_1 + \frac{1}{12(4\pi)^3}g_1\left((N-8)g_1^2 - 12g_1g_2 + g_2^2\right) \\
&- \frac{1}{432(4\pi)^6}g_1\left((536+86N)g_1^4 + 12(30-11N)g_1^3g_2 + (628+11N)g_1^2g_2^2 + 24g_1g_2^3 - 13g_2^4\right) \\
&+ \frac{1}{62208(4\pi)^9}g_1\{g_2^6(5195-2592\zeta(3)) + 12g_1g_2^5(-2801+2592\zeta(3)) \\
&- 8g_1^2g_2^4(1245+119N+7776\zeta(3)) + g_1^4g_2^2(-358480+53990N-3N^2-2592(-16+5N)\zeta(3)) \\
&+ 36g_1^5g_2(-500-3464N+N^2+864(5N-6)\zeta(3)) \\
&- 2g_1^6(125680-20344N+1831N^2+2592(25N+4)\zeta(3)) + 48g_1^3g_2^3(95N-3(679+864\zeta(3)))\} \\
\beta_2(g_1,g_2) = &-\frac{\epsilon}{2}g_2 + \frac{1}{4(4\pi)^3}\left(-4Ng_1^3 + Ng_1^2g_2 - 3g_2^3\right) \\
&+ \frac{1}{144(4\pi)^6}\left(-24Ng_1^5 - 322Ng_1^4g_2 - 60Ng_1^3g_2^2 + 31Ng_1^2g_2^3 - 125g_2^5\right) \\
&+ \frac{1}{20736(4\pi)^9}\{-48N(713+577N)g_1^7 + 6272Ng_1^2g_2^5 + 48Ng_1^3g_2^4(181+432\zeta(3)) \\
&- 5g_2^7(6617+2592\zeta(3)) - 24Ng_1^5g_2^2(1054+471N+2592\zeta(3)) \\
&+ 2Ng_1^6g_2(19237N-8(3713+324\zeta(3))) + 3Ng_1^4g_2^3(263N-6(7105+2448\zeta(3)))\}
\end{aligned}
$$



A test of the 5d F-theorem

- It was conjectured that for any odd-dimensional CFT the quantity IK, Pufu, Safdi

$$\tilde{F} = (-1)^{\frac{d+1}{2}} F = (-1)^{\frac{d-1}{2}} \log Z_{S^d}$$

should decrease under RG flow $\tilde{F}_{UV} > \tilde{F}_{IR}$

- Using our results on the d=5 critical O(N) models, we can provide a test of this 5d F-theorem. The two descriptions as either the IR fixed point of the cubic theory or UV fixed point of the quartic theory imply that F should satisfy

$$N\tilde{F}_{\text{free sc.}} < \tilde{F}_{\text{crit.}} < (N+1)\tilde{F}_{\text{free sc.}}$$

where $\tilde{F}_{\text{free sc.}} = \frac{\log 2}{128} + \frac{\zeta(3)}{128\pi^2} - \frac{15\zeta(5)}{256\pi^4} \simeq 0.00574$ is minus the free energy of a 5d free conformal scalar.

- In d=5, we can compute F_{crit} in the large N expansion using the Hubbard-Stratonovich auxiliary field. The result is *Giombi, IK, Safdi*

$$\tilde{F}_{\text{crit.}} = N\tilde{F}_{\text{free sc.}} + \frac{3\zeta(5) + \pi^2\zeta(3)}{96\pi^2} + \mathcal{O}\left(\frac{1}{N}\right)$$

- The O(1) correction is positive, so that the left side of the inequality $N\tilde{F}_{\text{free sc.}} < \tilde{F}_{\text{crit.}} < (N+1)\tilde{F}_{\text{free sc.}}$ is satisfied.
- The right side is also satisfied, because

$$\tfrac{3\zeta(5)+\pi^2\zeta(3)}{96\pi^2} \simeq 0.001601$$

is smaller than

$$\tilde{F}_{\text{free sc.}} = \frac{\log 2}{128} + \frac{\zeta(3)}{128\pi^2} - \frac{15\zeta(5)}{256\pi^4} \simeq 0.00574$$

Sphere free energy in continuous d

- We may study dimensional continuation of the sphere free-energies. Is there an interpolation between F-theorems in odd and *a*-theorems in even d?
- A natural quantity to consider is S. Giombi, IK

$$\tilde{F} = \sin(\pi d/2) \log Z_{S^d} = -\sin(\pi d/2) F$$

- In odd d, this reduces to

$$\tilde{F} = (-1)^{\frac{d+1}{2}} F = (-1)^{\frac{d-1}{2}} \log Z_{S^d}$$

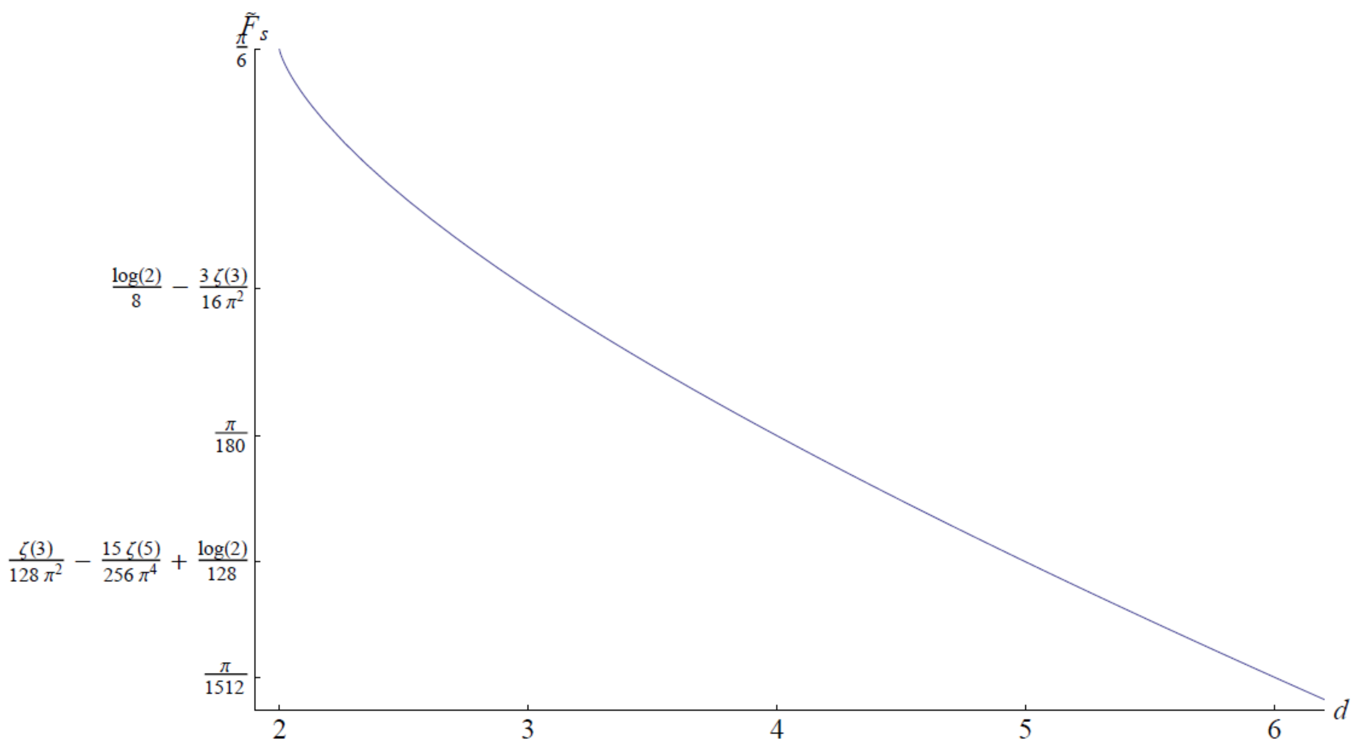
- In even d, log Z has a pole in dimensional regularization whose coefficient is the Weyl *a*-anomaly. The multiplication by $\sin(\pi d/2)$ removes it.
- Therefore, $\tilde{F}$ smoothly interpolates between *a*-anomaly coefficients in even and ``F-values" in odd d.

Free conformal scalar in continuous d

- For instance, for a free conformal scalar on S^d

$$\tilde{F}_s = \frac{1}{\Gamma(1+d)} \int_0^1 du\, u \sin \pi u\, \Gamma\left(\frac{d}{2}+u\right) \Gamma\left(\frac{d}{2}-u\right)$$

- This is positive for all d and smoothly interpolates between *a* and F



Generalized F-theorem in continuous d?

- Based on the known F- and a-theorems, it is natural to ask whether

$$\tilde{F}_{UV} > \tilde{F}_{IR}$$

holds in general dimension *d*.
- We have calculated $\tilde{F}$ in various examples of CFTs that can be defined in continuous dimension, including double-trace flows in large N CFTs and perturbative Wilson-Fisher fixed points in the epsilon-expansion.
- In all unitary examples that we considered, we find that $\tilde{F}$ indeed decreases under RG flow. For non-unitary fixed points, the inequality $\tilde{F}_{UV} > \tilde{F}_{IR}$ does not have to hold.

Cubic fixed points in d=6-ϵ

- For example, consider our cubic theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2} g_1 \sigma (\phi^i)^2 + \frac{1}{6} g_2 \sigma^3$$

After mapping to the sphere, the leading contribution to the free energy is given by the 2-point functions of the perturbing operators integrated over S^d . Using the integral *Cardy*

$$\int d^d x d^d y \sqrt{g_x} \sqrt{g_y} \frac{1}{s(x,y)^{2\Delta}} = (2R)^{2(d-\Delta)} \frac{2^{1-d} \pi^{d+\frac{1}{2}} \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma\left(\frac{1+d}{2}\right) \Gamma\left(d - \Delta\right)}$$

and going to the IR fixed point, we find

$$\tilde{F} = (N+1)\tilde{F}_s - \frac{\pi}{17280} \frac{3(g_1^*)^2 N + (g_2^*)^2}{(4\pi)^3} \epsilon + \mathcal{O}(\epsilon^3)$$

- For the unitary IR fixed points that exist for $N>N_{cr}$, we see that $\tilde{F}$ *decreases* from the UV fixed point (N+1 free scalars) to the IR.
- For non-unitary fixed points with imaginary couplings, such as in the single scalar model introduced by Michael Fisher to study the Yang-Lee edge singularity, the $\tilde{F}$ conjecture violated.
- Using the large N methods, one can also show that the inequalities $N\tilde{F}_{\text{free sc.}} < \tilde{F}_{\text{crit.}} < (N+1)\tilde{F}_{\text{free sc.}}$ hold to leading order in 1/N in the full range 4<d<6 (and are violated for d>6 where the CFT becomes non-unitary).

EE of EE

- The fact that $\tilde{F}$ is a smooth function of dimension suggests that, in the spirit of the Wilson-Fisher ϵ expansion, it may provide us with a useful tool to estimate the value of F for interacting CFTs.
- Consider the 3d Ising model, and more generally the O(N) Wilson-Fisher CFTs in d=3.
- They are strongly coupled CFTs in d=3, but they have a perturbative description in d=4-ϵ.
- We can compute the sphere free energy perturbatively and extrapolate the result to $\epsilon=1$ to estimate the value of F.

Wilson-Fisher O(N) theory in d=4-ϵ

$$S = \int d^d x \left(\frac{1}{2} \left(\partial_\mu \phi_0^i\right)^2 + \frac{\lambda_0}{4} (\phi_0^i \phi_0^i)^2 \right)$$

$$\beta = -\epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 - \frac{3(3N+14)}{64\pi^4}\lambda^3 + \dots$$

$$\lambda_* = \frac{8\pi^2}{N+8}\epsilon + \frac{24(3N+14)\pi^2}{(N+8)^3}\epsilon^2 + \dots$$

- Carrying out the renormalization procedure on the sphere (*Brown-Collins '80, Hathrell '82…*) we find

$$\tilde{F} = N\tilde{F}_s(\epsilon) - \frac{\pi}{576}\frac{N(N+2)}{(N+8)^2}\epsilon^3 - \frac{\pi}{6912}\frac{N(N+2)(13N^2+370N+1588)}{(N+8)^4}\epsilon^4 + \mathcal{O}(\epsilon^5)$$

- Extracting precise estimates from the ϵ–expansion typically requires a resummation technique, like Pade approximants

$$\mathrm{Pade}_{[m,n]}(x) = \frac{A_0 + A_1 x + A_2 x^2 + \ldots + A_m x^m}{1 + B_1 x + B_2 x^2 + \ldots + B_n x^n}$$

Disk EE for the 3d Ising model

- We expect that $\tilde{F}$ should be a smooth function of *d* , such that near d=4 it reproduces the perturbative ε–expansion, and in d=2 it reproduces the exact central charge of the 2d Ising model, c=1/2 (corresponding to $\tilde{F}=\pi/12$).
- The Pade approximants can be greatly improved if we impose the constraint that c=1/2 for d=2.
- Using this method, we get the estimate (Fei, Giombi, IK, Tarnopolsky, in progress)

$$\frac{F_{\rm 3d\ Ising}}{F_s} \approx 0.97$$

- The value of F (and hence of the disk entanglement entropy) for 3d Ising seems to be extremely close to the free field value!
- A similar result was found for C_T in the conformal bootstrap approach $c_T^{\rm 3d\ Ising}/c_T^{\rm 3d\ free\ scalar} \approx 0.9466$

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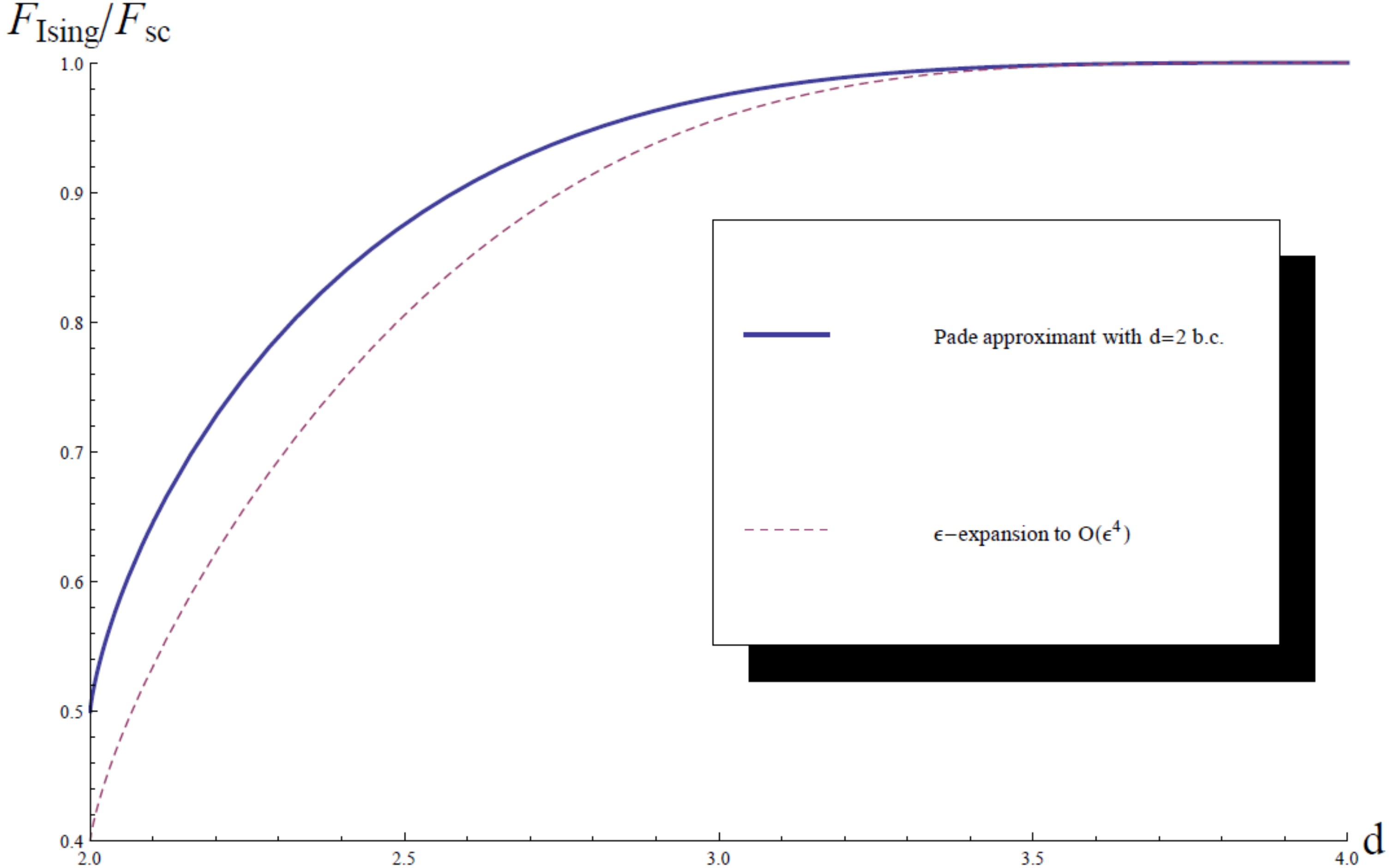

$$\tilde{F}_s = \frac{\pi}{180} + 0.0205991\epsilon + 0.0136429\epsilon^2 + 0.00690843\epsilon^3 + 0.00305846\epsilon^4 + \mathcal{O}(\epsilon^5)$$

$$\tilde{F}_s + \tilde{F}_{\rm int} = 0.0174533 + 0.0205991\epsilon + 0.0136429\epsilon^2 + 0.00670642\epsilon^3 + 0.00264884\epsilon^4 + \mathcal{O}(\epsilon^5)$$

Conclusion

- The universal term in the disk EE in 2+1 CFT is determined by the 3-sphere free energy F.
- It satisfies the F-theorem, and we reviewed its tests using conformal perturbation theory, free fields and Wilson-Fisher O(N) models.
- We tried to generalize the F-theorem to other dimensionalities.
- The 5d critical O(N) models can be used to provide a new test of the d=5 F-theorem.
- We found a new description of the UV fixed points of O(N) model in $4<d<6$ as IR fixed points of a cubic theory with N+1 fields. For $N>N_{crit}$, the IR fixed points are unitary and well-defined to all orders in 1/N.

- We studied dimensional continuation of the sphere free energy and provided evidence for a generalized F-theorem in continuous *d*, interpolating between F-theorems in odd and *a*-theorems in even *d*.
- The ϵ–expansion of

$$\tilde{F} = \sin(\pi d/2) \log Z_{S^d} = -\sin(\pi d/2) F$$

can be used to estimate the values of F for interesting 3d CFTs.

- For the critical Ising model it is only a few per cent lower than for the free conformal scalar.
- Can this result be compared with a numerical calculation of the EE for the Ising model?
- The disk geometry is needed, but a half-cylinder may be tried first as a warm-up. We expect the universal term to be close to that for a free massless scalar.