MPS of unitary and non-unitary fractional quantum Hall wavefunctions

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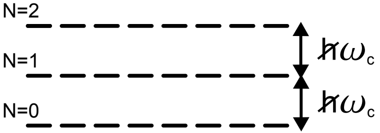
Outline

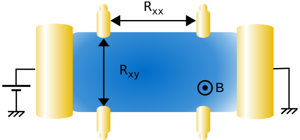
- FQH states and entanglement spectrum
- MPS for the FQH model states
- FQHE and topological entanglement entropy
- FQHE and non-unitary CFTs
- Probing the excitations

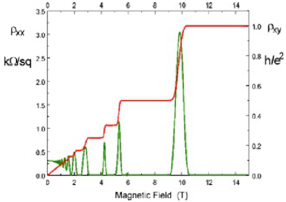
FQH states and entanglement spectrum

Fractional Quantum Hall effect

Landau levels (spinless case)







- Cyclotron frequency : $\omega_c = \frac{eB}{m}$,
- Filling factor : $\nu = \frac{hn}{eB} = \frac{N}{N_\Phi}$
- Partial filling + interaction $\rightarrow$ FQHE
- Lowest Landau level ($\nu < 1$) : $z^m \exp\left(-|z|^2/(4l_B^2)\right)$
- N-body wave function : $\Psi = P(z_1, ..., z_N) \exp(-\sum |z_i|^2/(4l_B^2))$
- What are the low energy properties ? Gapped bulk, Massless edge
- Strongly correlated systems, emergence of exotic phases :fractional charges, non-abelian braiding.

What can we do ? Numerical simulations, effective field theories, trial wavefunctions

The Laughlin wave function

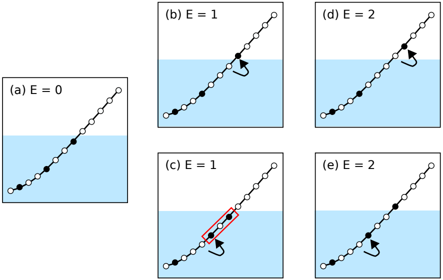
A (very) good approximation of the ground state at $\nu = \frac{1}{3}$

$$\Psi_L(z_1, ...z_N) = \prod_{i<j}(z_i - z_j)^3 e^{-\sum_i \frac{|z_i|^2}{4l^2}}$$

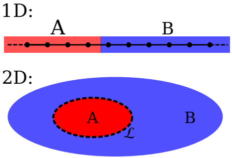
Excitations with fractional charge $\frac{\pm e}{3}$ and fractional statistics

Edge excitations

- A chiral $U(1)$ boson with a dispersion relation $E \simeq \frac{2\pi v}{L} n$
- The degeneracy of each energy level is given by the sequence $1, 1, 2, 3, ...$



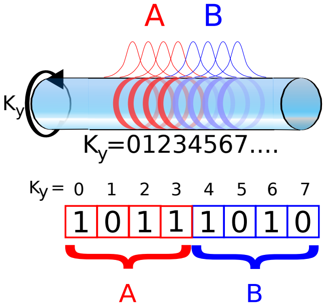
Entanglement entropy and entanglement spectrum



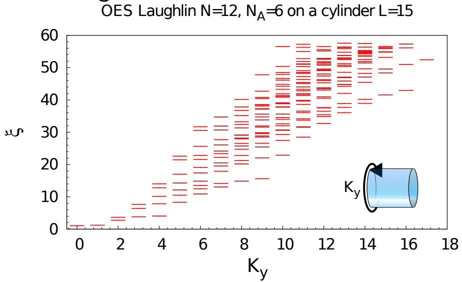
- Start from a quantum state $|\Psi\rangle$, create a bipartition of the system into A and B
- Reduced density matrix $\rho_A = \mathrm{Tr}_B |\Psi\rangle \langle\Psi|$
- Entanglement entropy $\mathcal{S}_A = -\mathrm{Tr}_A [\rho_A \ln \rho_A]$
  - For 2D topological phases : area law + topological constant correction (Levin/Wen, Kitaev/Preskill 06) : $\mathcal{S}_A \sim \alpha\mathcal{L} - \gamma$, $\mathcal{L}$ length of the boundary between A and B, γ only depends on the nature of the excitations.
  - Schmidt decomposition $|\Psi\rangle = \sum_i e^{-\xi_i/2} |A:i\rangle \otimes |B:i\rangle$ with $\langle A:i|A:j\rangle = \langle B:i|B:j\rangle = \delta_{i,j}$
  - $\rho_A = \sum_i e^{-\xi_i} |A:i\rangle \langle A:i|$
  - **Li and Haldane (2008)**
    - Look at the spectrum of ρ_A i.e. $\{\xi_i\}$ entanglement energies
    - Focus on specific blocks of ρ_A defined by their quantum numbers

Orbital entanglement spectrum

- FQHE on a cylinder (Landau gauge) : orbitals are labeled by k_y, rings at position $\frac{2\pi k_y}{L} l_B^2$
- Divide your orbitals into two groups A and B, keeping $N_{\mathrm{orb},A}$ orbitals : orbital cut $\simeq$ real space cut (fuzzy cut)



Laughlin state $N = 12$, half cut



- Fingerprint of the edge mode (edge mode counting) can be read from the ES. ES mimics the chiral edge mode spectrum.
- For FQH model states, nbr. levels is exp. lower than expected.

MPS for the FQH model states

Model states and CFT

- A large set of model wavefunctions can be written as a CFT correlator (Laughlin, Moore-Read, Read-Rezayi,...).

$$\Psi(z_1, \cdots, z_N) = \langle V(z_1) \cdots V(z_N) \rangle$$

  with electron operator $V(z)$ in some chiral $1+1$ CFT .
- Bulk-edge correspondence : The CFT used to describe the (gapped) bulk is identical to the CFT that describes the (gapless) edge
- **Laughlin state** :
  - $V(z) =: \exp(i\sqrt{m}\Phi(z)) :$, where $\Phi(z)$ is a free chiral boson
  - $\langle \Phi(z_1)\Phi(z_2) \rangle = -\log(z_1 - z_2)$
  - $\langle V(z_1) \cdots V(z_N) \rangle = \prod_{i<j}(z_i - z_j)^m$
- **Other states** : $V(z) =: \Psi(z) \otimes \exp(i\sqrt{q}\Phi(z)) :$ (i.e. neutral $\otimes$ $U(1)$ charge) (Moore-Read 91)

Matrix Product States

Any state can be written as

$$|\Psi\rangle = \sum_{\{m_i\}} \left(B^{[m_1]} \dots B^{[m_{N_{\mathrm{orb}}}]}\right)_{\alpha_l,\alpha_r} |m_1, \dots, m_{N_{\mathrm{orb}}}\rangle$$

where the $\{B^{[m]}\}$ is a set of matrices plus boundary conditions (here (α_l, α_r)).

The $B^{[m]}_{\alpha,\beta}$ matrices have two types of indices

- $[m]$ is the physical index (for FQH, occupied or empty orbital)
- (α, β) are the bond indices (auxiliary space), ranging from $1, \dots, \chi$.
- χ is related to the rank of the ES : the number of non-zero eigenvalues in the ES gives a lower bound to χ.
- In general χ is of the order of $\exp \mathcal{S}_A$ ($\mathcal{S}_A$ is the entanglement entropy) $\rightarrow$ for 2d topological phases, it grows exponentially with the perimeter of the cut (area law). An exponential improvement over the $\exp(\mathrm{surface})$ of ED...

Rewriting model states as MPS (Zaletel and Mong 2012)

- Insert a complete basis of states (continuous MPS - Dubail and Read (2012))

$$\sum_{\alpha_1,\cdots,\alpha_{N-1}} \langle 0|V(z_1)|\alpha_1\rangle\langle\alpha_1|V(z_2)|\alpha_2\rangle\cdots\langle\alpha_{N-1}|V(z_N)|0\rangle$$

$\langle\alpha|V(z)|\beta\rangle$ is a matrix with a continuous physical index (z).

- Project to $|m_1,\cdots,m_{N_{\rm orb}}\rangle$

One gets an infinite MPS on any genus 0 geometry

$$c_{(m_1,\cdots,m_{N_{\rm orb}})} = \left(B^{[m_1]}[1]\cdots B^{[m_{N_{\rm orb}}]}[N_{\rm orb}]\right)_{\alpha_L,\alpha_R}$$

$\langle\alpha'|B^{[0]}[j]|\alpha\rangle = \delta_{\alpha',\alpha}$ and $\langle\alpha'|B^{[1]}[j]|\alpha\rangle = \delta_{\Delta_{\alpha'},\Delta_\alpha+h+j}\langle\alpha'|V(1)|\alpha\rangle$

Site independent MPS

The previous formulation leads to B matrices that depend on the orbital. Can we make them site independent ?

Zaletel and Mong (2012) : if we can make the background charge uniform, then we obtain a site independent MPS

The CFT factorizes as $\mathcal{H} = \mathcal{H}_{\text{neutral}} \otimes \mathcal{H}_{U(1)}$, the product of a neutral CFT and a $U(1)$ chiral free boson.
What is needed for a numerical implementation ?

- To build the basis $|\alpha\rangle$ (i.e. the auxiliary space) and to have truncation scheme
- To compute the matrix elements $\langle\alpha'|B^{[m]}|\alpha\rangle$

Constructing the auxiliary CFT basis

We focus on the Laughlin state

- Electron operator $V(z) =: \exp(i\sqrt{m}\Phi(z)) :$.
- $\Phi(z) = \Phi_0 + ia_0 \log(z) + i\sum_{n\neq 0} \frac{1}{n} a_n z^n$.
- $|Q, \mu\rangle = \prod_{i=1}^{n} a_{-\mu_i} |Q\rangle$, $|Q\rangle$ vacuum with charge Q.
- $|\mu| = 0$, 1 state $|Q\rangle$, $\mu = \emptyset$.
- $|\mu| = 1$, 1 state $a_{-1}|Q\rangle$, $\mu = \{1\}$.
- $|\mu| = 2$, 2 states $a_{-1}^2|Q\rangle$, $\mu = \{1, 1\}$ and $a_{-2}|Q\rangle$, $\mu = \{2\}$.
- ...
- It is an infinite basis. $\rightarrow$ a truncation scheme is needed
- Analytical formula for the matrix elements $\langle Q, \mu | V(1) | Q', \mu' \rangle$

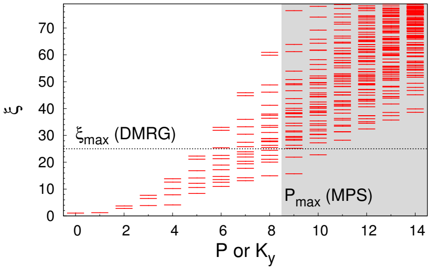
Beyond the Laughlin state, $|\Delta, \lambda\rangle \otimes |Q, \mu\rangle$, where $|\Delta, \lambda\rangle$ are the descendants of the primary field $|\Delta\rangle$ (non orthogonal, overcomplete basis).

Truncation of the auxiliary CFT basis

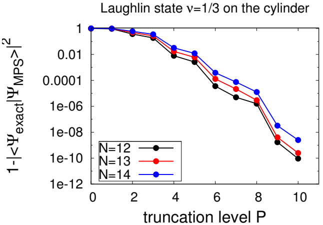
- The natural cut-off is the total level $P = |\lambda| + |\mu| \leq P_{\max}$ (cut-off at a given CFT level at $Q = 0$)
- Truncation over the momentum in the OES.
- In finite size, the truncated MPS becomes exact for $P_{\max}$ large enough.

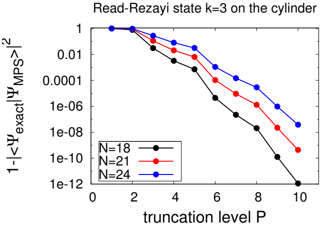
- DMRG : cut-off in ξ (remove the smallest weight of ρ_A).
- MPS : cut-off in momentum.

Equivalent if the ES mimics the chiral edge mode spectrum.

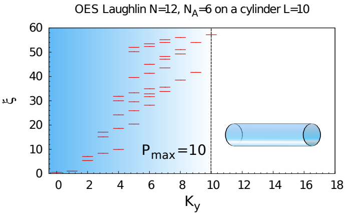


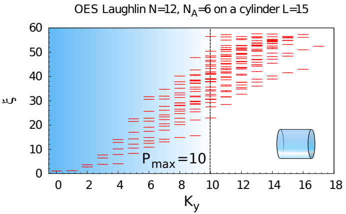
Benchmarking the MPS for the FQHE





MPS is a highly accurate approximation on the cylinder with a moderate perimeter ($\simeq 20 - 30 l_B$). A consequence of the area law.



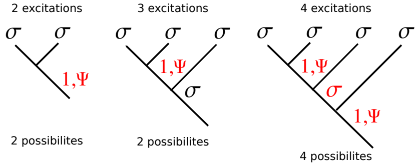


FQHE and topological entanglement entropy

Quantum dimension

Abelian states : one electronic wavefunction once the excitations are fixed
Non-abelian states : Nbr of wfs depends on the number of excitations $N_{\rm exc}$

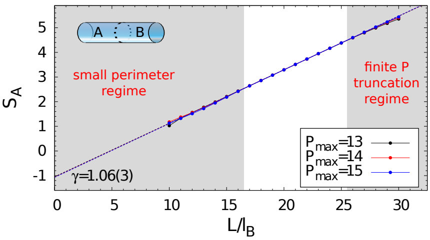
Fusion rules for the MR state :
$\sigma \times \sigma = 1 + \Psi$,
$1 \times \sigma = \sigma$ and $\Psi \times \sigma = \sigma$
Nbr of fusion trees = nbr of independent wfs
Here nbr of wfs $\simeq 2^{N_\sigma/2} = \sqrt{2}^{N_\sigma}$.



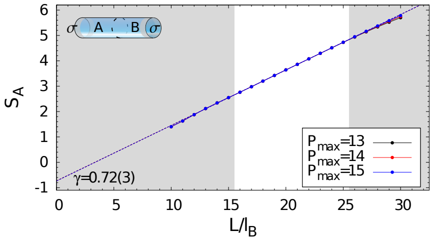
Quantum dimension d_a : for each topological sector a (i.e. type of excitations), the internal Hilbert space grows like $d_a^{N_\sigma}$.

Topological entanglement entropy

- 2d topological phase → area law : $\mathcal{S}_A \sim \alpha L - \gamma$
- Topological term $\gamma = \ln\left(\frac{\mathcal{D}}{d_a}\right)$
- Total quantum dimension $\mathcal{D} = \sqrt{\sum_a d_a^2}$, $d_a = 1$ for an abelian sector.
- For an abelian state at filling $\nu = \frac{p}{q}$, q abelian sectors.
- Moore-Read : 4 sectors with $d = 1$ and 2 sectors with $d = \sqrt{2}$



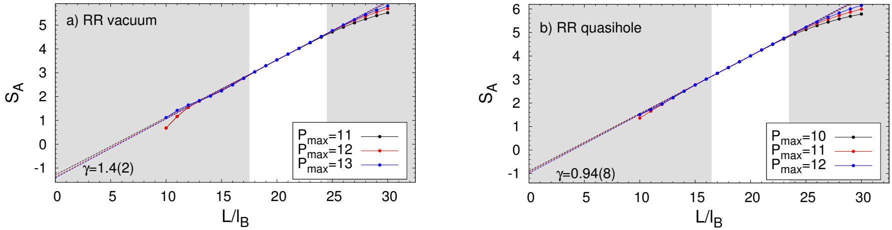
(CFT prediction $\ln\sqrt{8} \simeq 1.040$)



(CFT prediction $\ln\left(\sqrt{8}/\sqrt{2}\right) \simeq 0.693$)

Topological entanglement entropy

A highly non-trivial example : the $\mathbb{Z}_3$ Read-Rezayi state



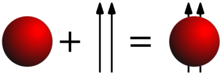
(CFT prediction $\simeq 1.44768$) (CFT prediction $\simeq 0.96647$)

- We can extract the quantum dimensions of all topological sectors.
- Good agreement despite using the orbital partition (versus real space cut)

FQHE and non-unitary CFTs

Composite fermions

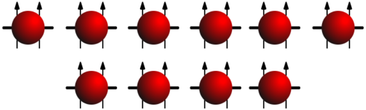
Jain's model (1989) :



Map FQHE into an integer quantum Hall effect for these composite fermions.

$$\nu^* = N/N_\phi^* = p \quad \longrightarrow \quad \nu = \frac{p}{2p+1}$$

Example : two filled effective Landau levels :



The $\nu = 2/5$ Jain's state is an abelian state with 5 topological sectors. No simple CFT expression (see Hermanns et al.)

Non-unitary CFTs and FQHE

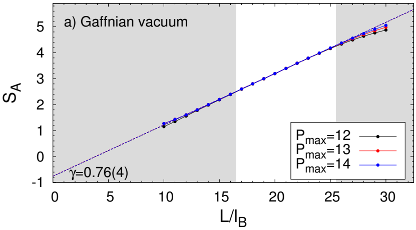
- Non-unitary CFTs have at least one primary field with a negative conformal dimension. Some correlation function for the edge may diverge

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = |x-y|^{-2h_{\mathcal{O}}}$$

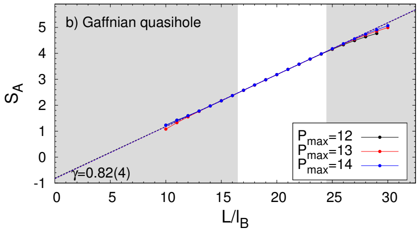
- But looking at finite size numerics, the bulk wavefunction seems OK .
- Hard to tell numerically if the bulk is gapless.
- Example : Gaffnian state at $\nu = 2/5$, non-abelian excitations (like the Moore-Read state), built from the $M(3,5)$ minimal model, quasihole field $h_\sigma = -1/20$. High overlap with the $\nu = 2/5$ Jain's state.
- Can the MPS probe the pathology of this state ?

Topological entanglement entropy

What about the Gaffnian state ? ED/Jack $L \simeq 17 l_B$



(CFT prediction $\simeq$ 1.44768)



(CFT prediction $\simeq$ 0.96647)

- γ barely depends on the topological sector i.e. $d_a = 1$.
- For an abelian state at $\nu = \frac{2}{5}$, $\gamma = \ln\left(\sqrt{5}\right) \simeq 0.805$.
- We only capture the abelian excitations.
- Is there something wrong about our calculation ?

iMPS - Transfer matrix

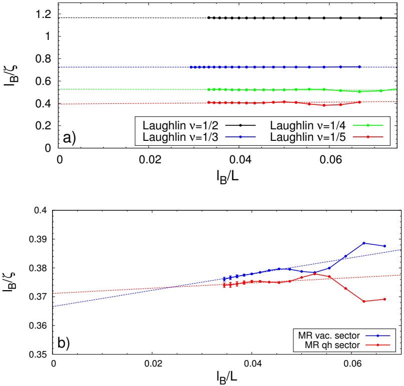
Transfer matrix : $E = \sum_m B^{[m]*} \otimes B^{[m]}$.

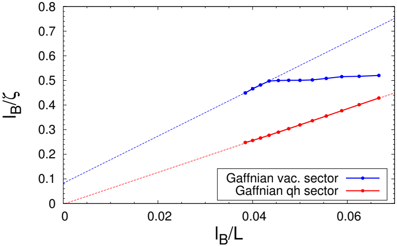
- How to compute the normalization of an MPS state ?
  $\langle \Psi | \Psi \rangle = \left(E^{N_{\mathrm{orb}}} \right)_{(\alpha,\alpha),(\beta,\beta)}$.
- For a large number of orbitals, only the eigenstate with the largest eigenvalue will matter.
- The transfer matrix allows to do several calculations in a simple and elegant manner for an infinitely long system (iMPS) including the entanglement entropy.
- We can extract the correlation length from the ratio between the two largest eigenvalues λ_1 and λ_2 :

$$\langle \mathcal{O}^\dagger(x)\mathcal{O}(0)\rangle - \langle \mathcal{O}^\dagger(x)\rangle\langle \mathcal{O}(0)\rangle \propto \exp\left(-\frac{|x|}{\zeta}\right)$$

with $\zeta^{-1} = \frac{L}{2\pi l_B^2} \ln(\frac{\lambda_1}{\lambda_2})$

Correlation length ζ





The Gaffnian has a critical behavior in the qh sector of the bulk, (where qh operator has a negative scaling dimension). The MR state has the same corr. length in both sectors $\zeta \simeq 2.7 l_B$.

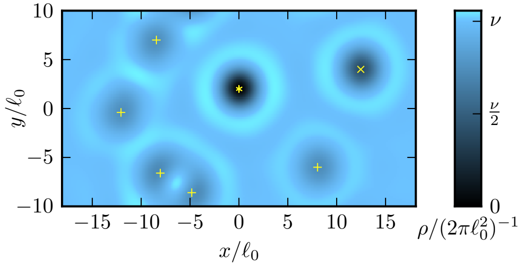
Probing the excitations

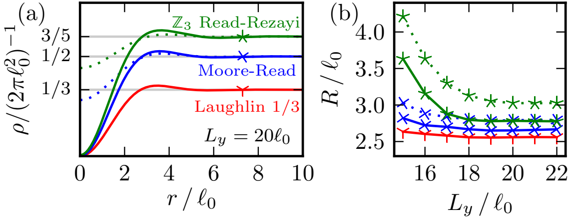
Quasihole Sizes for $\mathbb{Z}_{k\leq 3}$ Read-Rezayi

density profile for $\mathbb{Z}_3$ Read-Rezayi quasiholes

$+ : \frac{e}{5}$, $x : \frac{2e}{5}$, $* : \frac{3e}{5}$

- isotropic
- fusion-channel indep. when well separated





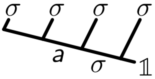
quasihole radius comparable between different k ($\simeq 3l_B$)

First numerical estimate for Read-Rezayi quasihole radius.

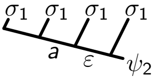
$l_B \sim 10$ nm $\Rightarrow R \sim 30$ nm ; interferometer qh separation ~ 100 nm

Braiding

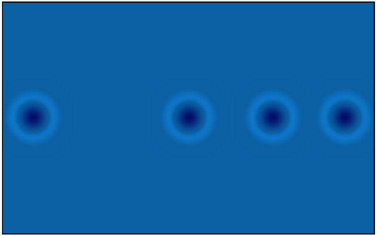
conformal block basis

$\mathrm{MR}: |\Psi_a\rangle =$ 

$, \quad a = 1, \psi.$

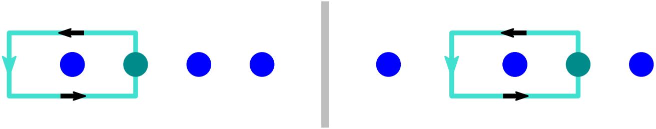
$\mathrm{RR}: |\Psi_a\rangle =$ 

$, \quad a = \psi_1, \sigma_2.$



compute Wilson loop by integrating non-Abelian Berry connection

$$\mathcal{A}_{ab}(\eta; \mathrm{d}\eta) \equiv e^{-i\mathrm{d}\eta A_{ab}(\eta)} \equiv \frac{\langle \Psi_a(\eta + \mathrm{d}\eta) | \Psi_b(\eta) \rangle}{\| \Psi_a(\eta + \mathrm{d}\eta) \| \cdot \| \Psi_b(\eta) \|},$$



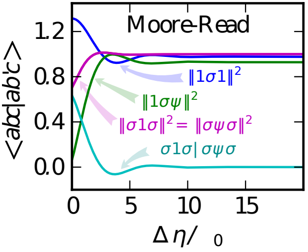
Branch cuts and test of the screening

- Usual assumptions : Link between $2+1$ TQFT and $1+1$ CFT
  - degeneracy = number of conformal blocks
  - braiding = monodromies
- Branch cuts : monodromy, tested MR/RR (10^{-4})
- Non-singular part : the non-universal could spoil everything
  Controlled by overlap matrix $\langle \Psi_a | \Psi_b \rangle$ for fixed qh positions

Screening condition and Berry connection :
at large quasihole separation $|\Delta\eta|$, the overlap converges exponentially fast to a constant diagonal matrix,

$$\langle \Psi_a | \Psi_b \rangle = C_a \delta_{ab} + \mathcal{O}(e^{-|\Delta\eta|/\xi_{ab}})$$

then the Berry connection vanishes up to an exp. small correction $A_{ab}(\eta) \sim \mathcal{O}(e^{-|\Delta\eta|/\xi_{ab}})$ after subtracting Aharonov-Bohm phase.



Conclusion

- MPS can be constructed for a large class of FQH model states.
- Quantum dimensions can be extracted from the microscopic wavefunction (MR and RR).
- MPS allows to directly probe the pathology some model wave functions built from the non-unitary CFT.
- First numerical estimate of the RR quasihole size.
- Microscopic verification that the $\mathbb{Z}_3$ Read-Rezayi quasiholes are Fibonacci anyons, without assuming screening.