Scaling laws for the dynamo

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What magnetic field should your planet produce?

- Given the radius and mass of your planet and its orbital period, how large a magnetic field can you expect?
  - The magnetostrophic idea: a force balance argument.
  - Scaling laws from numerical simulations: a hot topic!
  - Exploring a new approach: turbulent scaling laws.
- How often should its magnetic field reverse?

Dynamo-generated magnetic field

Gauss $\nabla \cdot \mathbf{B} = 0$

Faraday $\partial_t \mathbf{B} = -\nabla \times \mathbf{E}$

Ohm $\dfrac{\mathbf{j}}{\sigma} = \mathbf{E} + \mathbf{U} \times \mathbf{B}$

Ampère $\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$

Induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \eta \Delta \mathbf{B}$$

$$\eta = \frac{1}{\mu_0 \sigma}$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \frac{1}{Rm} \Delta \mathbf{B}$$

$$Rm = \frac{UL}{\eta}$$

Magnetic Reynolds number

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1) Magnetostrophic equilibrium

Navier-Stokes equation

$$\partial_t \mathbf{U} + \left(\mathbf{U}\cdot\nabla\right)\mathbf{U} = -2\mathbf{\Omega}\times\mathbf{U} - \nabla P + \mathbf{j}\times\mathbf{B} - \alpha T\mathbf{g} + \nu\Delta\mathbf{U}$$

In the Earth’s core, we expect the dominant terms to be the Coriolis term and the Lorentz force, which should balance each other: this is the magnetostrophic equilibrium, which leads to:

$$\Lambda = \frac{jB}{2\Omega U} \approx 1$$

Elsasser number

Magnetostrophic field intensity (1)

If you estimate *j* from Ohm's law, you get:

$$\Lambda_{Ohm} = \frac{jB}{2\Omega U} \approx \frac{\sigma U B^2}{2\Omega U} \approx 1$$

=> $V_{Alfven} \approx \sqrt{\eta\Omega}$

where: $V_{Alfven} = B \big/ \sqrt{\rho\mu_0}$

i.e., the intensity of the magnetic field only depends on the rotation rate of the planet, irrespective of the available power.

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Magnetostrophic field intensity (2)

If you estimate *j* from Ampère's law, which is more appropriate in the dynamo regime, you get:

$$\Lambda_{Ampère} = \frac{jB}{2\Omega U} \approx \frac{B^2}{2\Omega U \mu_0} \frac{L}{\ell_B} \approx 1$$

=>

$$V_{Alfven} \approx \sqrt{\eta \Omega}\, Rm^{1/4}$$

i.e., the intensity of the magnetic field still depends on the rotation rate of the planet, but increases with the available power (because Rm increases).

Cardin et al, 2002

A word on Alfvén waves (1942)

Waves that propagate along field lines of an imposed magnetic field **B**

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \left(\frac{\mathbf{B}}{\mu}\cdot\nabla\right)\mathbf{b} + \rho\nu\nabla^2\mathbf{u},$$

linearized Navier-Stokes eq^n

$$\frac{\partial \mathbf{b}}{\partial t} = (\mathbf{B}\cdot\nabla)\,\mathbf{u} + \frac{1}{\mu\sigma}\nabla^2\mathbf{b}$$

linearized induction eq^n

Introducing Elsasser variables: $\mathbf{u}^{\pm} = \mathbf{u} \pm \mathbf{b}/\sqrt{\rho\mu}$ yields:

$$\frac{\partial \mathbf{u}^+}{\partial t} = -\nabla\frac{p}{\rho} + \left(\frac{\mathbf{B}}{\sqrt{\rho\mu}}\cdot\nabla\right)\mathbf{u}^+ + \nu\nabla^2\mathbf{u} + \frac{1}{\mu\sigma}\nabla^2\frac{\mathbf{b}}{\sqrt{\rho\mu}},$$

$$\frac{\partial \mathbf{u}^-}{\partial t} = -\nabla\frac{p}{\rho} - \left(\frac{\mathbf{B}}{\sqrt{\rho\mu}}\cdot\nabla\right)\mathbf{u}^- + \nu\nabla^2\mathbf{u} - \frac{1}{\mu\sigma}\nabla^2\frac{\mathbf{b}}{\sqrt{\rho\mu}}$$

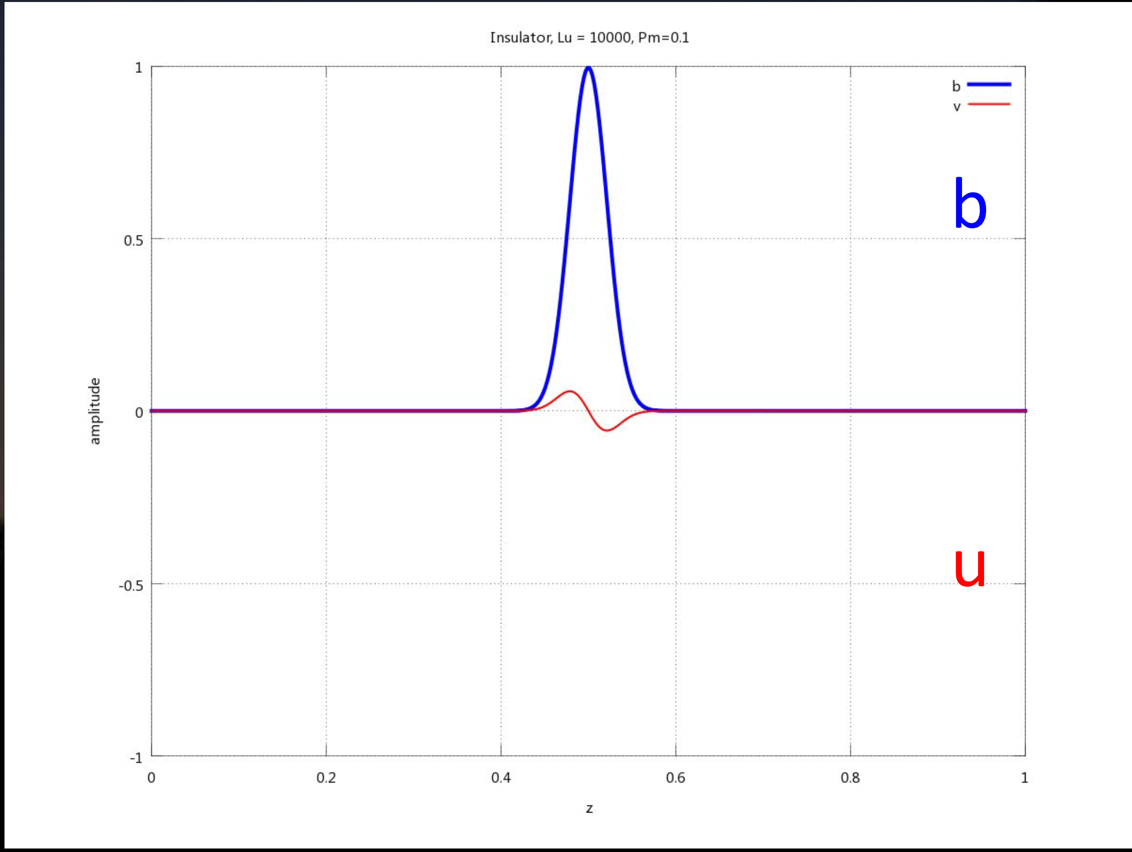
from Alboussière et al, 2011

Properties of ideal Alfvén waves

$$\frac{\partial \mathbf{u}^{\pm}}{\partial t} = \pm\left(\frac{\mathbf{B}}{\sqrt{\rho\mu}}\cdot\nabla\right)\mathbf{u}^{\pm}$$

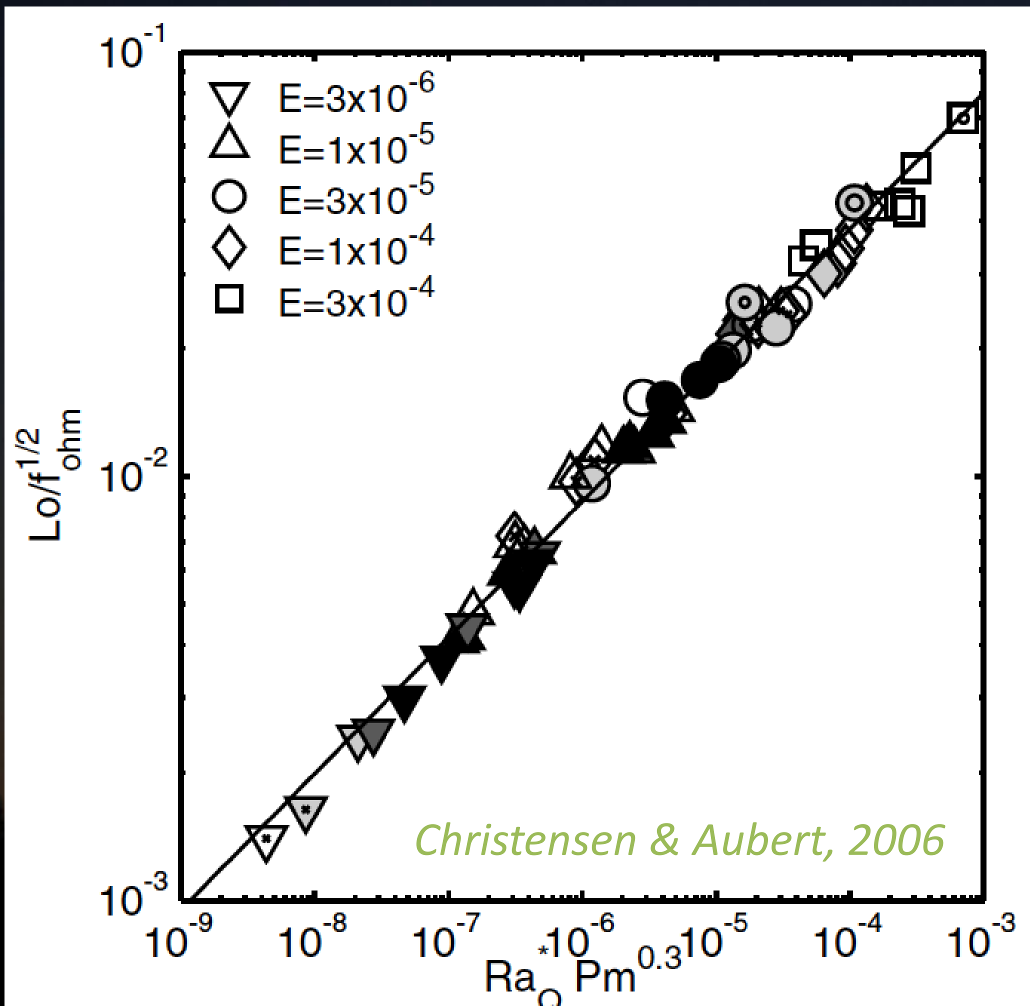
With vanishing diffusivities

- Transverse
- Non-dispersive
- Alfvén velocity
- Energy equipartition



courtesy Ph. Cardin

2) numerical simulations scaling laws



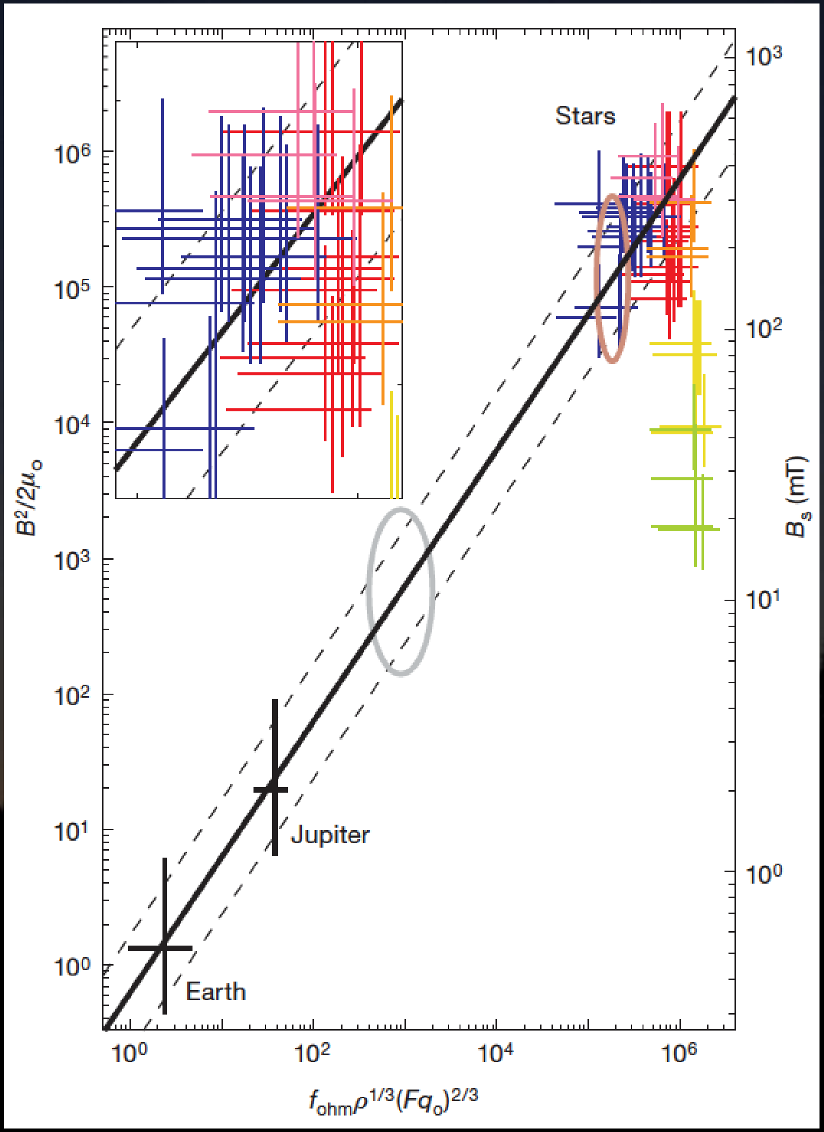
$$V_{Alfvén} \approx a \left[\frac{P}{M_{oc}} R_{oc} \right]^{1/3}$$

Gaidos et al, 2010

Predicts a magnetic field intensity that only depends on the available power *P*.

Christensen & Aubert, 2006
Olson & Christensen, 2006
Christensen, 2010

Widely used…



Christensen et al, 2009

…but severely criticized!

- Log(a*x) vs log(b*x) deceiving alignment *(X)*
- Shingling effect *(Cheng & Aurnou, 2015)*
- Circular non-predictive law *(Oruba & Dormy, 2014)*
- Non-magnetostrophic viscous regime *(King & Buffett, 2013; Oruba & Dormy, 2014)*

From the *same* simulation results, Oruba & Dormy (2014) deduce an alternative scaling:

$$V_{Alfvén} \approx \sqrt{\nu\Omega}\sqrt{\tilde{R}-\tilde{R}_d}$$

with:

$$\tilde{R} = Ra/Ra_c$$

3) Turbulent scaling laws

- In liquid metals, Re ~ 10^6 Rm *must* be large
- Turbulence involves a large range of spatial and temporal scales. Numerical simulations can't cover that range. Neither do observations, but phenomena we observe at the large scales do depend upon the unseen scales.
- Let's try to see what turbulence can tell us about the intensity of the magnetic field.

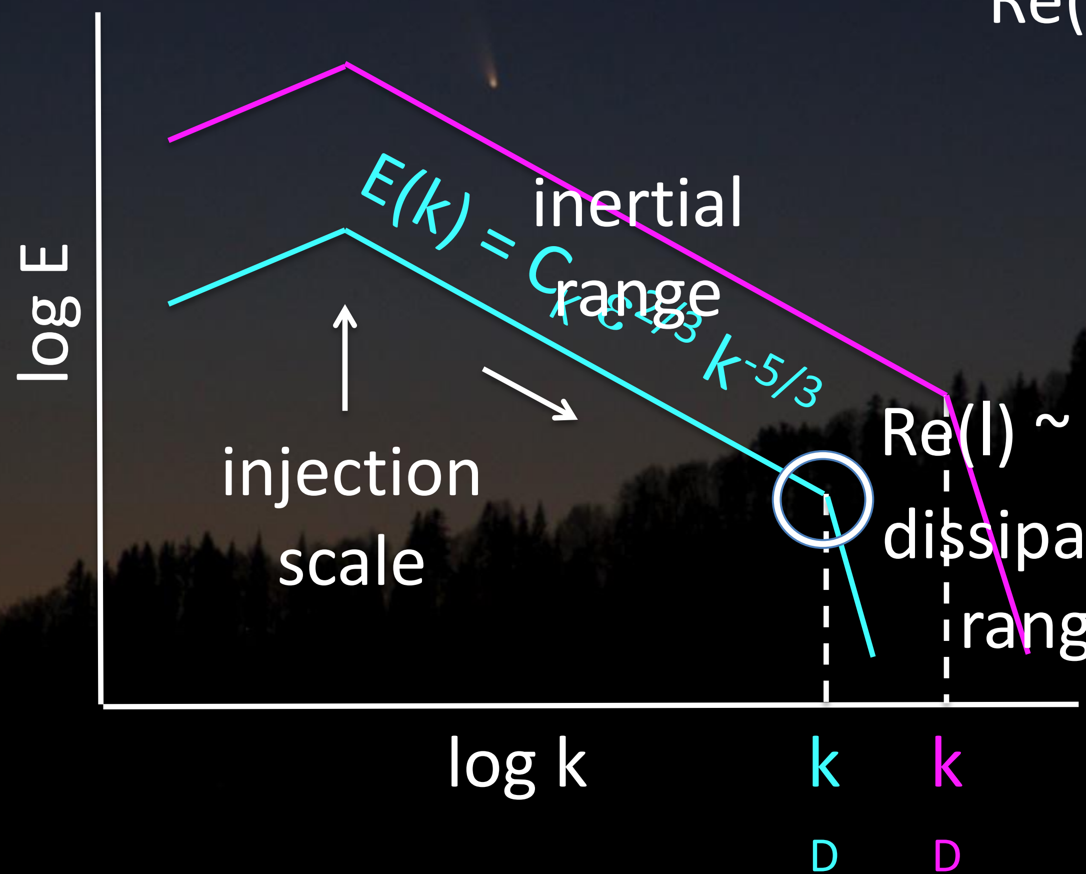
What do we know about turbulence?

- Most of what we know relates to hydrodynamic turbulence governed by the non-linear $(\mathbf{u}.\nabla)\mathbf{u}$ term in the Navier-Stokes equation.
- The role of turbulence appears very dull !
  It transfers energy from some ‘large scale’ at which it is injected to ‘small scales’ at which it can be dissipated by viscosity.

Kolmogorov's universal turbulence

Re(l) ~ U(l) l/ν

l ~ 1/k



What turbulence in planetary cores?

→ Introducing "τ-l regime diagrams"

"Turbulence in the core", H-C. Nataf and N. Schaeffer, in *Treatise on Geophysics*, volume 9 "the Core" 2nd edition, Ed P. Olson, Elsevier, to appear in May 2015.

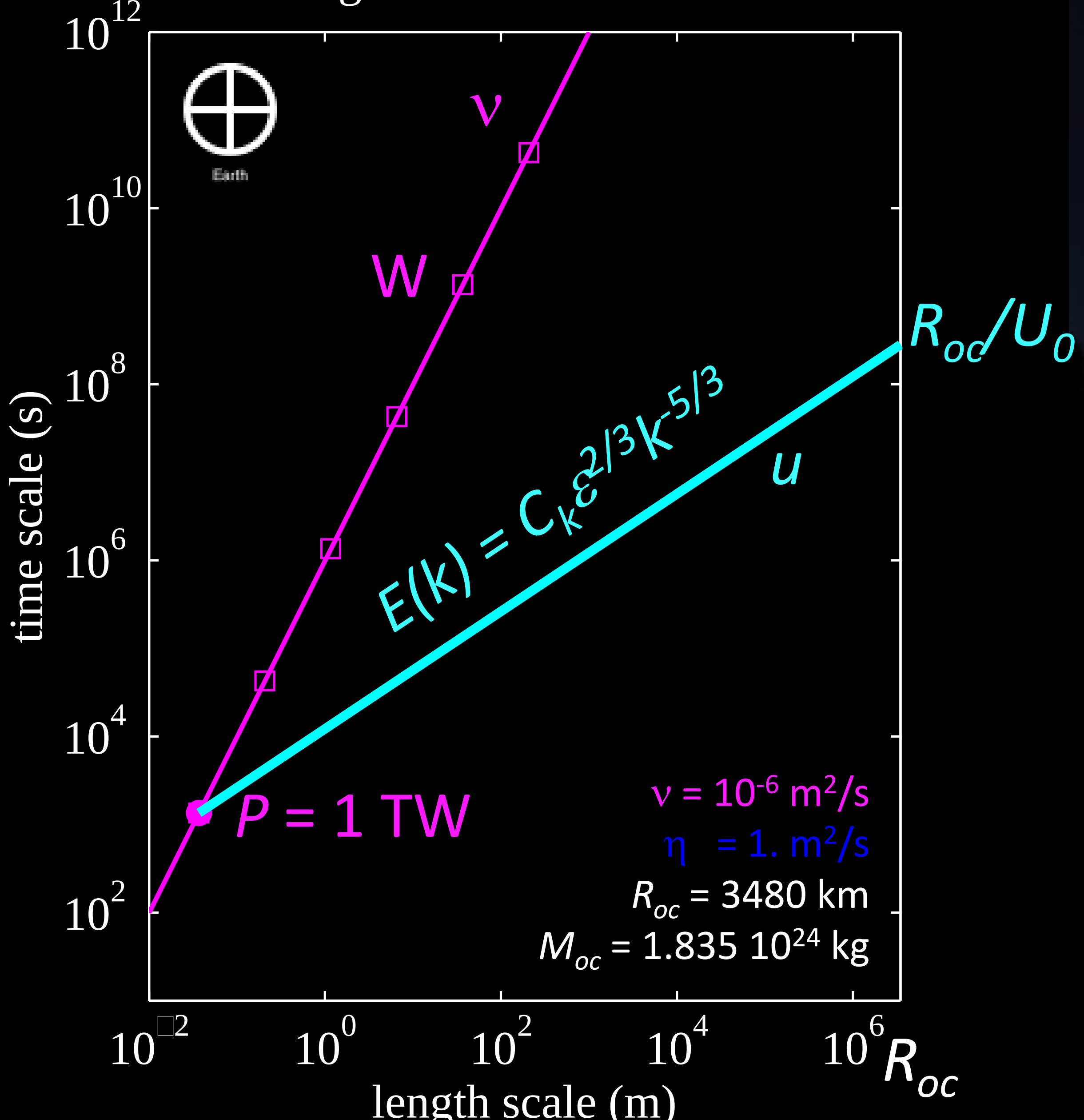
τ-l regime diagrams assumptions

- Turbulence involves a wide range of scales
- Time-scales τ and length-scales l are related by various physical processes
- Regime changes occur when τ(l) lines intersect
- Dimensionless numbers can we written as time-scale ratios
- The scale l at which a dimensionless number is ~1 is more important than the value of that number at the integral scale
- Turbulent dynamics is controlled by the shortest time-scale process

Kolmogorov universal turbulence

A flow of intensity U_0 injected at scale R_{oc} cascades down to viscous dissipation scale where it dissipates P given by:

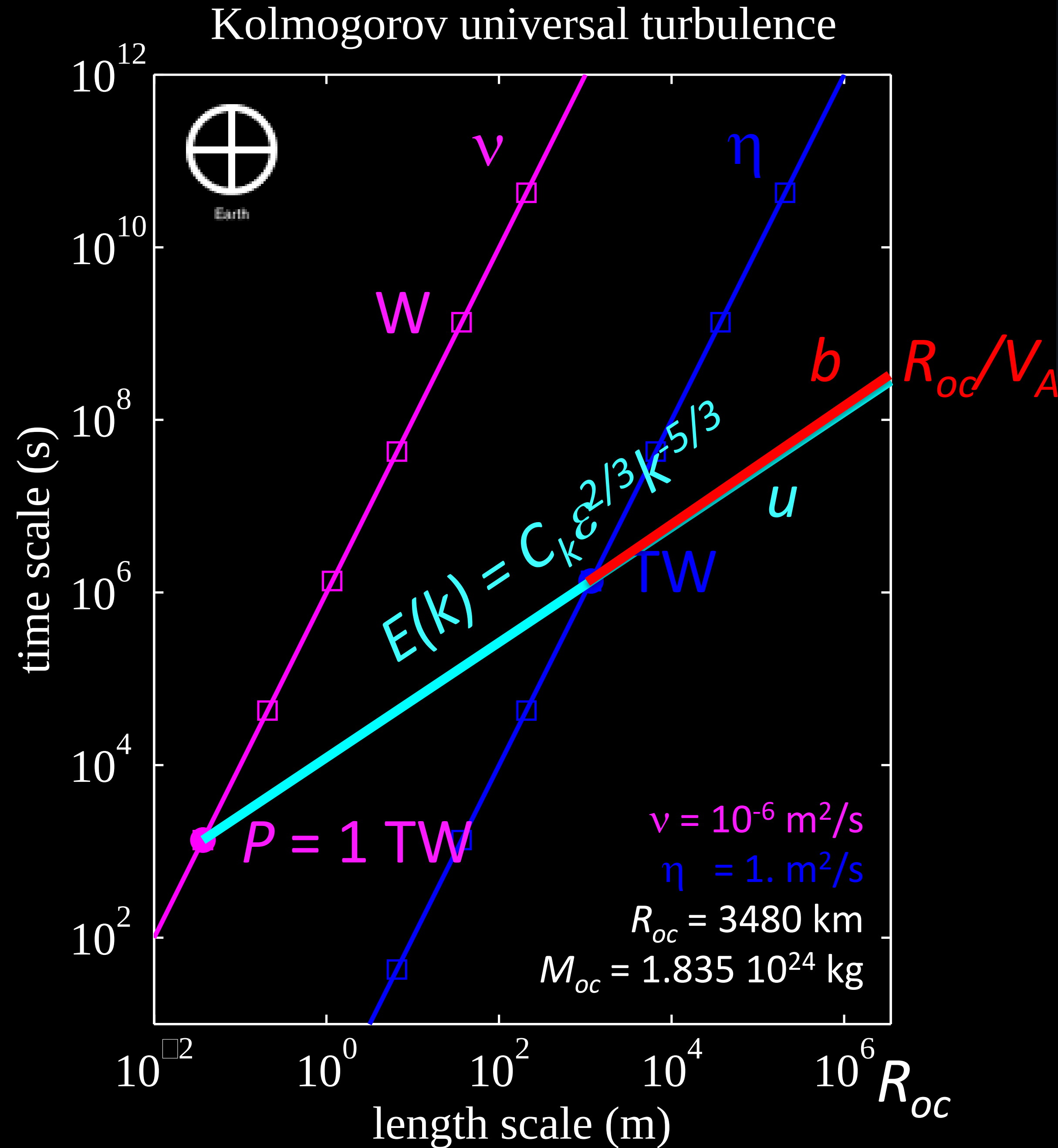
$$U_0 \approx \left[\frac{P}{M_{oc}} R_{oc} \right]^{1/3}$$



The same reasoning can be made with the magnetic field yielding the same scaling law as was derived from the numerical simulations:

$$V_{Alfvén} \approx \left[\frac{P}{M_{oc}} R_{oc} \right]^{1/3}$$

Note the equipartition in both ***energy*** and ***dissipation*** despite Pm = 10^{-6}



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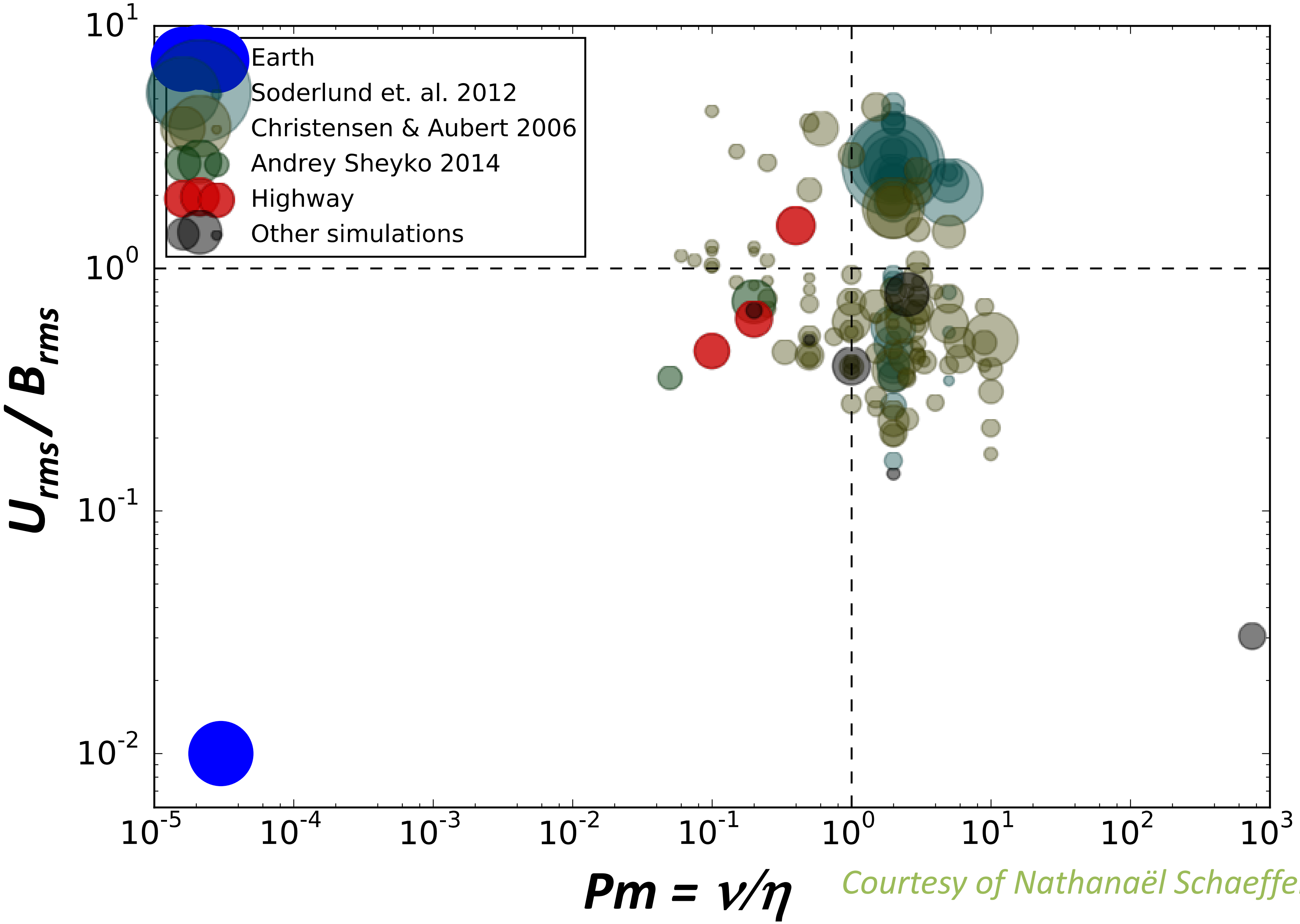
Problems…

- In the Earth's core, U_0 is some 100 hundred times slower than $V_{Alfvén}$ (*i.e.*, $E_M / E_K \sim 10^4$)



Courtesy of Nathanaël Schaeffer, 2015

Problems…

- In the Earth's core, U_0 is some 100 hundred times slower than $V_{Alfvén}$ (*i.e.*, $E_M / E_K \sim 10^4$)
- Turbulence should be strongly modified by the presence of such a strong magnetic field
- Turbulence should also be strongly modified by the rotation of the planet

=> We have built τ-l diagrams to help us assess when, how and how much

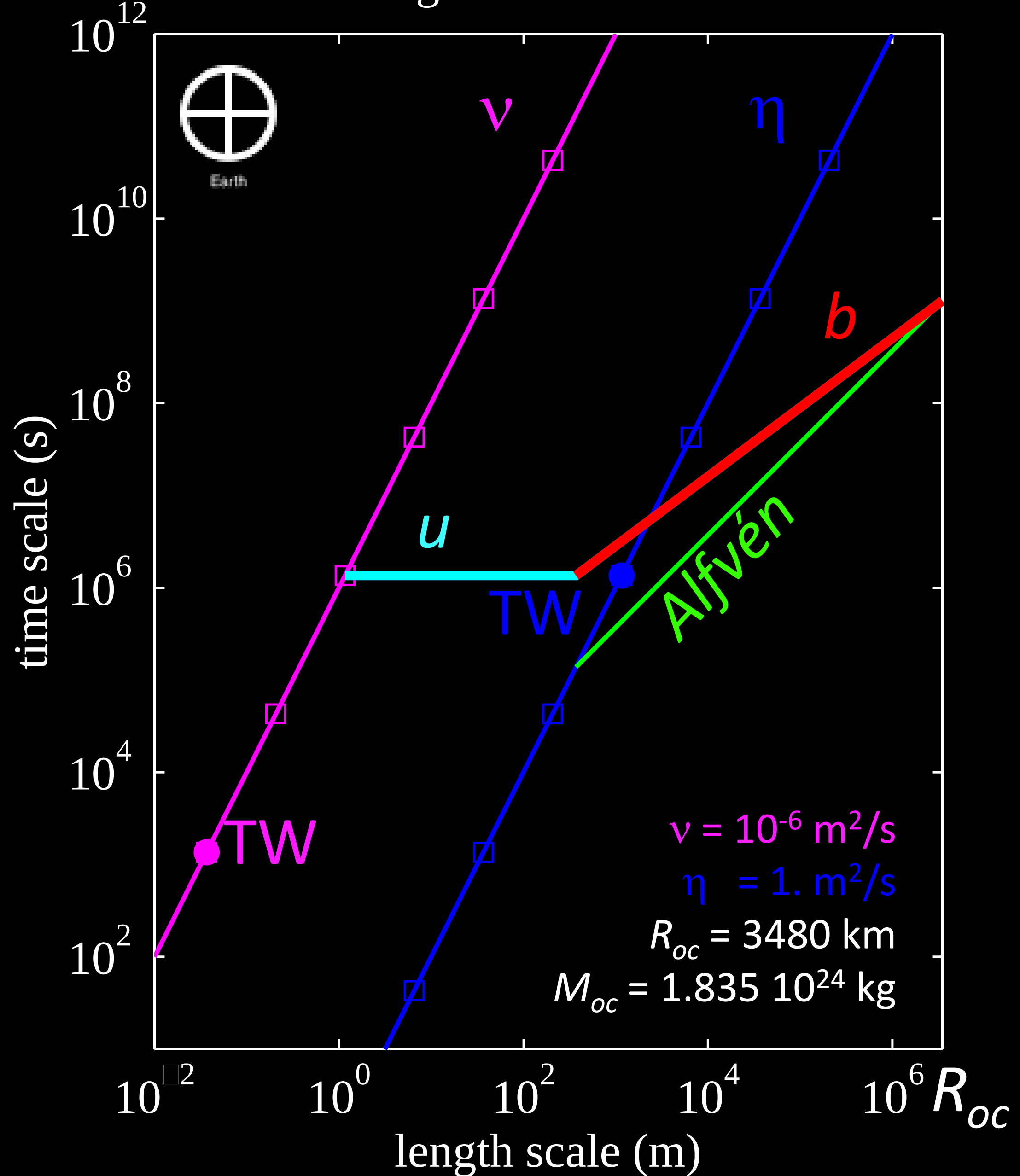
MHD turbulence

- While eddies are the building bricks of hydrodynamic turbulence, it is believed that Alfvén waves are those of magnetohydrodynamic turbulence, governed by the non-linear $(\mathbf{B}.\nabla)\mathbf{b}$ term in the Navier-Stokes equation and $(\mathbf{B}.\nabla)\mathbf{u}$ term in the induction equation.

Tobias et al, 2013

$$V_A = \frac{B}{\sqrt{r m_0}}$$

strong MHD turbulence



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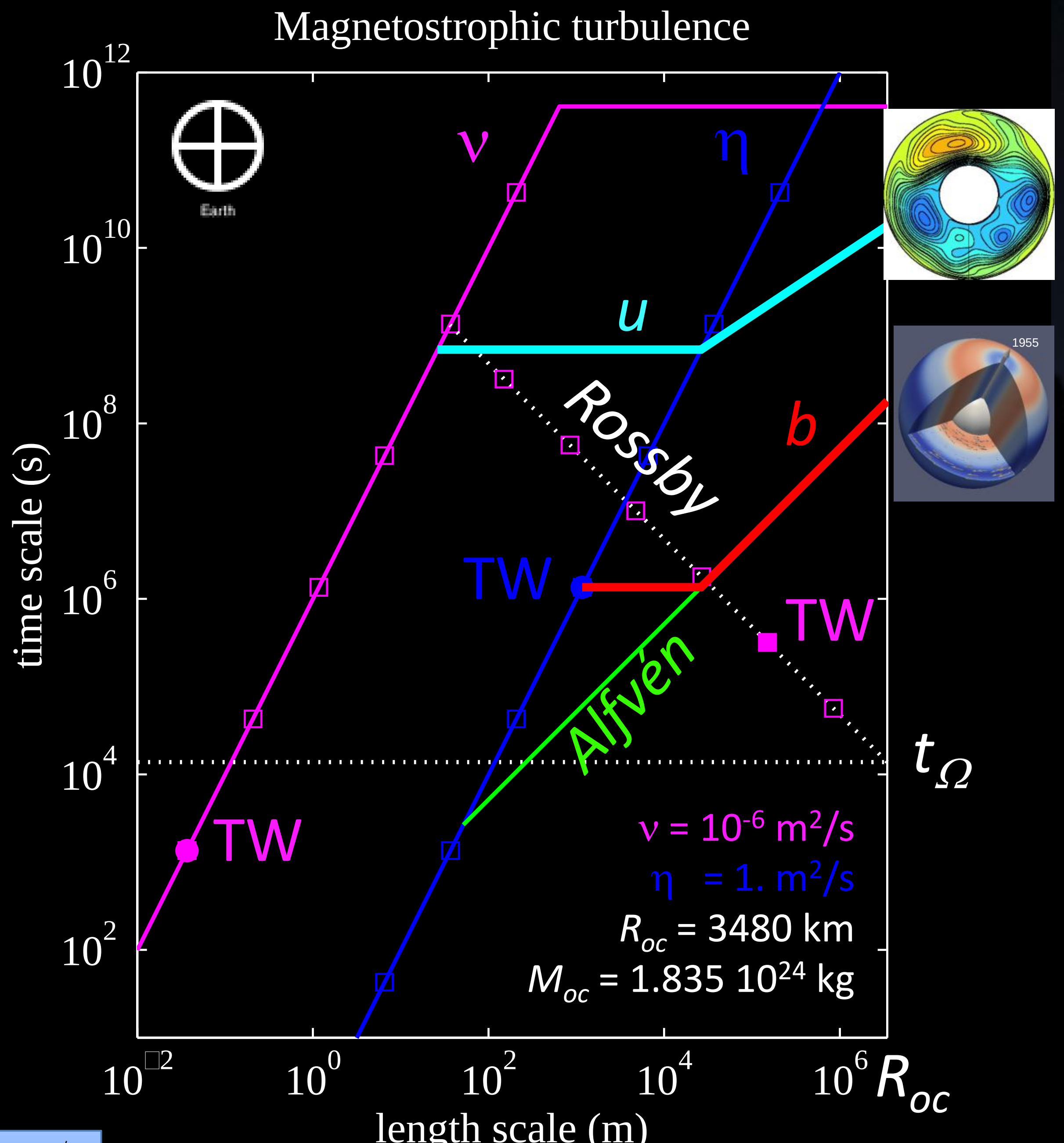
When the spin period of the planet is smaller than the Alfvén wave time, Alfvén wave turbulence is inhibited. We build an alternative scenario, with U_0/V_A given by $Pm=\nu/\eta$. Dissipation occurs at much larger scales, and ohmic dissipation dominates.

The resulting magnetic field intensity is not very different from the numerical simulations.



$$V_{magnetostrophy} \approx Pm^{-1/7} V_{simu}^{6/7} V_h^{1/7}$$

Magnetic reversals

- The magnetic field of the Earth reverses
- The magnetic field of the Sun reverses
- Magnetic reversals are observed in the VKS dynamo experiment *(Berhanu et al, 2007)*

=> How often should the magnetic field of your planet reverse?

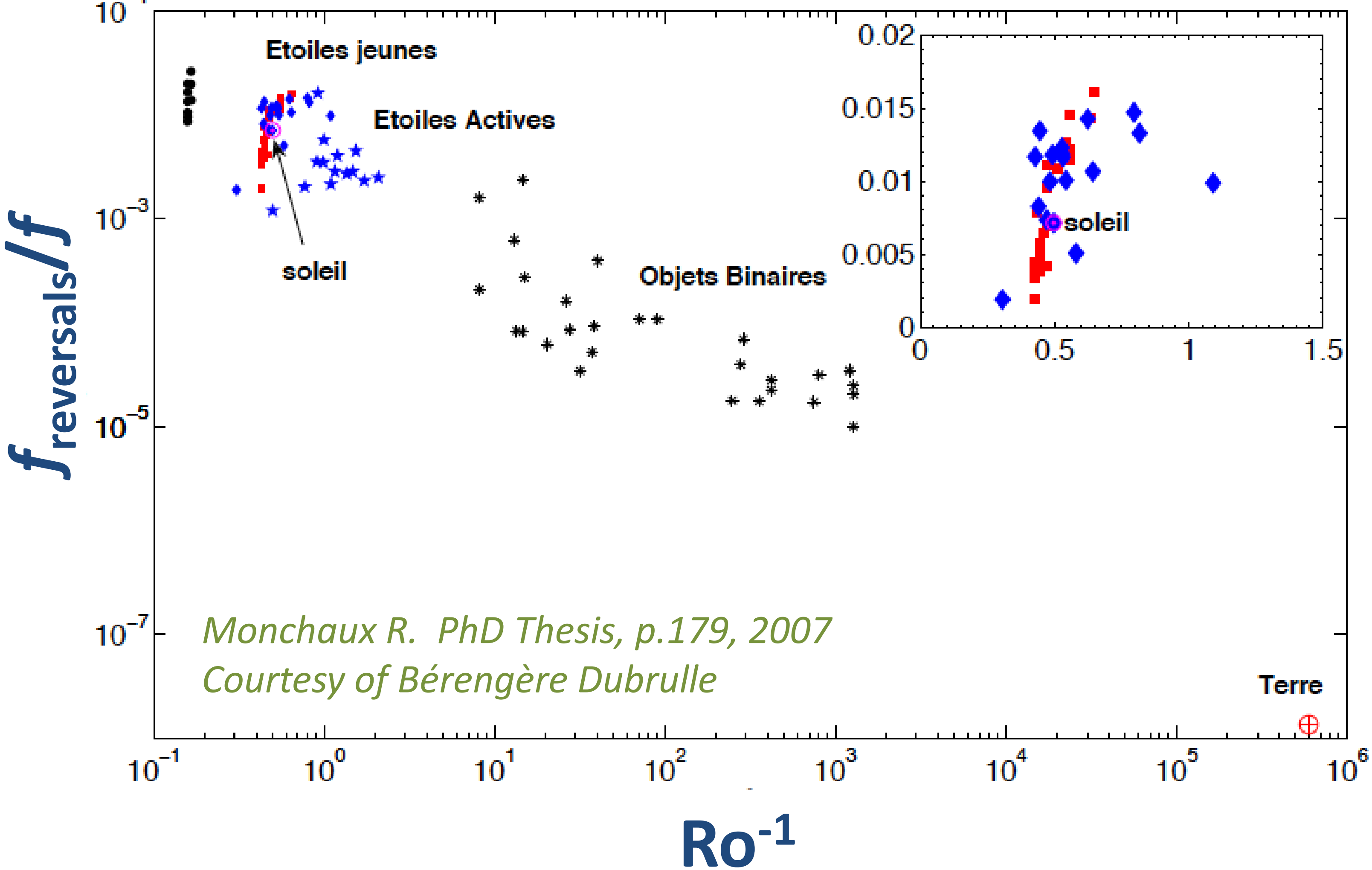


FIG. 9.21: Fréquence d'oscillation du champ magnétique F_{dyn} adimensionnée par la fréquence de rotation de l'objet F_{rot} en fonction du paramètre θ pour divers objets astrophysiques en échelle logarithmique. Diamants bleus : étoiles jeunes dont le soleil (⊙ violet), étoiles bleues : étoiles actives, étoiles noires : objets binaires, ⊕ rouge : la Terre. Les carrés rouges correspondent aux régimes oscillants de VKS, les cercles noirs aux renversements. L'insert présente un diagramme linéaire-linéaire sur lequel les étoiles jeunes et VKS sont représentées.

Thank you!