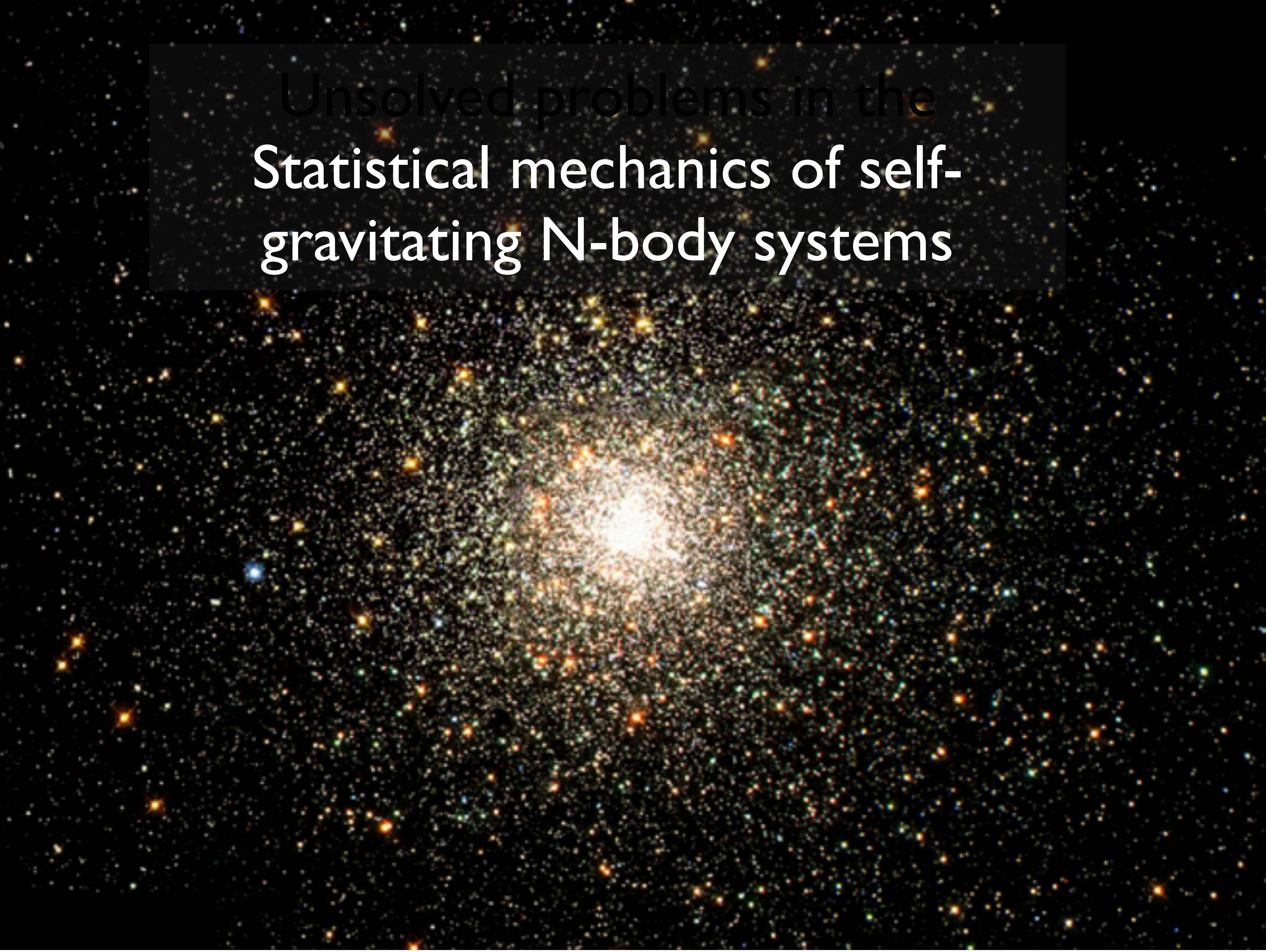

Unsolved problems in the
Statistical mechanics of self-gravitating N-body systems

gas in a box	stellar system
molecules, m ~ 10^{-24} g	stars, m ~ 10^{33} g
N ~ 10^{23}	N ~ 10^2-10^5 (star clusters), ~10^5-10^{12} (galaxies)
short-range forces	long-range forces (gravity)
confined in a box	confined by self-gravity
mean free path << system size (Knudsen number Kn << 1)	mean free path >> system size (Kn >> 1)

Equations of motion for N-body system are

$$\ddot{\mathbf{r}}_i = \sum_{j=1}^{N} G m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Then

$$I = \frac{1}{2}\sum_{i=1}^{N} m_i r_i^2,$$

$$\ddot{I} = \sum_{i=1}^{N} m_i v_i^2 + \sum_{i \neq j} G m_i m_j \frac{\mathbf{r}_i \cdot (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j|^3} = \sum_{i=1}^{N} m_i v_i^2 + \sum_{i>j} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

$$= 2K + W = K + E$$

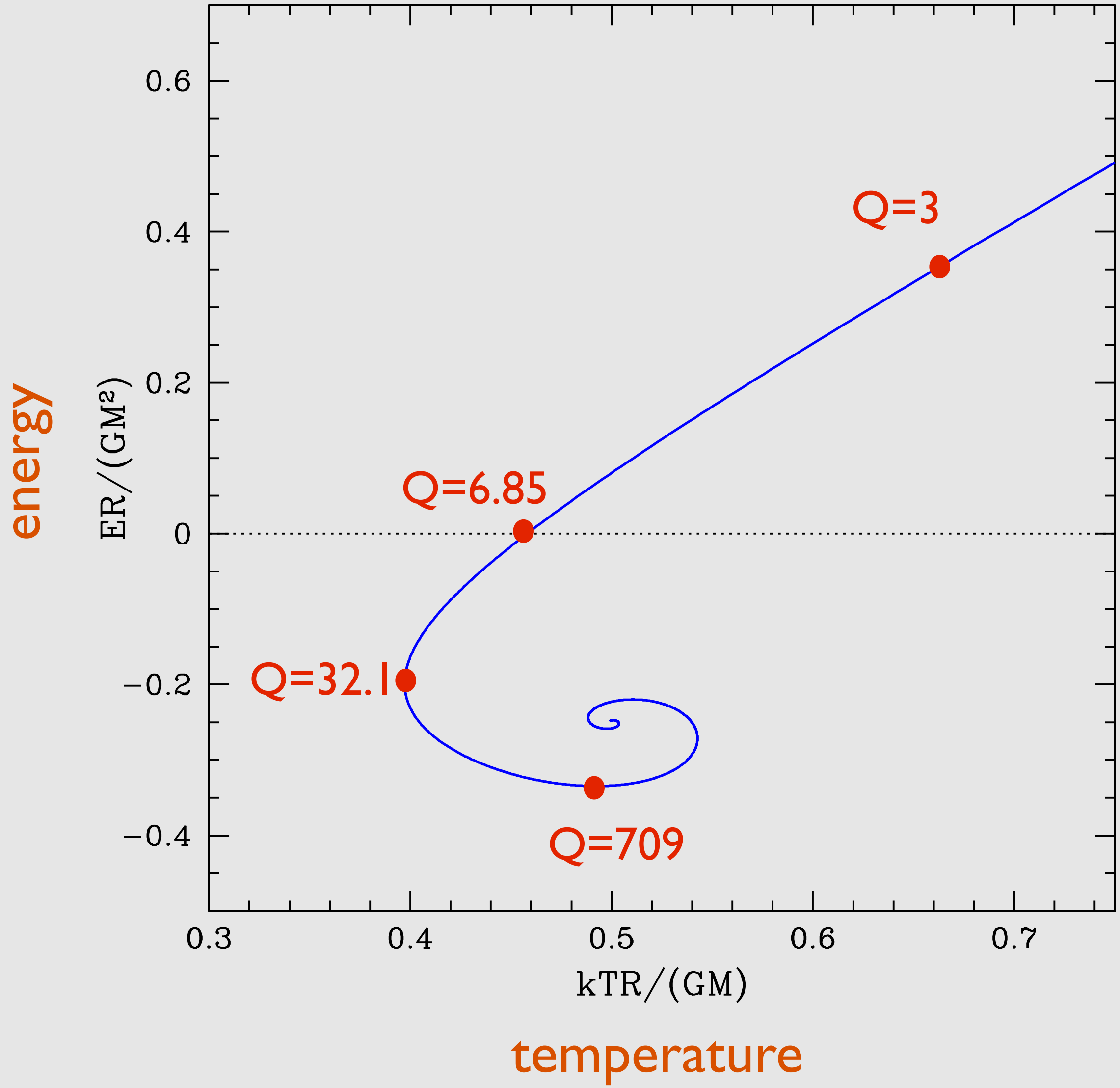
K = kinetic energy
W = potential energy
E = K+W = total energy

So in a steady state

$$E = -K.$$

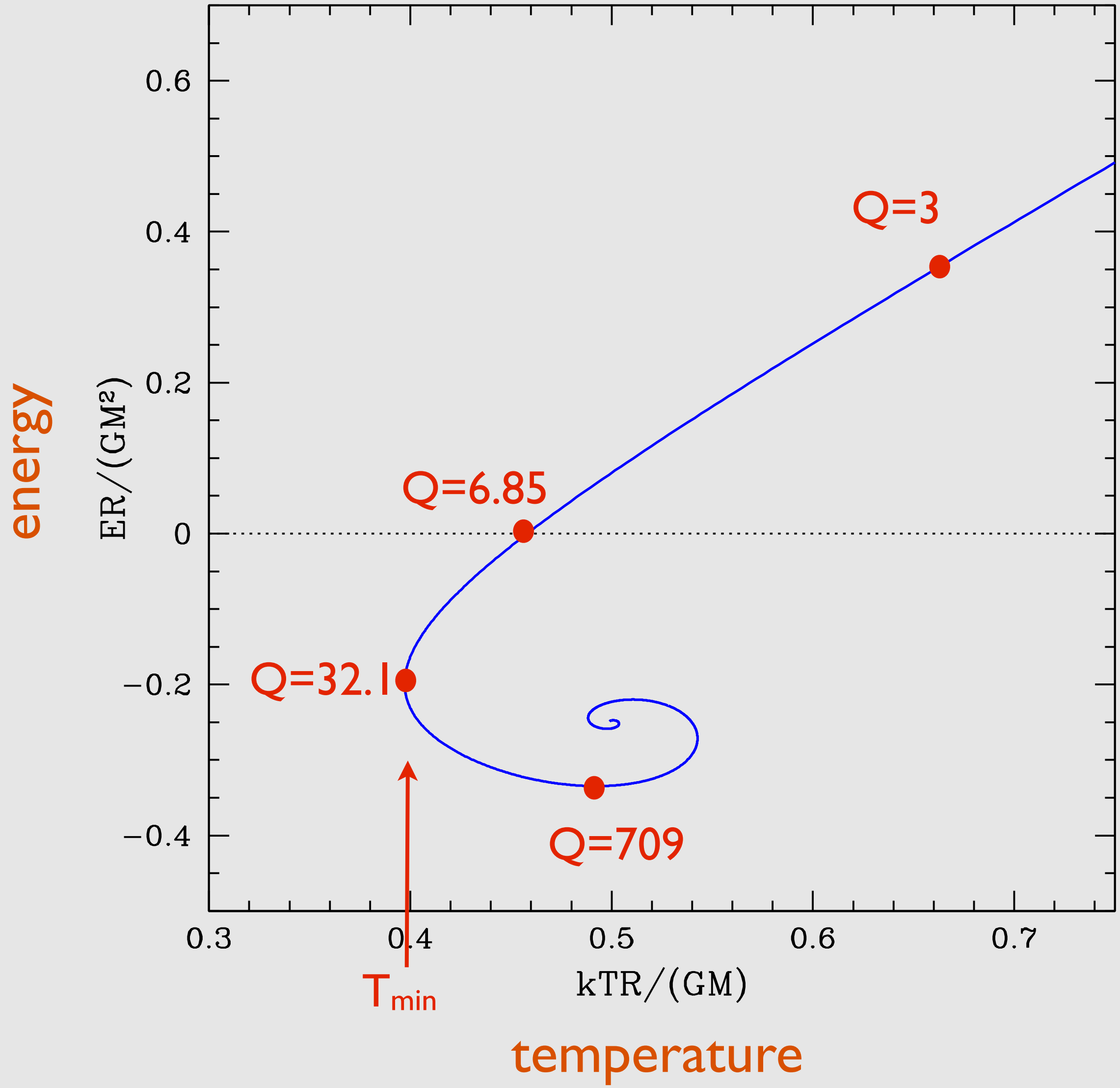
If isothermal $K = \frac{3}{2} NkT$ so heat capacity is

$$C = \frac{dE}{dT} = -\frac{3}{2} Nk.$$

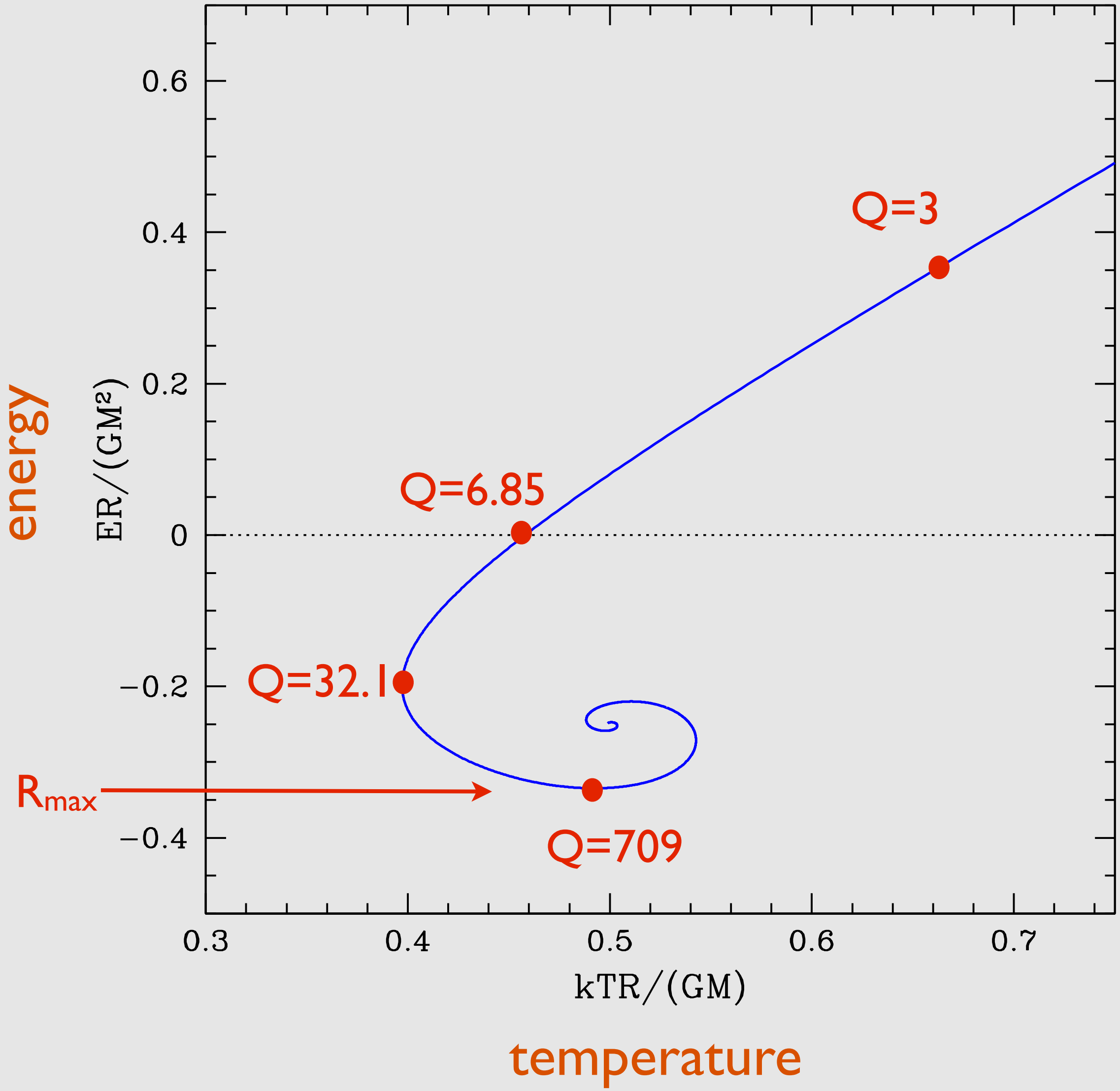


- self-gravitating gas of mass M in a rigid spherical container of radius R
- solutions parametrized by density contrast Q = ρ(0)/ρ(R)
- heat capacity at constant volume C = dE/dT = slope

Antonov (1962)
Lynden-Bell & Wood (1968)
Thirring (1970)
Katz (1978)



- place box in contact with a heat bath at temperature T and slowly reduce T
- below T_{min} there is no equilibrium state
- systems between Q=32.1 and Q=709 are unstable equilibria (entropy is a saddle point, not a maximum)



- insulate box and suddenly expand its radius R
- E is conserved so if E<0 $ER/(GM^2)$ becomes more negative
- for $R > R_{max}$ there is no equilibrium state
- for Q > 709 all equilibrium states are unstable

- isolated self-gravitating systems have negative heat capacity
- there is no thermodynamic equilibrium state for self-gravitating systems unless they are enclosed in a sufficiently small box
- there is no "heat death" of the Universe

- there is no thermodynamic equilibrium state for self-gravitating systems unless they are enclosed in a sufficiently small box
  ⇒ stellar systems cannot survive much longer than the equipartition or relaxation time due to gravitational encounters between stars
- for a spherical system of N stars with crossing time t_{cross}

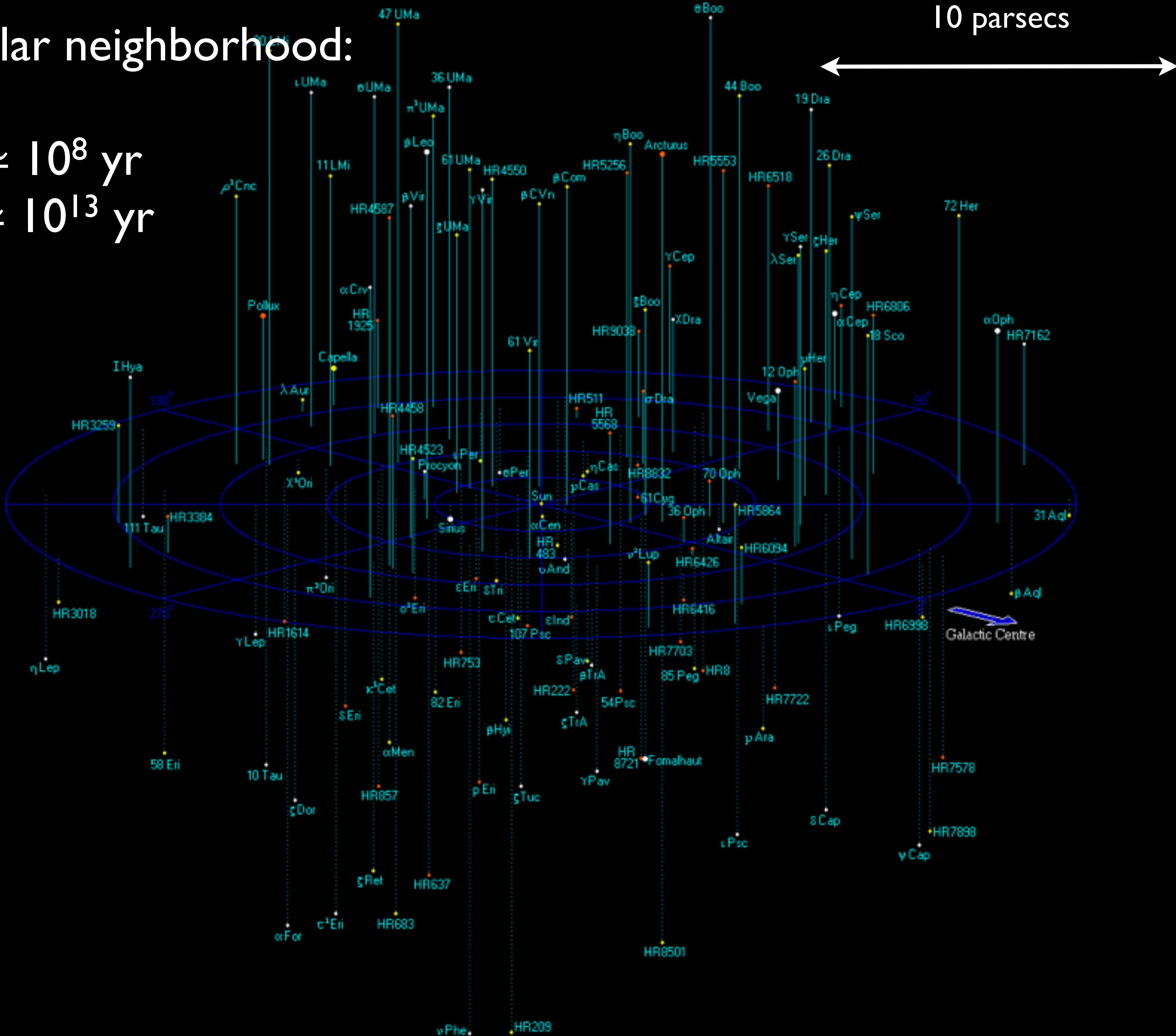
$$t_{relax} \simeq 0.1\, t_{cross}\, N / \log N$$

- relaxation time t_{relax} is similar to the equipartition time

the solar neighborhood:

$t_{cross} \simeq 10^8$ yr

$t_{relax} \simeq 10^{13}$ yr



- stars in the solar neighborhood exhibit "random" velocities of 5-50 km/s in addition to common rotational velocity of ~220 km/s
- more massive stars have smaller random velocities
- rms velocity vs mass is roughly consistent with equipartition
- timescale required to reach equipartition due to gravitational encounters between stars is ~10^{13} yr ⇒ universe must be *at least* this old

TABLE I.—EQUIPARTITION OF ENERGY IN STELLAR MOTIONS.

Type of Star.	Mean Mass, M.	Mean Velocity, C.	Mean Energy, $\frac{1}{2}MC^2$.	Corresponding Temperature.
				Degrees.
Spectral type $B3$.	$19{\cdot}8 \times 10^{33}$	$14{\cdot}8 \times 10^5$	$1{\cdot}95 \times 10^{46}$	$1{\cdot}0 \times 10^{62}$
,, $B8{\cdot}5$.	12·9	15·8	1·62	0·8
,, $A0$.	12·1	24·5	3·63	1·8
,, $A2$.	10·0	27·2	3·72	1·8
,, $A5$.	8·0	29·9	3·55	1·7
,, $F0$.	5·0	35·9	3·24	1·6
,, $F5$.	3·1	47·9	3·55	1·7
,, $G0$.	2·0	64·6	4·07	2·0
,, $G5$.	1·5	77·6	4·57	2·2
,, $K0$.	1·4	79·4	4·27	2·1
,, $K5$.	1·2	74·1	3·39	1·7
,, $M0$.	1·2	77·6	3·55	1·7

Jeans (1928)

- stars in the Milky Way disk exhibit "random" velocities of 5-50 km/s in addition to common rotational velocity of ~220 km/s
- more massive stars had smaller random velocities, consistent with equipartition
- timescale required to reach equipartition due to gravitational encounters between stars is ~10^{13} yr ⇒ universe must be *at least* this old

- in fact random velocities arise from gravitational interactions with interstellar clouds and spiral arms, and more massive stars have smaller velocities because they are younger

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giant galaxies:
$N \simeq 10^{11}$
$t_{cross} \simeq 10^8$ yr
$t_{relax} \simeq 10^{19}$ yr

giant galaxies:

$N \simeq 10^{11}$

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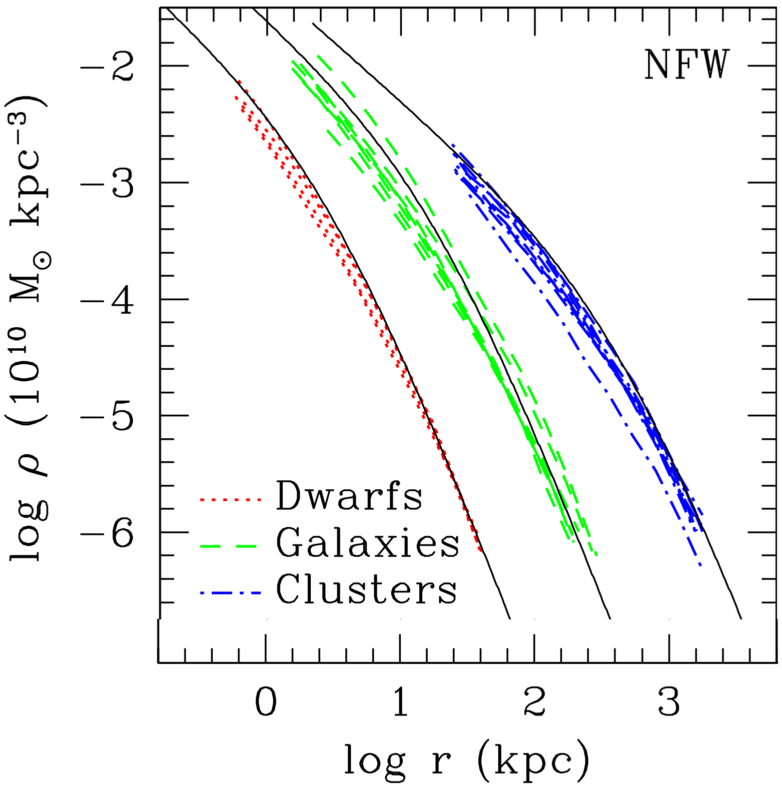
- the distribution of stars is similar, apart from scale, in all galaxies
- the distribution of stellar velocities is close to Maxwellian
- how is this achieved if the relaxation time is much longer than the age?

Answer:

- large-scale fluctuations in the mean gravitational field during collapse of the galaxy drive the distribution of stars towards an (approximately) universal form ("violent relaxation", Lynden-Bell 1967)

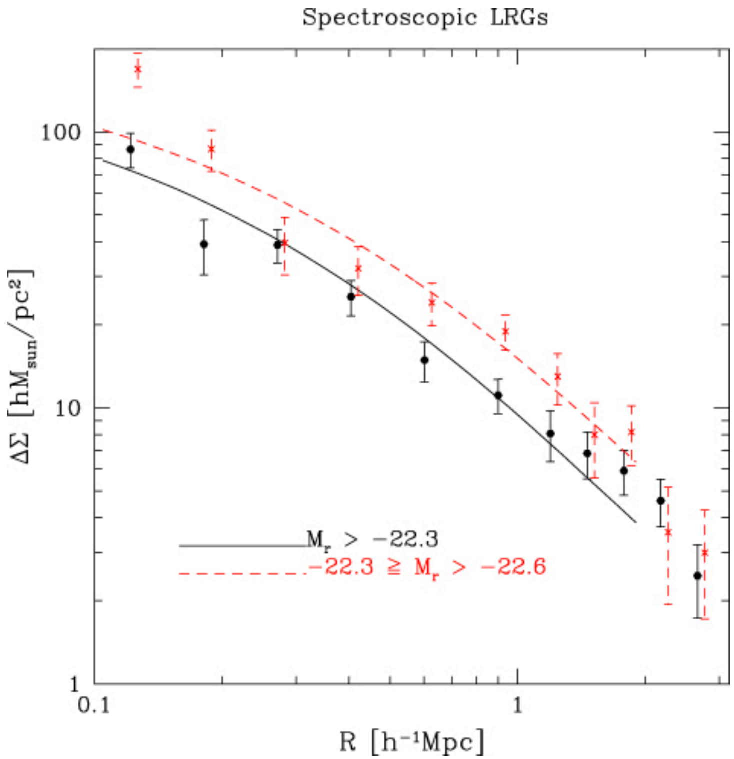


Navarro et al. (2004)

- density profiles of dark-matter halos in simulations are well fit over > 3 orders of magnitude in radius, > 5 orders of magnitude in mass, and a wide variety of initial conditions by simple empirical formulae
- e.g., Navarro-Frenk-White (NFW) profile

$$\rho(r) = \rho_0 \frac{a^3}{r(r+a)^2}$$

- suggests that there is some simple physics that determines the density profile and other halo properties



Mandelbaum et al. (2008)

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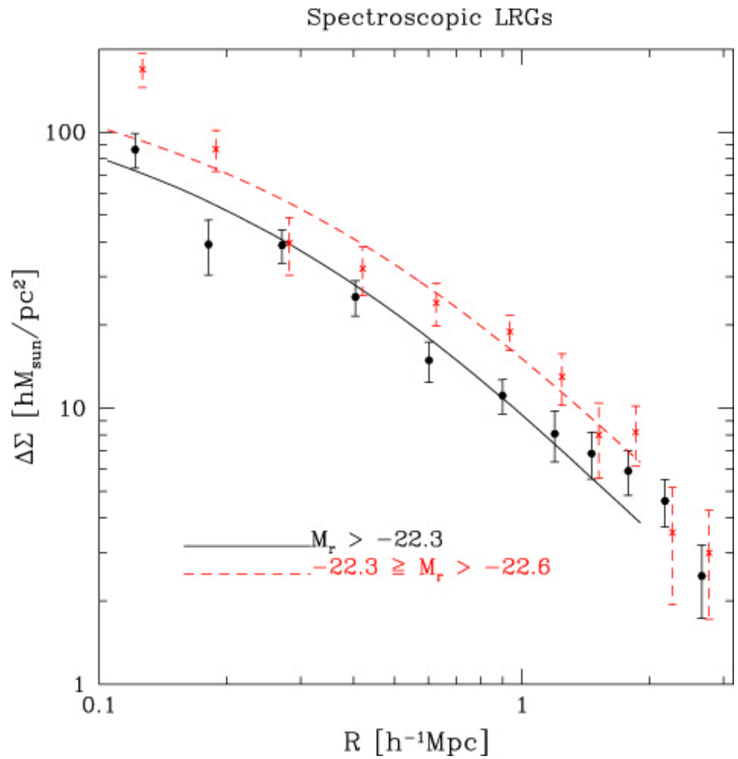

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Lynden-Bell (1967)
Binney (1982)
Madsen (1987)
Shu (1987)
Stiavelli & Bertin (1987)
Williams & Hjorth (2010)
Dalal et al. (2010)
Pontzen & Governato (2013)
Beraldo e Silva et al. (2014)
Alard (2014)

In most dark matter models the phase-space density $f(\mathbf{x}, \mathbf{v})$ satisfies the collisionless Boltzmann equation (a.k.a. Vlasov equation, Liouville equation, continuity equation in phase space)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

and the Poisson equation

$$\nabla^2 \Phi = 4\pi G \int d\mathbf{v} \, f(\mathbf{x}, \mathbf{v}, t).$$

The natural first approach is to assume that violent relaxation leads to a final state that maximizes the entropy

$$S = -\int d\mathbf{x} d\mathbf{v} \, f \log f + \text{constant}$$

at fixed mass and energy.

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$$S = -\int d\mathbf{x} d\mathbf{v}\, f \log f + \text{constant} \qquad ✗$$

at fixed mass and energy.

The primary feature of entropy in statistical mechanics is that it satisfies Boltzmann's H theorem, i.e. molecular collisions imply that

$$\frac{dH}{dt} \leq 0 \quad \text{where} \quad H = -S = \int d\mathbf{x} d\mathbf{v} \, f \log f.$$

Relaxation is a Markov process in phase space defined by the probability p_{ji} that a particle in cell i transitions to cell j after time Δt. If all cells have the same size then time-reversibility implies $p_{ji} = p_{ij}$. Then

$$\frac{dH}{dt} \leq 0 \quad \text{where} \quad H = \int d\mathbf{x} d\mathbf{v} C(f)$$

and $C(f)$ is any convex function, $C''(f) \geq 0$, e.g.,

$$C(f) = f \log f, \quad C(f) = f^2, \quad C(f) = -\log f, \text{ etc.}$$

Maximum-entropy arguments in violent relaxation do not lead to a unique final state. An initial phase-space distribution f(x,v) can only evolve into a final one f′(x,v) if *all* possible H-functions are smaller for f′ than for f.

A simpler criterion:

$$\int d\mathbf{x} d\mathbf{v} \max[f(\mathbf{x},\mathbf{v}) - \phi, 0] \geq \int d\mathbf{x} d\mathbf{v} \max[f'(\mathbf{x},\mathbf{v}) - \phi, 0] \quad \text{for all } \phi > 0$$

Dehnen (2005)

Unfortunately for cold dark matter the left side diverges...

⇒ some physics other than maximum entropy is needed to understand violent relaxation

Statistical mechanics of planetary systems

There are many bad examples of attempts to explain the spacing and other properties of planetary orbits from first principles 

Nevertheless there are reasons to try again:

- *Kepler* has provided a large statistical sample of planet candidates, and multi-planet systems
- N-body integrations can routinely follow the evolution of hundreds of systems for 100 Myr and these capture a "giant impact phase" lasting from a few Myr to a few tens of Myr
- there are also hints of interesting behavior from studies of the stability of the solar system:
  - the orbits of all of the planets in the solar system are chaotic, with Liapunov (e-folding) times of ~10^7 yr (Sussman & Wisdom 1988, 1992, Laskar 1989, Hayes 2008)
  - the outer solar system is "full" in the sense that no stable orbits remain between Jupiter and Neptune (Holman 1997)
  - there is a 1% chance that Mercury will be lost from the solar system before the end of the Sun's life in ~ 7 Gyr
- ejected planets are likely to be detected in the near future by microlensing surveys

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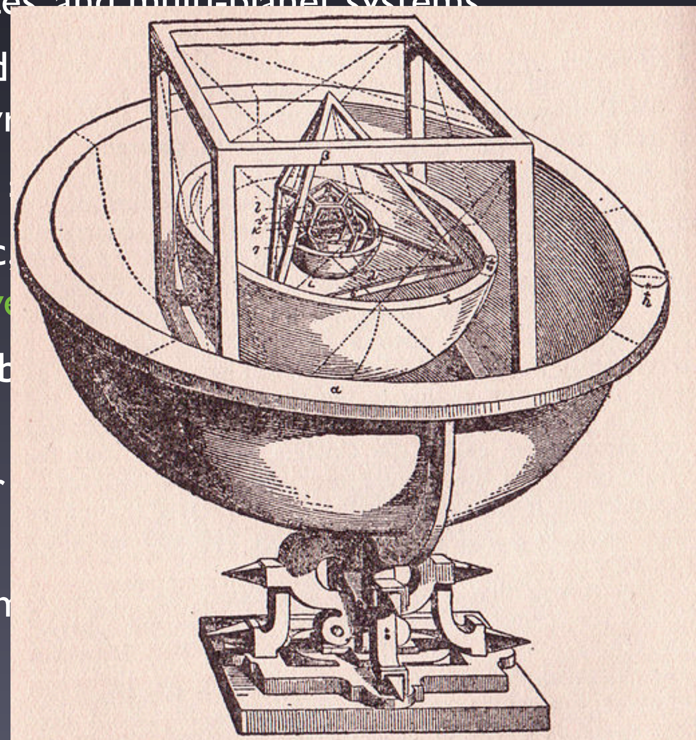

Fig. 37 —Kepler's Analogy of the Five Solids and the Five Worlds.

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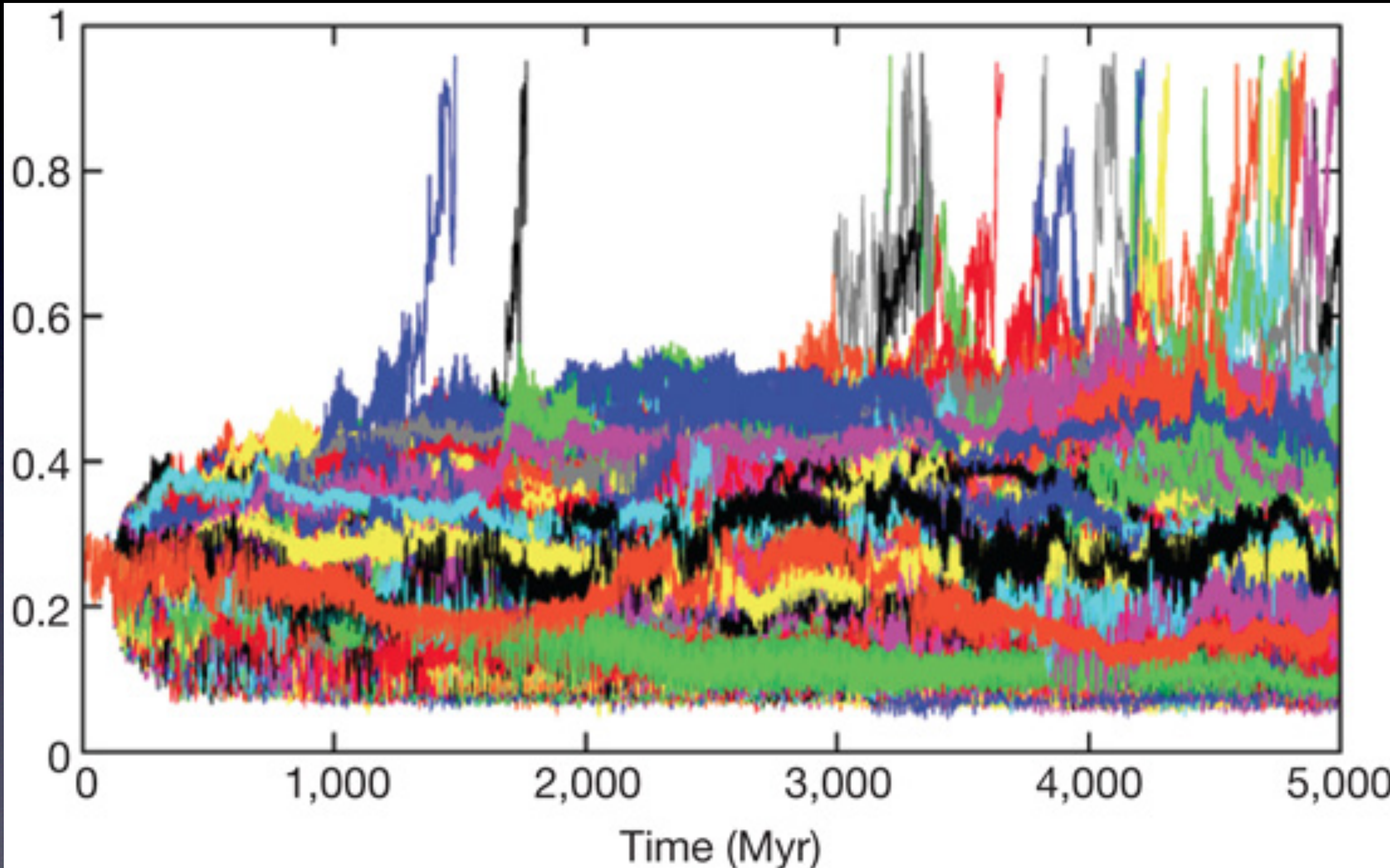
eccentricity of Mercury for 2500 nearby initial conditions



Laskar & Gastineau (2009)

Statistical mechanics of planetary systems

The range of strong interactions from a planet of mass m orbiting a star of mass M in a circular orbit of radius a is the Hill radius

$$r_H = a\left(\frac{m}{3M}\right)^{1/3}.$$

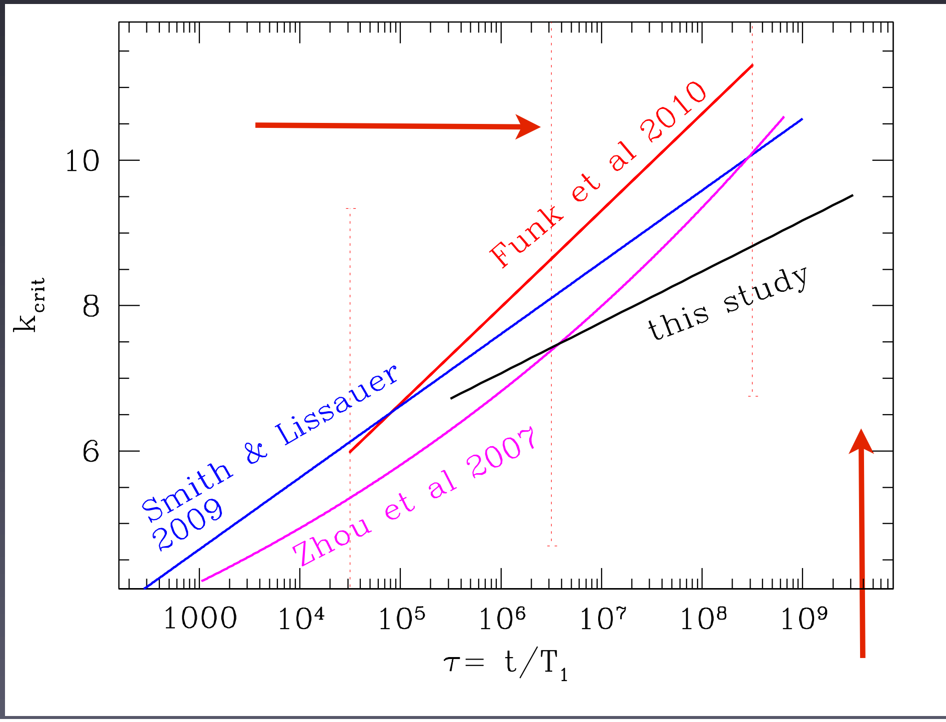
Numerical integrations show that planets of mass m, m′ on circular orbits are stable for N orbital periods if

$$|a' - a| > k(N) r_H \quad \text{where} \quad r_H = a\left(\frac{m+m'}{3M}\right)^{1/3}.$$

typically $k(10^{10}) \simeq 11 \pm 1$

Generalization to eccentric orbits: pericenter of outer orbit and apocenter of inner orbit must be separated by at least k Hill radii

Pu & Wu (2014)



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Ansatz: planetary systems fill uniformly the region of phase space allowed by stability (~ ergodic hypothesis)

N ... table

fo ...

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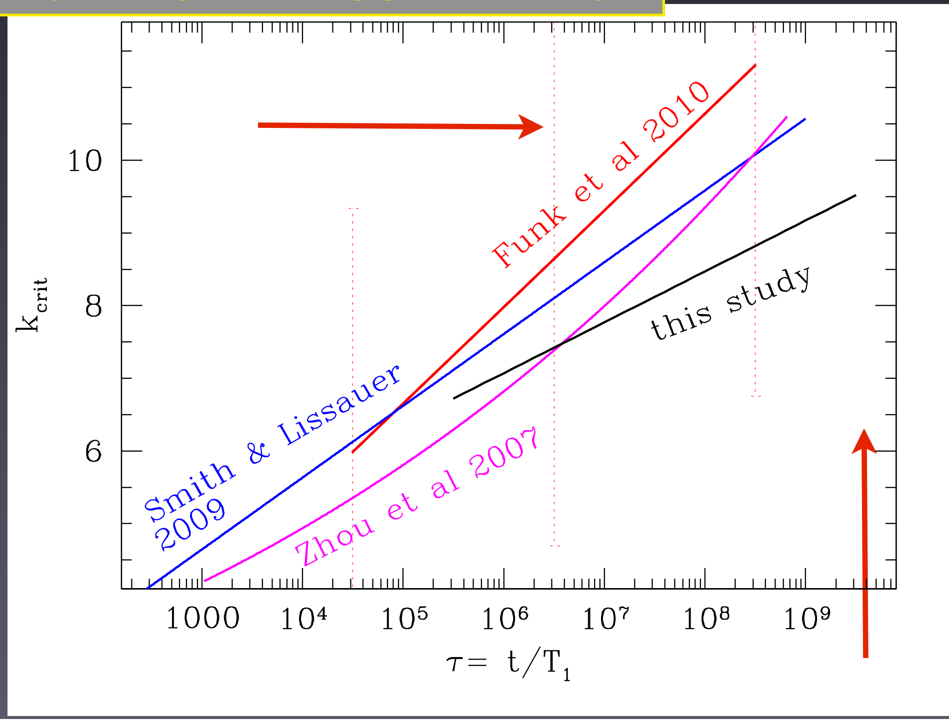

Statistical mechanics of planetary systems

Ansatz: planetary systems fill uniformly the region of phase space allowed by stability

Leads to an N-planet distribution function

phase-space volume

apocenter and pericenter must be separated by k Hill radii

$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} \overbrace{da_i de_i^2}\, H[a_{i+1}(1-e_{i+1}) - a_i(1+e_i) - k r_H]$$

step function

where $H(\cdot)$ is the step function, $k = 11\pm1$, and $r_H = \bar{a}(m_i+m_{i+1})^{1/3}/(3M_\star)^{1/3}$.

For comparison the distribution function for a one-dimensional gas of hard rods of length L (Tonks 1936) is

$$p(a_1, \ldots, a_N) \propto \prod_{i=1}^{N} da_i H(a_{i+1} - a_i - L).$$

In both systems the partition function depends only on the filling factor

$$F = \frac{k\langle r_H\rangle}{\langle a_{i+1} - a_i\rangle}, \qquad F = \frac{L}{\langle a_{i+1} - a_i\rangle}.$$

Statistical mechanics of planetary systems

N-planet distribution function

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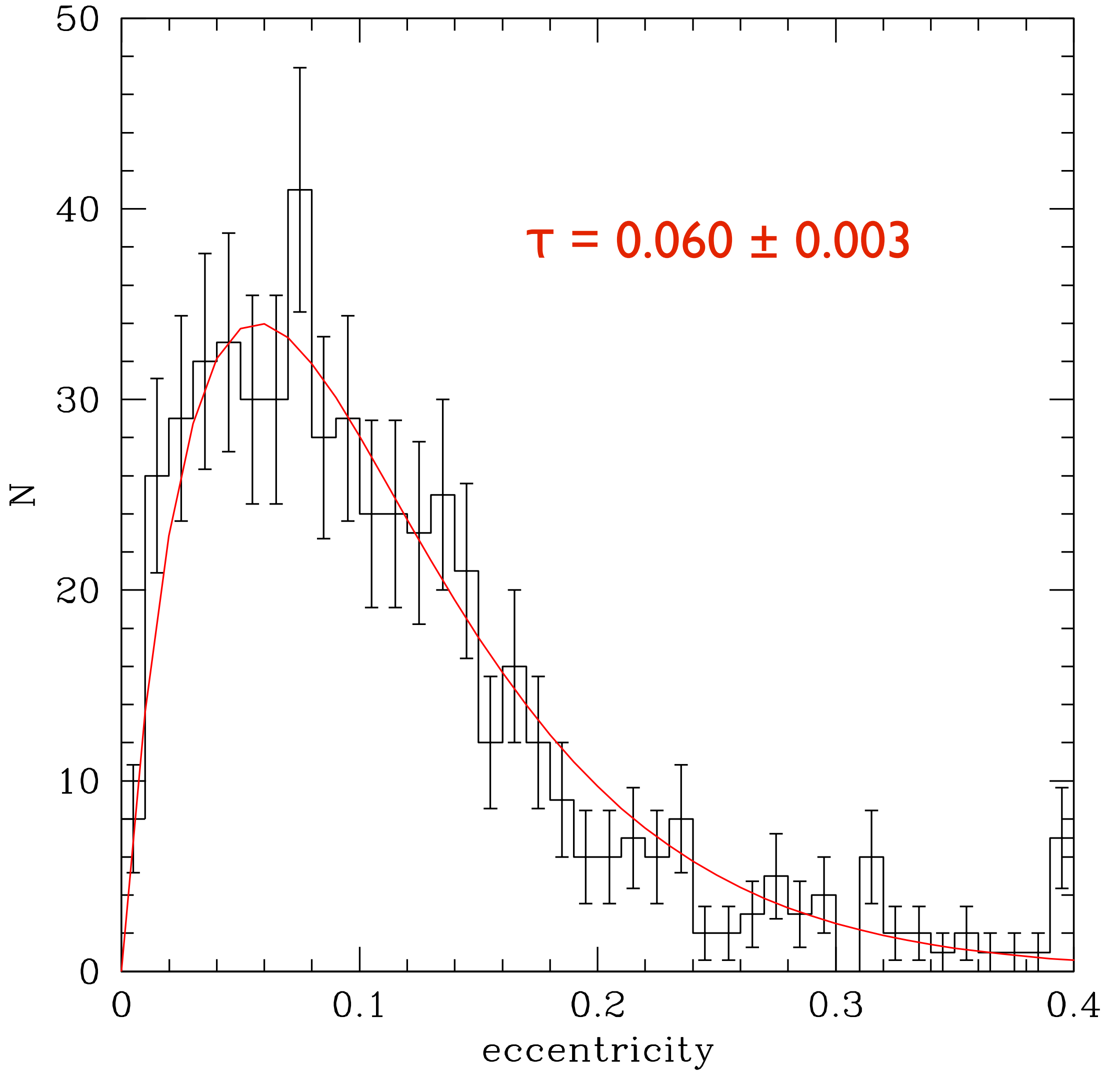
Take limit N>>1 and assume each planetary system is a subsystem

Predictions:

- eccentricity distribution:

$$p(e) = \frac{e}{\tau^2} \exp\left(-\frac{e}{\tau}\right)$$

where τ is a free parameter determined by the filling factor e.g., N-body simulations of planet growth by Hansen & Murray (2013)



Statistical mechanics of planetary systems

N-planet distribution function

$$p(a_1, e_1, \ldots, a_N, e_N) \propto \prod_{i=1}^{N} da_i de_i^2\, H[a_{i+1} - a_i - \bar{a}(e_{i+1} + e_i) - kr_H]$$

where $H(\cdot)$ is the step function, $k = 11 \pm 1$, and $r_H = \bar{a}(m_i + m_{i+1})^{1/3}/(3M_\star)^{1/3}$.

Take limit N>>1 and assume each planetary system is a subsystem

Predictions:

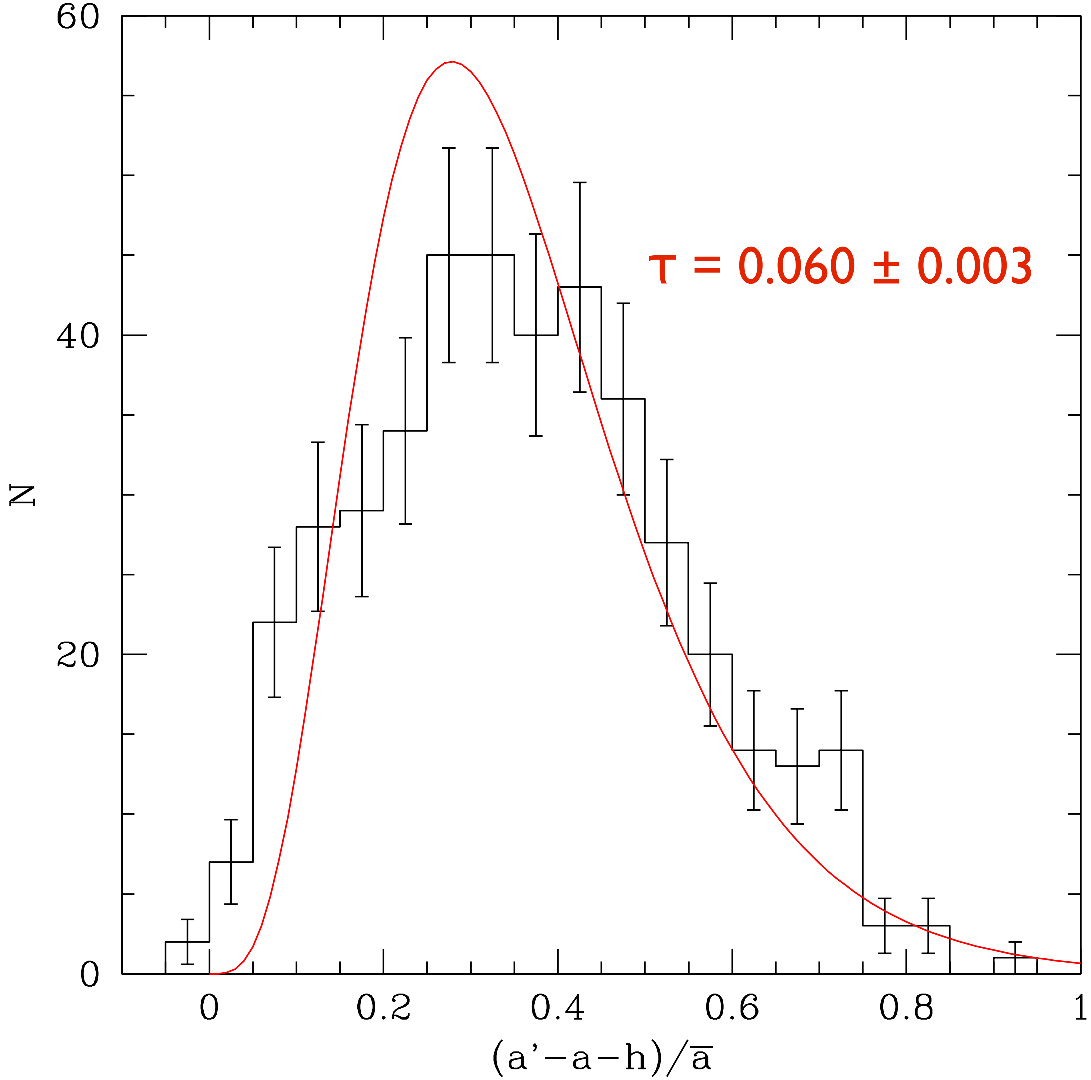
- eccentricity distribution
- distribution of semi-major axis differences between nearest neighbors:

$$p(a' - a) = \frac{4}{2\bar{a}\tau} G\left(\frac{a' - a - kr_H}{2\bar{a}\tau}\right)$$

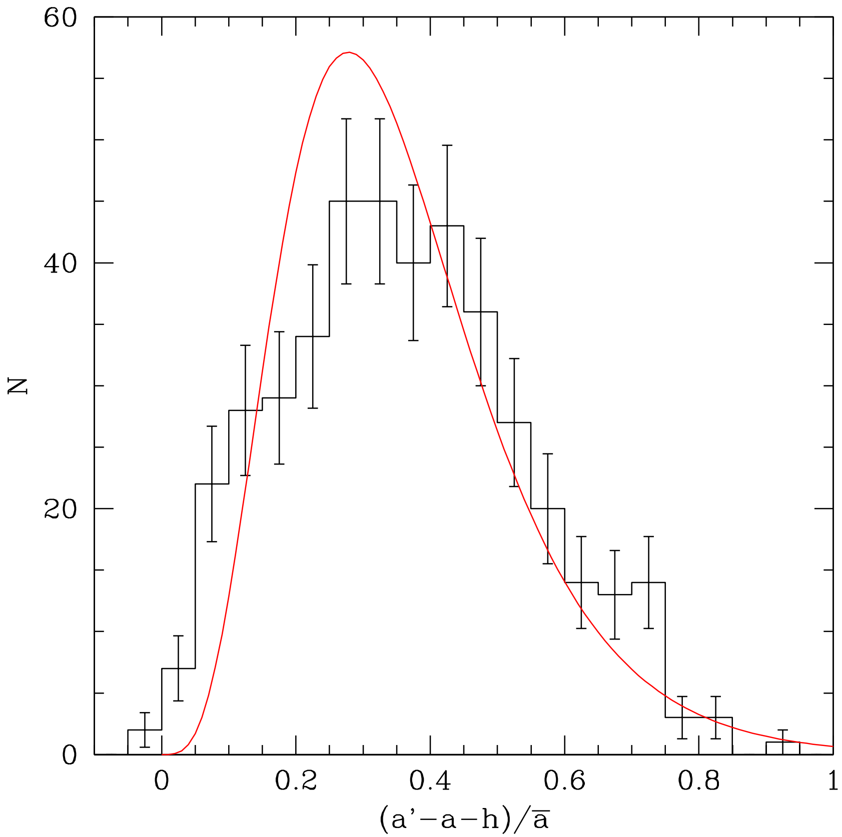
where

$$G(x) = 6\exp(-x) - \exp(-2x)(x^3 + 3x^2 + 6x + 6).$$

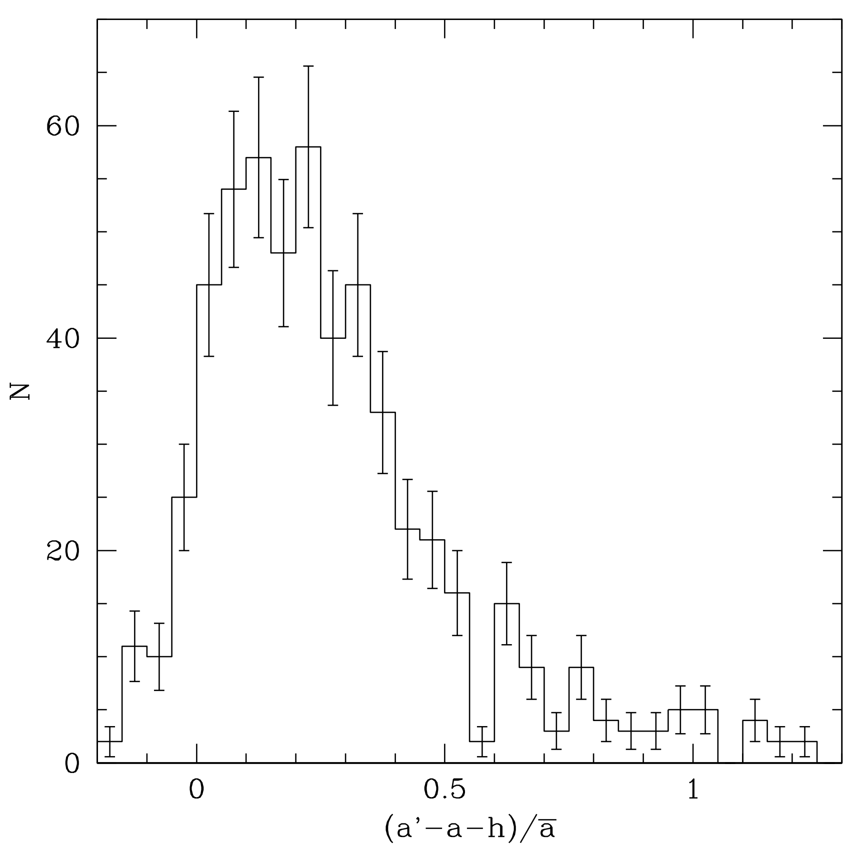
e.g., N-body simulations of planet growth by Hansen & Murray (2013)



Hansen & Murray (2013) simulations



Kepler planets, using Weiss & Marcy (2014) mass-radius relation:



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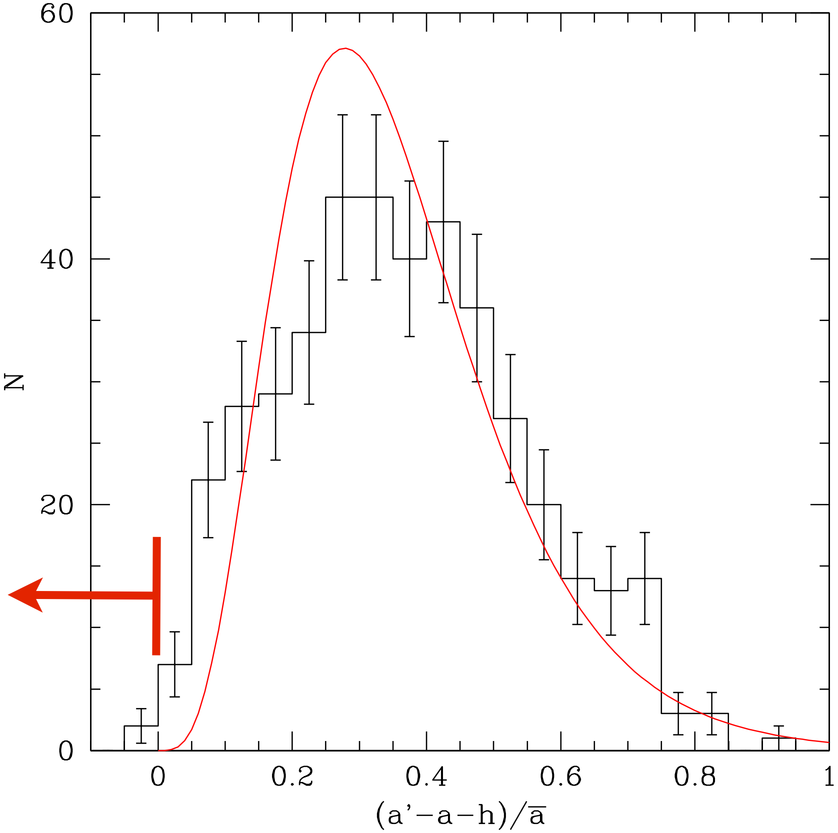

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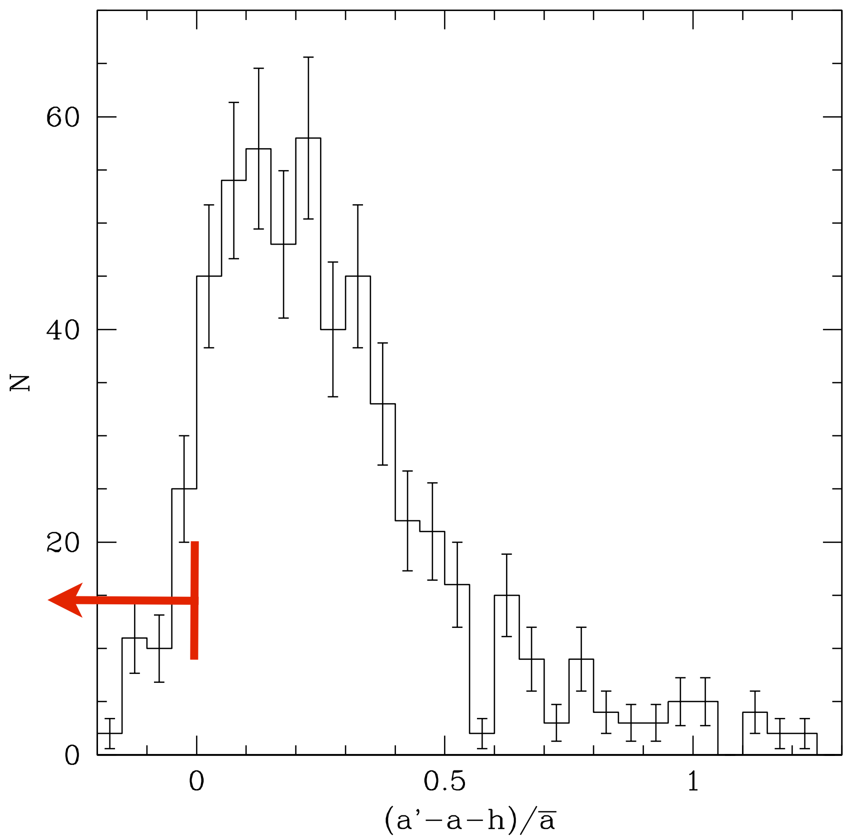

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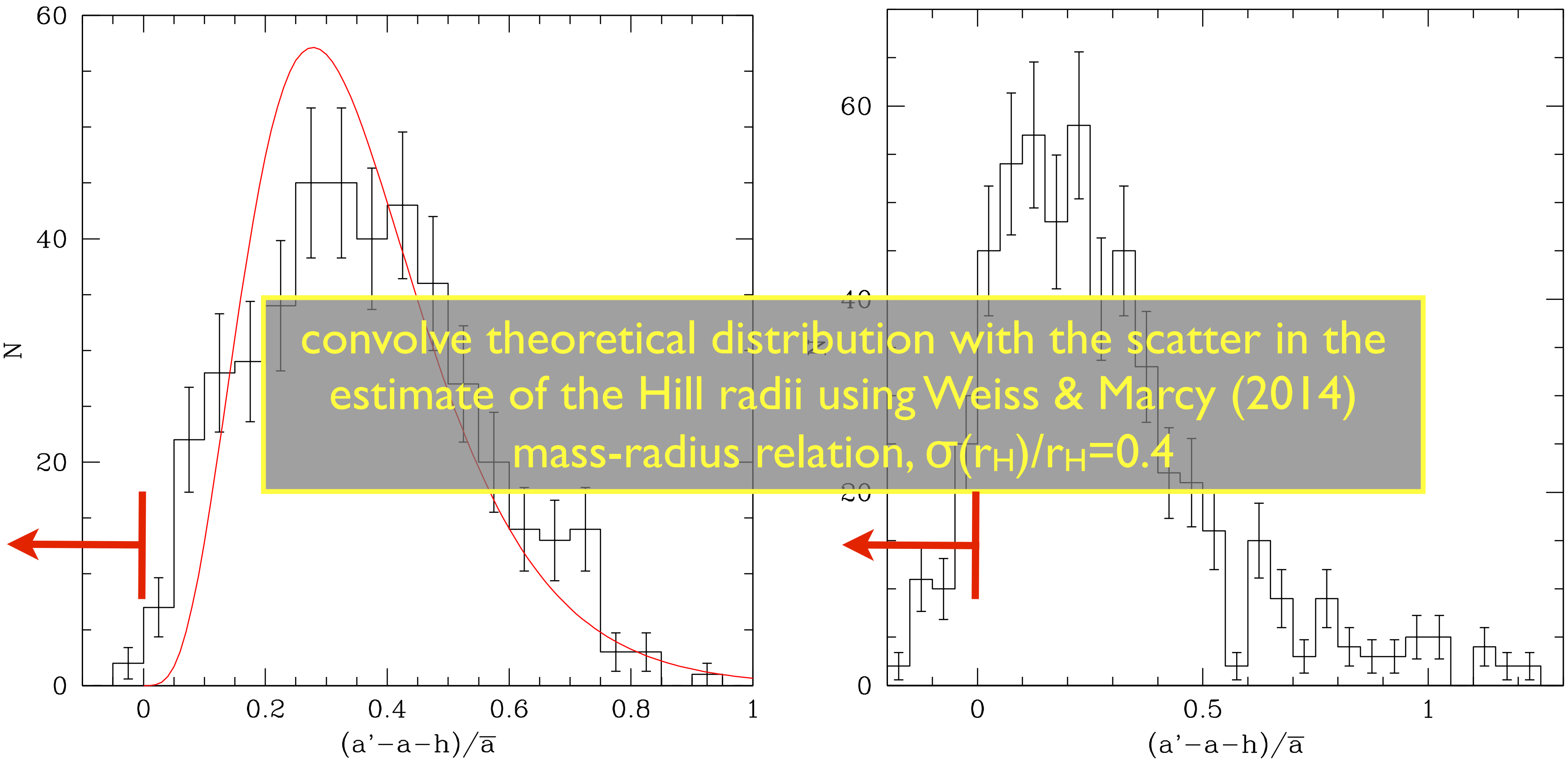
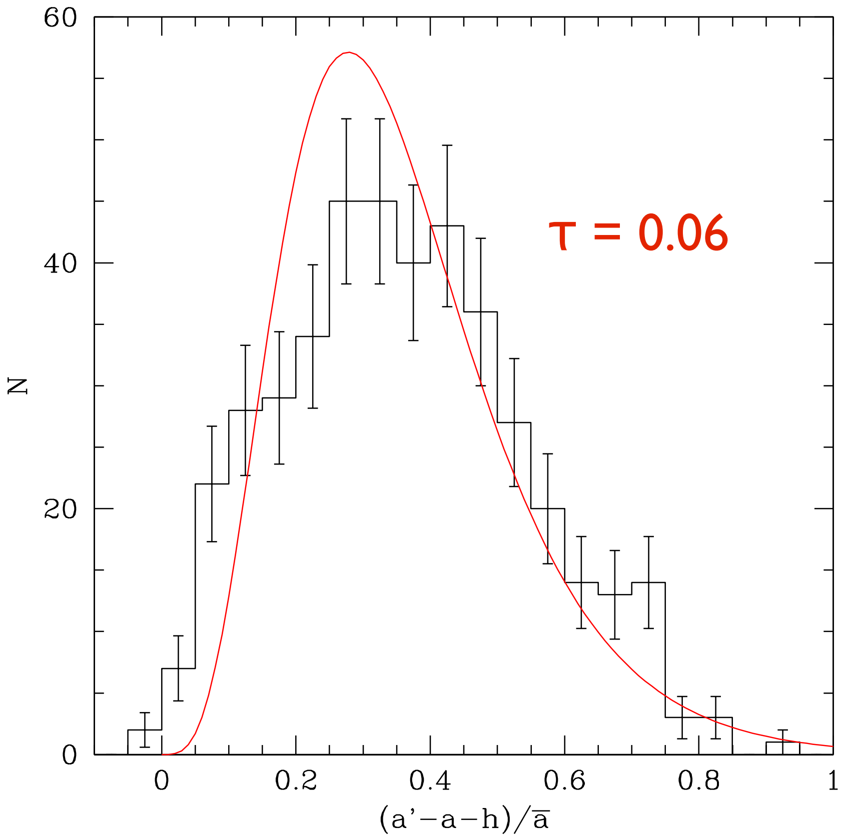


convolve theoretical distribution with the scatter in the estimate of the Hill radii using Weiss & Marcy (2014) mass-radius relation, $\sigma(r_H)/r_H=0.4$

Hansen & Murray (2013) simulations



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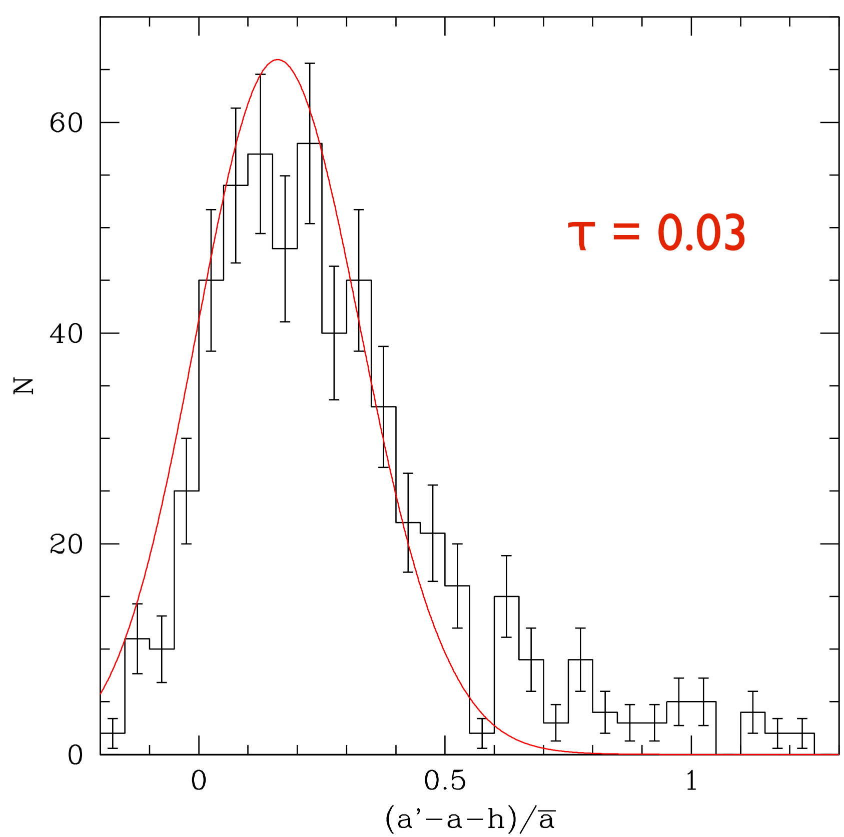

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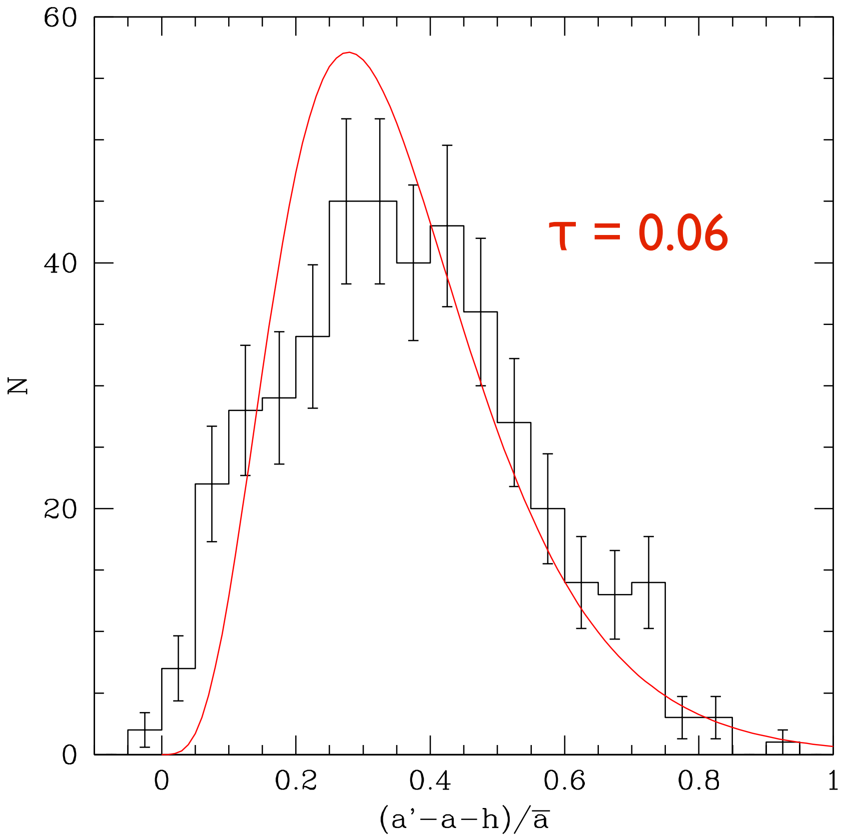

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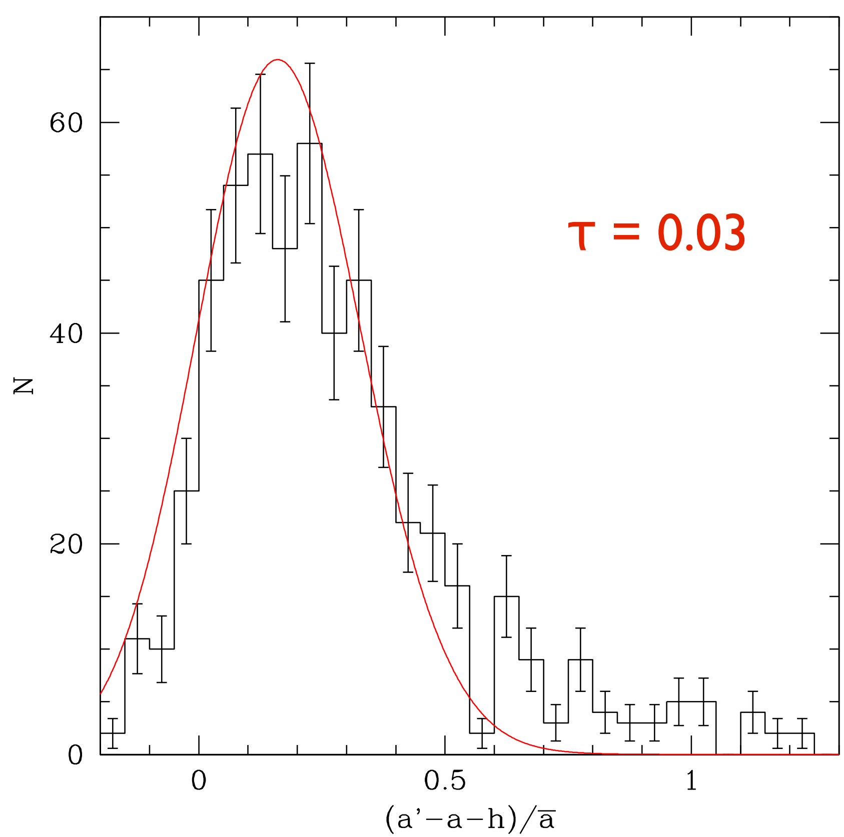


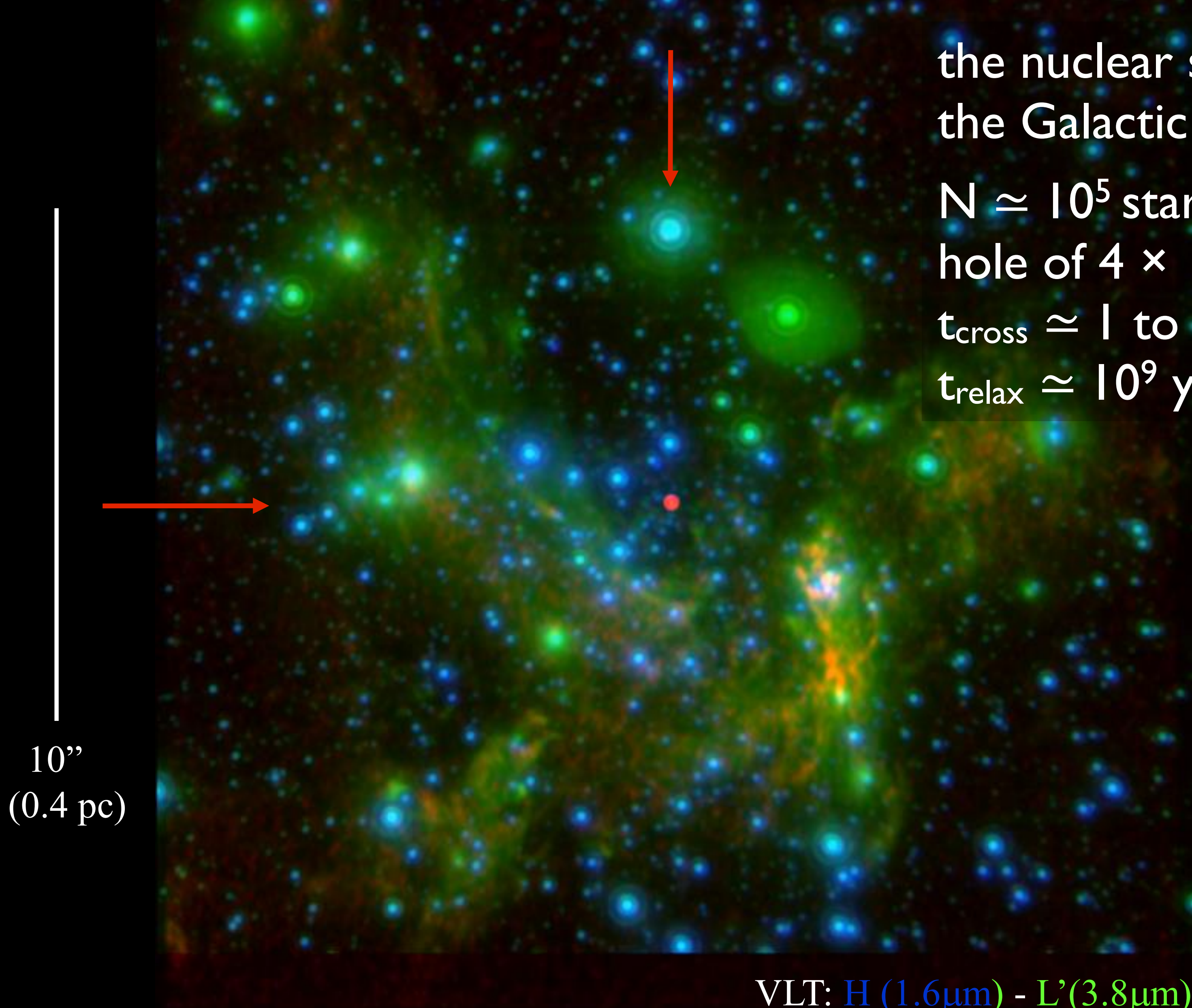
- predicts <e>=0.06, larger than Hadden & Lithwick (2014), <e>=0.023±0.006
- predicts no correlation between mass and eccentricity in a given system

the nuclear star cluster at the Galactic center:

$N \simeq 10^5$ stars plus a black hole of $4 \times 10^6\ M_\odot$

$t_{cross} \simeq 1$ to 10^4 yr

$t_{relax} \simeq 10^9$ yr

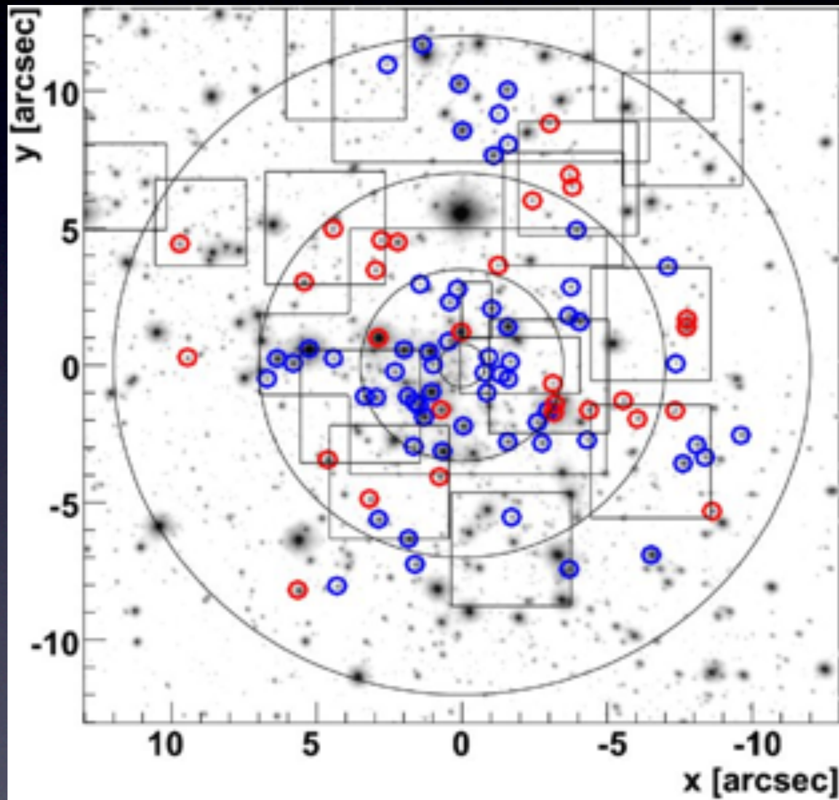


Genzel (2015)

The stellar disk(s) in the Galactic center

1 pc = 25"

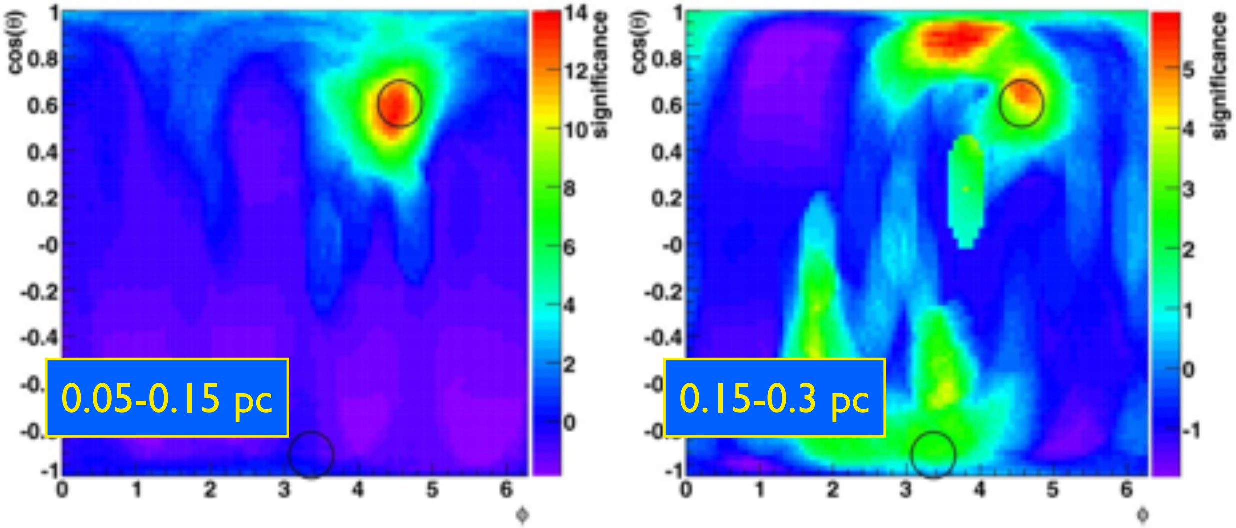
- ~ 100 massive young stars found in the central parsec
- age 6 Myr; implied star-formation rate is so high that it must be episodic
- line-of-sight velocities measured by Doppler shift and angular velocities measured by astrometry (five of six phase-space coordinates)
- velocity vectors lie close to a plane, implying that many of the stars are in a disk or perhaps 2 disks (Levin & Beloborodov 2003)



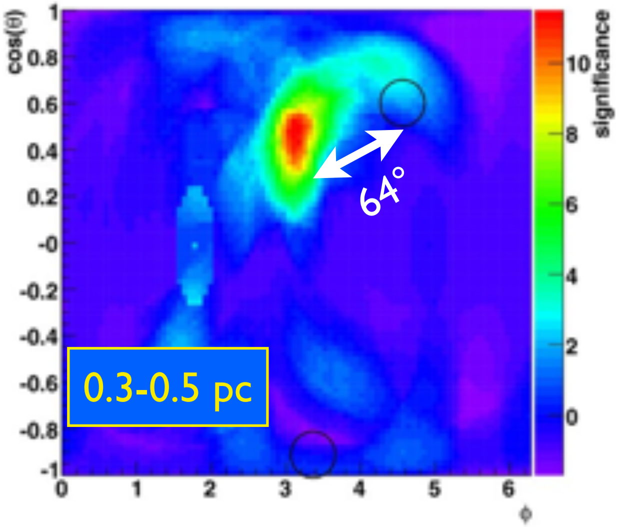
Bartko et al. (2009)

blue = clockwise rotation (61 stars)

red = counter-clockwise rotation (29 stars)



- ~100 massive stars in central 0.5 pc of the Milky Way
- plots show distribution of orbit normals



1 pc = 25”

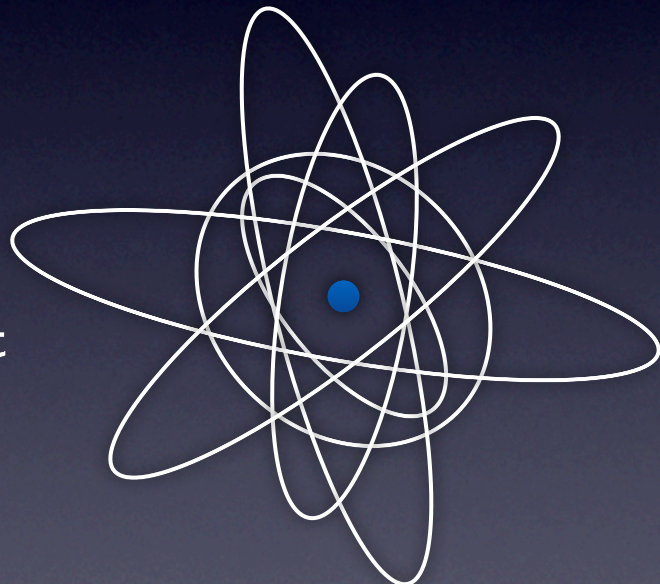
Bartko et al. (2009)

- **clockwise disk:**
  - warped (best-fit normals in inner and outer image differ by 64°)
  - disk is less well-formed at larger radii
- **counter-clockwise disk:**
  - weaker evidence
  - localized between 0.1 and 0.3 pc

- disks are embedded in a spherical cluster of old, fainter stars with M(0.1 pc) ~ 1×10^5 $M_\odot$ compared to $M_\bullet = 4\times10^6\ M_\odot$

Resonant relaxation

- inside ~0.5 pc gravitational field is dominated by the black hole ($M_{stars} < 10^5\ M_\odot$, $M_\bullet \sim 4\times10^6\ M_\odot$) and therefore is nearly spherical
- on timescales longer than the apsidal precession period each stellar orbit can be thought of as a disk or annulus
- each disk exerts a torque on all other disks
- mutual torques can lead to relaxation of orbit normals or angular momenta
- energy (semi-major axis) and scalar angular momentum (or eccentricity) of each orbit is conserved, but orbit normal is not



Rauch & Tremaine (1996)

Resonant relaxation

Interaction energy between stars i and j is $m_i m_j f(a_i, a_j, e_i, e_j, \cos\mu_{ij})$ where μ_{ij} is the angle between the orbit normals

masses

semi-major axes

eccentricities

Toy model:

Simplify this drastically by assuming equal masses, equal semi-major axes, circular orbits, and neglecting all harmonics other than quadrupole

Resulting interaction energy between two stars i and j is just

$$-C\cos^2\mu_{ij}$$

where $\boldsymbol{\mu}_{ij}$ is the angle between the two orbit normals $\mathbf{n}_i$ and $\mathbf{n}_j$

$$\frac{d\hat{\mathbf{n}}_i}{dt} = -\frac{2C}{\sqrt{GM_\bullet a}}\sum_{j\neq i}(\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j)\,\hat{\mathbf{n}}_i\times\hat{\mathbf{n}}_j$$

Maier-Saupe model

Resonant relaxation

Interaction energy between two stars is

H = -C cos² μ

where **μ** is the angle between the two orbit normals

- 800 stars
- each point represents tip of orbit normal
- orbit normals initially in northern hemisphere are yellow, south is red

$$\frac{d\mathbf{n}_i}{dt} = -\frac{2C}{\sqrt{GMa}} \sum_{j \neq i} (\mathbf{n}_i \cdot \mathbf{n}_j) \mathbf{n}_i \times \mathbf{n}_j$$

animation by B. Kocsis

animation by B. Kocsis

- integrate orbit-averaged equations of motion
- yellow = disk stars, blue-red = stars in spherical cluster, colored by increasing radius
- direction and radius of each point represents direction of angular-momentum vector and semi-major axis of star
- 8192 stars
- each point represents tip of orbit normal

animation by B. Kocsis

animation by B. Kocsis

"All models are wrong, but some are useful"

Box & Draper (1987)