Q-based Models (& their use)

Equilibrium tide model

Darwin 1879, 1880 ← 179 pages in Phil. Trans. Roy Soc. Lon !

Jeffreys 1961

Goldreich 1963 → Q introduced

Goldreich & Soter 1966 "Q in the Solar System"

- rates of change of eccentricity & semimajor axis to first-order in e.
- ASSUMES CONSTANT LAG ANGLES

Hut 1981 "Tidal Evol^n in Close Binary Systems"

- ASSUMES CONSTANT LAG TIMES (as did Darwin & Jeffries).

Eggleton et al 1998 "The Equil^m Tide Model for Tidal Friction"

Equilibrium tide model:

Assumes hydrostatic equilibrium

- for a fluid contours of constant density coincide with contours of constant gravitational potential:

$$\underset{\sim}{0} = -\frac{1}{\rho}\nabla P - \nabla\Phi$$
$$= -\nabla\left[\int\frac{dP}{\rho} + \Phi\right]$$

Thus if one knows the potential a "fluid" body moves in, one can deduce the shape the body will assume.

The assumption of hydrostatic equilibrium is questionable when the potential is time-varying – the "natural" modes of vibration of the body will be excited – this is the <u>dynamical tide</u>.

<u>Analogy</u>: forced oscillator

$$\ddot{x} + \omega^2 x = A \cos \Omega t$$

$$x(t) = \underbrace{C_1 \cos \omega t + C_2 \sin \omega t}_{\text{dynamical "tide"}} + \underbrace{\frac{A}{\omega^2 - \Omega^2} \cos \Omega t}_{\text{equilibrium "tide"}}$$

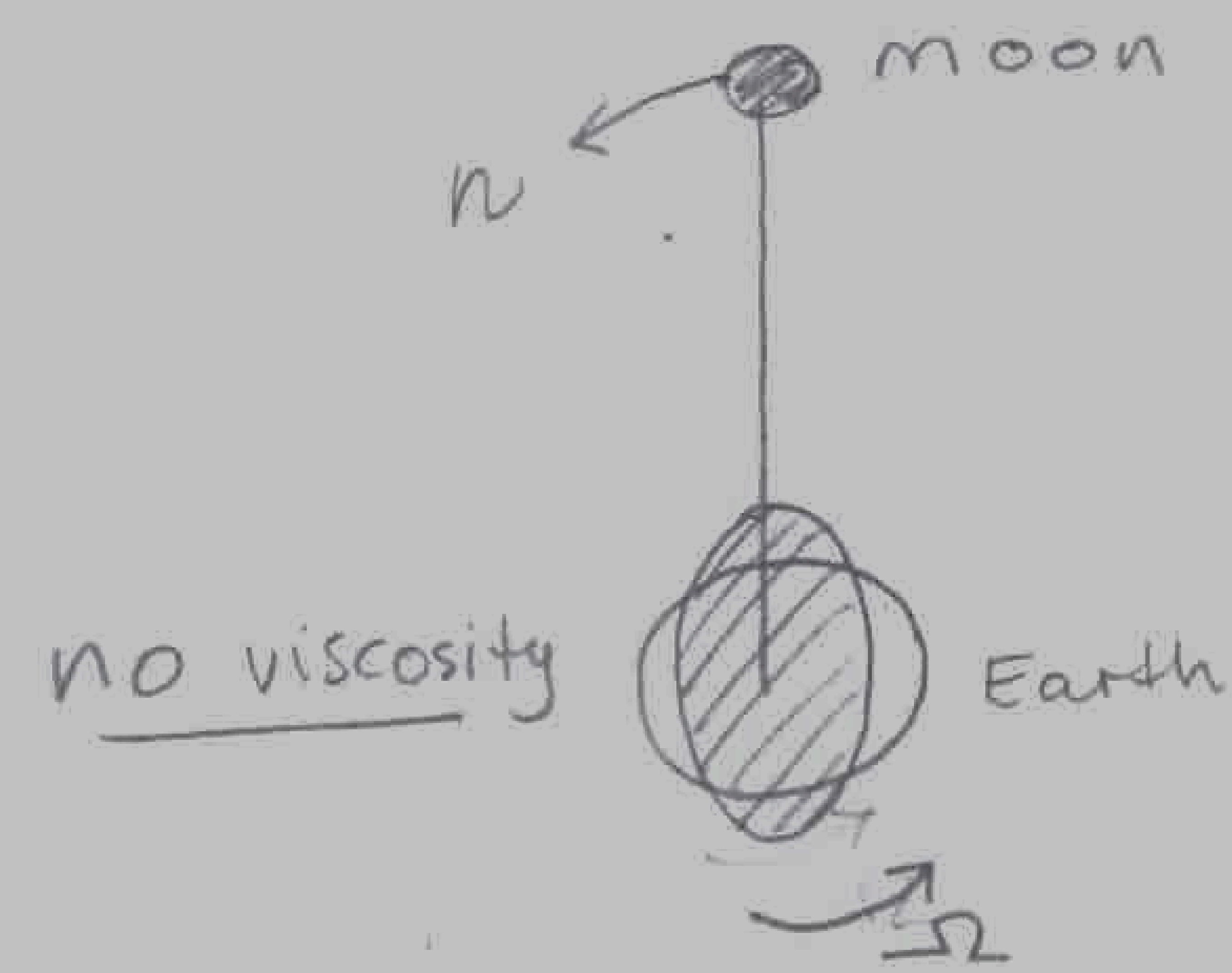
<u>End state of tidal evolution</u>:

- Synchronous rotation of both bodies
- Spins aligned with orbit normal
- Circular orbit

– in this case the equilibrium tide is static
– there is true hydrostatic equilibrium

Darwin → Jeffreys → Goldreich
⇓
Earth-Moon

Simplifying assumptions: • zero obliquity
• small eccentricity



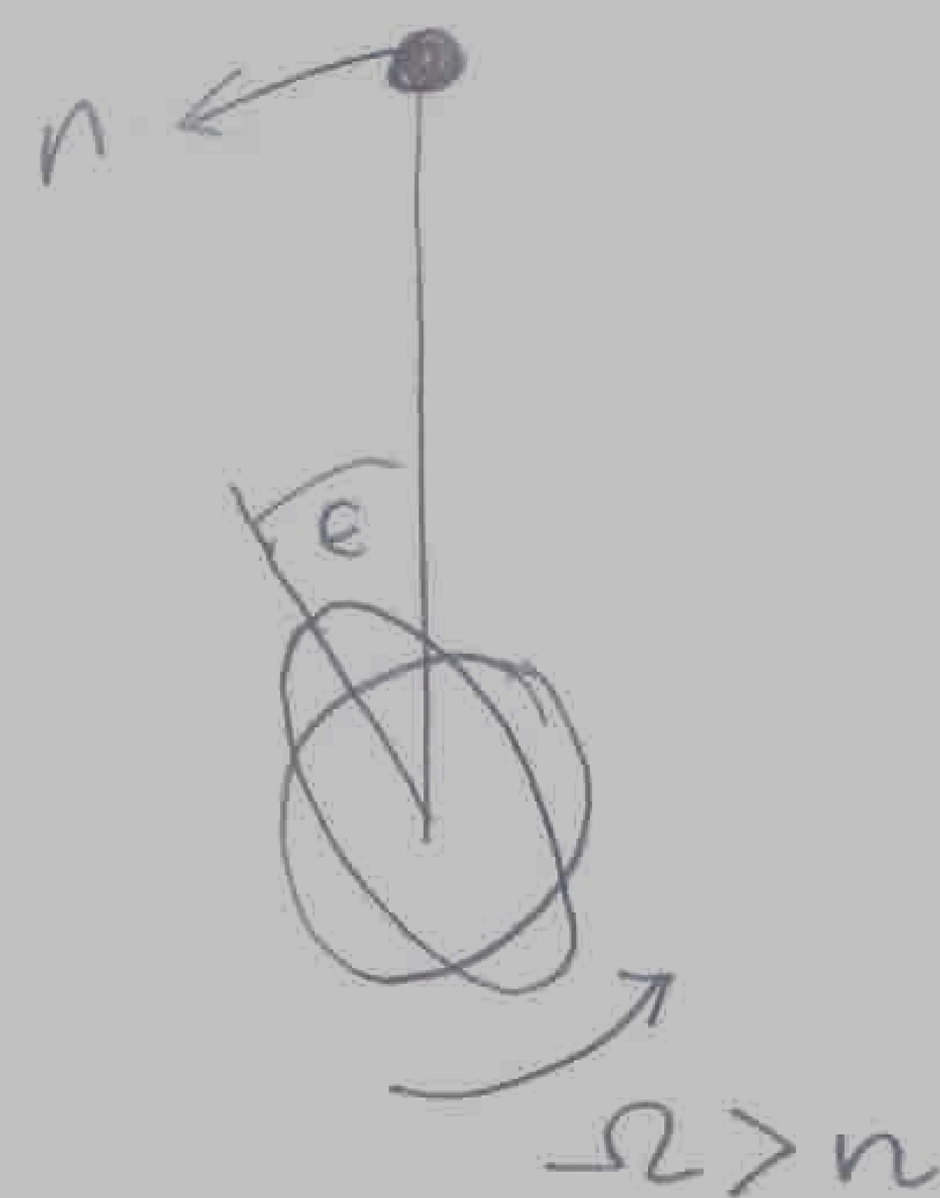
For a circular orbit, each fluid particle rises and falls with a frequency $2|\Omega - n|$

– there are internal shear stresses which lead to dissipation:

Darwin 1879

Now suppose that at any instant the fluid becomes endowed with friction; then those particles of fluid which happen at that instant to be in the equatoreal protuberances cannot fall so quickly as they did before the friction began to act. Thus the particles linger in the protuberances longer they than should do, and the particles which ought to be rising do not rise so quickly as they should. The obvious consequence of this is that the equatoreal protuberances and depressions are carried on along with the planet's rotation, and the longer axis of the equator now points in advance of the satellite in its orbit. This condition is perpetuated as long as the friction lasts. Tides of this kind are said to lag; and the angle of lagging may be defined as the angle between the longer axis of the equator and the direction of the satellite.

Another result of the supposed internal friction or viscosity of the fluid is, that the oscillations of the particles are impeded, and that they have not time to move so far as they would have done without friction. Hence if the fluid be viscous, the tides not only lag but are reduced in height.



with viscosity:
for $\Omega > n$, tidal bulge leads line of centres.

For eccentric orbits, there are additional tidal components, each with their own lag angle.

Theory

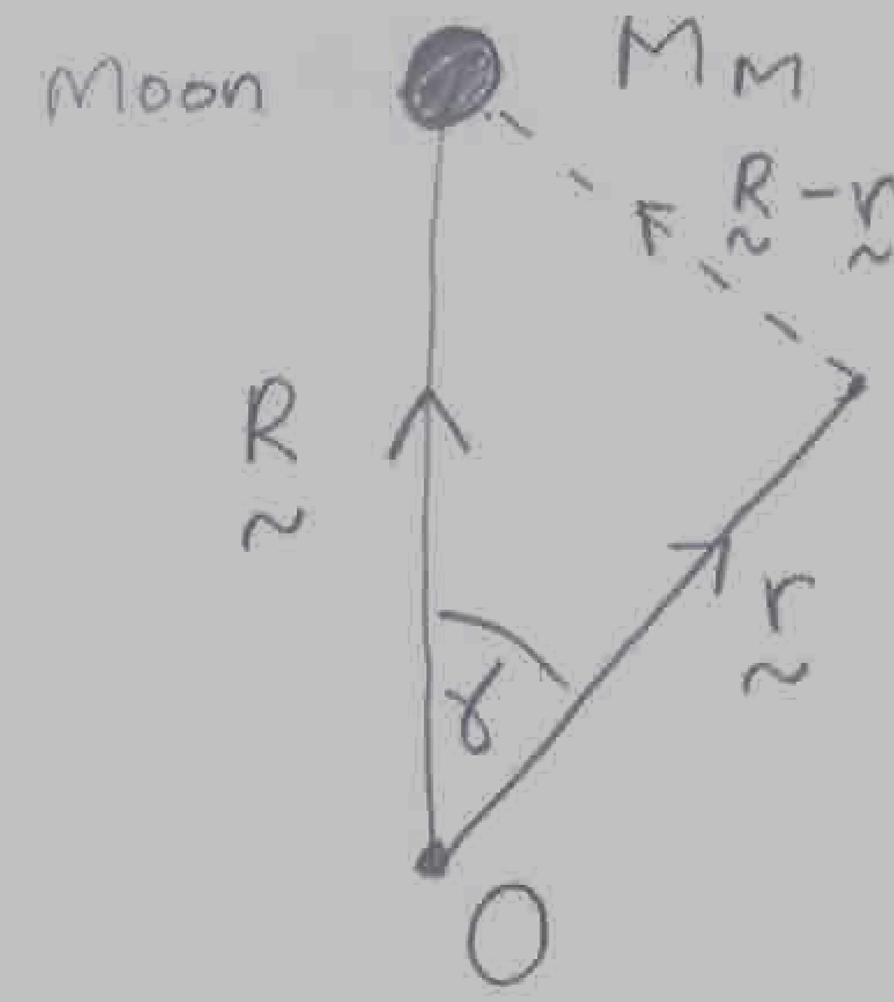
Step 1 Write down the potential at every point in space due to the perturber, using the centre of the distorted body as origin.

Step 2 Calculate the distorted shape of the disturbed body due to its own potential (including bulge) and the potential of the perturber.

Step3 Calculate the back effect on the orbit of the perturber.

Step 1

The perturber potential



$|\underset{\sim}{r}| < |\underset{\sim}{R}|$

$$\Phi_{moon}(\underset{\sim}{r}) = \frac{-GM_m}{|\underset{\sim}{R}-\underset{\sim}{r}|}$$

$$= -\frac{GM_m}{R}\sum_{l=0}^{\infty}\left(\frac{r}{R}\right)^{l} P_l(\cos\gamma) \qquad \cos\gamma = \hat{\underset{\sim}{r}}\cdot\hat{\underset{\sim}{R}}$$

$$= -\frac{GM_m}{R} - \frac{GM_m}{R}\left(\frac{r}{R}\right)\cos\gamma - \frac{GM_m}{R}\left(\frac{r}{R}\right)^2 P_2(\cos\gamma) + \dots$$

$$-\nabla_r\Phi = \underset{\sim}{0} \underbrace{- \frac{GM_m}{R^2}\hat{\underset{\sim}{R}}}_{\text{2-body acceleration}} \underbrace{- \frac{GM_2}{R^2}\nabla_r\left(r^2 P_2(\hat{\underset{\sim}{r}}\cdot\hat{\underset{\sim}{R}})\right)}_{\text{tidal acceleration}} - \dots$$

$$\Phi_{tide}(\underset{\sim}{r}) = -\frac{GM_m}{R}\left(\frac{r}{R}\right)^2 P_2(\cos\gamma) = \text{tidal potential.}$$

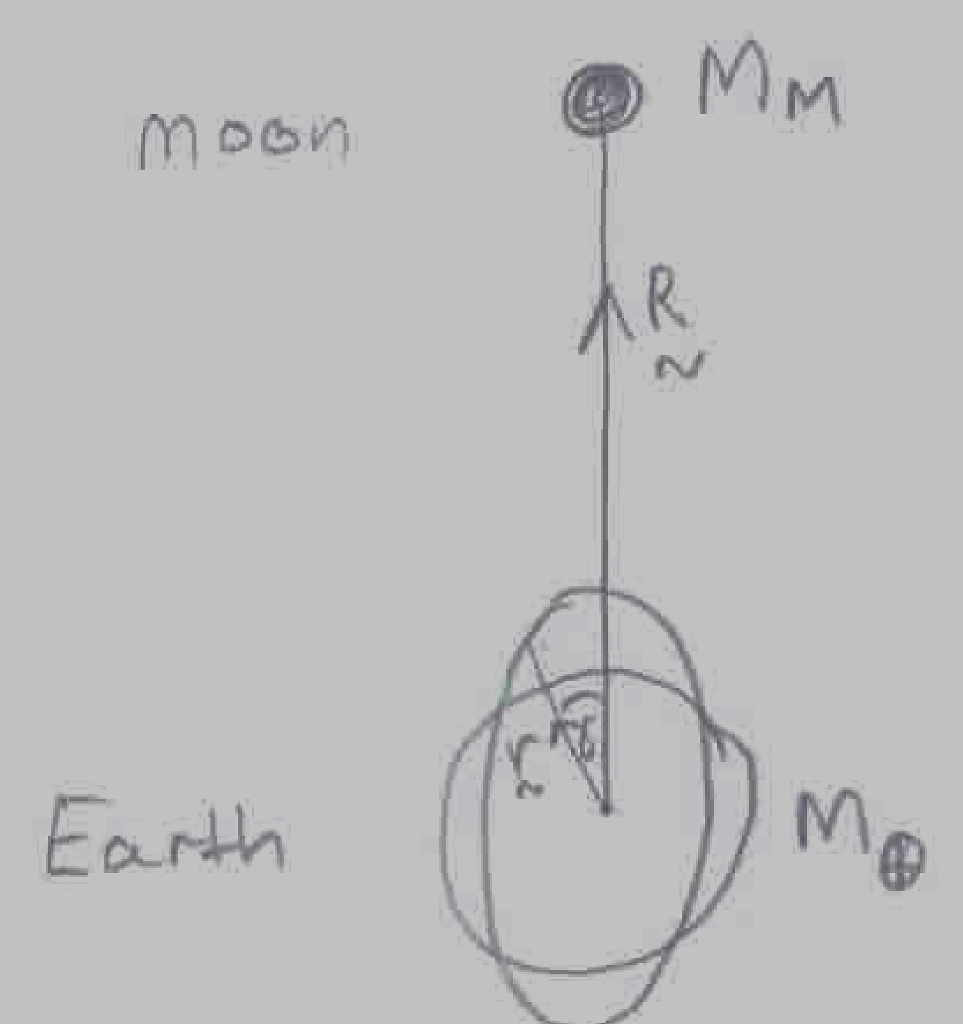
(disturbing potential)

Step 2 : tidal distortion

no viscosity

hydrostatic equilibrium : the distorted body will assume the shape of the tidal potential :

$$\Phi_{bulge} \propto \Phi_{tide}$$



Each Earth element feels total Φ

$$= \Phi_{sphere} + \Phi_{bulge} + \Phi_{moon}$$

Bulge shape :

$$r = r_\oplus + \delta r_\oplus \, P_2(\cos\gamma)$$

– can work out $\delta r_\oplus$ using Newton's third thm:
potential at surface should be independent of γ and equal $-\frac{GM_\oplus}{r_\oplus}$.

Such an analysis gives

$$\frac{\delta r_\oplus}{r_\oplus} = (k_2 + 1) \left(\frac{M_M}{M_\oplus}\right) \left(\frac{r_\oplus}{R}\right)^3$$

k_2 = (quadrupole) tidal Love number
(= twice apsidal motion constant).

<u>Step 3</u> The back effect on the orbit:

how does the Moon's orbit evolve in response given the Earth is not spherical?

- <u>no dissipation</u>: tidal bulge aligned with line of centres
  → rotation of the orbit in its plane
  = apsidal motion (<u>not</u> precession!)
  = rotation of orbital plane)
  (also occurs due to spin oblateness).

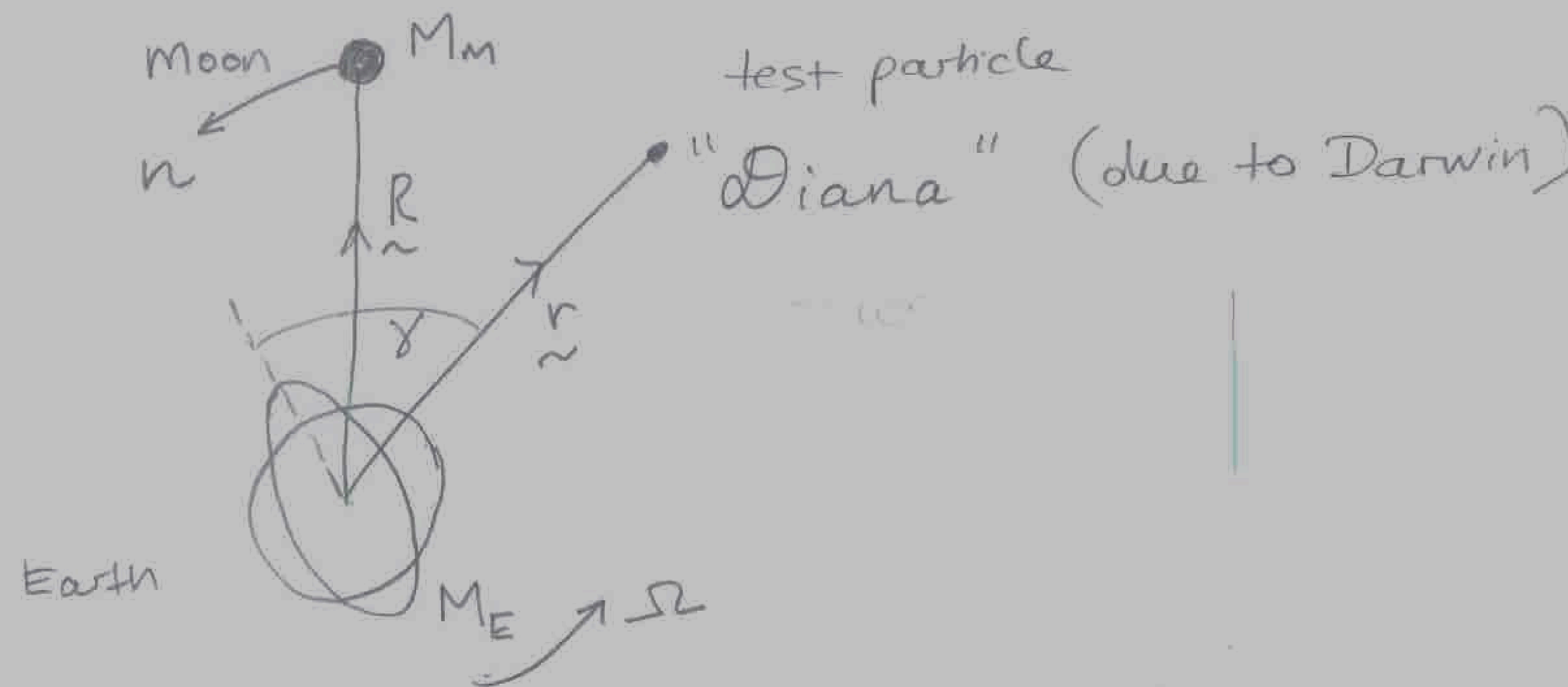
- <u>dissipation</u> → tidal bulge <u>not</u> aligned with line of centres
  - tidal torque transfers angular momentum between spin & orbit
  - dissipation remove energy from spin & orbit

$$\frac{d\varpi}{dt} \neq 0, \quad \frac{de}{dt} \neq 0, \quad \frac{da}{dt} \neq 0, \quad \frac{d\Omega}{dt} \neq 0$$

The potential outside the Earth due to the bulge is

$$\Phi_{bulge}(\underset{\sim}{r}) \propto \Phi_{tide}$$

$$= -\zeta \frac{GM_\oplus}{r_\oplus} \left(\frac{r_\oplus}{r}\right)^3 P_2(\cos\gamma)$$



Constant of proportionality ζ is

$$\zeta = k_2 \left(\frac{M_M}{M_\oplus}\right)\left(\frac{r_\oplus}{R}\right)^3$$

"Diana" is necessary because we need to know how the moon responds to the tidal bulge, independent of the fact that it is responsible for it. After we have used Lagrange's planetary eq^{n}s (or worked out $\nabla\Phi_{bulge}$ or whatever we need to do) we put $\underset{\sim}{r} = \underset{\sim}{R}$.

Addition theorem:

spherical polar coods

$$P_\ell(\cos\gamma) = \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \underbrace{Y_{\ell m}(\theta_D, \varphi_D)}_{\text{Diana}} \underbrace{Y^*_{\ell m}(\theta_B, \varphi_B)}_{\text{Bulge}}$$

Take $\theta_D = \theta_B = \frac{\pi}{2}$

Then

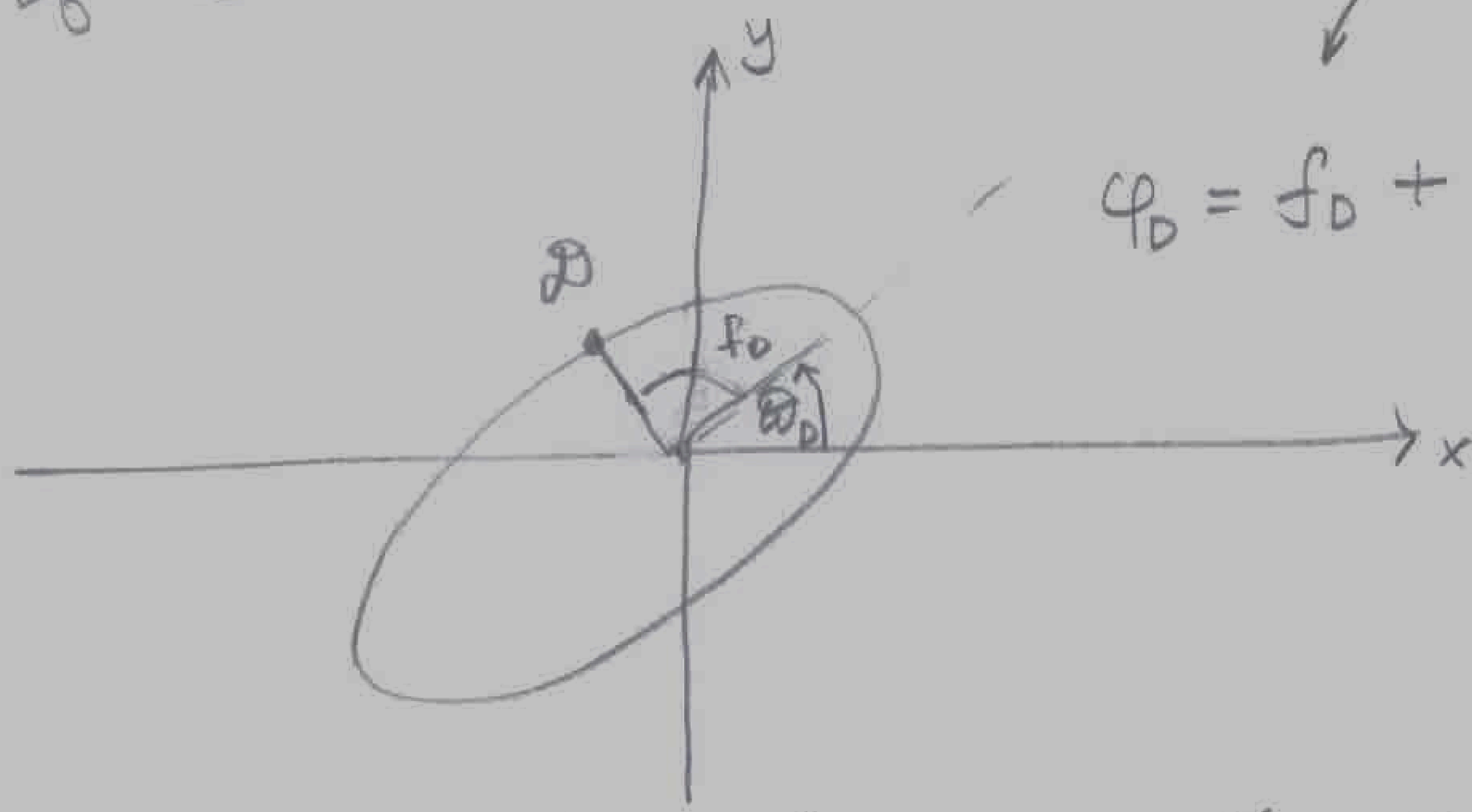
$$Y_{\ell m}\left(\tfrac{\pi}{2}, \varphi_D\right) Y^*_{\ell m}\left(\tfrac{\pi}{2}, \varphi_B\right) = \text{const.}\; e^{im(\varphi_D - \varphi_B)}$$

Orbit of Diana:

true anamoly

longitude of perigee.

$$\varphi_D = f_D + \varpi_D$$



$$P_2(\cos\gamma) = \frac{3}{8} e^{2i(\varphi_D - \varphi_B)} + \frac{3}{8} e^{-2i(\varphi_D - \varphi_B)} + \frac{1}{4}$$

What is φ_B?

$$\frac{\Phi_{bulge}}{GM_\oplus / r_\oplus} = k_2 \left(\frac{M_M}{M_\oplus}\right)\left(\frac{r_\oplus}{R}\right)^3\left(\frac{r_\oplus}{r}\right)^3 P_2(\cos\gamma) \qquad (*)$$

$$= k_2 \left(\frac{M_M}{M_\oplus}\right)\left(\frac{r_\oplus}{R}\right)^2\left(\frac{r_\oplus}{r}\right)^3\left[\frac{3}{8}e^{i(\varphi_D-\varphi_B)} + \frac{3}{8}e^{-i(\varphi_D-\varphi_B)} + \frac{1}{4}\right]$$

For eccentric orbits, we need to Fourier-decompose the various contributions — we will then "lag" each bulge component individually.

$\boxed{\text{Diana}}$: $\varphi_D = f_D + \varpi_D$, $r = \dfrac{a_D(1-e_D^2)}{1+e_D\cos f_D}$

$\underline{l=m=2}$

$$\frac{e^{2i\varphi_D}}{(r/a_D)^3} = \sum_{j=-\infty}^{\infty} F_j^{(22)}(e_D)\, e^{ijM_D}, \qquad M_D = \lambda_D - \varpi_D$$

mean anomaly of Diana ; mean longtitude

$$= -\frac{1}{2}e_D e^{iM_D} + e^{2iM_D} + \frac{7}{2}e_D e^{3iM_D} + O(e_D^2)$$

$$\frac{e^{-2i\varphi_D}}{(r/a_D)^3} = -\frac{1}{2}e_D e^{-iM_D} + e^{-2iM_D} + \frac{7}{2}e_D e^{-3iM_D} + O(e_D^2)$$

$$\left(\frac{a_D}{r}\right)^3 = \sum_{j=-\infty}^{\infty} F_j^{(20)}(e_D)\, e^{ijM_D}$$

$$= \frac{3}{2}e_D e^{-iM_D} + \underbrace{1}_{\left(\text{close form actually } (1-e_D^2)^{-3/2}\right)} + \frac{3}{2}e_D e^{iM_D} + O(e_D^2)$$

$\boxed{\text{Bulge}}$ $\qquad \varphi_B = f_M + \epsilon$

bulge tracks moon but with a lag

$$R = \frac{a_M(1-e_M^2)}{1+e_M\cos f_M}$$

mean anomaly of moon

lag of $(2,2,j)$ compt.

$$\frac{e^{2i\varphi_B}}{(R/a_M)^3} = \sum_{j=-\infty}^{\infty} F_j^{(22)}(e_D)\, e^{ij(M_M+\epsilon_j^{(22)})}$$

$$= -\frac{1}{2}e_M\, e^{i(M_M+\epsilon_1^{(22)})} + e^{2i(M_M+\epsilon_2^{(22)})} + \frac{7}{2}e_M\, e^{3i(M_M+\epsilon_3^{(22)})}$$

$$\frac{e^{-2i\varphi_B}}{(R/a_M)^3} = \text{complex conjugate}$$

$$\left(\frac{a_M}{R}\right)^3 = \sum_{j=-\infty}^{\infty} F_j^{(20)}(e_D)\, e^{ij(M_M+\epsilon_j^{(20)})}$$

$$= \frac{3}{2}e^{-i(M_M+\epsilon_1^{(20)})} + 1 + \frac{3}{2}e^{i(M_M+\epsilon_1^{(20)})}$$

Putting it all together in (*), p11, and assuming lag angles are small gives

Putting $M_D = \lambda_D - \varpi_D$

$M_M = nt$

$$\frac{\Phi_{bulge}}{(GM_\oplus/r_\oplus)} = k_2 \left(\frac{m_M}{M_\oplus}\right)\left(\frac{r_\oplus}{a_M}\right)^3\left(\frac{r_\oplus}{a_D}\right)^3 \Bigg\{$$

$$\frac{3}{4}\epsilon_0\left[2\sin(nt-2\lambda_D) - e_D\sin(2nt-\lambda_D-\varpi_D) + 7e_D\sin(2nt-3\lambda_D+\varpi_D)\right]$$

$$+\frac{63}{16}\epsilon_1\Big[-2e_M\sin(3nt-2\lambda_D) + e_D e_M\sin(3nt-\lambda_D-\varpi_D) -$$
$$- 7e_D e_M\sin(3nt-3\lambda_D+\varpi_D)\Big]$$

$$+\frac{3}{16}\epsilon_2\Big[2e_M\sin(nt-2\lambda_D) - e_D e_M\sin(nt-\lambda_D-\varpi_D) + 7e_D e_M\sin(nt-3\lambda_D+\varpi_D)$$

$$-\frac{3}{8}\epsilon_3\Big[2e_M\sin(nt) + 3e_D e_M\sin(nt+\lambda_D-\varpi_D)$$
$$+ 3e_D e_M\sin(nt-\lambda_D+\varpi_D)\Big]\Bigg\},$$

where we have used Jeffreys/Goldreich notation for the lag angles:

$$\epsilon_0 = \epsilon_2^{(22)},\quad \epsilon_1 = \epsilon_3^{(22)},\quad \epsilon_2 = \epsilon_1^{(22)},\quad \epsilon_3 = \epsilon_1^{(20)}$$

and $\cos(nt-2\lambda+\epsilon_2) \simeq -\epsilon_2\sin(nt-2\lambda)$ etc.

Now we have a disturbing f^n for Diana's motion – we can use Lagrange's planetary eq^{n}s to calculate $\frac{de_D}{dt}$ etc.

To first order in e_D,

$$\frac{de_D}{dt} = -\frac{e_D}{2n_D a_D^2}\frac{\partial \Phi_{bulge}}{\partial \lambda_D} - \frac{1}{n_D a_D^2 e_D}\frac{\partial \Phi_{bulge}}{\partial \varpi_D}$$

Putting $n_D = n$, $e_D = e_M = e$, $\varpi_D = \varpi_M = 0$, $a_D = a_M = a$ and averaging over short-period terms gives (from Goldreich – not checked from above):

$$\left.\frac{1}{e}\frac{de}{dt}\right|_{\substack{\text{Earth}\\ \text{tide}}} = \frac{3}{2} n k_2 \left(\frac{M_m}{M_\oplus}\right)\left(\frac{r_\oplus}{a}\right)^5 \underbrace{\left[\epsilon_0 - \frac{49}{4}\epsilon_1 + \frac{1}{4}\epsilon_2 + \frac{3}{2}\epsilon_3\right]}_{\beta}$$

Darwin vs. Goldreich
(Hut)

$\boxed{\text{Darwin}}$ argues that lag angles should be proportional to their "speeds".

For example, a bulge particle moves ~vertically with frequency $2\Omega - 2n$ due to tide component

$$e^{2iM_m} = e^{2int}$$

(e.g. if spin synchronous with $\Omega = n$, particle is stationary relative to neighbouring particles).

— standard argument for frictional forces being proportional to speed of fluid particle relative to neighbouring particles → shear.

Thus

semi-diurnal tide → $\epsilon_0 = \epsilon_2^{(22)} \propto 2\Omega - 2n$

$\epsilon_1 = \epsilon_3^{(22)} \propto 2\Omega - 3n$

diurnal tide only for non-zero obliquity $\epsilon_2 = \epsilon_1^{(22)} \propto 2\Omega - n$

monthly tide → $\epsilon_3 = \epsilon_1^{(20)} \propto n$

in general

$$\epsilon_j^{(lm)} \propto m\Omega - jn$$

Constant of proportionality $= \tau$
$= \underline{\text{lag time}}$

Darwin: tide components have <u>constant lag times</u>

This then gives sum of lag angles

$$\beta = (22\Omega - 36n)\tau$$

$$\rightarrow \frac{de}{dt} > 0 \quad \text{if} \quad n > \frac{11}{18}\Omega$$

$\boxed{\text{Goldreich}}$ makes the following argument.

First he defines Q:

$$Q = \frac{2\pi E_{peak}}{\oint \frac{dE}{dt}\,dt}$$

where $\frac{dE}{dt}$ is the work done by the tidal force on the fluid.

Thus
$$\frac{dE}{dt} = \int_V \rho \underset{\sim}{v} \cdot (-\nabla \Phi_{tide})\, dV$$

Since
$$\nabla \cdot (\Phi \rho \underset{\sim}{v}) = \rho \underset{\sim}{v} \cdot \nabla \Phi + \Phi \underbrace{\nabla \cdot (\rho \underset{\sim}{v})}_{-\frac{\partial \rho}{\partial t} = 0}$$

we can write

$$\frac{dE}{dt} = -\int_V \nabla \cdot (\Phi \rho \underset{\sim}{v})\, dV$$

$$= -\int_{\substack{\text{Sphere}\\ \text{surface}}} \Phi \rho \underset{\sim}{v} \cdot \underset{\sim}{\hat{n}}\, dS \quad , \quad \underset{\sim}{\hat{n}} = \underset{\sim}{\hat{r}}$$

The velocity of a fluid element $\underset{\sim}{v}$ is

$$\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt}, \quad \text{where } \underset{\sim}{r} = r\,\hat{\underset{\sim}{r}}$$

Fluid particle oscillates with frequency ν due to tidal component with freq ν_i (eg $2\Omega - 2n$).

Thus $r \propto \cos(\nu_i t + \epsilon_i)$ for each ν_i & $\Phi_i \propto \cos \nu_i t$

work done by tidal compt i

$$\text{Thus} \quad \frac{dE_i}{dt} = \underbrace{\left(-\int_S \rho \nu_i r_* \Phi_i^* \, dS \right)}_{\nu_i C_i} \cos \nu_i t \, \sin(\nu_i t + j\epsilon_i)$$

$j = 2$ for $\epsilon_0, \epsilon_1, \epsilon_2$, $j = 0$? for ϵ_3

$$r = r_* \cos(\nu_i t + \epsilon_i), \quad \Phi_i = \Phi_i^* \cos \nu_i t$$

(+ other comps)

Thus over a complete osc^n cycle,

$$\oint \frac{dE_i}{dt} dt = \pi C_i \sin j\epsilon_i \quad \text{to first order in } \epsilon_i$$

& peak energy is

$$E_{peak} = \int_0^{\frac{\pi/2}{\nu_i}} \frac{dE_i}{dt} dt = \frac{1}{2} C_i \cos j\epsilon_i$$

Thus

$$Q_i = \frac{1}{\tan j\epsilon_i}$$

For small ϵ_i,

$$\boxed{Q_i^{-1} = j\epsilon_i} = 2\epsilon_i \quad \text{for } \epsilon_0, \epsilon_1, \epsilon_2$$

Goldreich argues that Q varies by no more than a factor of 4 over the frequency range

1 cycle per second → 1 cycle per year !

Thus he takes $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3$ to get

$$\frac{1}{e}\frac{de}{dt}\bigg|_{\oplus} = \frac{171}{16}\, n \underbrace{\left(\frac{2}{3}\frac{k_2}{Q}\right)}_{\frac{1}{Q'}} \left(\frac{m_M}{m_\oplus}\right)\left(\frac{r_\oplus}{a}\right)^5 \sigma$$

$$\left(\frac{171}{16} \times \frac{2}{3} = \frac{57}{8}\right)$$

where $\sigma = \text{sign}\underbrace{(2\Omega - 3n)}_{\text{frequency of tidal component with largest contribution to } \frac{de}{dt}\text{!}}$ $\left(-\frac{49}{4}\epsilon_1 : \text{p } 14\right)$

For tides raised on moon by planet,

$\Omega = n$ so that

$$\epsilon_0 \propto 2\Omega - 2n = 0$$

$$\epsilon_1 \propto 2\Omega - 3n \propto -n$$

$$\epsilon_2 \propto 2\Omega - n \propto n$$

- Goldreich then puts $\epsilon_1 = -\epsilon_2$
  - this assumes ϵ_1 & ϵ_2 have the same lag angle!!

Even though $\epsilon_3 \propto n$, Goldreich keeps this separate and puts

$$\beta = \frac{25}{2}\epsilon_2 + \frac{3}{2}\epsilon_3 .$$

Putting $\epsilon_2 = \epsilon_3$ gives

$$\left.\frac{de}{dt}\right|_{moon} = -\frac{63}{4} n \left(\frac{2}{3}\frac{k_2}{Q}\right)\left(\frac{M_\oplus}{M_m}\right)\left(\frac{r_M}{a}\right)^5$$

$$= -\frac{21}{2} n \left(\frac{k_2}{Q}\right)\left(\frac{M_\oplus}{M_m}\right)\left(\frac{r_M}{a}\right)^5$$

Hut

Replaces tidal bulge by small point masses with masses (for distorted Earth)

$$\mu = \frac{1}{2} k_2 M_\oplus \left(\frac{r_\oplus}{R}\right)^3$$

line of centres catches up with current bulge line after time τ.
Lag angle
$\epsilon = (\Omega - \dot{f})\tau$

μ is in general time dependent (for $e \neq 0$) and bulges lag line of centres by a time τ. Hut then puts

$$\mu(t) = \frac{1}{2} k_2 M_\oplus r_\oplus^3 \left[R(t-\tau)\right]^{-3}$$

$$\simeq \frac{1}{2} k_2 M_\oplus r_\oplus^3 \left(1 + 3\frac{\dot{R}}{R}\tau\right).$$

He then writes down the eq^n governing orbital motion and identifies the tidal force $\underset{\sim}{F}$. He uses this to calculate the energy and angular momentum exchanged (including (energy lost) and from there gives (orbit-averaged) $\frac{de}{dt}$, $\frac{da}{dt}$, $\frac{d\Omega}{dt}$. Also given is $\frac{di}{dt}$ for small i, where i is the obliquity of the tidally distorted object.

(Note that Hut's k_2 is the apsidal motion constant). Hut's expressions are exact to all orders in e.

His tidal-damping timescale T is defined via the lag time τ:

$$T = \frac{r_\oplus^3}{GM_\oplus} \cdot \frac{1}{\tau}$$

For sychronous rotation we can match Goldreich & Hut to first-order in e:

$$\underbrace{\frac{1}{T}\left(1+\frac{M_m}{M_\oplus}\right)\left(\frac{R_\oplus}{a}\right)^3}_{\text{Hut}} = \underbrace{\frac{n}{Q}}_{\text{Goldreich}}$$

or

$$\boxed{nT = \left(\frac{M_\oplus + M_M}{M_\oplus}\right)\left(\frac{r_\oplus}{a}\right)^3 Q}$$

One could generalize Huts lag-time analysis to have different lag times (and a corresponding μ) for each tidal component. This would provide closed-form expressions for the rates of change of the elements.

Note the Eggleton provides expressions for all obliquities, but introduces yet another tidal damping parameter (which can be related to Q as above). He effectively introduces the concept of fluid shear via inertia tensors but in the end, his analysis assumes constant time-lags $\equiv$ DARWIN.

Note that constant time lags preserve P_2 shape of bulge.