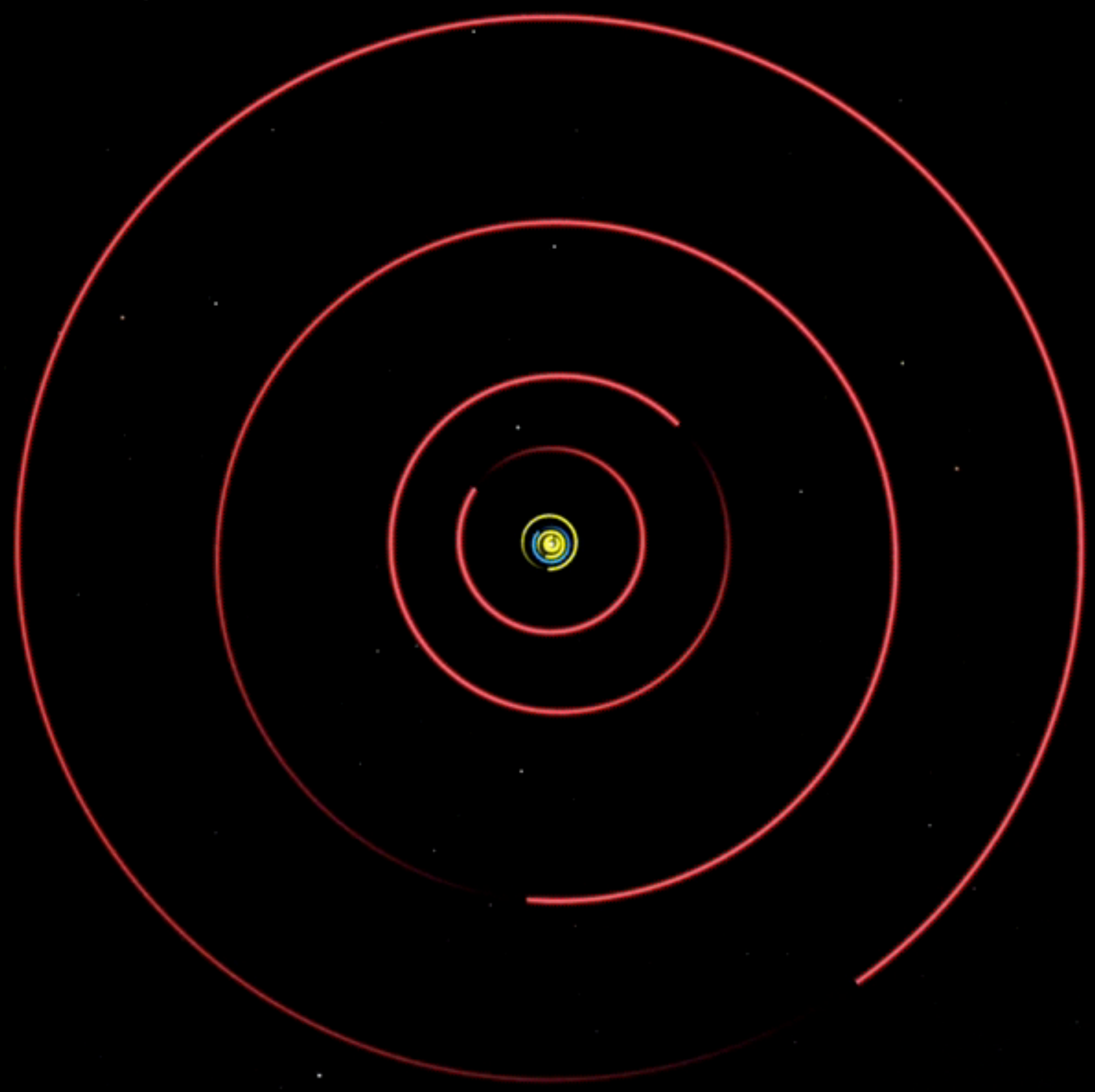
Hyperbolas and Stuff

Konstantin Batygin (Caltech) Fred Adams (UMich) Erik Petigura (UCLA)

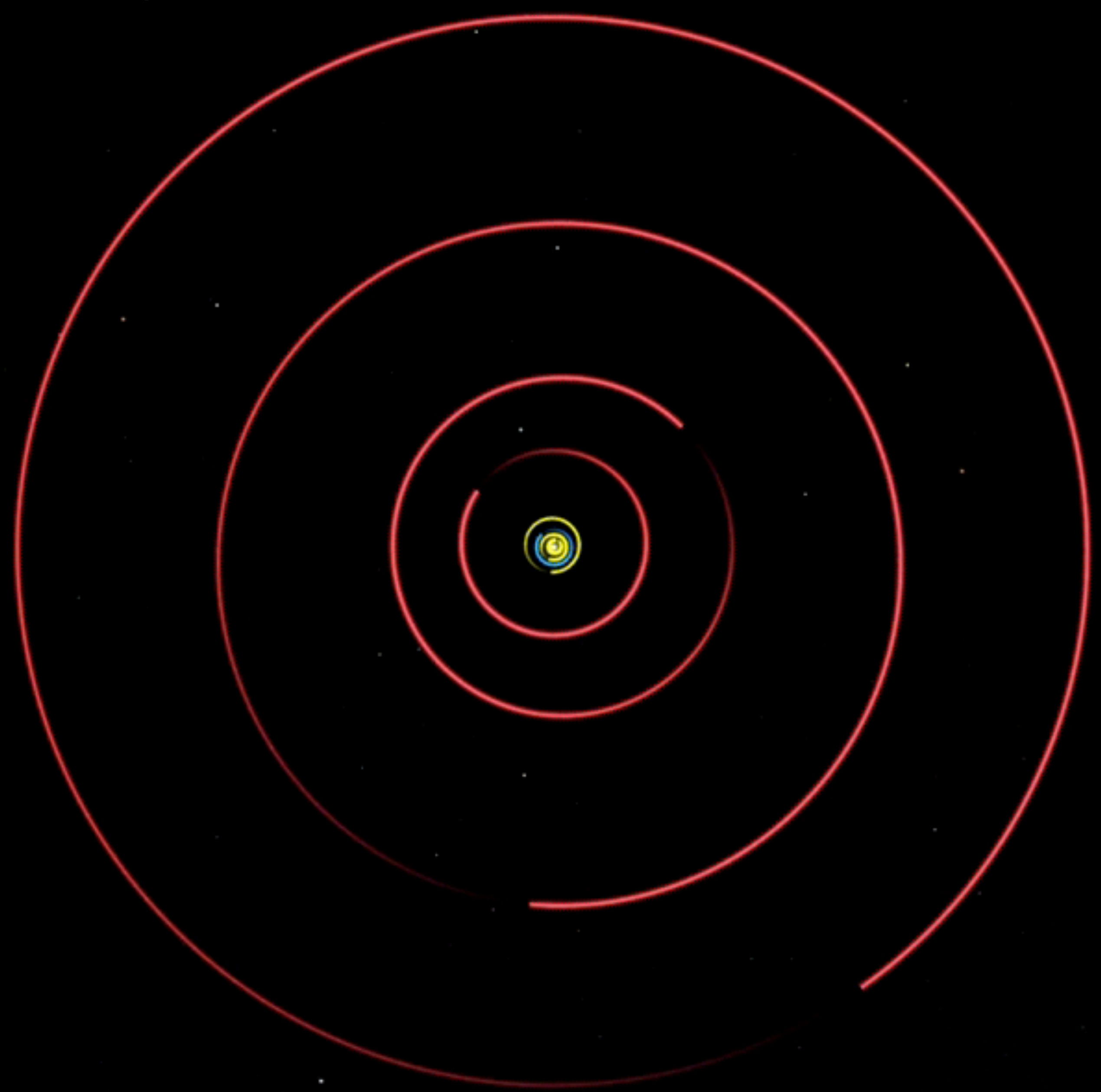

Orion Nebula Cluster

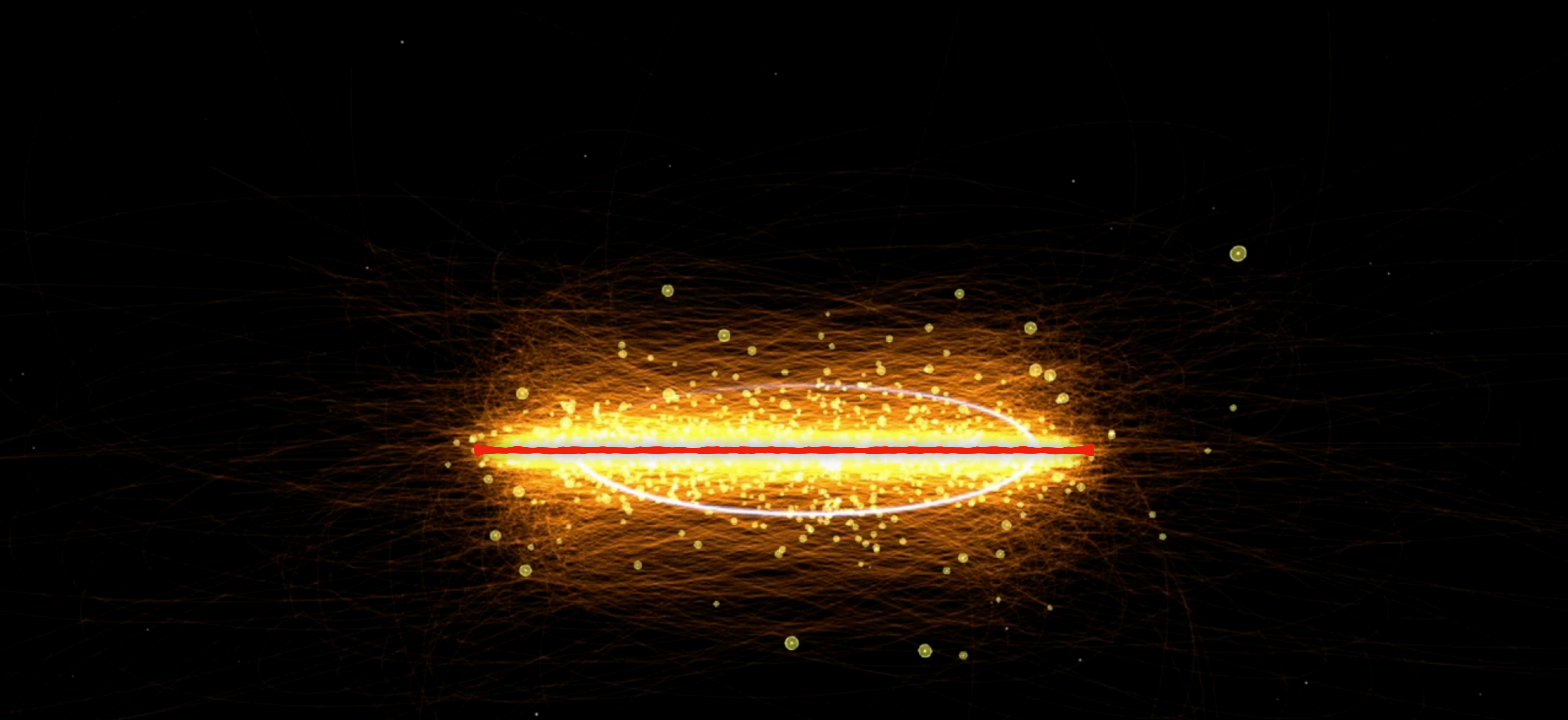

Pleiades Cluster

outer edge of the solar system as a probe of cluster properties



outer edge of the solar system as a probe of cluster properties



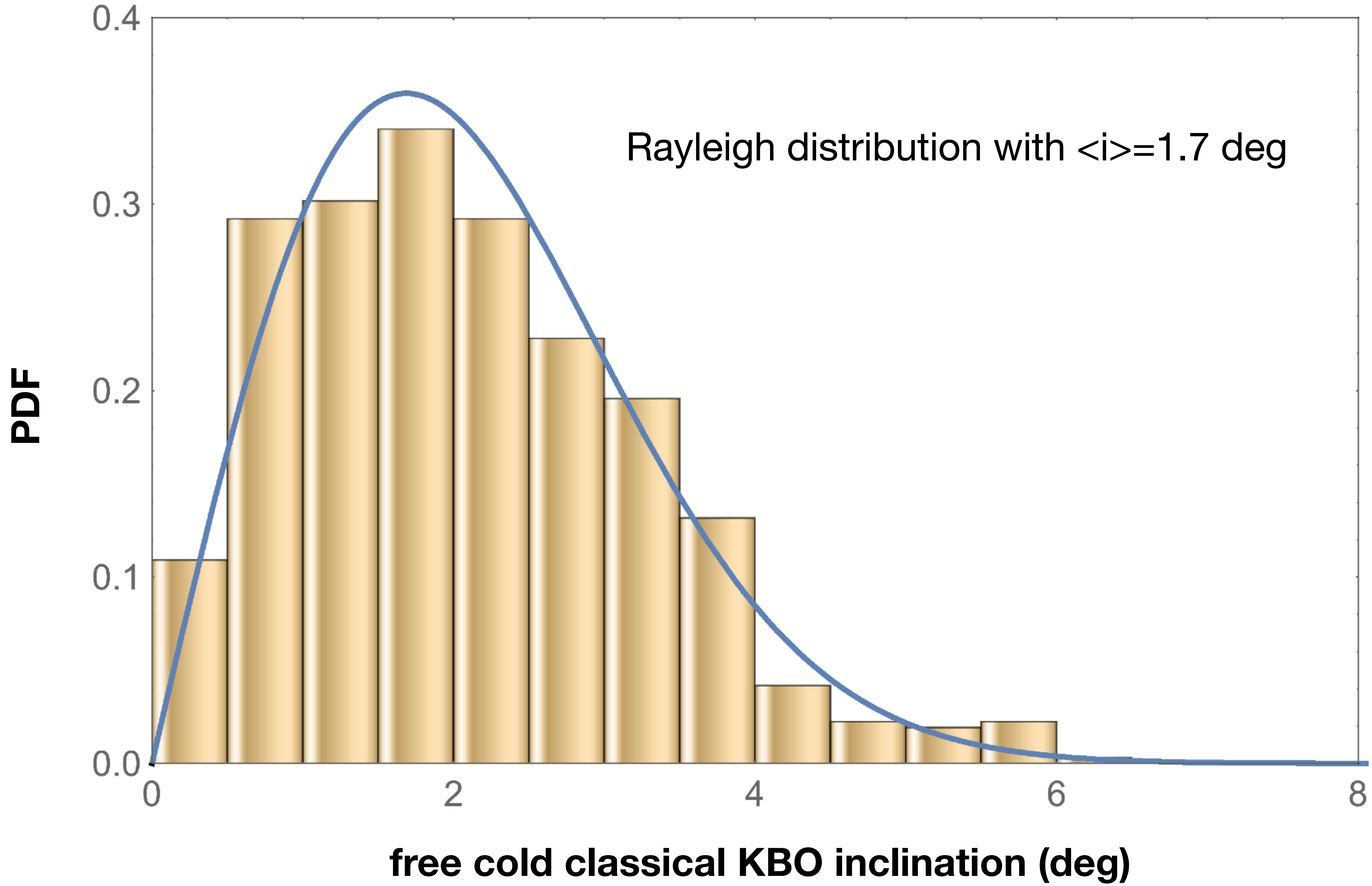


cold classical Kuiper belt: a primordial, flat (∂i~5 deg) population of KBOs

unsolved: retention is a constraint on cluster properties

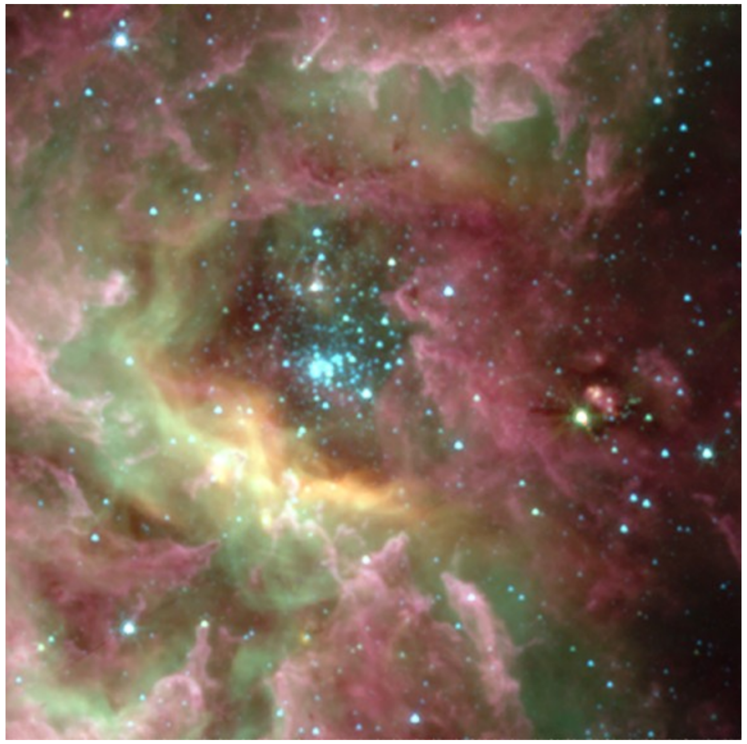
Forced-free decomposition (Brouwer and van Woerkom, 1950):

$$i\, e^{\imath\,\Omega} \approx i_{\rm free}\, e^{\imath\,\Omega_{\rm free}} - \frac{B_8}{B} I_{5,8} e^{\imath\,\Omega_5} - \frac{B_8}{B - f_8} I_{8,8} e^{\imath\,\Omega_8} + \ldots$$



almost all stars are born in clusters….

embedded t ~ 10 Myr



N ~ 100

open t ~ 100 Myr

N ~ 10,000

globular t ~ Gyr

N ~ 1,000,000

Number of clusters ~ $1/N^2$

Bigger clusters = more stars: the probability of being born in a cluster of size N ~ 1/N.

But cumulative probability of being born in a cluster of size N goes like P \sim \int (1/N)dN \sim \ln(N)

So clusters in each decade in N produce (more or less) equal numbers of stars.

Enrichment of meteorites in ^{26}Al: N~1,000-10,000

Let's instead start from scratch…

dynamical effects of the star cluster

mean cluster potential

stellar fly-by's

mean cluster potential

cluster potential: generalization of Hernquist, Plummer, etc

$$\Phi = \frac{\mathcal{G}\,M}{c}\left(\frac{1}{1+\xi^{\gamma}}\right)^{1/\gamma}$$

integer

core-radius

dimensionless radius r/c

orbit-averaged Hamiltonian is reminiscent of the Kozai-Lidov Hamiltonian

$$\mathcal{H} = \frac{\mathcal{G}\,M}{32\,c}\,\frac{\xi^{\gamma}\left(a/(\xi\,c)\right)^{2}}{\left(1+\xi^{\gamma}\right)^{2+1/\gamma}}\left[\left(2+3\,e^{3}\right)\left(\xi^{\gamma}-3\gamma-2\right)+\left(3\xi^{\gamma}-\gamma+2\right)\left(\left(2+3\,e^{2}\right)\cos(2\,i)+10\,e^{2}\sin^{2}(i)\cos(2\omega)\right)\right]$$

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Secular Perturbations of Asteroids with High Inclination and Eccentricity

Yoshihide Kozai*
Smithsonian Astrophysical Observatory, Cambridge, Massachusetts

(Received August 29, 1962)

dynamical effects of the star cluster

mean cluster potential

generalized Kozai effect

stellar fly-by’s

stellar fly-by's

usual method: impulse approximation

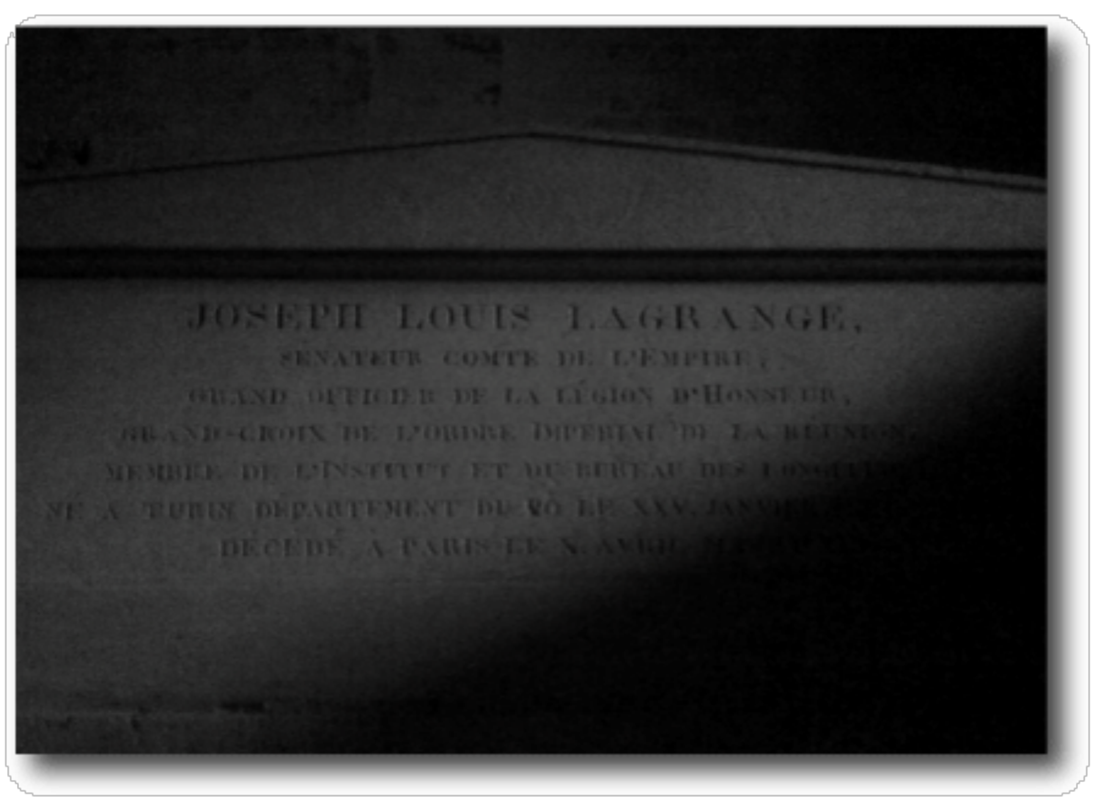
key insight: encounters in the cluster are secular!

1000AU

$$\tau \sim \frac{b}{v} \sim 5000\,\mathrm{yrs} \gg \mathrm{P_{orb}} \sim 300\,\mathrm{yrs}$$

1 km/s

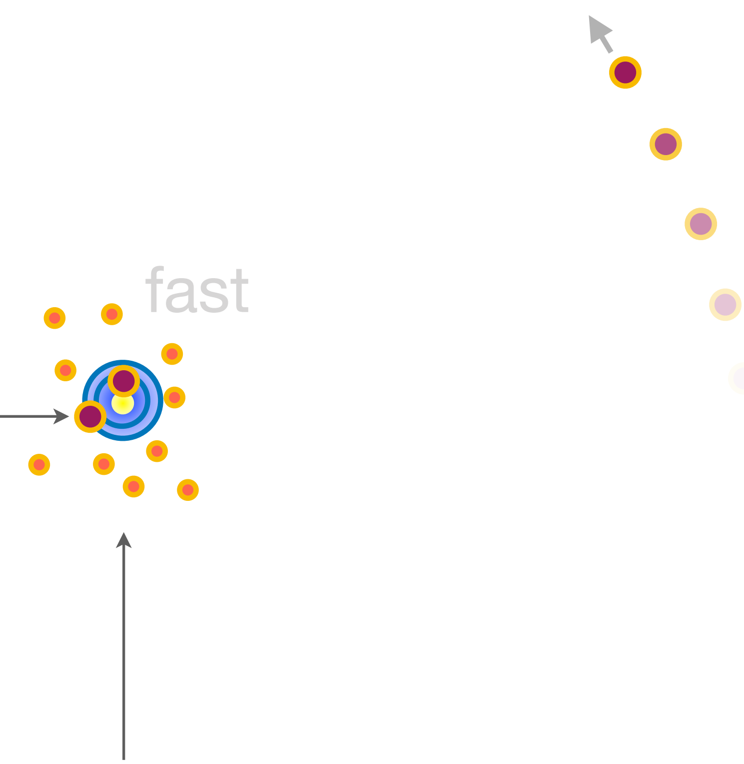
cold classical KBO



average away Keplerian motion

stellar fly-by's

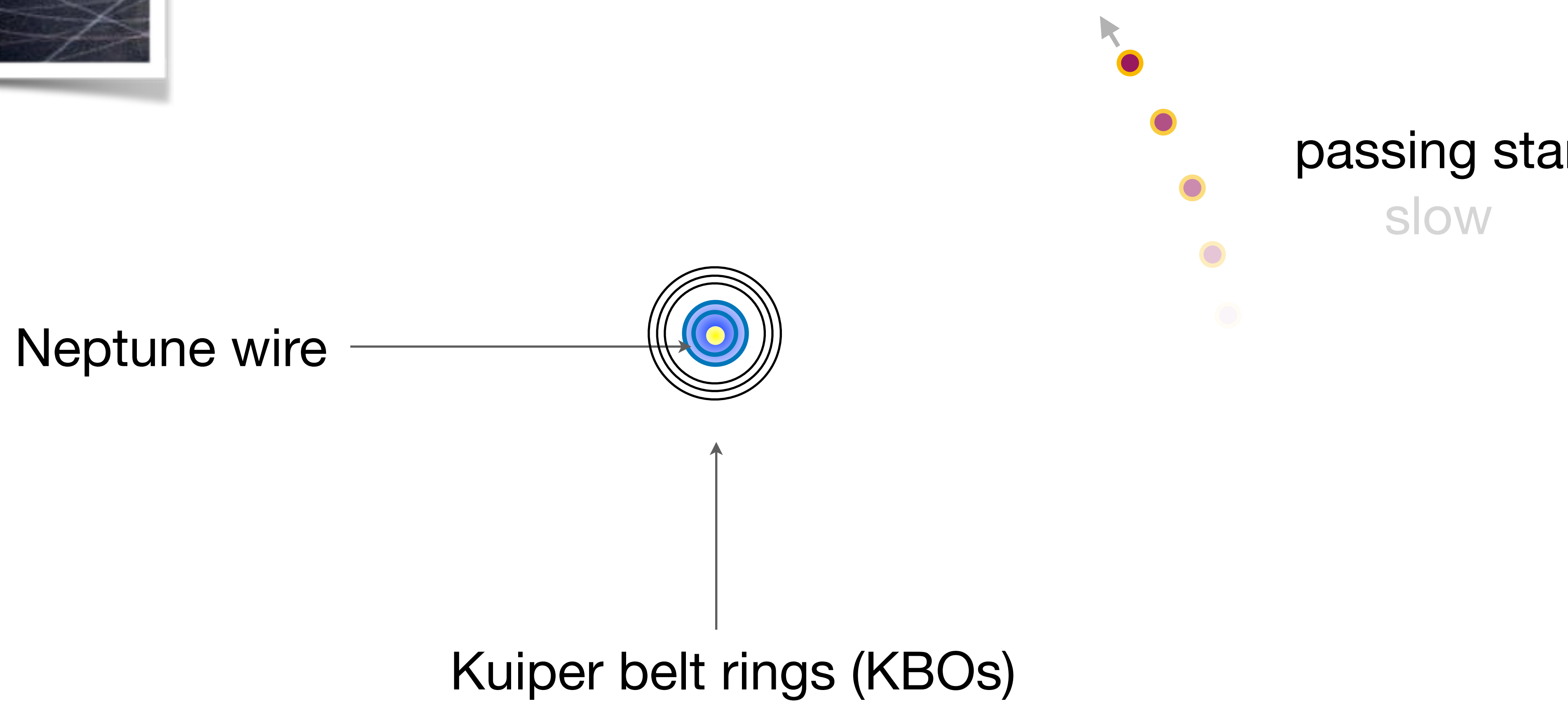
gravitational N-body problem



stellar fly-by's

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\gamma}} = \sum_{\ell=0}^{\infty} \frac{r'^{\ell}}{r^{\ell+1}} P_\ell(\cos\gamma).$$

gravitational wire-body problem



Quadrupole-level expansion of the Hamiltonian, averaged over the particle's orbital period

$$\bar{\mathcal{H}} = -\frac{1}{2\pi}\oint \mathcal{R}^{(2)}\left(1 - e\cos(\mathcal{E})\right)d\mathcal{E}$$

$$= \frac{\mathcal{G}\, m'\, \alpha^2}{4\, a'\, (e' \cosh(\mathcal{W}') - 1)^3}\Bigg[\left(\frac{e' - \cosh(\mathcal{W}')}{e'\cosh(\mathcal{W}') - 1}\right)^2$$

$$\times \Big(3\left(1 - e^2\right)\left(\cos(i)\cos(\omega)\sin(\Omega)\right.$$

$$\left. + \sin(\omega)\cos(\Omega)\right)^2 + \left(12\, e^2 + 3\right)\left(\cos(\omega)\cos(\Omega)\right.$$

$$\left. - \cos(i)\sin(\omega)\sin(\Omega)\right)^2\Big) - (2 + 3e^2)$$

$$+ \left(\frac{3\left(\cosh(\mathcal{W}' - e')\right)\sinh(\mathcal{W}')\sqrt{e'^2 - 1}}{2\left(e'\cosh(\mathcal{W}') - 1\right)^2}\right)$$

$$\times \Big(\cos^2(i)\sin(2\,\Omega)\left(5e^2\cos(2\,\omega) - 3e^2 - 2\right)$$

$$+ 10\, e^2\cos(i)\sin(2\,\omega)\cos(2\,\Omega) + \sin(2\,\Omega)\big(5\, e^2\cos(2\,\omega)$$

$$+ 3\, e^2 + 2\big)\Big) + \left(\frac{3\,\sinh^2(\mathcal{W}')\,(1 - e'^2)}{2\left(e'\cosh(\mathcal{W}') - 1\right)^2}\right)$$

$$\times \Big(\cos^2(i)\cos^2(\Omega)\left(5\, e^2\cos(2\omega) - 3\, e^2 - 2\right)$$

$$- 5\, e^2\cos(i)\sin(2\omega)\sin(2\Omega)$$

$$- \sin^2(\Omega)\left(5\, e^2\cos(2\omega) + 3e^2 + 2\right)\Big)\Bigg]$$

$$
\begin{aligned}
\bar{\bar{\mathcal{K}}} &= \int_{-\infty}^{\infty} \bar{\mathcal{K}}\, dt = \frac{1}{n'} \int_{-\infty}^{\infty} \bar{\mathcal{K}}\left(e' \cosh(\mathcal{W}' - 1)\right) d\mathcal{W}' \\
&= \frac{\alpha^3}{16\, e'^2\, (e'^2 - 1)^{3/2}} \frac{n}{n'} \frac{m'}{M} \\
&\times \Big[\left(3\, e^2 + 2\right) e'^2\, \kappa\, (3 \cos(2i) + 1) \\
&+ 30\, e^2\, e'^2\, \kappa\, \sin^2(i) \cos(2\,\omega) \\
&+ 2 \left(3\, e^2 + 2\right) \left(e'^2 - 1\right)^{3/2} \sin^2(i) \cos(2\,\Omega) \\
&+ 5 e^2 \left(e'^2 - 1\right)^{3/2} (\cos(i) + 1)^2 \cos(2(\omega + \Omega)) \\
&+ 5 e^2 \left(e'^2 - 1\right)^{3/2} (\cos(i) - 1)^2 \cos(2(\omega - \Omega)) \Big],
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa = 2 \Bigg[&\frac{\sqrt{e'^2 - 1}}{2} + \arctan\left(\frac{1}{\sqrt{e'^2 - 1}}\right) \\
&+ \arctan\left(\frac{e' - 1}{\sqrt{e'^2 - 1}}\right) \Bigg] \approx e' + \frac{\pi}{2} + \frac{1}{2\, e'}.
\end{aligned}
$$

An important second-level approximation: effective secular evolution is slow compared to crossing time

timescale hierarchy

KBO orbit

crossing

secular

+ low-ecc

shorter ↑

$$\bar{\bar{\mathcal{K}}} = \int_{-\infty}^{\infty} \bar{\mathcal{K}}\, dt = \frac{1}{n'} \int_{-\infty}^{\infty} \bar{\mathcal{K}} \left(e' \cosh(\mathcal{W}' - 1)\right) d\mathcal{W}'$$

$$= \frac{\alpha^3}{16\, e'^2\, (e'^2 - 1)^{3/2}} \frac{n}{n'} \frac{m'}{M}$$

$$\times \Big[\left(\cancel{3e^2} + 2\right) e'^2\, \kappa\, (3\cos(2i) + 1)$$

$$\cancel{+\, 30\, e^2\, e'^2\, \kappa \sin^2(i) \cos(2\,\omega)}$$

$$+\, 2\left(\cancel{3e^2} + 2\right) \left(e'^2 - 1\right)^{3/2} \sin^2(i) \cos(2\,\Omega)$$

$$\cancel{+\, 5e^2 \left(e'^2 - 1\right)^{3/2} (\cos(i) + 1)^2 \cos(2(\omega + \Omega))}$$

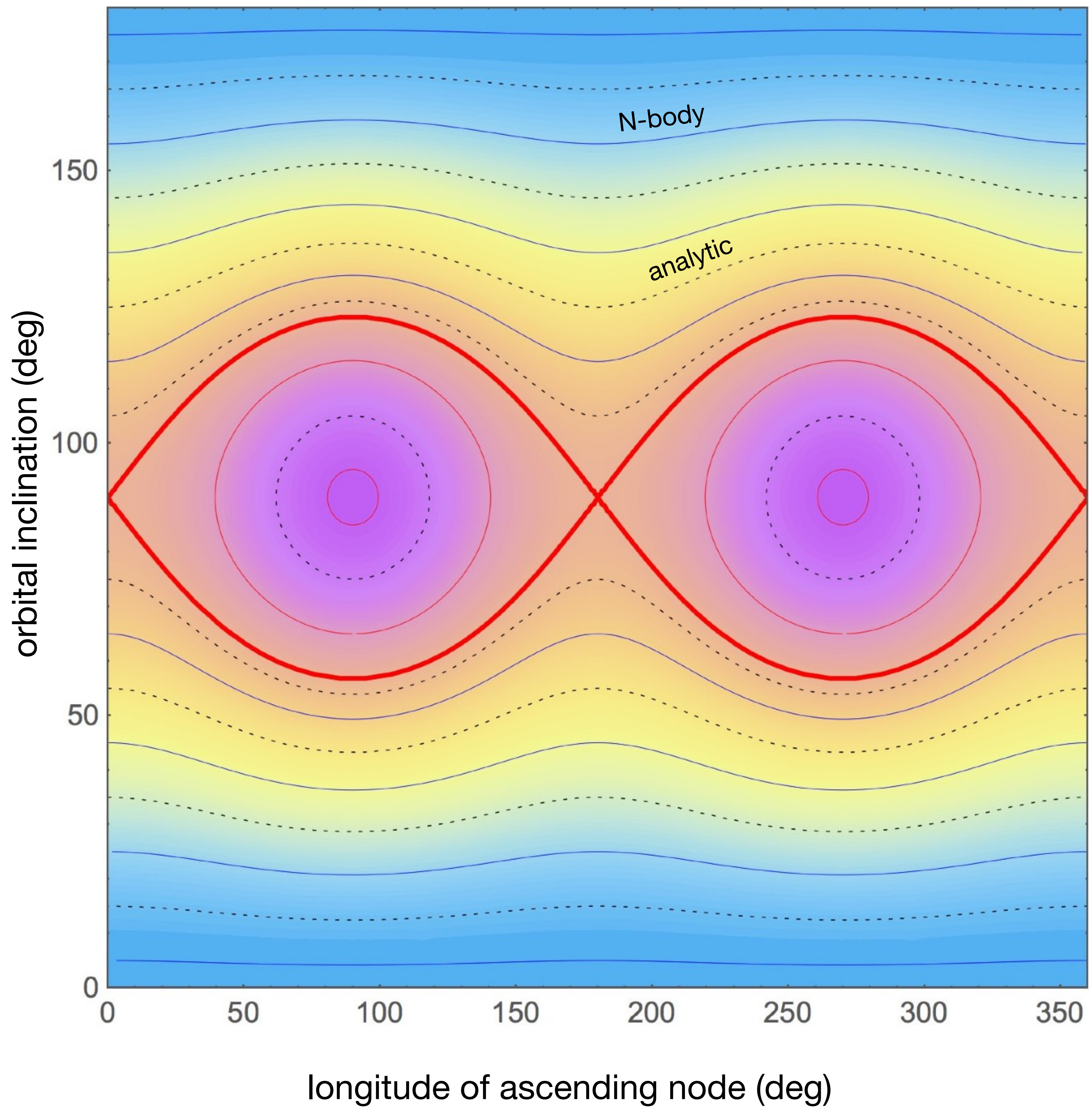
$$\cancel{+\, 5e^2 \left(e'^2 - 1\right)^{3/2} (\cos(i) - 1)^2 \cos(2(\omega - \Omega))} \Big],$$

where

$$\kappa = 2\left[\frac{\sqrt{e'^2 - 1}}{2} + \arctan\left(\frac{1}{\sqrt{e'^2 - 1}}\right) + \arctan\left(\frac{e' - 1}{\sqrt{e'^2 - 1}}\right)\right] \approx e' + \frac{\pi}{2} + \frac{1}{2\, e'}.$$

An important second-level approximation: effective secular evolution is slow compared to crossing time

$$\bar{\bar{\mathcal{K}}} = \frac{\alpha^3}{4\,e'^2}\,\frac{n}{n'}\,\frac{m'}{M}\left[3\,H^2 + \left(1 - H^2\right)\,\cos(2\,h)\right]$$



Let's instead start from scratch…

dynamical effects of the star cluster

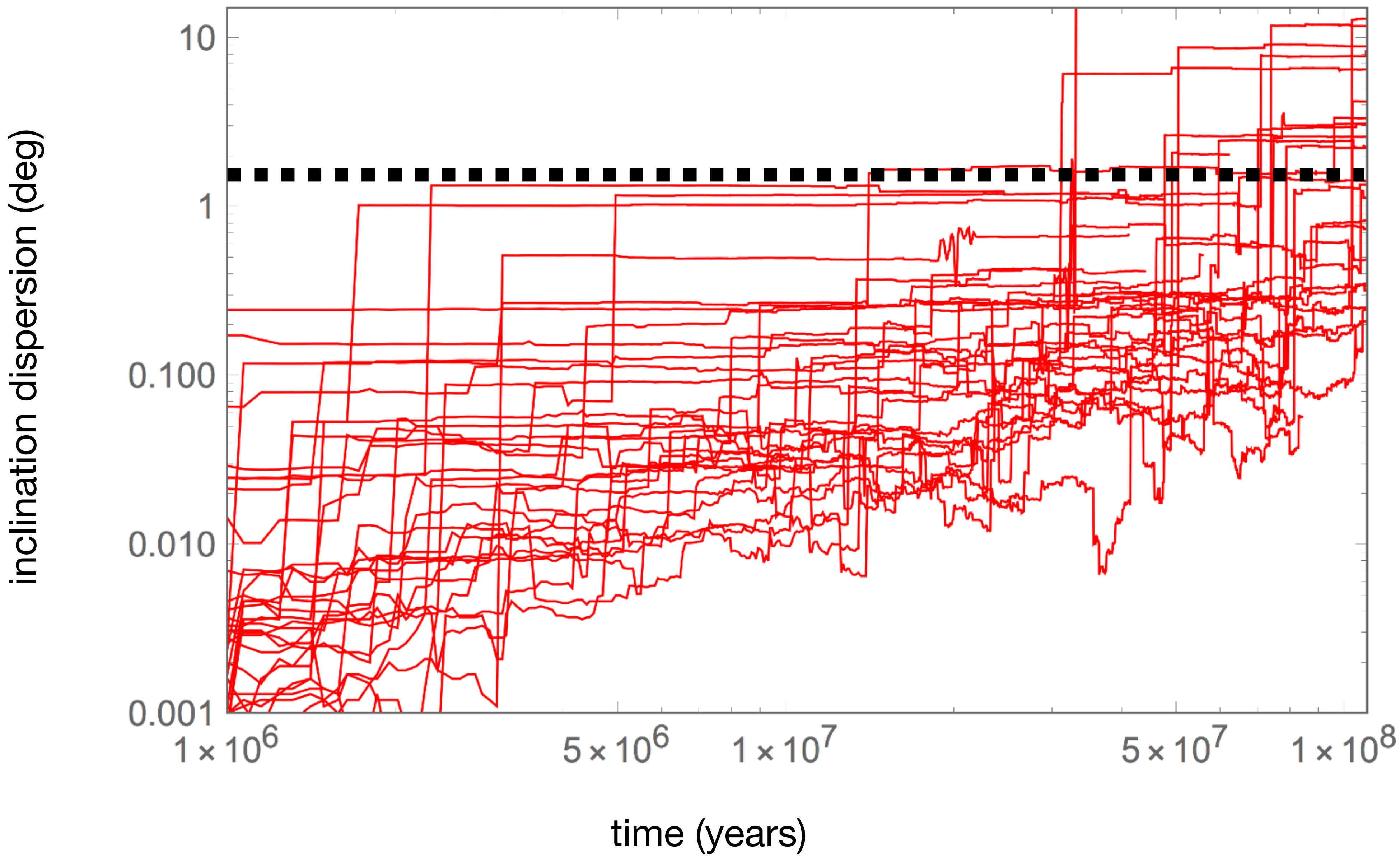
mean cluster potential

generalized Kozai effect

stellar fly-by's

secular impulses

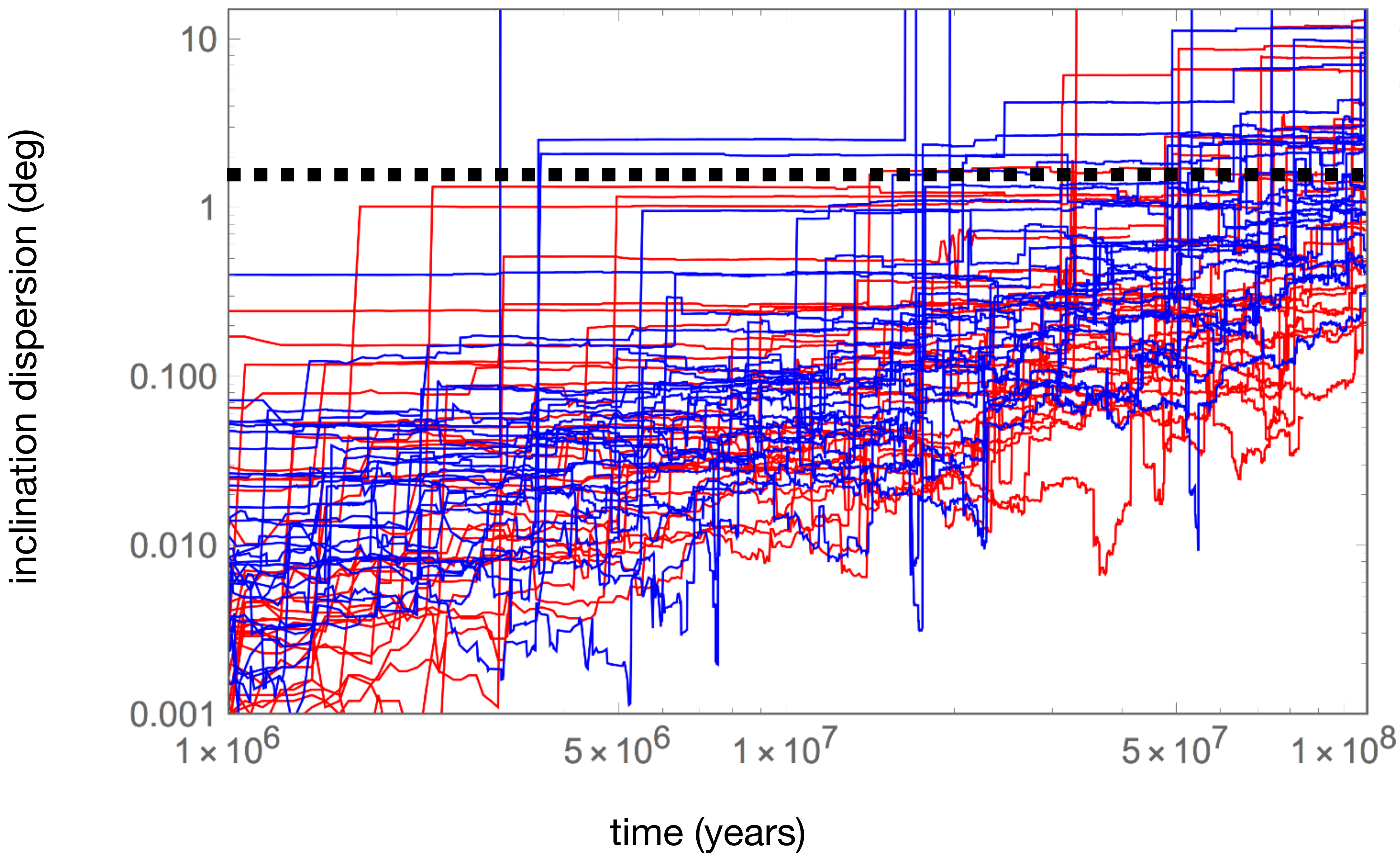
evolution of the cold classical Kuiper belt in a cluster



<n>~100/pc³
<v>~1 km/s

N-body results

evolution of the cold classical Kuiper belt in a cluster



<n>~100/pc^3

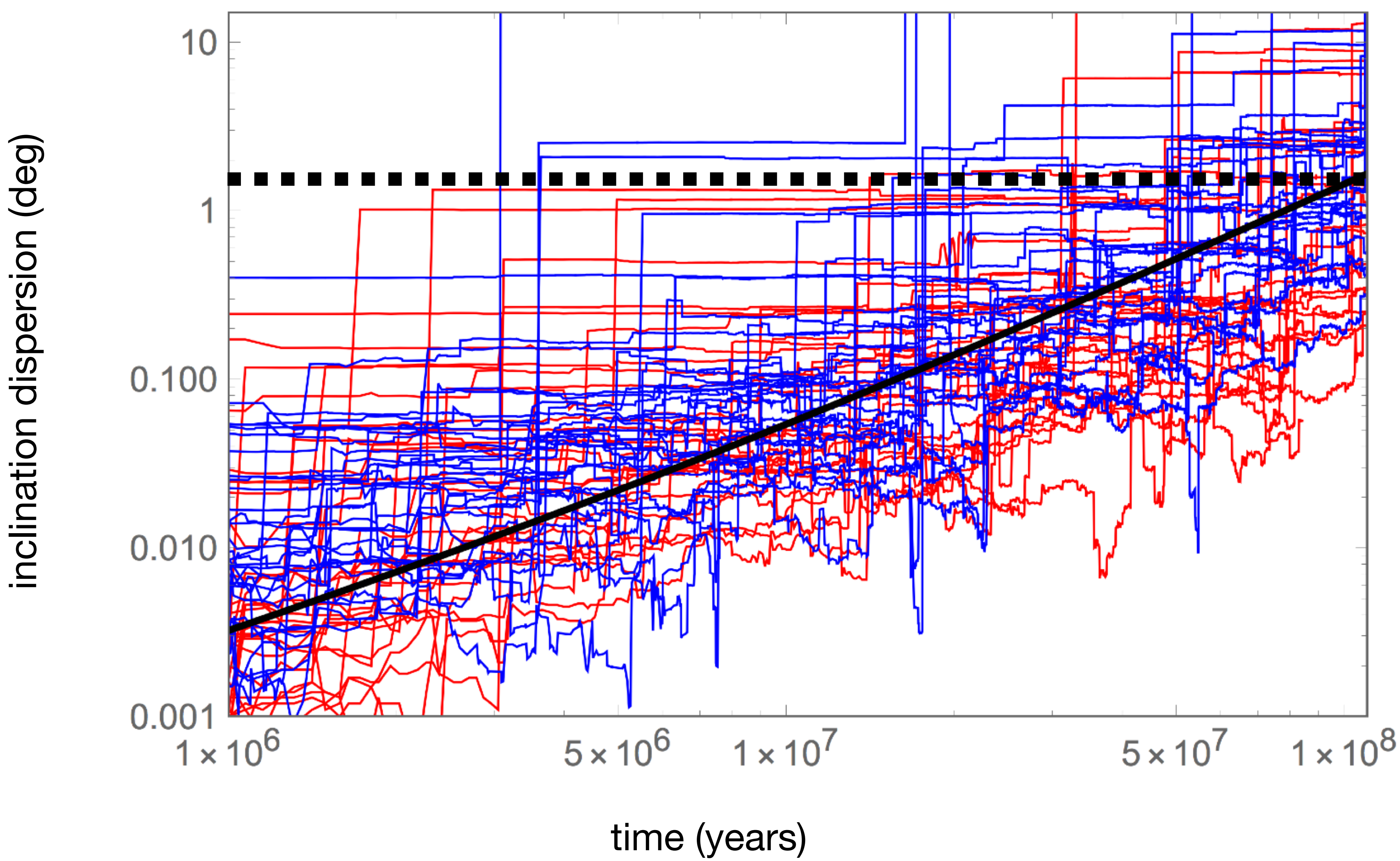
<v>~1 km/s

N-body results

analytic mapping

evolution of the cold classical Kuiper belt in a cluster

$$\langle i \rangle = \left(\frac{a}{a' \sqrt{e'^2 - 1}} \right)^3 \frac{m'}{M} \frac{n}{n'} \frac{\sqrt{(e'^2 - 1)^3 + 12\, e'^4\, \kappa^2}}{2\sqrt{2}\, e'^2}$$



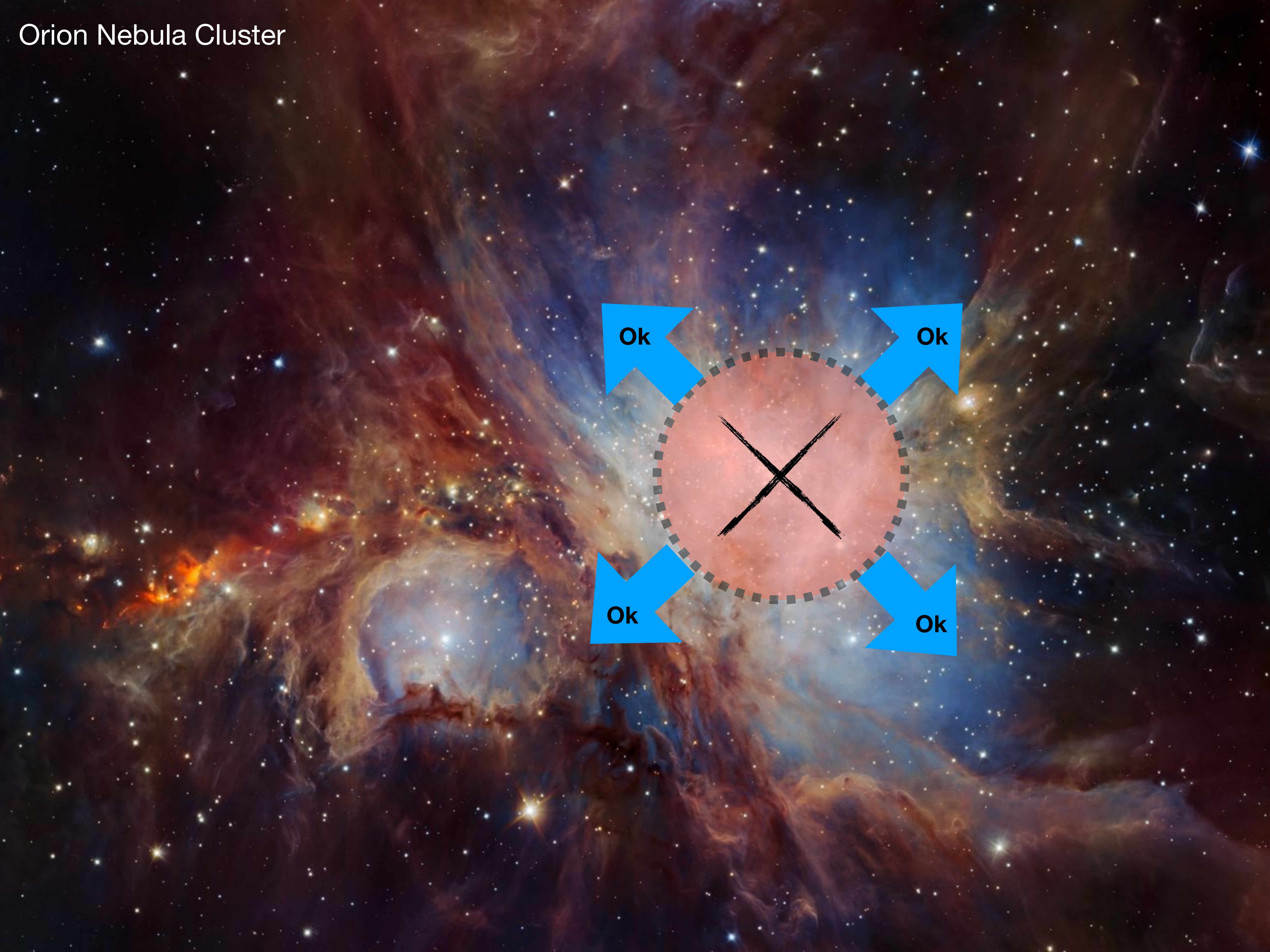
<n>~100/pc^3

<v>~1 km/s

N-body results

analytic mapping

Above equation


Orion Nebula Cluster
Ok
Ok
Ok
Ok

Academic entertainment: i=0 case

$$
\begin{aligned}
\bar{\bar{\mathcal{K}}} &= \int_{-\infty}^{\infty} \bar{\mathcal{K}}\, dt = \frac{1}{n'} \int_{-\infty}^{\infty} \bar{\mathcal{K}}\left(e' \cosh(\mathcal{W}' - 1)\right) d\mathcal{W}' \\
&= \frac{\alpha^3}{16\, e'^2\, (e'^2 - 1)^{3/2}} \frac{n}{n'} \frac{m'}{M} \\
&\times \Big[\left(3\, e^2 + 2\right) e'^2\, \kappa\, (3 \cos(2i) + 1) \\
&+ 30\, e^2\, e'^2\, \kappa \sin^2(i) \cos(2\,\omega) \\
&+ 2 \left(3\, e^2 + 2\right) \left(e'^2 - 1\right)^{3/2} \sin^2(i) \cos(2\,\Omega) \\
&+ 5 e^2 \left(e'^2 - 1\right)^{3/2} (\cos(i) + 1)^2 \cos(2(\omega + \Omega)) \\
&+ 5 e^2 \left(e'^2 - 1\right)^{3/2} (\cos(i) - 1)^2 \cos(2(\omega - \Omega)) \Big], \qquad (13)
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa = 2 \Bigg[\frac{\sqrt{e'^2 - 1}}{2} + \arctan\left(\frac{1}{\sqrt{e'^2 - 1}} \right) \\
+ \arctan\left(\frac{e' - 1}{\sqrt{e'^2 - 1}} \right) \Bigg] \approx e' + \frac{\pi}{2} + \frac{1}{2\, e'}. \qquad (14)
\end{aligned}
$$

An important second-level approximation: effective secular evolution is slow compared to crossing time

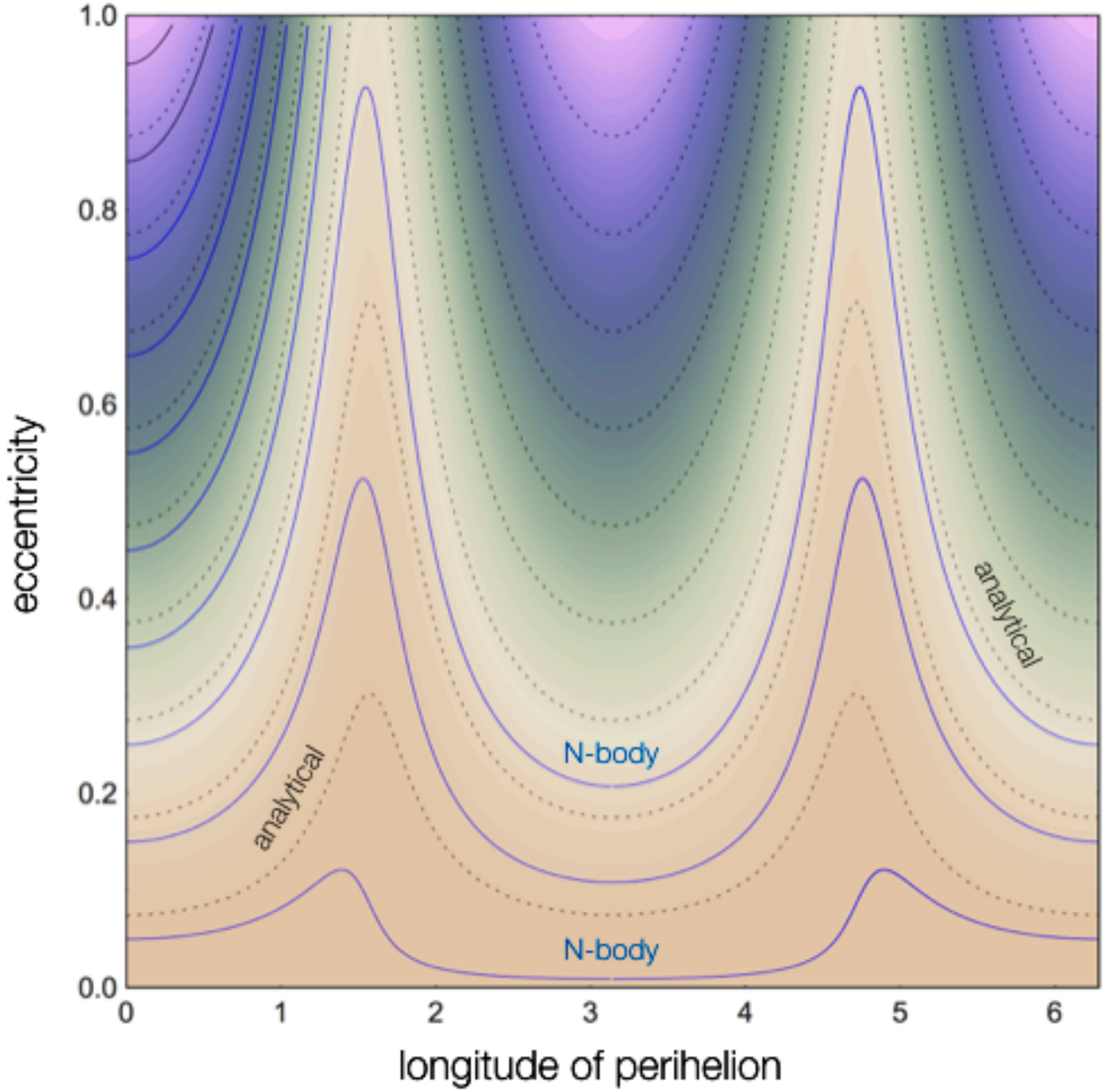
Integrability attained!

$$\bar{\bar{\mathcal{K}}} = \frac{\alpha^3}{4\,e'^2\,(e'^2-1)^{3/2}}\,\frac{n}{n'}\,\frac{m'}{M}\Bigg[3\,e^2\,e'^2\,\kappa + 5\,e^2\left(e'^2-1\right)^{3/2}\cos(2\,\varpi)\Bigg], \qquad (15)$$

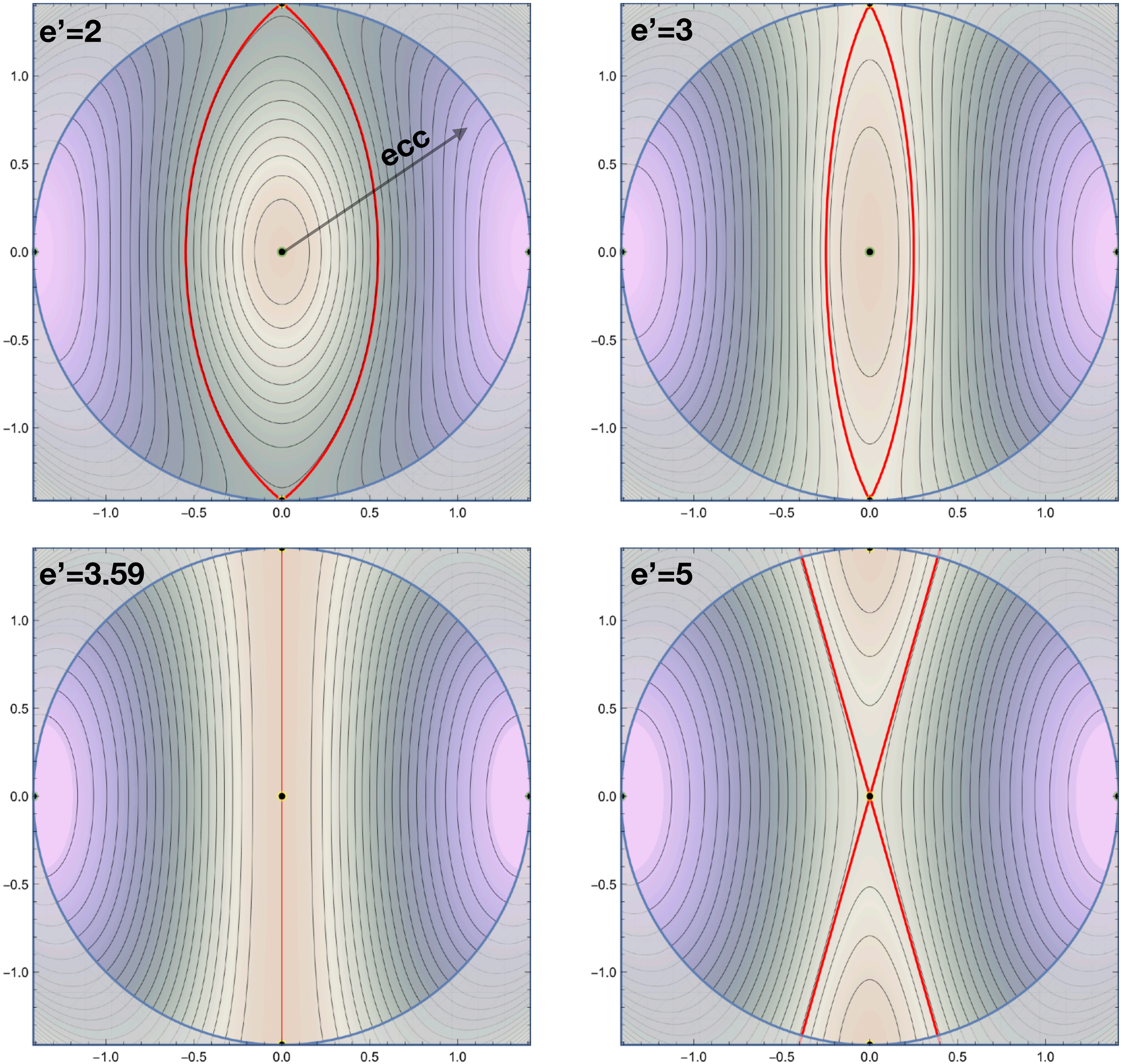
Special case: planar encounters

(1) circular orbit is only secularly unstable for e’>3.59

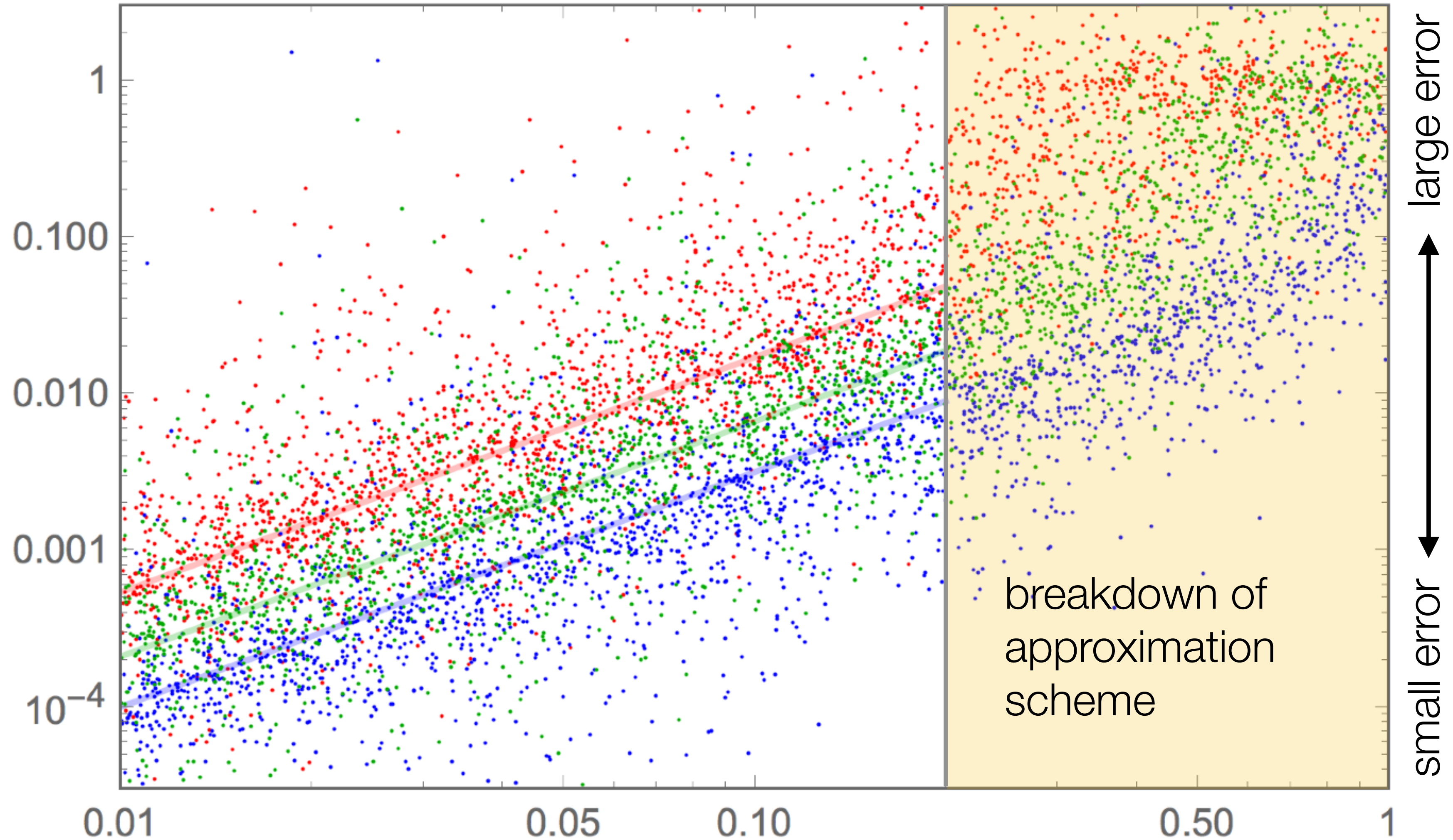
(2) perturbation amplitude goes as 1/e’^5!!!!!



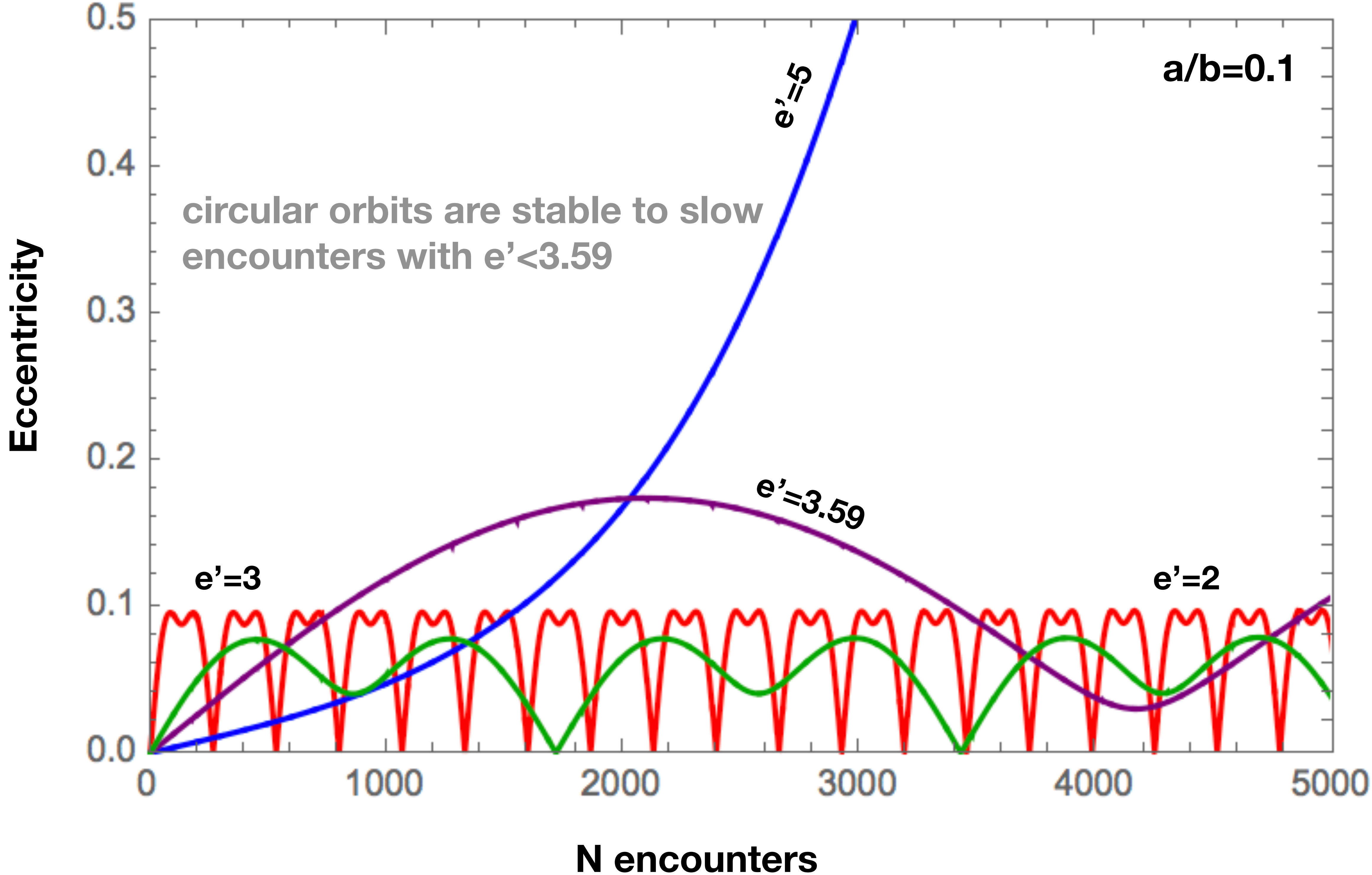
eccentricity
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
6
longitude of perihelion
analytical
analytical
N-body
N-body



e'=2
ecc
e'=3
e'=3.59
e'=5



1
0.100
0.010
0.001
10^{-4}
0.01
0.05
0.10
0.50
1
breakdown of
approximation
scheme
large error
small error



a/b=0.1
e’=5
circular orbits are stable to slow
encounters with e’<3.59
e’=3.59
e’=3
e’=2
Eccentricity
N encounters
0.0
0.1
0.2
0.3
0.4
0.5
0
1000
2000
3000
4000
5000