Quantum Loop Gas Approach to Topological States of Matter

M. Freedman, C.N., K. Shtengel, K. Walker, Z. Wang

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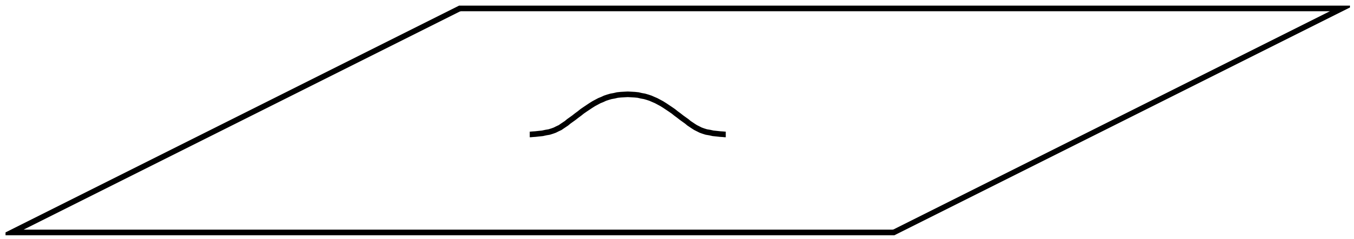
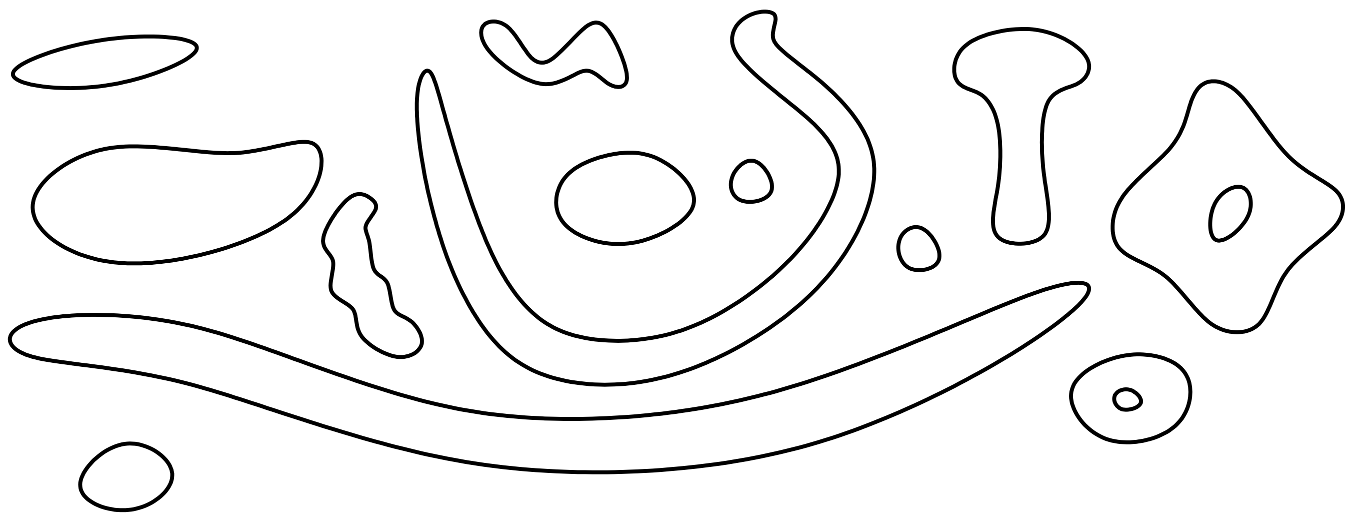
An elastic medium is a simple caricature describing many states of matter:



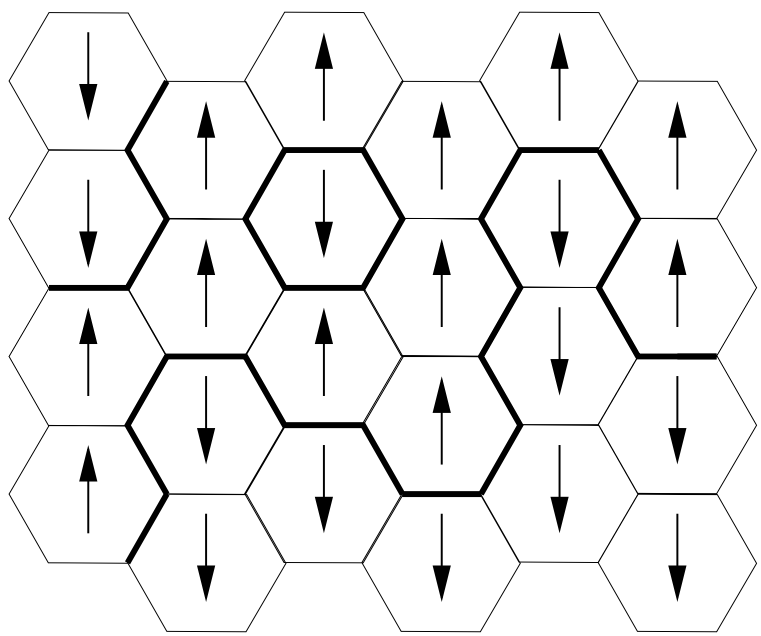
The ground state breaks a symmetry, and the low-lying excitations can be thought of as ripples in the medium which tend to restore the symmetry. They are gapless because in the long-wavelength limit, such a ripple is a symmetry operation.

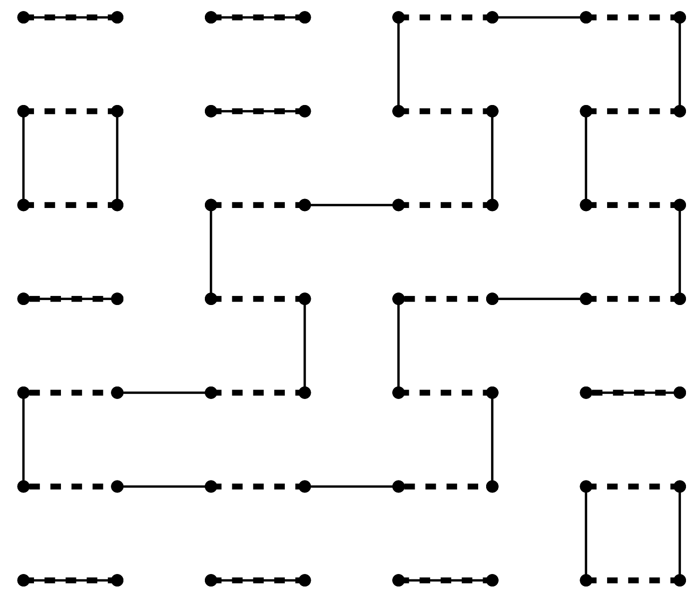
Quantum Loop Gases

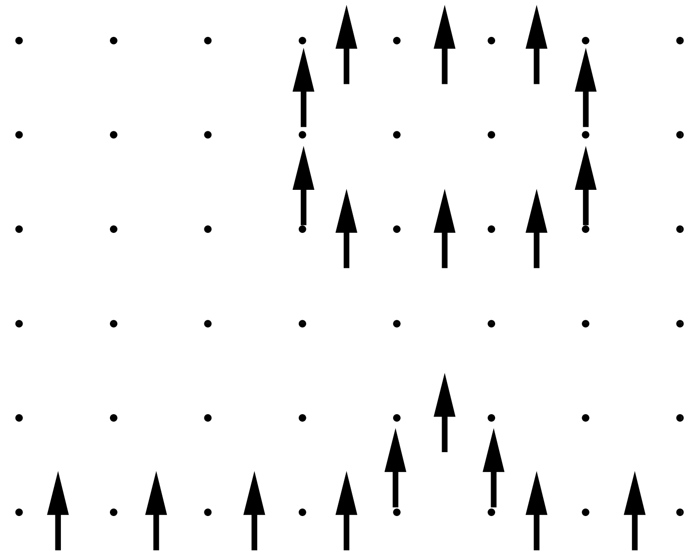
In this talk, I will describe states of matter for which the appropriate caricature is a sea of fluctuating loops.



- The loops may arise as domain walls, dimers, chains of up-spins or occupied sites, etc.







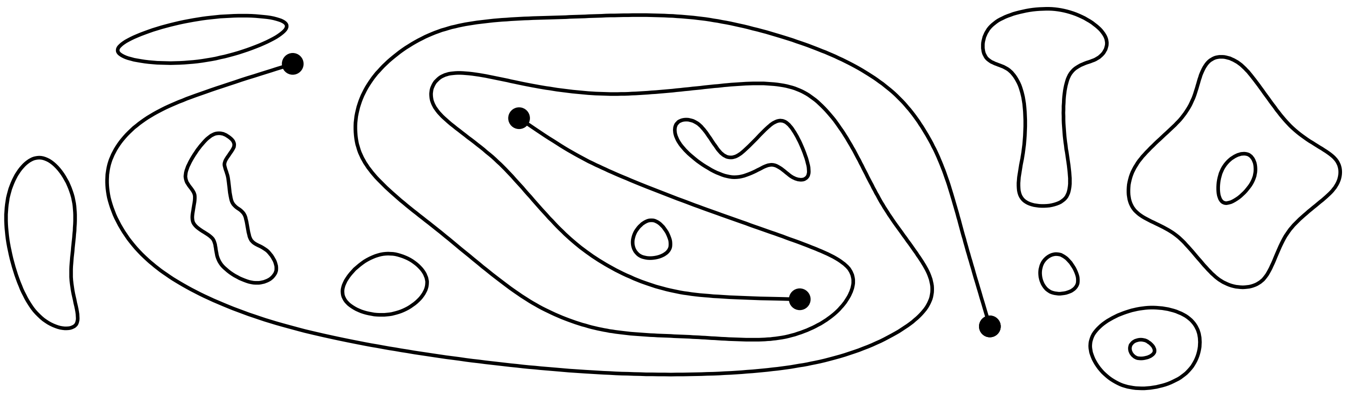
Kitaev:

$$H = -J_1 \sum_i A_v - J_2 \sum_p F_p$$

$$\begin{aligned} A_v &\equiv \Pi_{\alpha \in \mathcal{N}(v)} \sigma^z_\alpha \\ F_p &\equiv \Pi_{\alpha \in p} \sigma^x_\alpha \end{aligned}$$

- The loops obey a certain quantum dynamics; depending on the topological rules it imposes, the state may be a stable, gapped topological state or a gapless critical point.

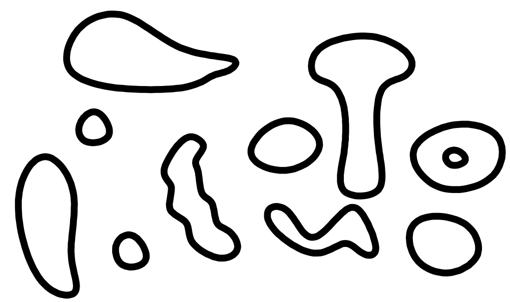
- Excitations are violations of these rules, e.g. broken loops:



- The rules obeyed by loops determine the braiding properties of the quasiparticles, ground state degeneracy, etc.

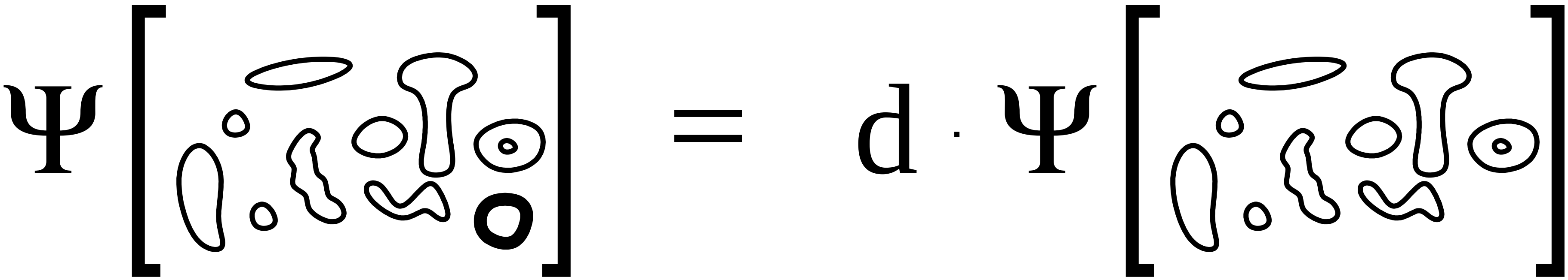
Basic Structure of a Class of Theories

- Wavefunctions $\Psi[\alpha]$ on multi-loops α which are invariant under smooth deformations of the loops.

$\Psi\Big[$  $\Big] = \Psi\Big[$  $\Big]$

We would expect this for any topological phase.

- A ‘fugacity’ d for small, contractible loops.

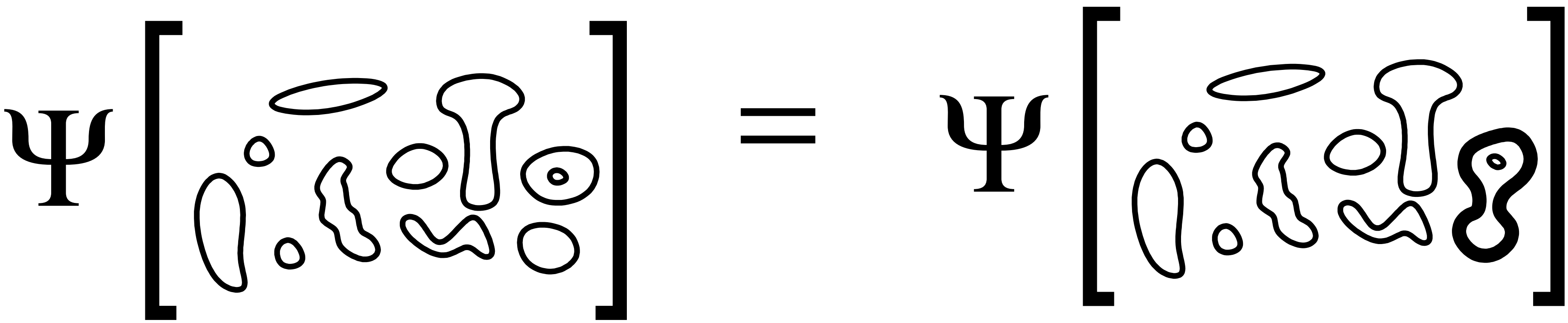


In Kitaev’s model, $d = 1$.

Without such a relation, the ground state would be degenerate even on the sphere.

- Invariance of the wavefunction under a 'surgery relation' which cuts and rejoins loops,

e.g.

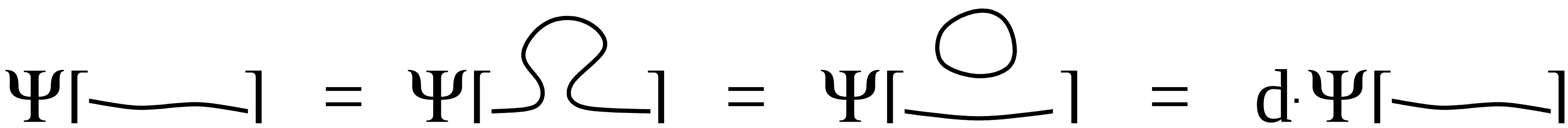


Without such a relation, the ground state would be infinitely degenerate on the torus.

By generalizing the latter two conditions, we will construct a family of topological states of matter, all of which can be described as quantum loop gases.

Consistency Conditions for Quantum Loop Gases

If $d \neq 1$, then the surgery relation must be modified or else there is a contradiction:



Hence, we must look at surgery relations involving $3, 4, \ldots$ curves.

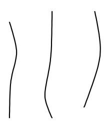
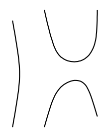
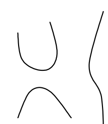
Important Mathematical Result: For almost all d, there is no consistent surgery relation.

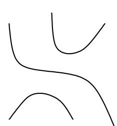
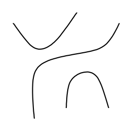
Consistent surgery relations can be found only for

$$d = 2\cos\left(\frac{\pi}{k+2}\right)$$

(Jones-Wenzl projectors)

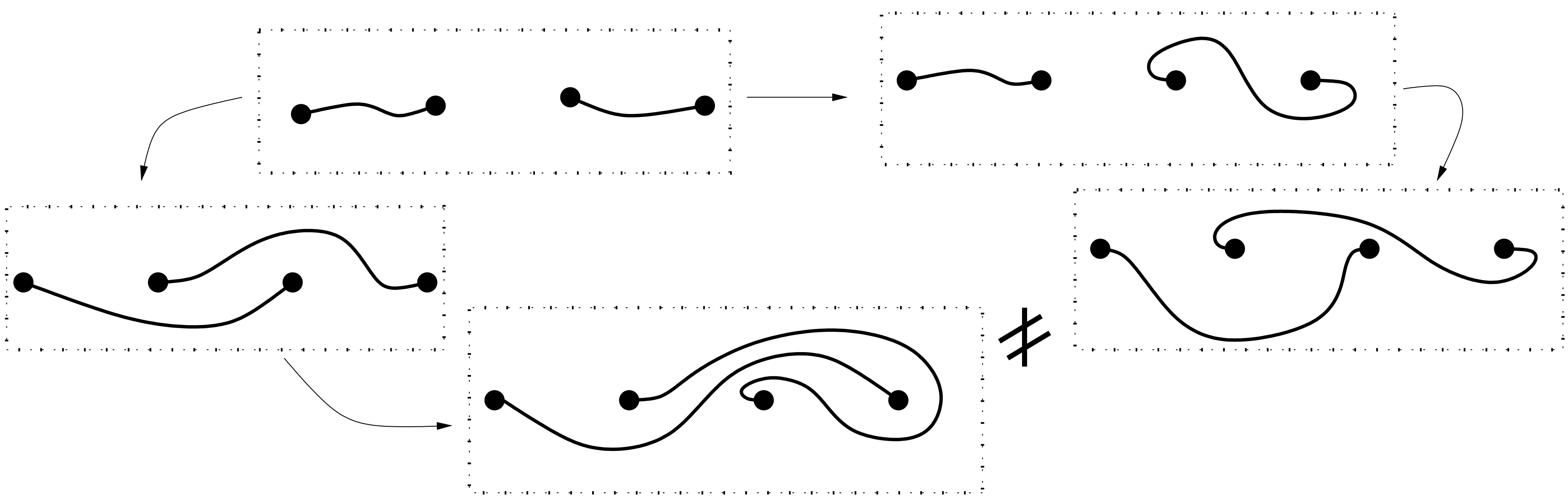
e.g. for $d = \sqrt{2}$,

$\Psi[$  $] - \sqrt{2}\,\Psi[$  $] - \sqrt{2}\,\Psi[$  $]$

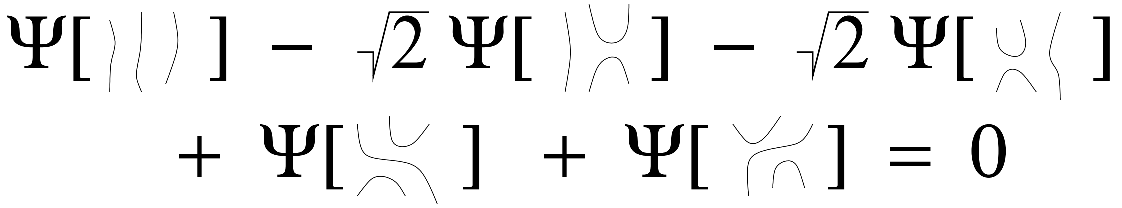
$+\ \Psi[$  $] + \Psi[$  $] = 0$

Surgery and Physical Properties

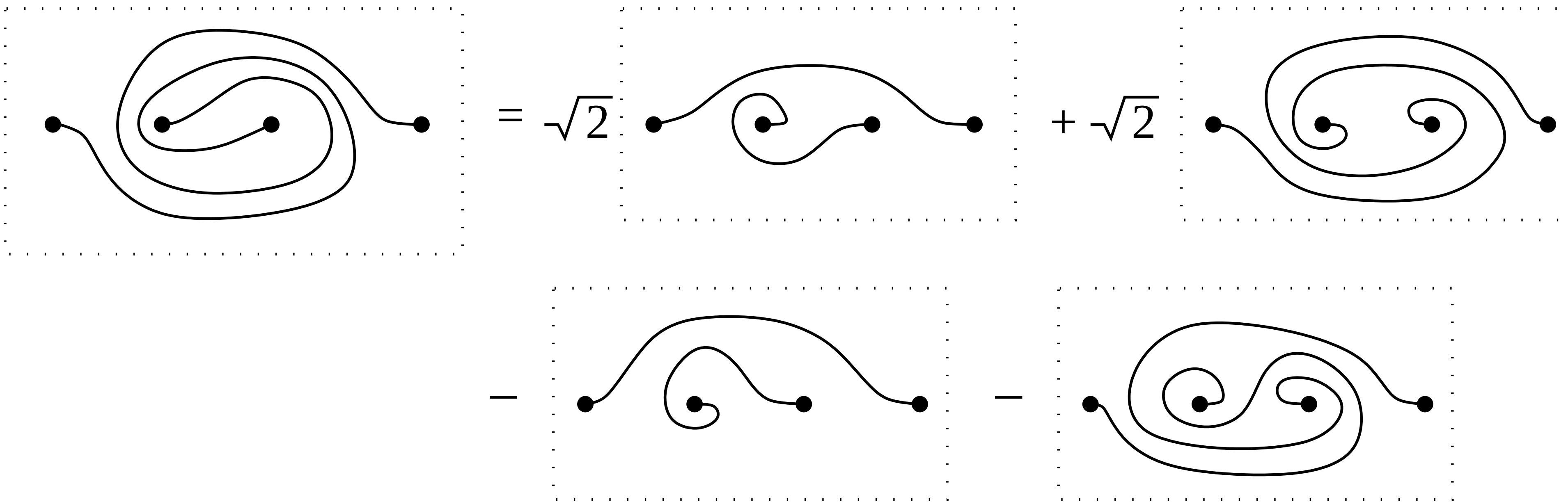
For a given k, the value d assigned to a contractible loop and the associated $k+1$-curve surgery relation defines a **topological state**.



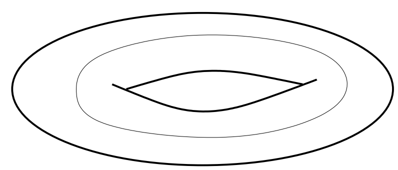
e.g. for k=2,



$\Rightarrow$



9 Ground States on T^2:



Field Theoretic Description

- The associated field theories are gauge theories.
- Braiding statistics from the generalized Aharonov-Bohm effect
- Wilson loop operators act in a simple ‘pictorial’ manner on the argument of wavefunctions.
- Unoriented loops are a feature of SU(2).

‘Doubled’ $SU(2)_k$ Chern-Simons theories.

$$S_{\mathrm{CS}} \;=\; \frac{k}{4\pi}\int \mathrm{tr}\left(a \wedge da + \frac{2}{3} a \wedge a \wedge a\right)$$

Important gauge-invariant operators:

$$W[\gamma] \equiv \mathrm{tr}\left(\mathcal{P} e^{i \oint_\gamma \mathbf{a}^c T^c \cdot d\mathbf{l}}\right)$$

Their commutator algebra:

$$[W[\gamma], W[\gamma']] = 2 \sin\left(\frac{\pi}{2(k+2)}\right) \sum_i \left(W[\gamma \circ_i \gamma'] - W[\gamma' \circ_i \gamma]\right)$$

Algebra of Wilson Loops

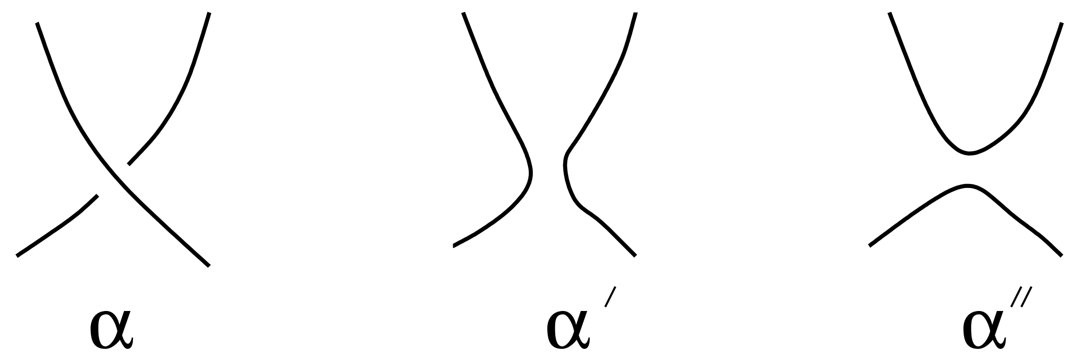
Can be represented on isotopy, d, surgery-invariant $\Psi[\alpha]$ if:

$$W_+[\gamma]\,\Psi[\beta] = \Psi[\beta{\star}\gamma]$$

where $\alpha \star \gamma = \alpha \cup \gamma$ with intersections resolved by:

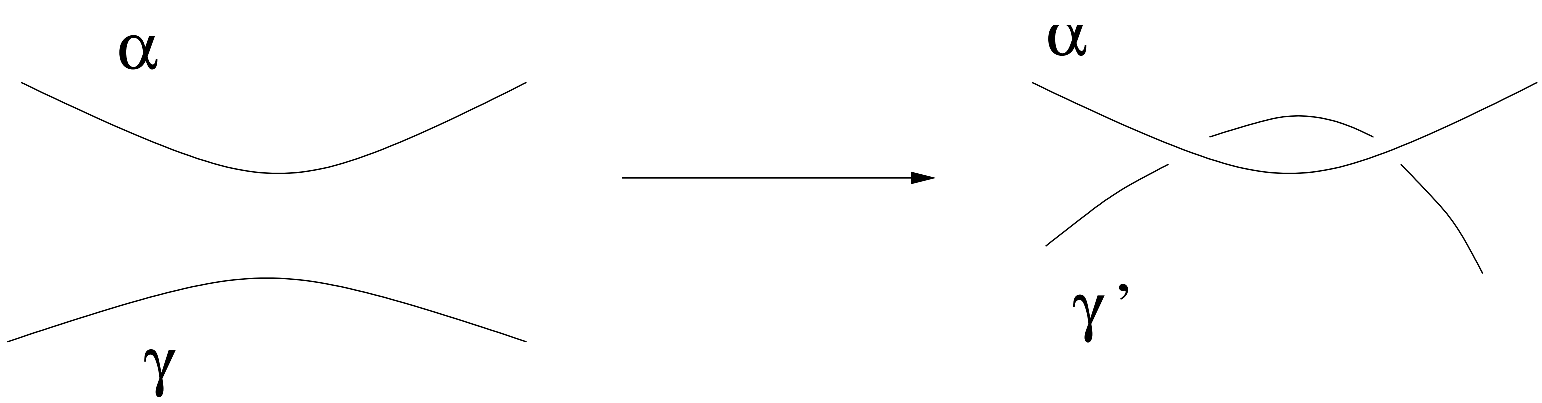
$$\Psi[\alpha] = A\Psi[\alpha'] + A^{-1}\Psi[\alpha'']$$

$$A = i\exp\left(\pi i/2(k+2)\right)$$



This guarantees that the desired commutation relations are obeyed. It also fixes d.

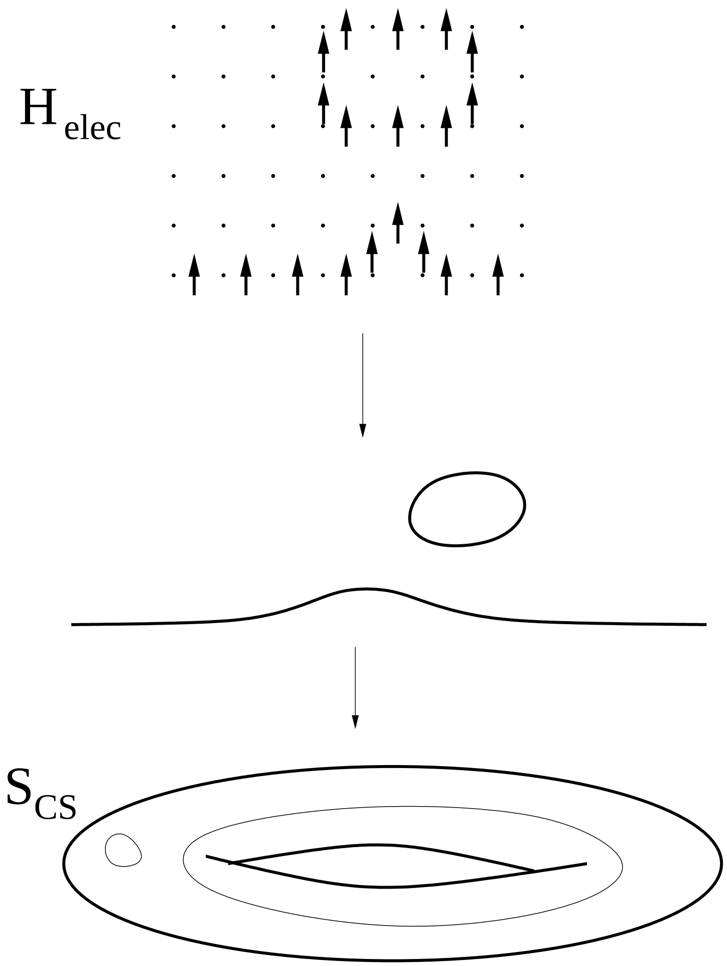
Suppose we deform γ into γ' which has two new intersections with α,



Using the resolution of crossings, we see that $\Psi[\alpha\star\gamma] = \Psi[\alpha\star\gamma']$ iff

$$d = -A^2 - A^{-2} = 2\cos\left(\frac{\pi}{k+2}\right)$$

Descriptions at Different Scales



Short scales: electrons/spins at points (0-D)

Intermed. scales: fluctuating curves/loops (1-D)

Long scales: degenerate ground states on genus-g surfaces (2-D)

Intermediate Scales $\sim$ Nearby Critical Point

Intermed. length scale physics: **‘d-isotopy’**.

Long-wavelength physics: **Jones-Wenzl surgery relations** restrict winding numbers and det. the energy gap.

A nearby critical point might determine intermediate length scale physics.

d-isotopy Hamiltonians

Spins on the links of the honeycomb lattice:

$$H_{\mathrm{d-iso}} = \sum_v \left(1 + \prod_{i\in\mathcal{N}(v)} \sigma_i^z\right) + \sum_p \Bigg[\frac{1}{d^2}\left(F_p^0\right)^\dagger F_p^0 + \left(F_p^0\right)^\dagger F_p^0 - \frac{1}{d}F_p^0 - \frac{1}{d}\left(F_p^0\right)^\dagger$$

$$+ \left(F_p^1\right)^\dagger F_p^1 + \left(F_p^1\right)^\dagger F_p^1 - F_p^1 - \left(F_p^1\right)^\dagger \left(F_p^2\right)^\dagger F_p^2 + \left(F_p^2\right)^\dagger F_p^2 - F_p^2 - \left(F_p^2\right)^\dagger$$

$$\left(F_p^3\right)^\dagger F_p^3 + \left(F_p^3\right)^\dagger F_p^3 - F_p^3 - \left(F_p^3\right)^\dagger\Bigg]$$

$$F_p^0 = \sigma_1^-\sigma_2^-\sigma_3^-\sigma_4^-\sigma_5^-\sigma_6^-\ , \qquad F_p^1 = \sigma_1^+\sigma_2^-\sigma_3^-\sigma_4^-\sigma_5^-\sigma_6^- + \text{cyclic perm.}$$

$$F_p^2 = \sigma_1^+\sigma_2^+\sigma_3^-\sigma_4^-\sigma_5^-\sigma_6^- + \text{c. p.}\ , \qquad F_p^3 = \sigma_1^+\sigma_2^+\sigma_3^+\sigma_4^-\sigma_5^-\sigma_6^- + \text{c. p.}$$

Up-spins again form closed loops which satisfy d-isotopy, but without surgery.

Ground State and Stat. Mech. Analogy

$$|\Psi_0\rangle = \sum_\alpha d^{n_\alpha} |\alpha\rangle$$

Can be interpreted as a Loop Gas of fugacity d^2:

$$\boxed{\sum_\alpha |\Psi[\alpha]|^2 = Z_{\mathrm{O(n)}}(x = n)} \qquad \text{where} \qquad n = d^2$$

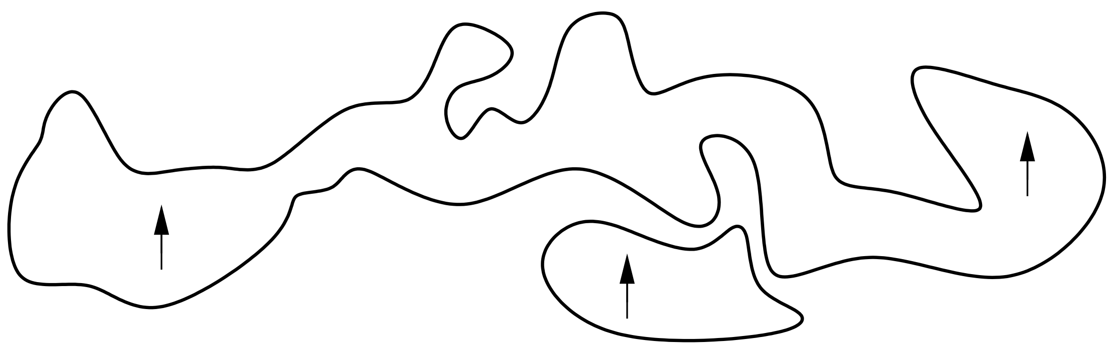
$$Z_{\mathrm{O(n)}}(x) = \int \prod_i d\hat{S}_i \prod_{\langle i,j\rangle} (1 + x\hat{S}_i \cdot \hat{S}_j) = \sum_\alpha \left(\frac{x}{n}\right)^{\ell_\alpha} n^{n_\alpha}$$

Ground State Properties

For $x = n$, the $O(n < 2)$ loop model is in its critical low-temperature phase.

- Loops meander over long distances with exponents $\eta_k = \frac{g}{4}k^2 - \frac{1}{g}(1-g)^2$ where $n = -2\cos(\pi g)$
- The $x \to \infty$ limit is the FK rep. of the *critical* $q = n^2$ state Potts model, which has the same exponents as the *low-temp.* O(n) model.

- The ground state of $H_{\mathrm{d-iso}}$ contains long loops characterized by exponents η_k for $d \leq \sqrt{2}$, which arise in correlators of **non-local operators** – *referring to the same loop*.



- However, correlation functions of local ops. $\vec{\sigma}$ are **short-ranged**.

- **A ‘Quasi-Topological Phase’.**

Low-Energy Excitations

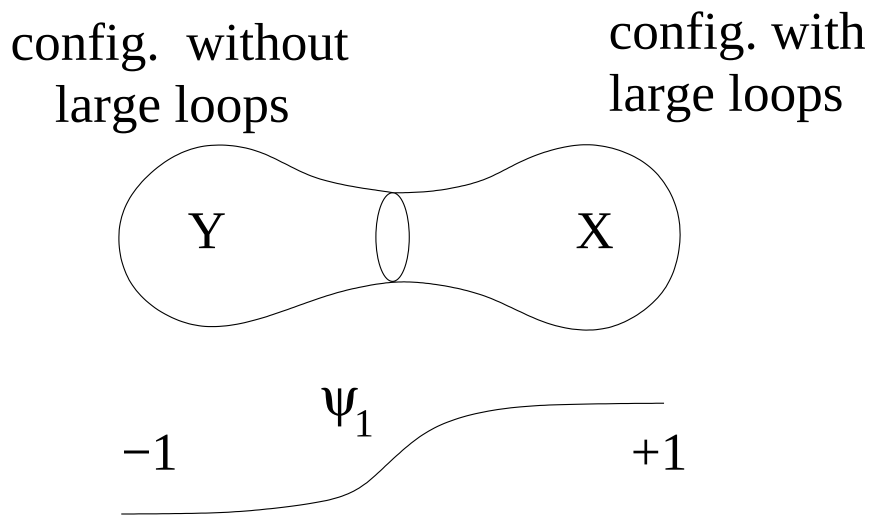
Trial wavefunction:

$$|\Psi_1\rangle = \sum_{\alpha\in X} d^{n_\alpha} |\alpha\rangle - \sum_{\alpha\in Y} d^{n_\alpha} |\alpha\rangle$$

$X =$ configs. with long loops; $Y =$ without.

Since the O(n) model is critical for $n \le 2$, we can define ‘long’ so that the prob. of a config. with a long loop is $1/2$. Then $\langle\Psi_1|\Psi_1\rangle = 0$.

$\langle\Psi_1|H_{\mathrm{d-iso}}|\Psi_1\rangle = 0$ because the two sectors of configuration space are not directly connected by the Hamiltonian, i.e. there is a bottleneck.



This is a critical line parametrized by $d \leq \sqrt{2}$.

Low-Energy Field Theory

An effective field theory would help us address stability, dynamics, etc.

- $\omega \sim k^2$
- SU(2) gauge theory
- Local operators equal-time correlations are short-ranged, but non-local operators have power-laws $\eta_k = \frac{g}{4}k^2 - \frac{1}{g}(1-g)^2$.

The first two requirements motivate the guess:

$$S = \frac{1}{g^2} \int d^2x \, d\tau \left(E_i^a \partial_\tau A_i^a + A_0^a D_i E_i^a + \frac{1}{2} E_i^a D^2 E_i^a + \frac{1}{2} B^a B^a \right)$$

But is this interacting theory actually critical?

The first two requirements motivate the guess:

$$S = \frac{1}{g^2} \int d^2x \, d\tau \left(E_i^a \partial_\tau A_i^a + A_0^a D_i E_i^a + \frac{1}{2} E_i^a D^2 E_i^a + \frac{1}{2} B^a B^a \right)$$

But is this interacting theory actually critical?

At one-loop,

$$\frac{dg}{d\ell} = 0$$

This theory is also on a critical line.

For g small, the perturbations $\lambda_1(E_i^a E_i^a)^2 + \lambda_2\big(E_i^a E_j^a\big)^2$ have runaway flows. This presumably corresponds to $d \geq 2$. *The classical limit is massive, as in the q-state Potts/O(n) models*.

If the conjecture is correct, then for g sufficiently large, these become irrelevant, and in this regime we expect

$$\langle W[\gamma]\rangle = d$$

$$\left\langle \mathrm{tr}\left(E_i(x)\,\mathcal{P}e^{i\int_0^x a}\,E_j(0)\,\mathcal{P}e^{i\int_x^0 a}\right)\right\rangle \sim \frac{1}{|x|^{\eta_2}}\,\delta_{ij}$$

Future Directions

With the pictorial-combinatorial description of topological phases in hand, there are many open questions which one hopes to address.

- Electrons, topology, statistical mechanics, gauge theories ... computer science.
- The pictorial representation motivates certain types of microscopic models.

Perturbing away from soluble models, towards more realistic ones.

- Imposing the Jones-Wenzl relations.
- *Stability of d-isotopy critical line.*

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