

Few-body hadronic resonances in lattice QCD



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Outline

First half:

- Elastic two-body scattering

Second half:

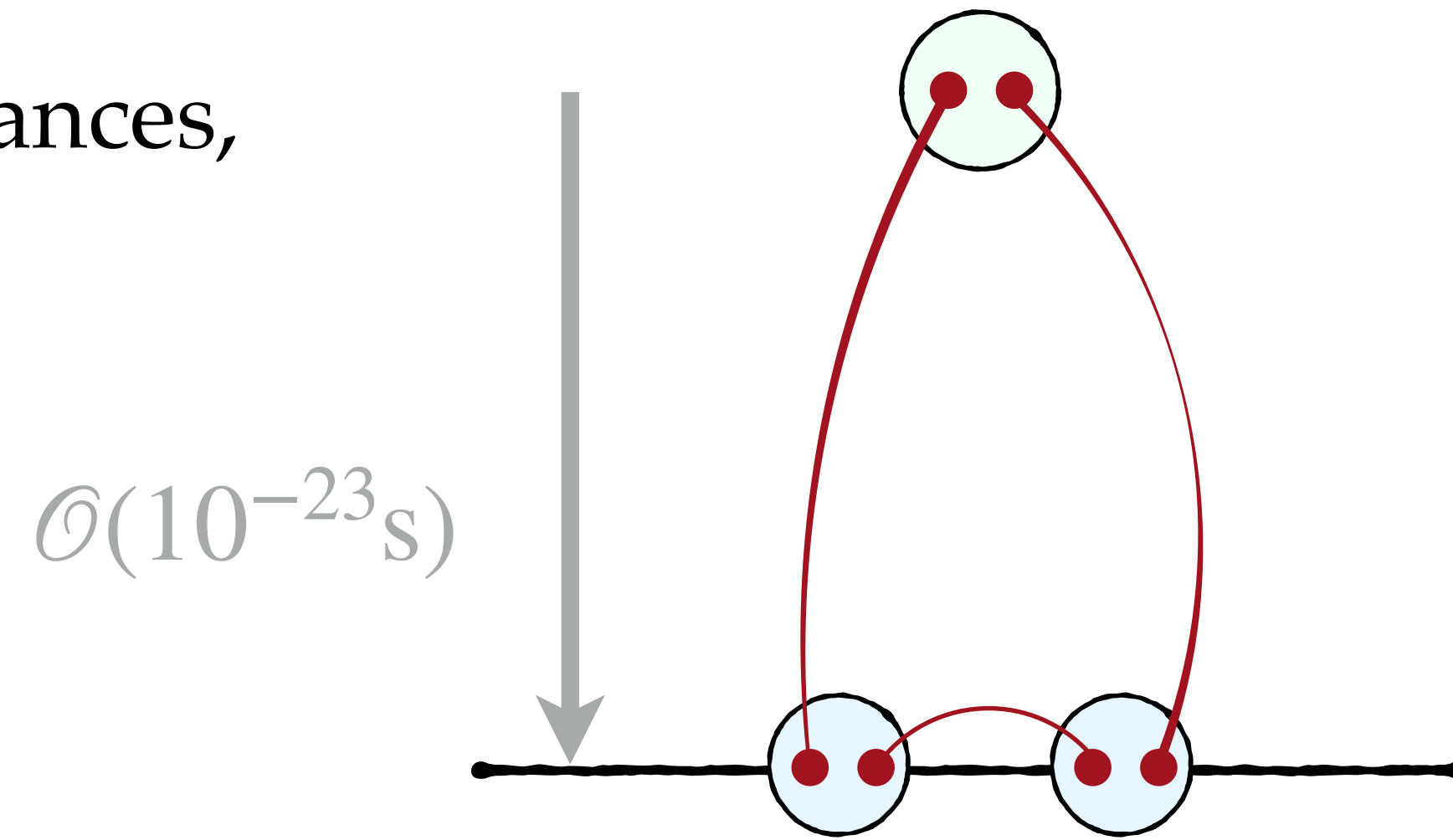
- Two-body, coupled-channel scattering

- Three-body scattering

few-body physics in QCD

Big picture:

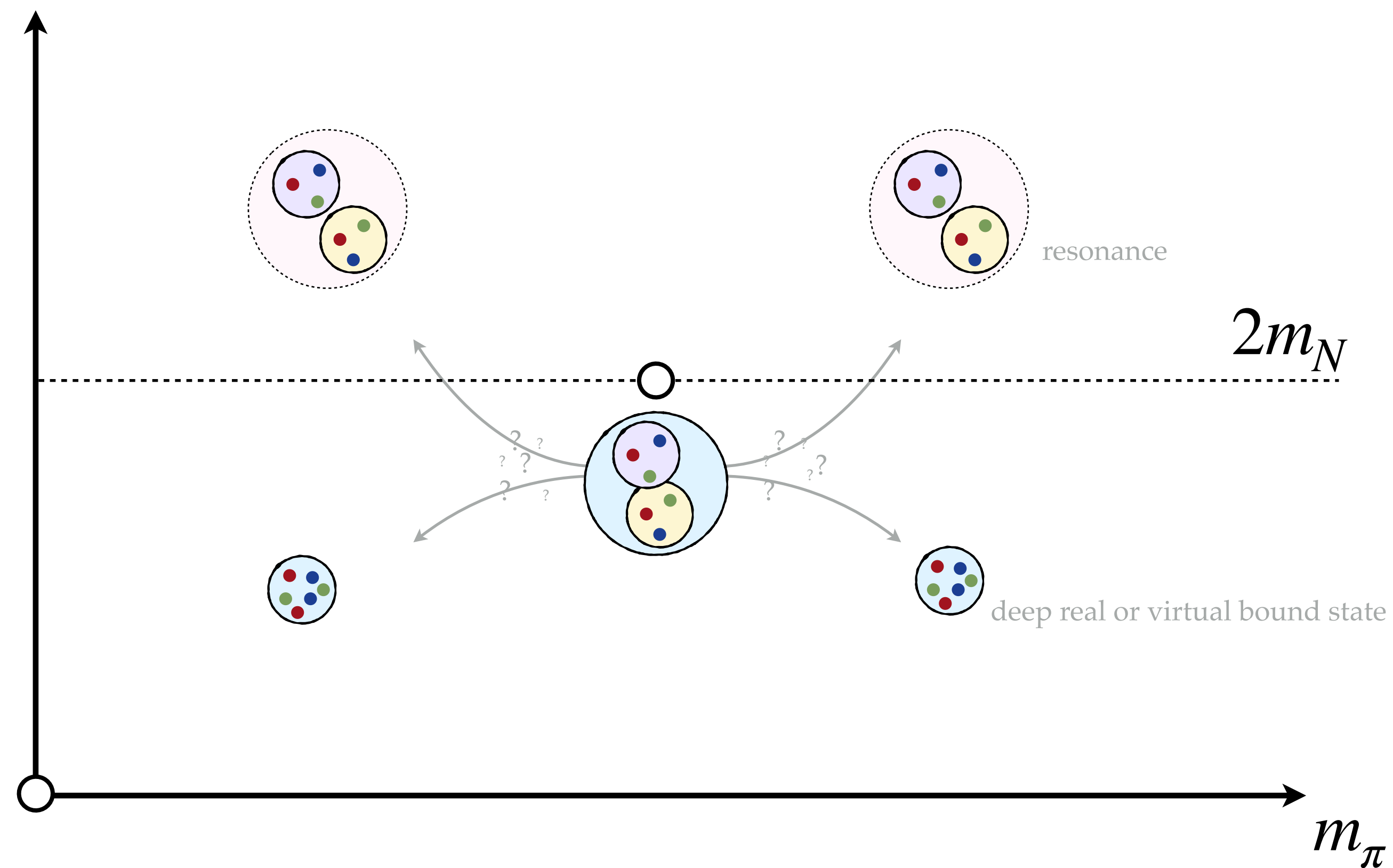
most states in QCD are unstable resonances,



few-body physics in QCD

Big picture:

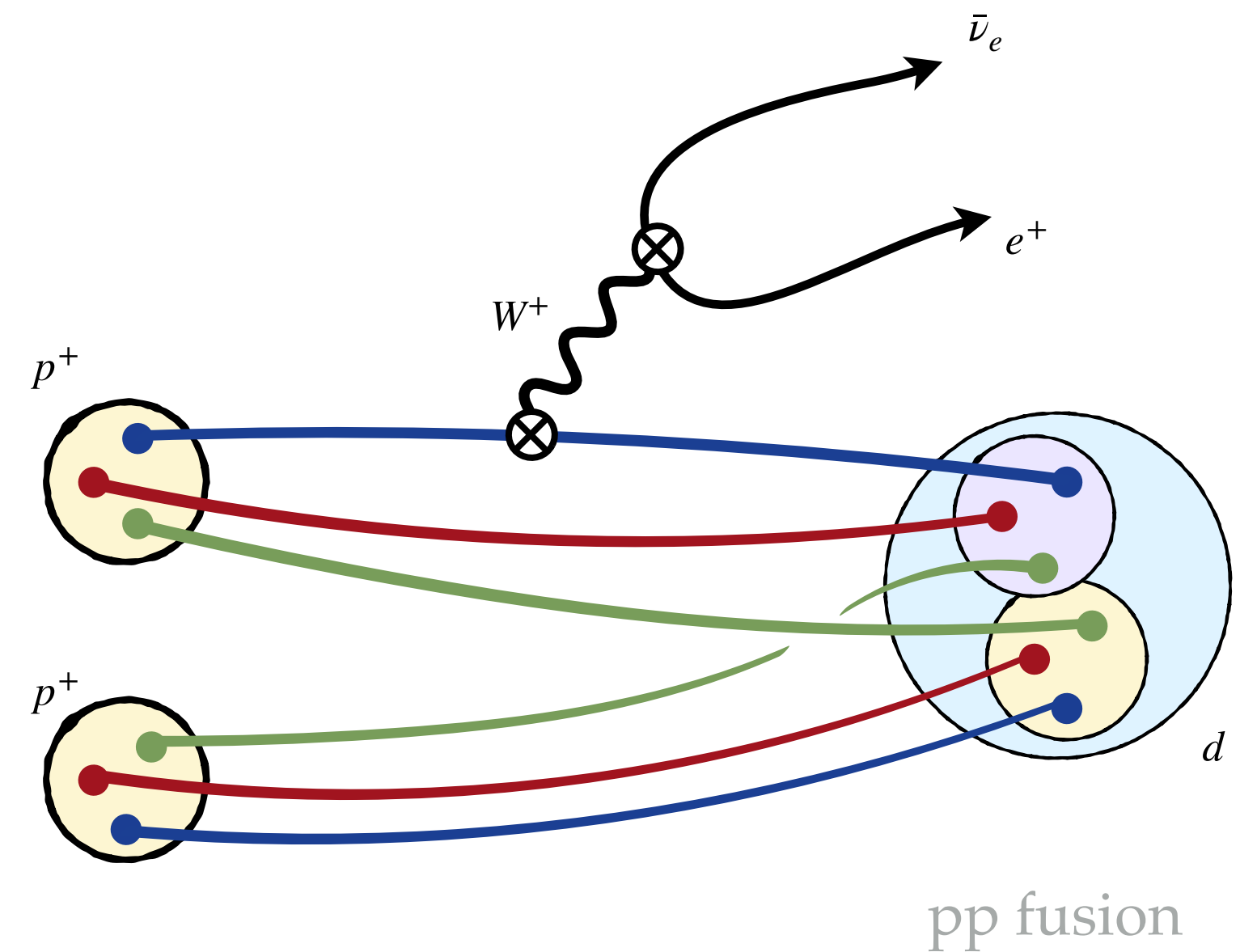
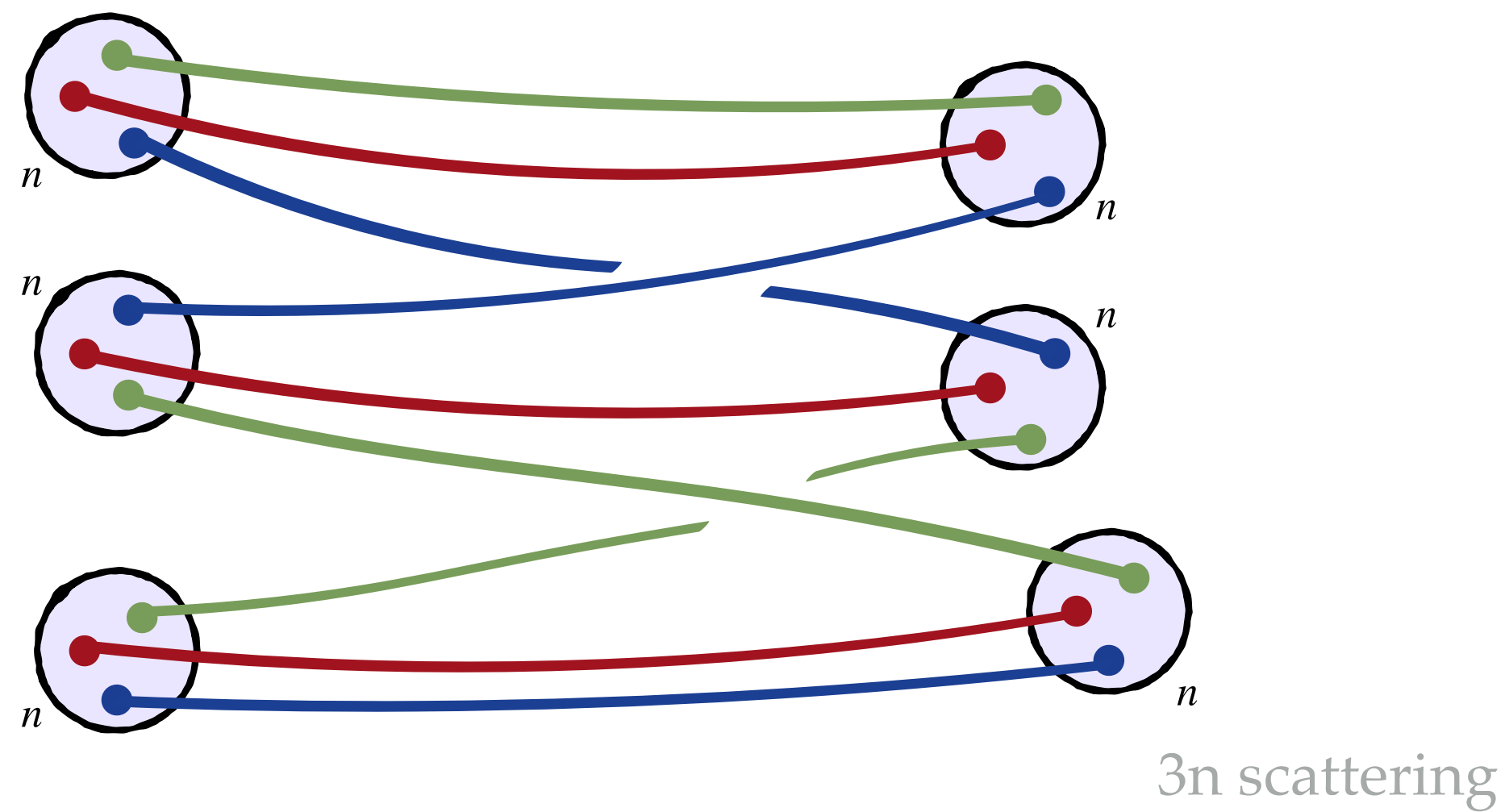
- ☑ most states in QCD are unstable resonances,
- ☑ nuclear physics dependence on nature's fundamental parameters,



few-body physics in QCD

Big picture:

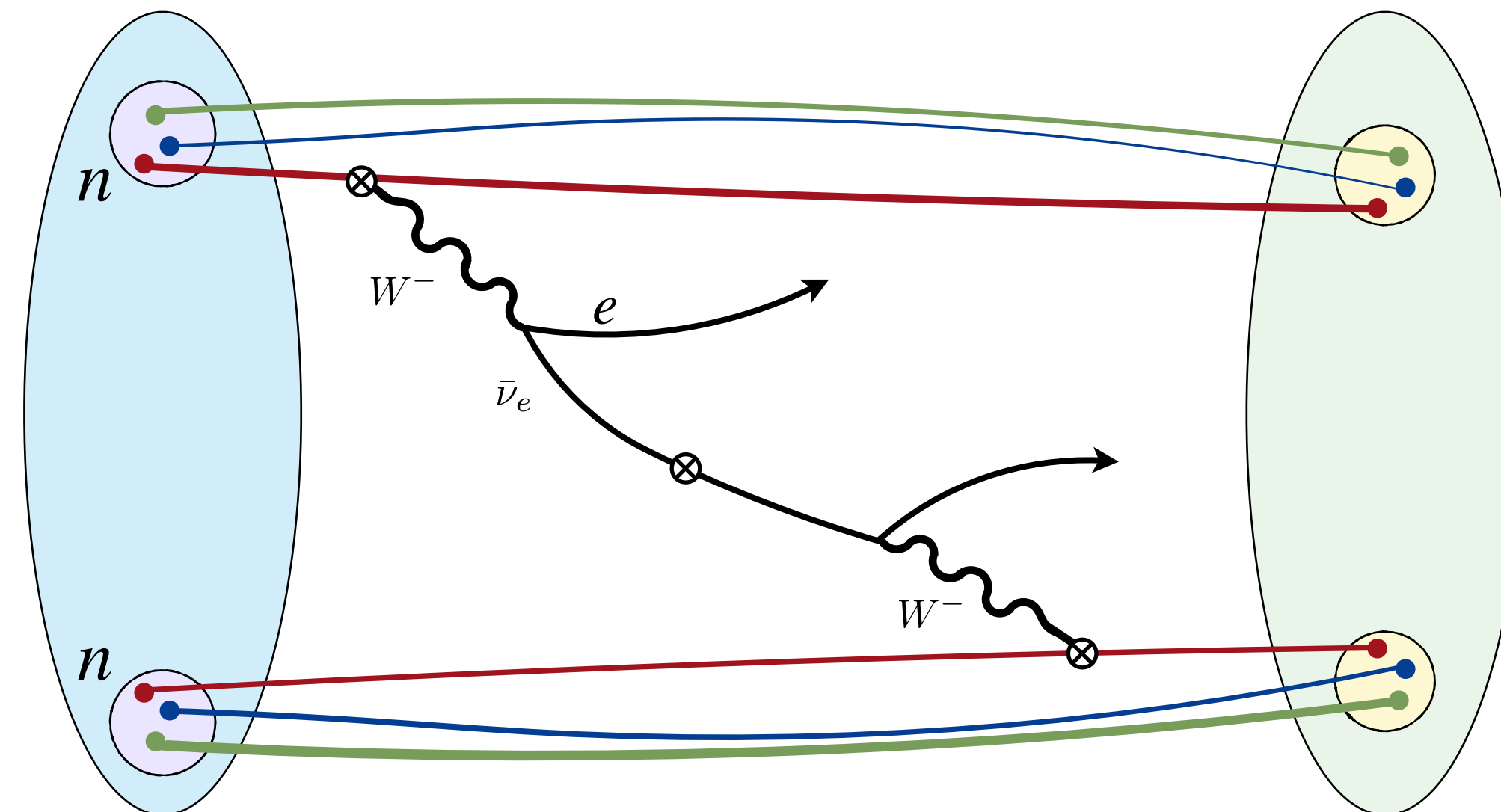
- ☑ most states in QCD are unstable resonances,
- ☑ nuclear physics dependence on nature's fundamental parameters,
- ☑ determining experimentally inaccessible quantities,



few-body physics in QCD

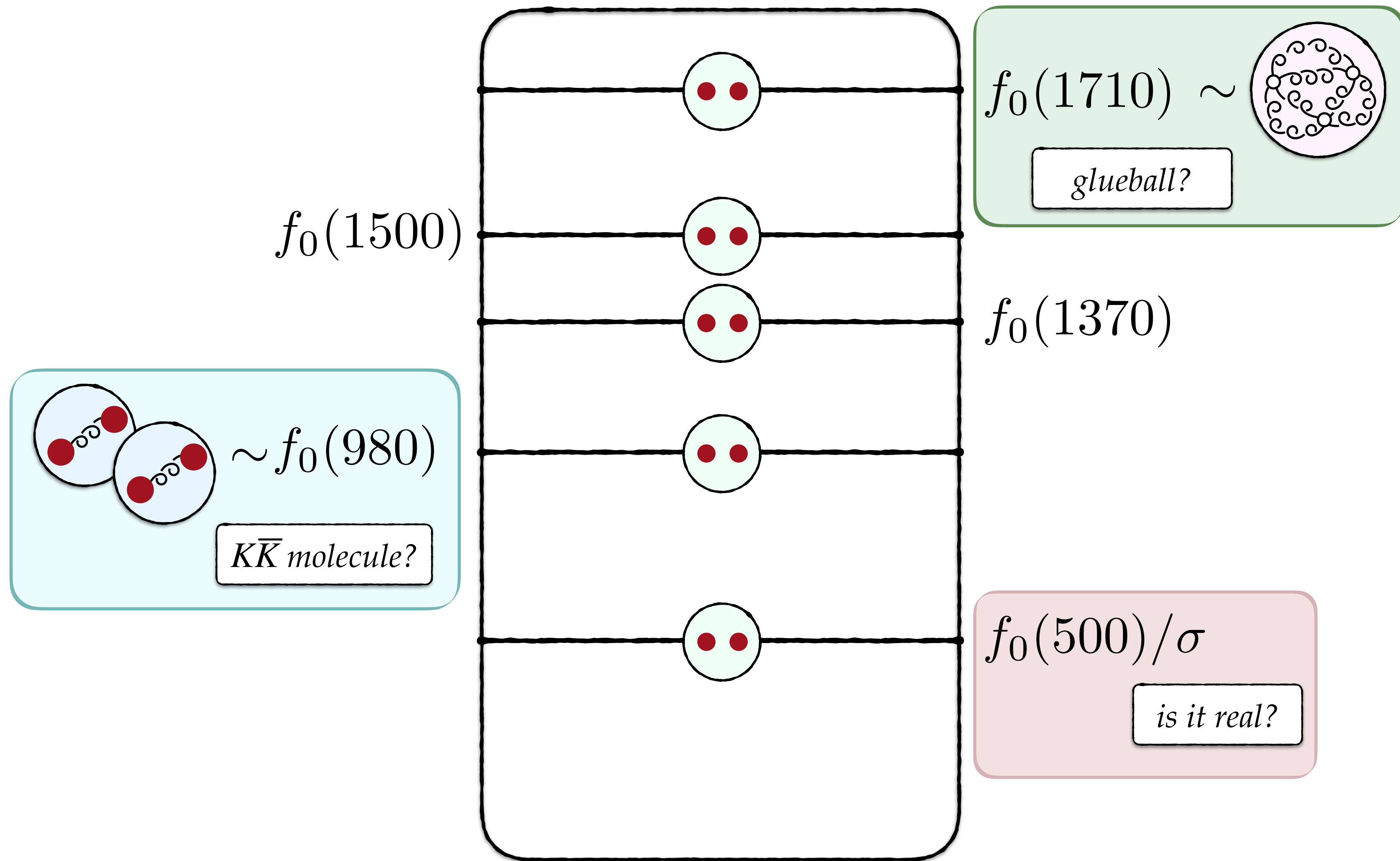
Big picture:

- ☑ most states in QCD are unstable resonances,
- ☑ nuclear physics dependence on nature's fundamental parameters,
- ☑ determining experimentally inaccessible quantities,
- ☑ determining QCD contributions to BSM physics,
- ☑ ...

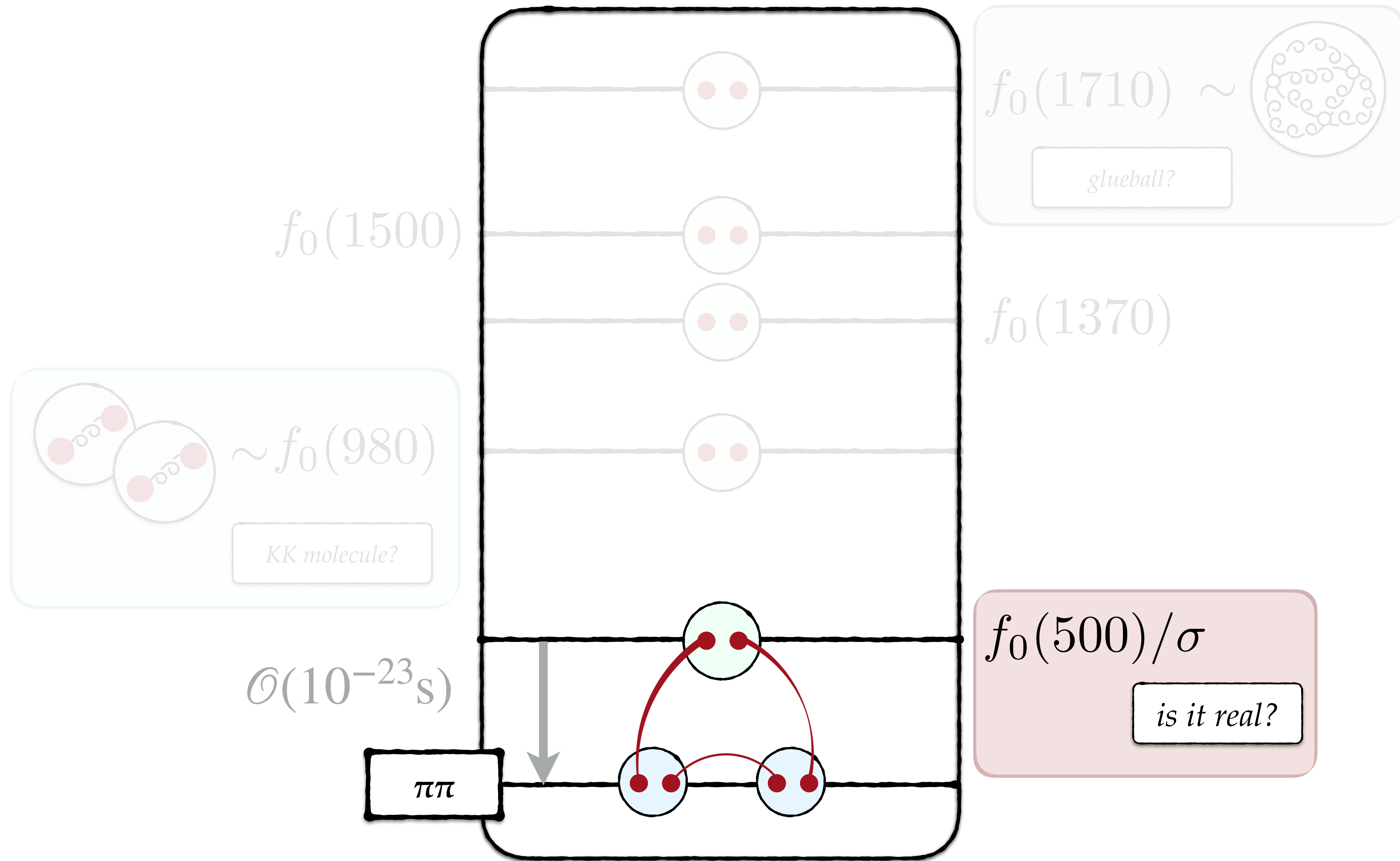


neutrinoless double-beta decay

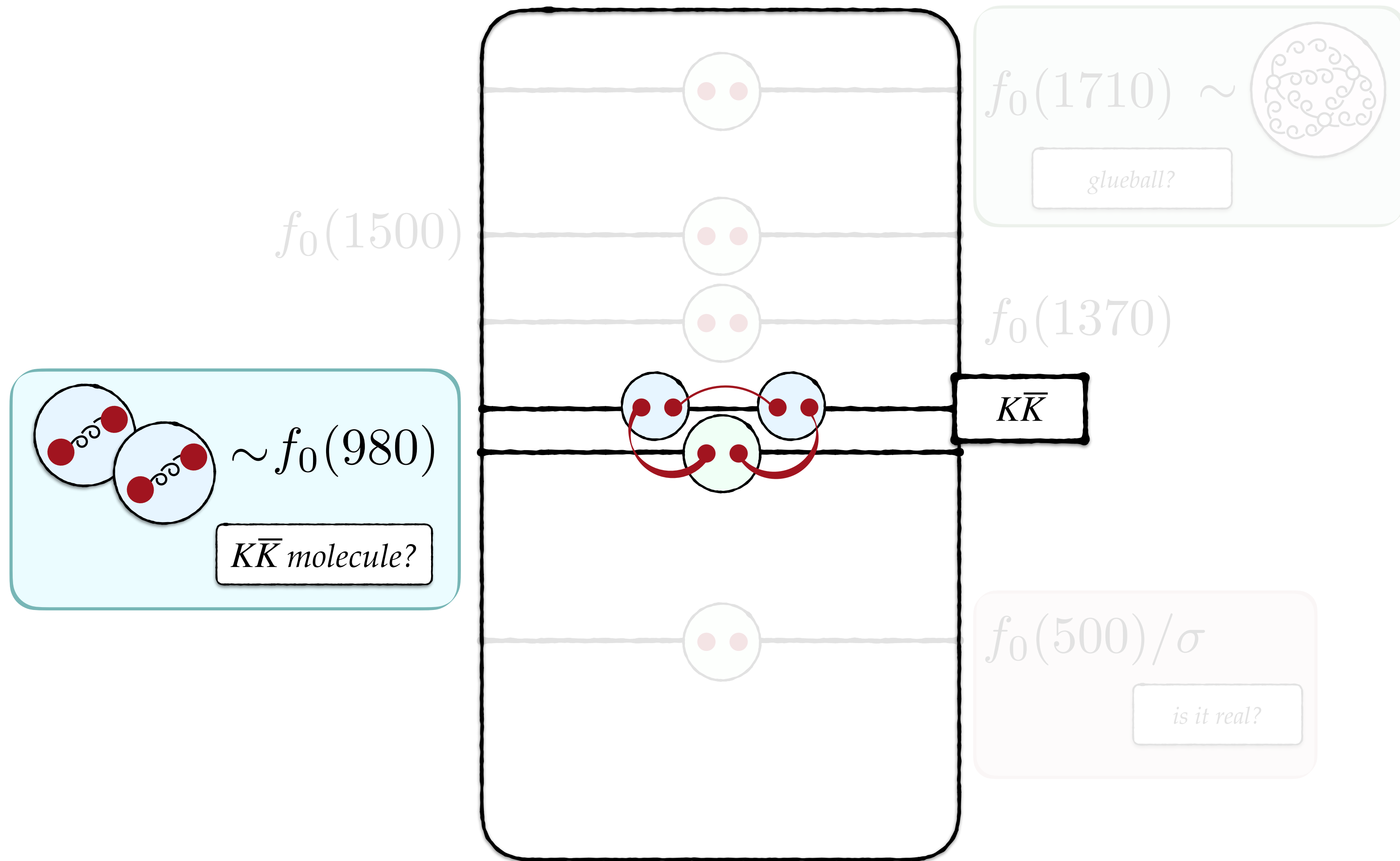
The vacuum channel



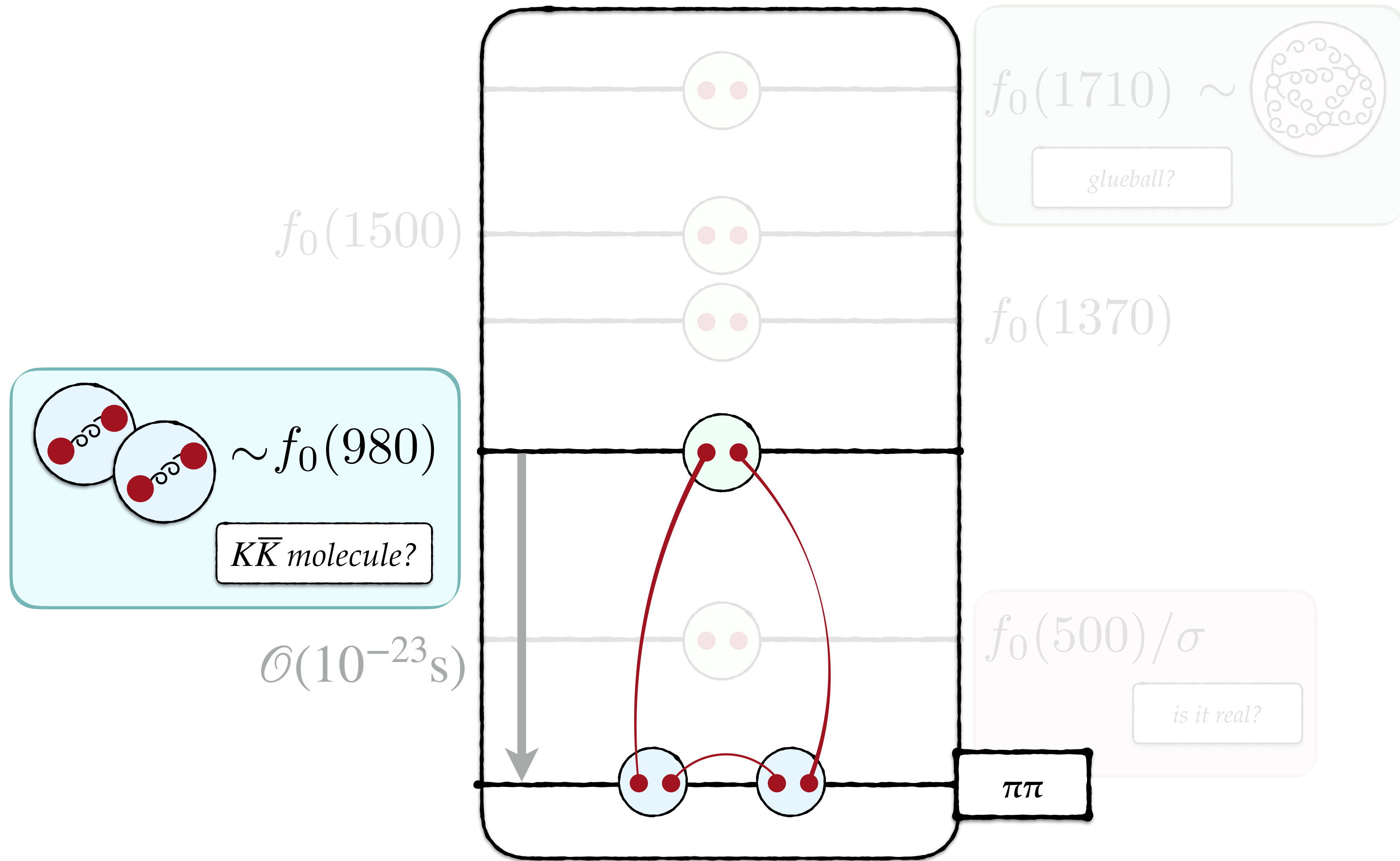
The vacuum channel



The vacuum channel



The vacuum channel

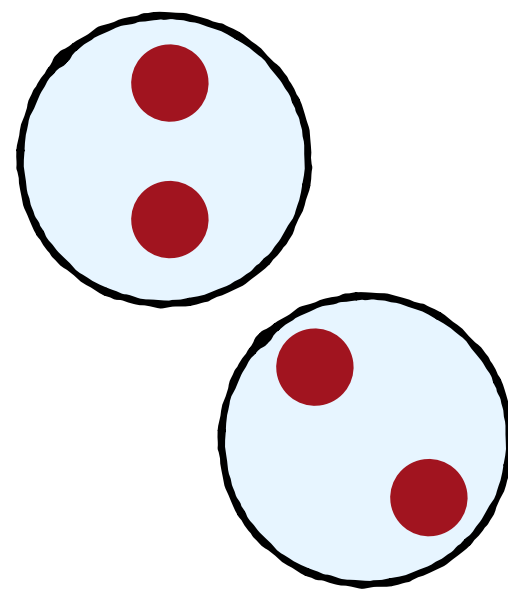


Lattice QCD in a nut shell

Non-perturbative tool to study non-perturbative QCD phenomena:

- ☑ broad and narrow resonances,
- ☑ strongly coupled systems,
- ☑ structure of few-body systems,
- ☑ ...

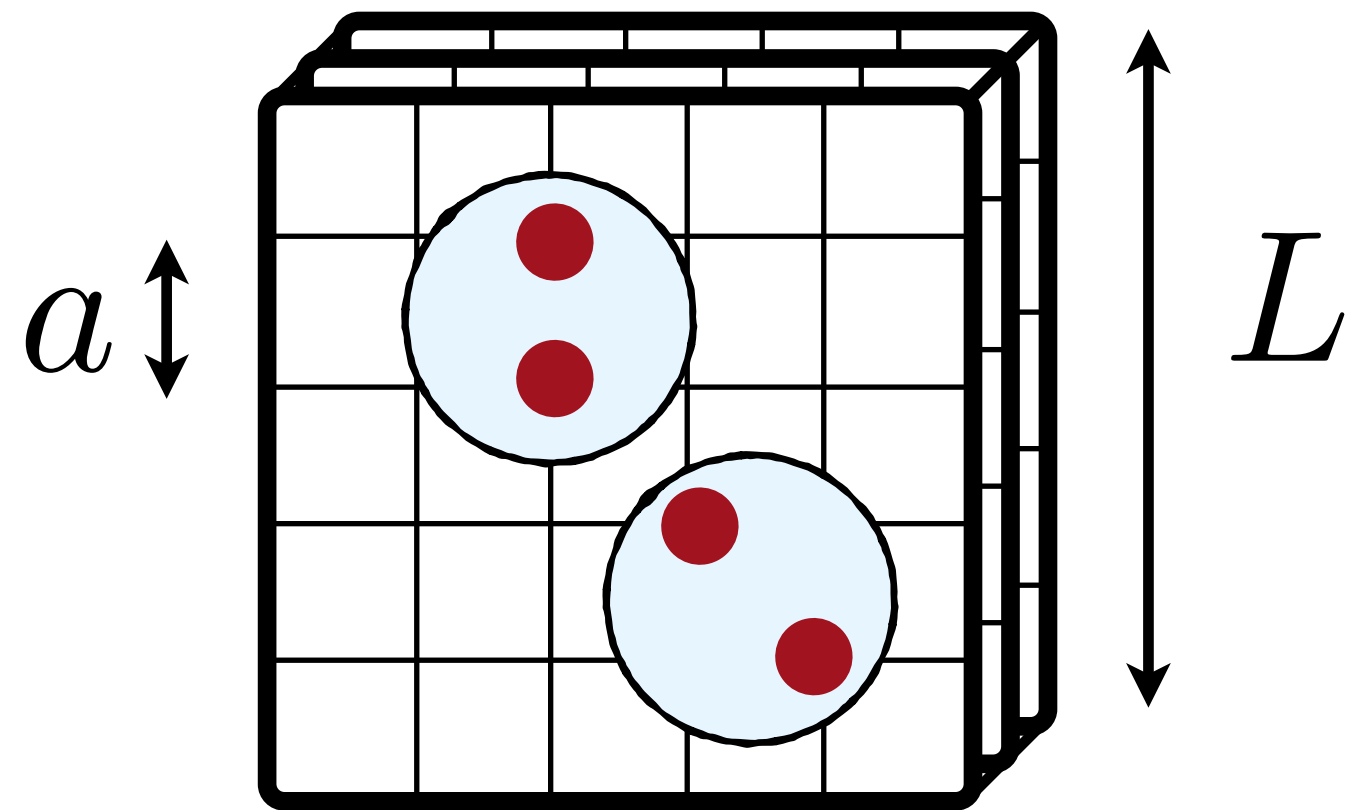
Lattice QCD in a nutshell



let's say you want to study
how two mesons interact...

Lattice QCD in a nut shell

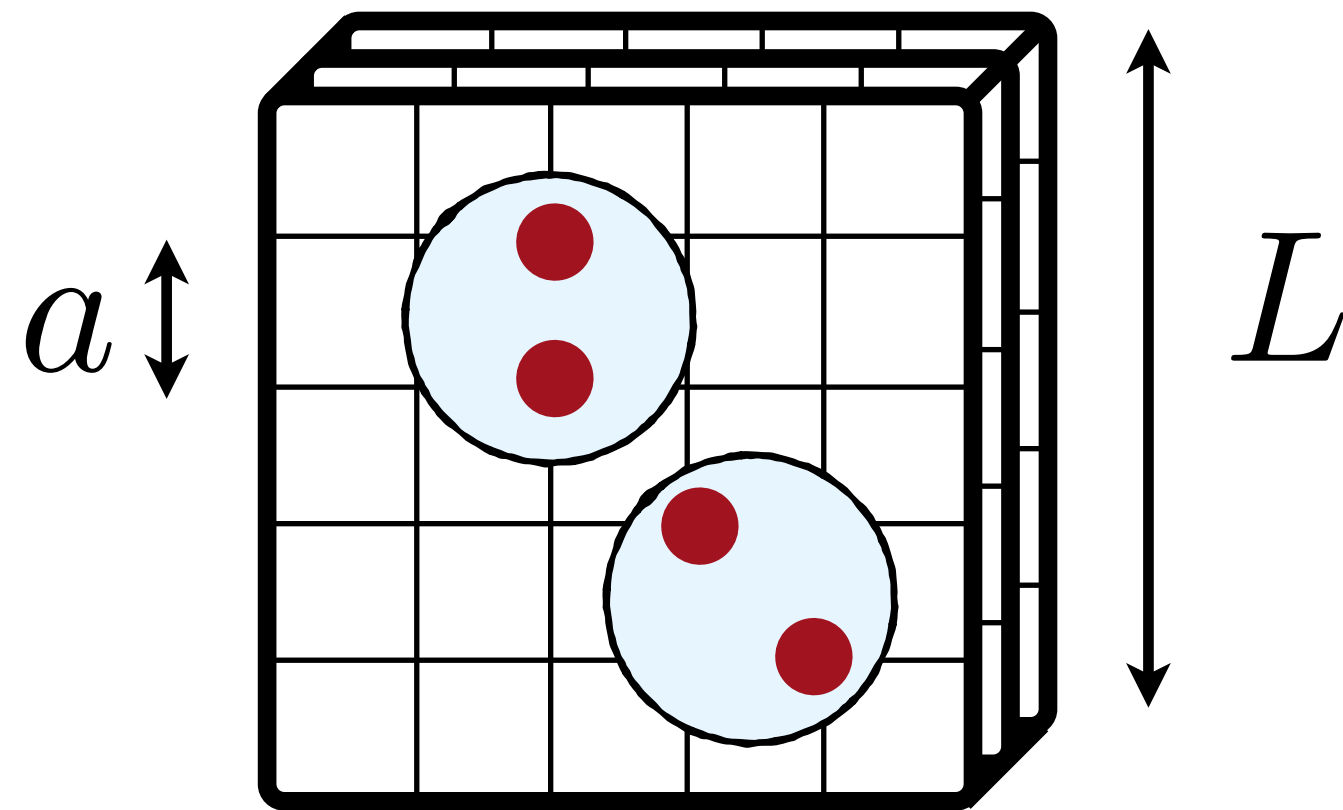
- lattice spacing: $a \sim 0.03 - 0.12$ fm
- finite volume: $L \sim 6 - 12$ fm



$$D_{\mu} = \left(\quad \right) \updownarrow (L/a)^3 \times (T/a)$$

Lattice QCD in a nut shell

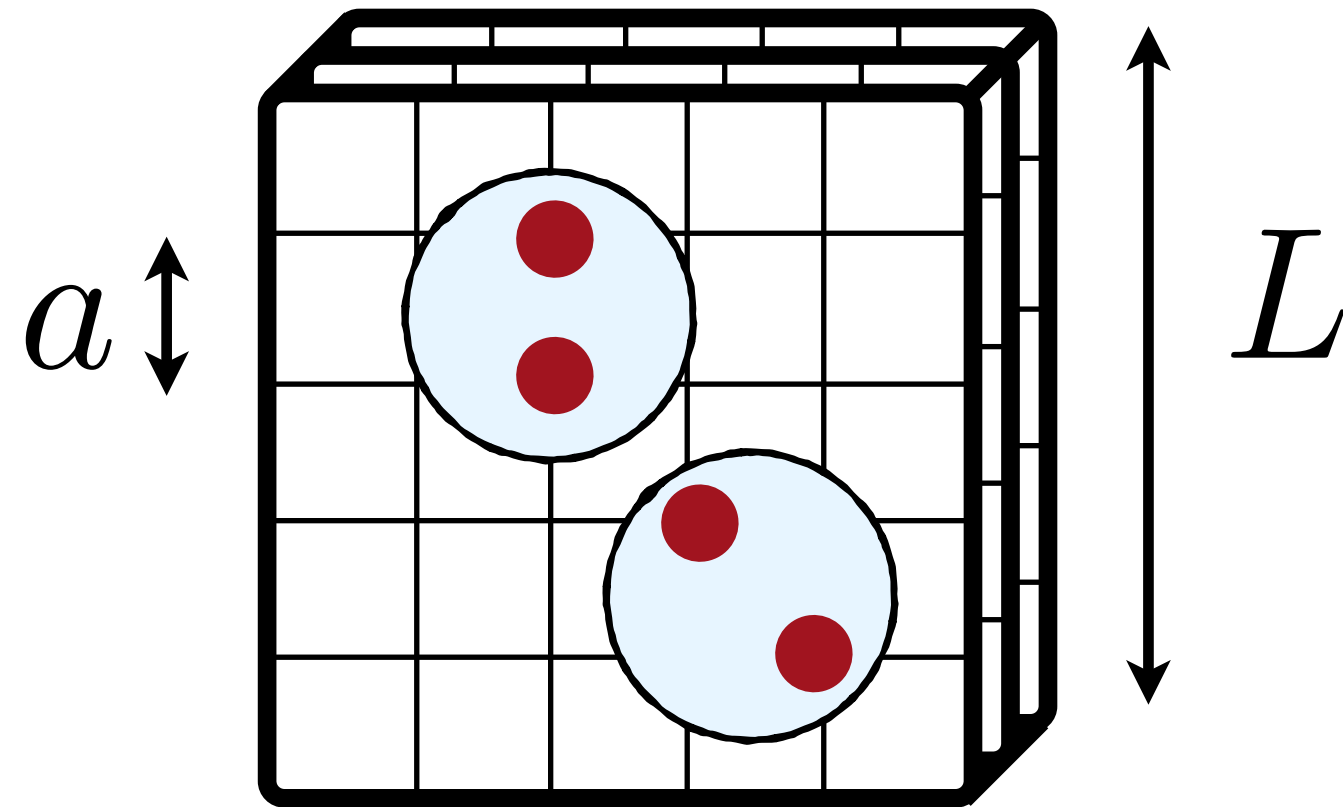
- lattice spacing: $a \sim 0.03 - 0.12$ fm
- finite volume: $L \sim 6 - 12$ fm
- quark masses



Advantage over experiment!

Lattice QCD in a nut shell

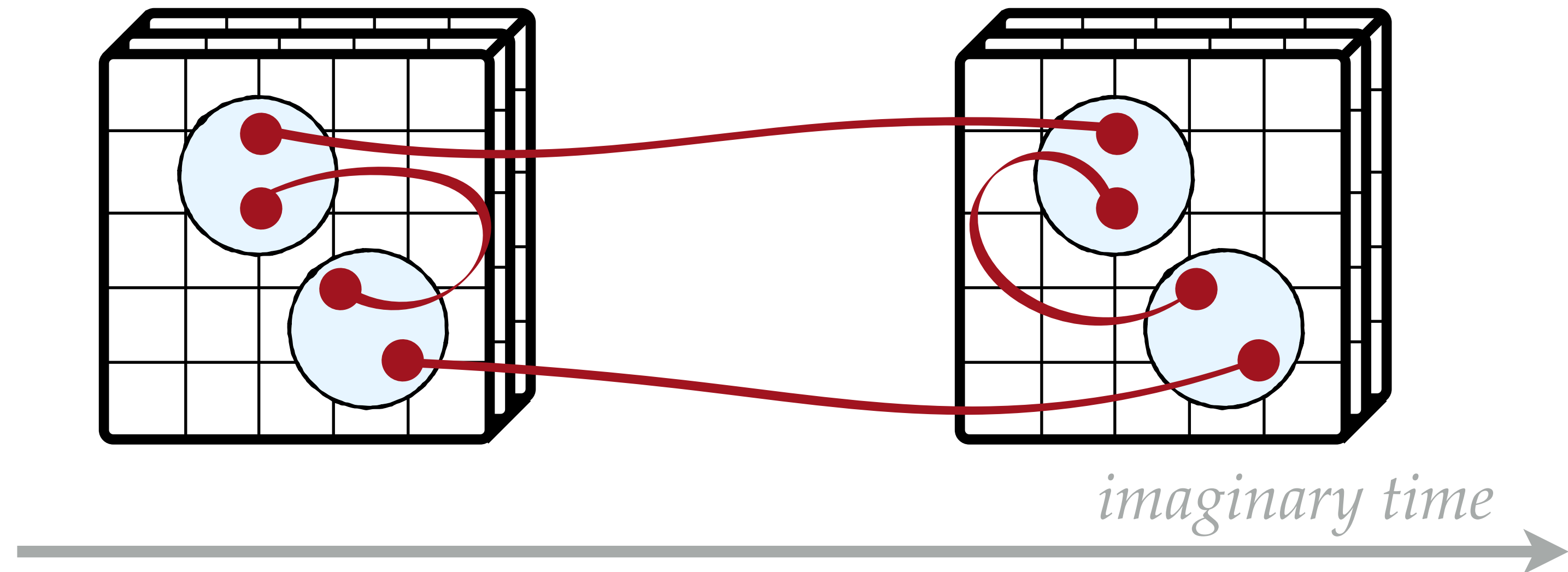
- lattice spacing: $a \sim 0.03 - 0.12$ fm
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- quark masses
- Euclidean spacetime: $t_M \rightarrow -it_E$
- Monte Carlo sampling



Lattice QCD in a nutshell

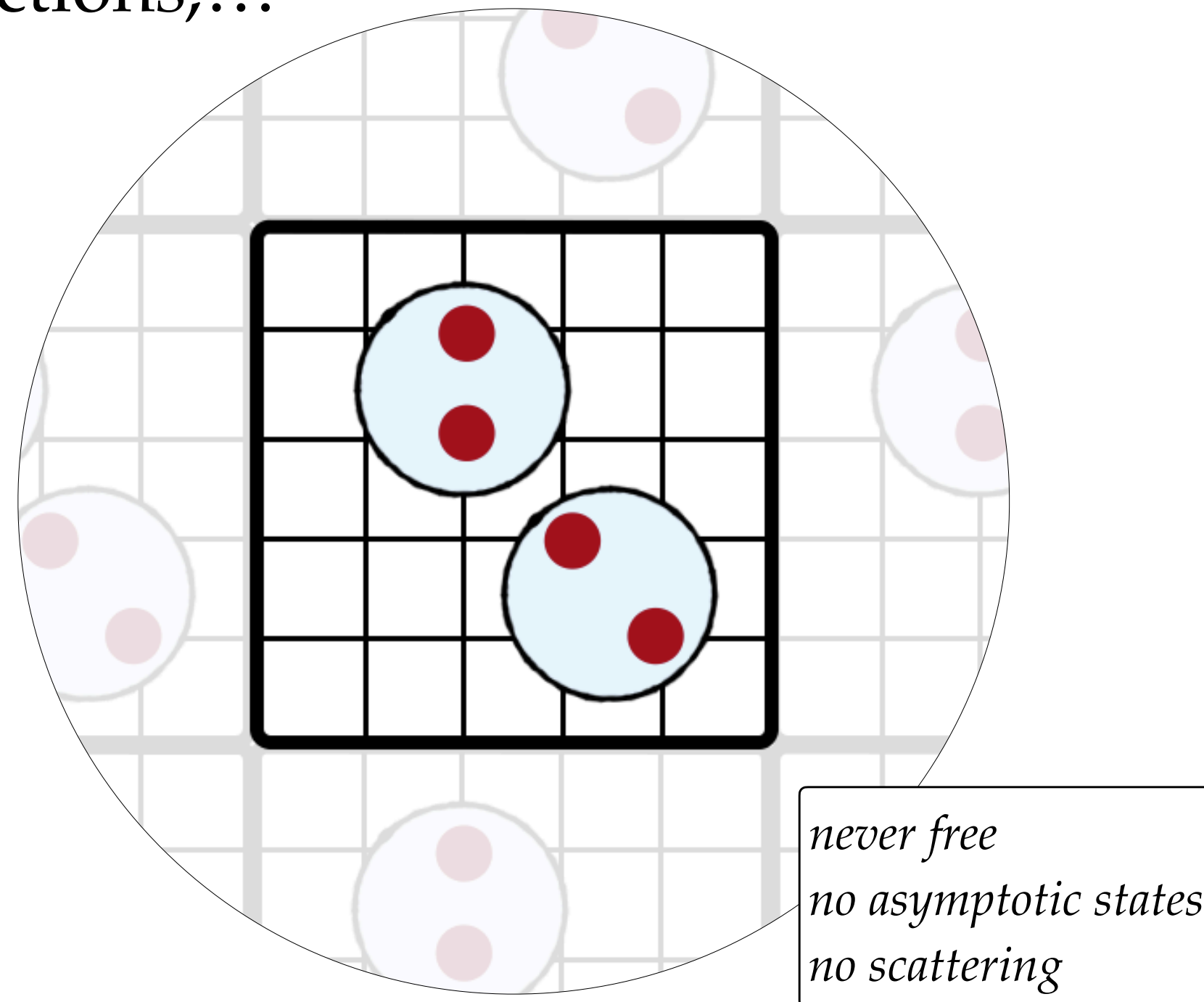
- lattice spacing: $a \sim 0.03 - 0.12$ fm
- finite volume: $L \sim 6 - 12$ fm
- quark masses
- Euclidean spacetime: $t_M \rightarrow -it_E$
- Monte Carlo sampling
- Correlation functions, ...

$$C^{2pt.}(t_E) \equiv \langle 0 | \mathcal{O}(t_E) \mathcal{O}^\dagger(0) | 0 \rangle = \sum_n c_n e^{-E_n t_E}$$



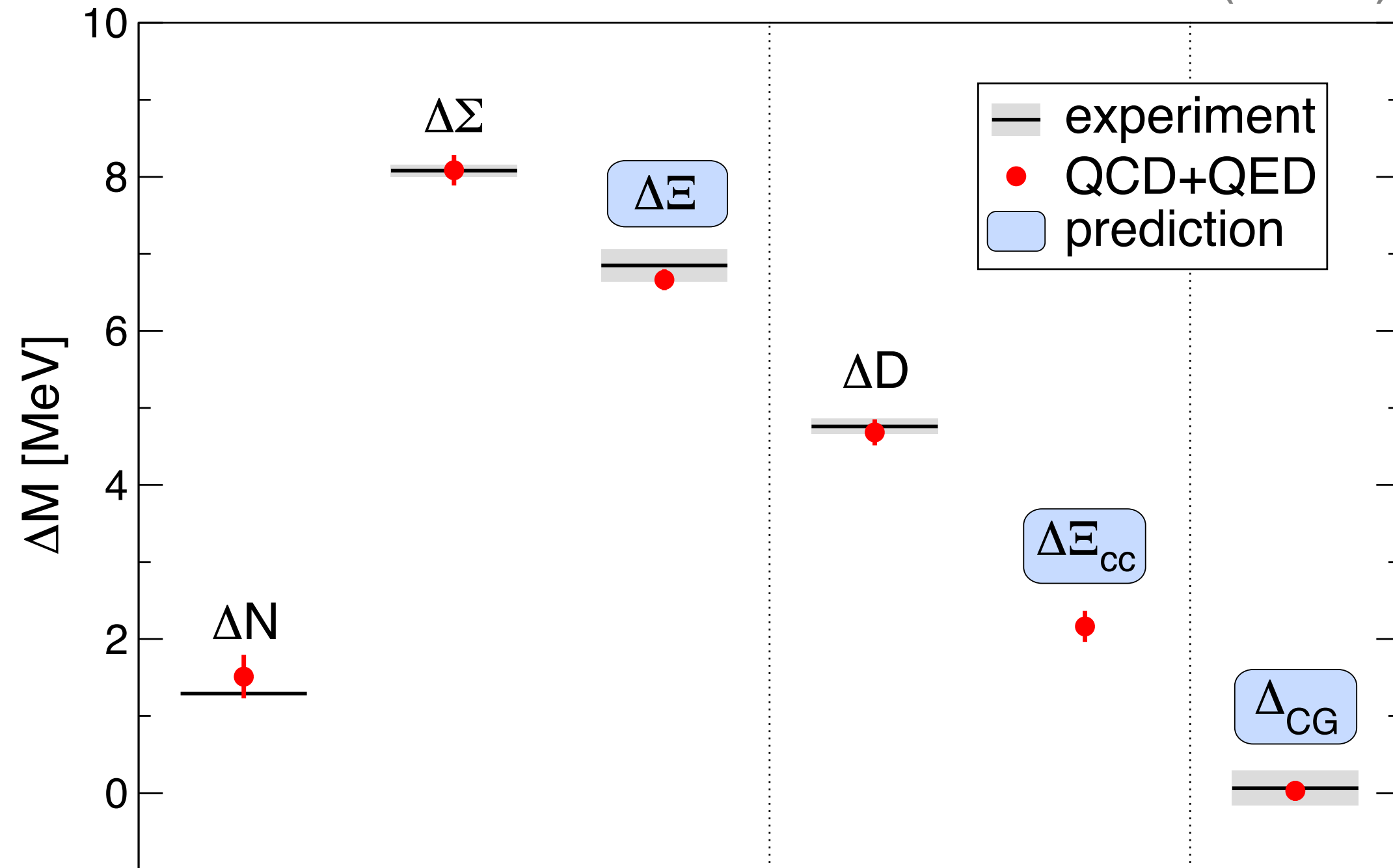
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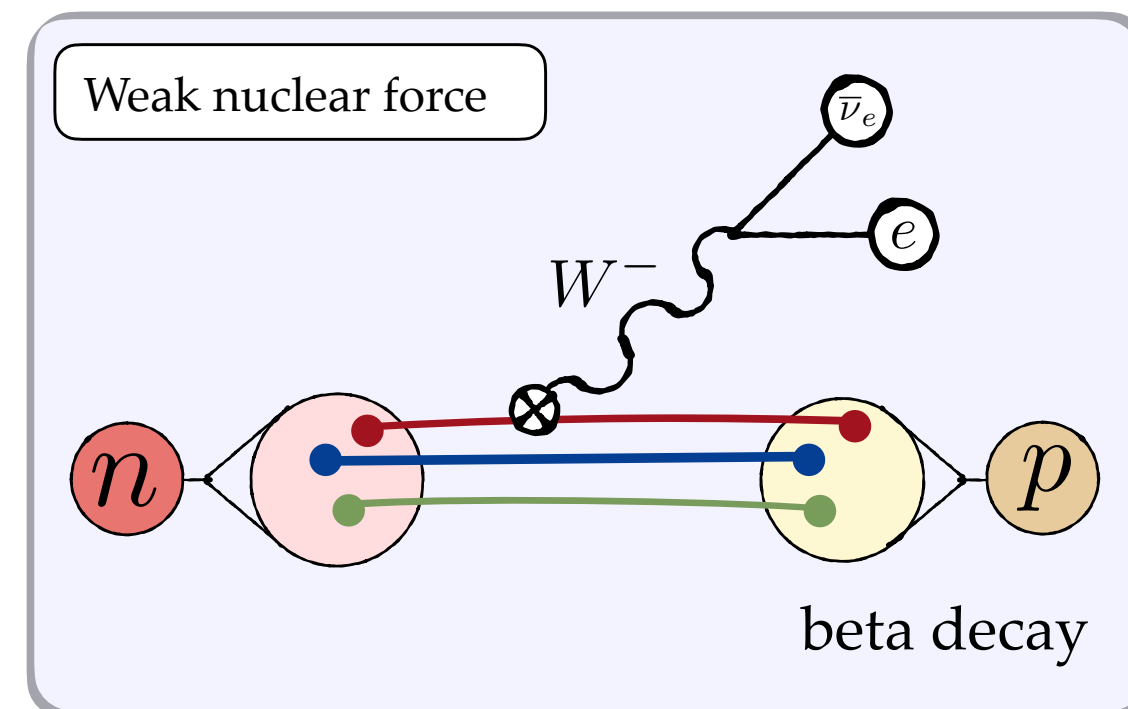
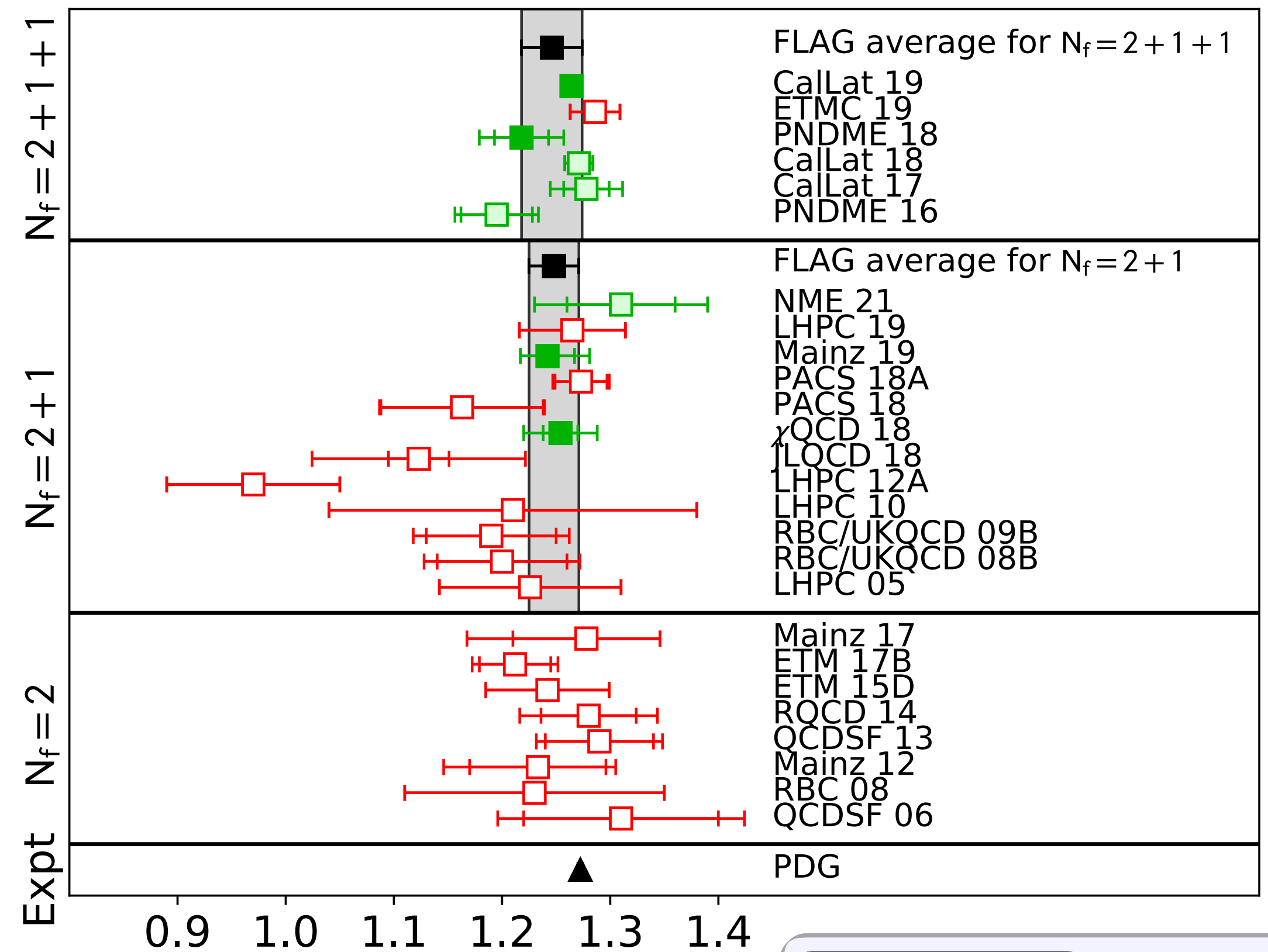
"Simple quantities" from lattice QCD

BMW Collaboration (2015)



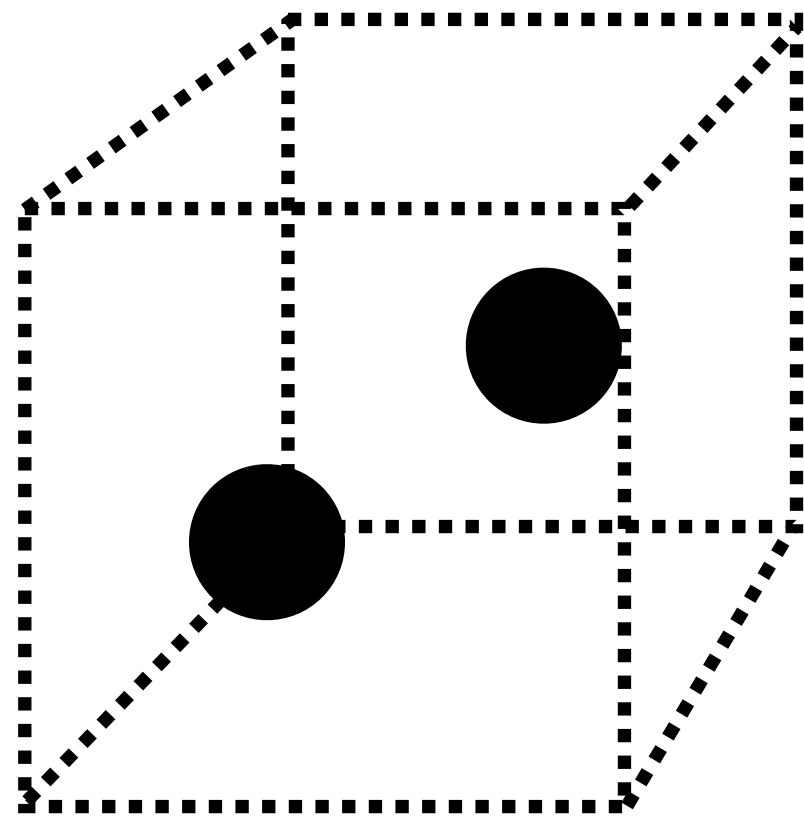
g_A^{u-d}

FLAG (2021)



Coleman-Glashow mass difference $\Delta_{CG} = \Delta M_N - \Delta M_\Sigma + \Delta M_E$

Two-hadron systems

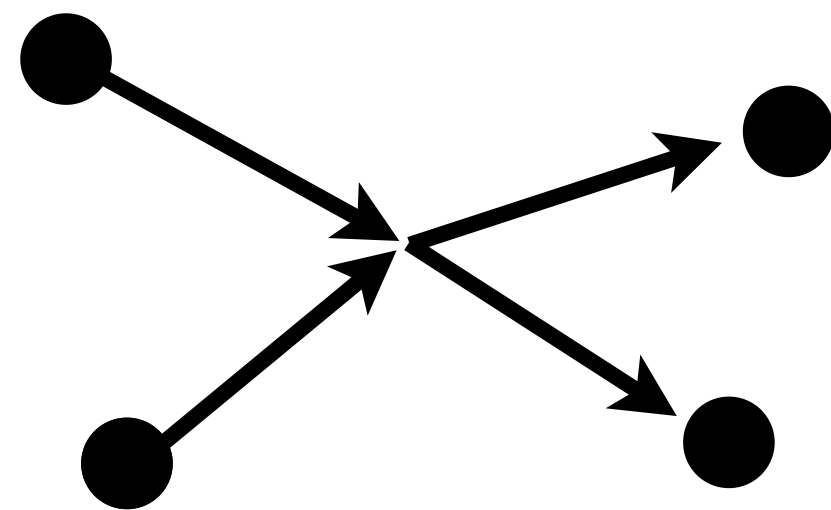


finite-volume
spectroscopy

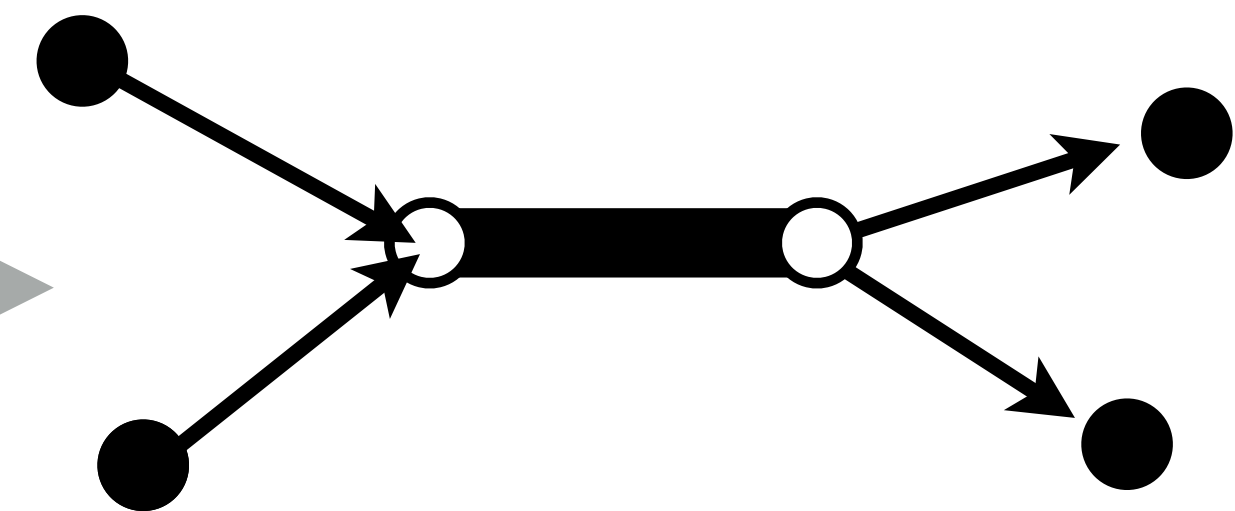
$$\det [F(E_L, L) + \mathcal{M}^{-1}(E_L)] = 0$$



Lüscher (1986, 1991)
Rummukainen & Gottlieb (1995)
Kim, Sachrajda, & Sharpe (2005)
Christ, Kim & Yamazaki (2005)
Feng, Li, & Liu (2004)
Hansen & Sharpe (2021)
RB & Davoudi (2012)
RB (2014)



infinite-volume
scattering amplitudes



bound state and
resonance poles

2 minimum requirements

Two “musts” for few-body systems:

☑ Generalized eigenvalue problem (GEVP),

☑ large basis of ops,

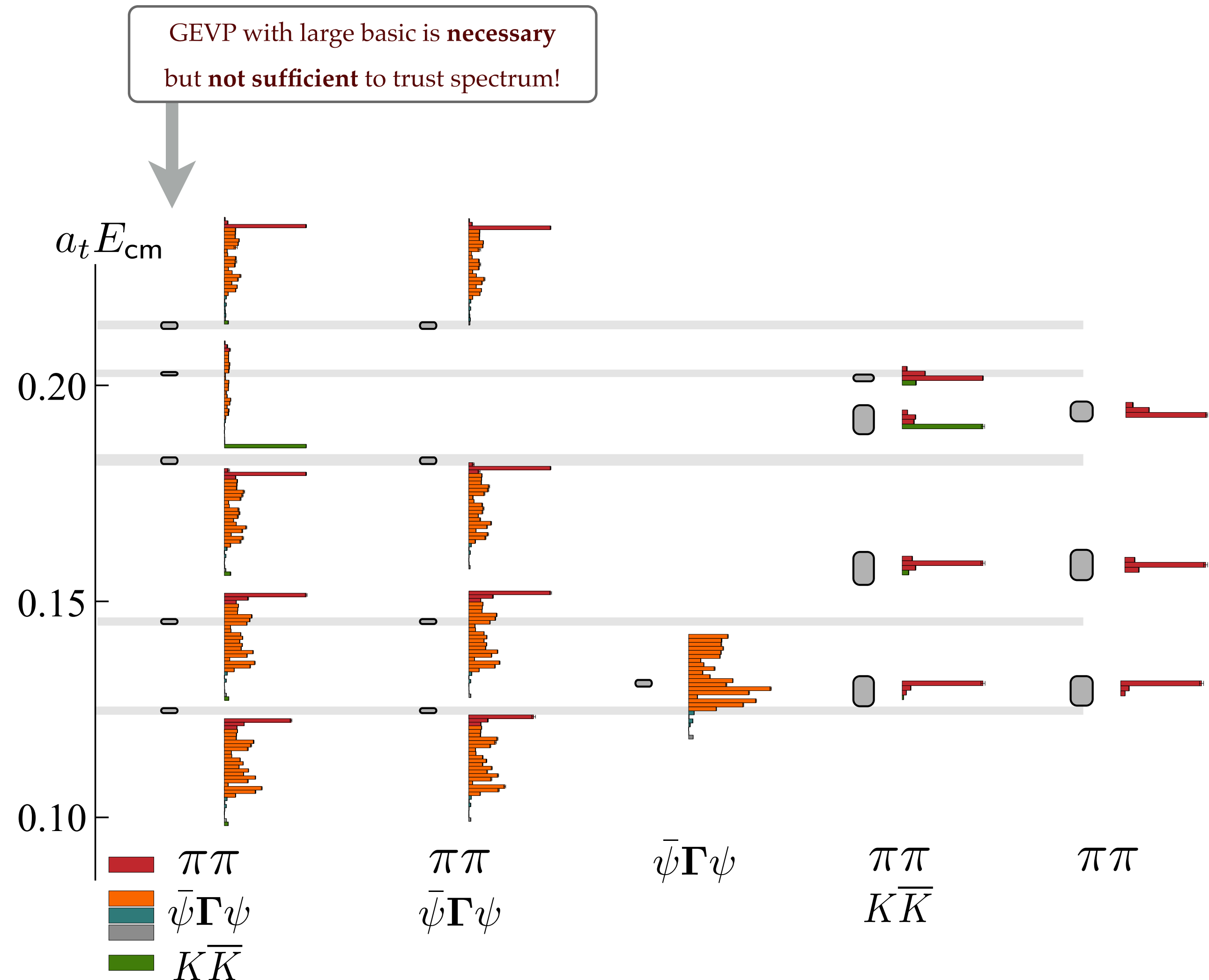
$$\mathcal{O}_b \sim \bar{q} \Gamma_b q, \pi\pi, K\bar{K}, \dots,$$

☑ diagonalization,

$$C_{ab}^{2pt.}(t, \mathbf{P}) \equiv \langle 0 | \mathcal{O}_b(t, \mathbf{P}) \mathcal{O}_a^\dagger(0, \mathbf{P}) | 0 \rangle = \sum_n Z_{b,n} Z_{a,n}^* e^{-E_n t}$$

$$C(t) \vec{v}^{(n)}(t, t_0) = \lambda_n(t, t_0) C(t_0) \vec{v}^{(n)}(t, t_0)$$

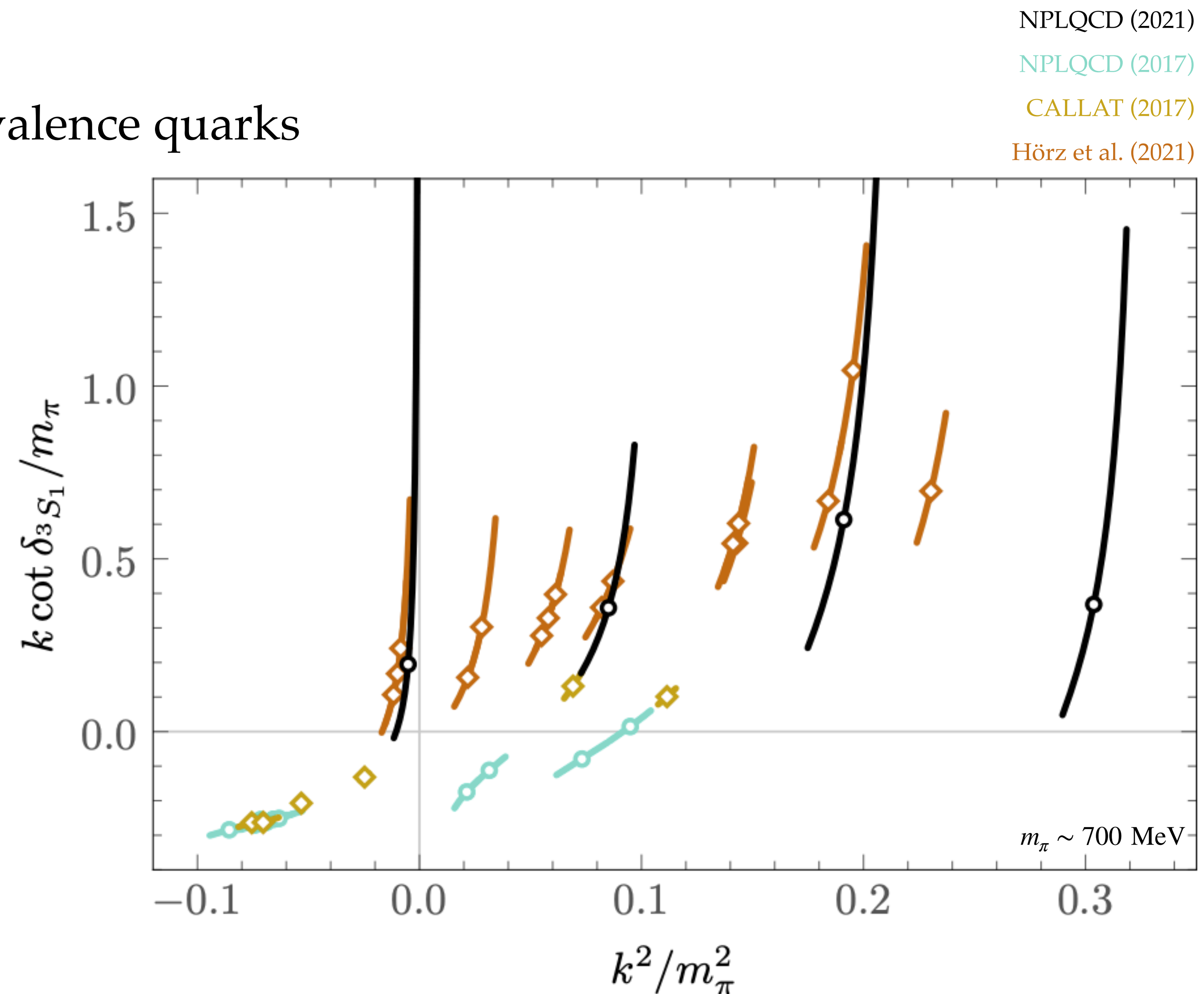
☑ Finite-volume formalisms.



Why no-nucleons?

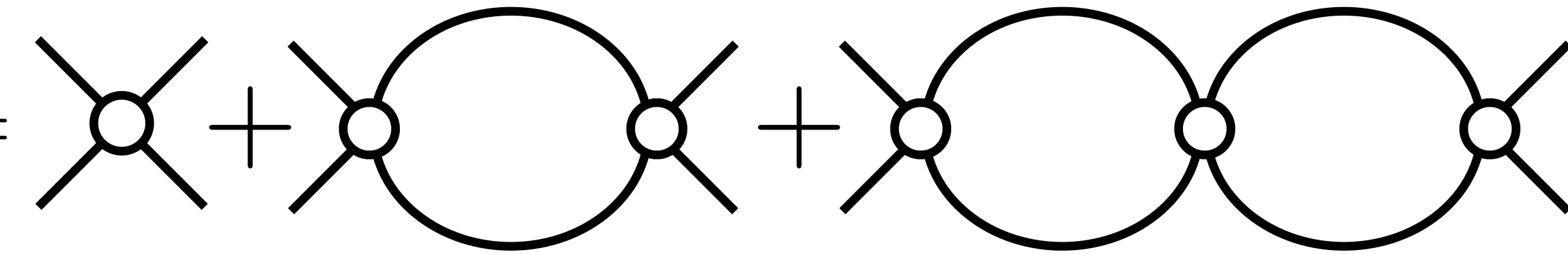
Nucleons introduce many challenges

- ☑ More valence quarks
 - ☑ Computational cost grows with number of valence quarks
- ☑ Noise grows with baryon number
 - ☑ more statistics needed
- ☑ Noise grows for lighter quark masses
- ☑ **Hard...not impossible**

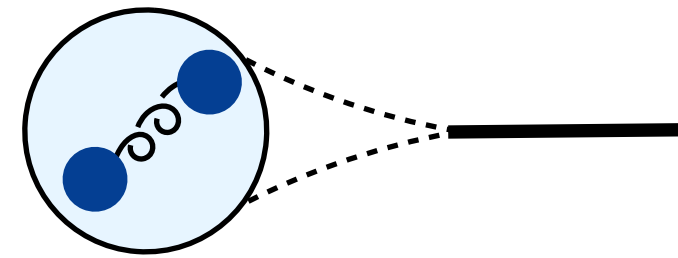


Two-hadron scattering

Unitarity using all orders perturbation theory:

$$i\mathcal{M} = \text{tree} + \text{one-loop} + \text{two-loop} + \dots$$
The equation shows a series of Feynman diagrams representing the imaginary part of the scattering amplitude $i\mathcal{M}$. The first term is a tree-level diagram with four external lines meeting at a central vertex. The second term is a one-loop diagram consisting of a tree-level vertex connected to a loop, which then connects to another tree-level vertex. The third term is a two-loop diagram with two loops connected in series between the external lines. The series continues with an ellipsis.

IR limit of QCD, only interested in hadronic d.o.f.



Two-hadron scattering

Unitarity using all orders perturbation theory:

$$i\mathcal{M} = \underbrace{\text{tree} + \text{one-loop} + \text{two-loop} + \dots}_{\left\{ \text{non-perturbative kernel} + \dots \right\}}$$

*non-perturbative kernel including
all diagrams not shown...*

"yep, the left hand cut is there"

Two-hadron scattering

Unitarity using all orders perturbation theory:

$$i\mathcal{M} = \text{tree} + \text{one-loop} + \text{two-loop} + \dots$$

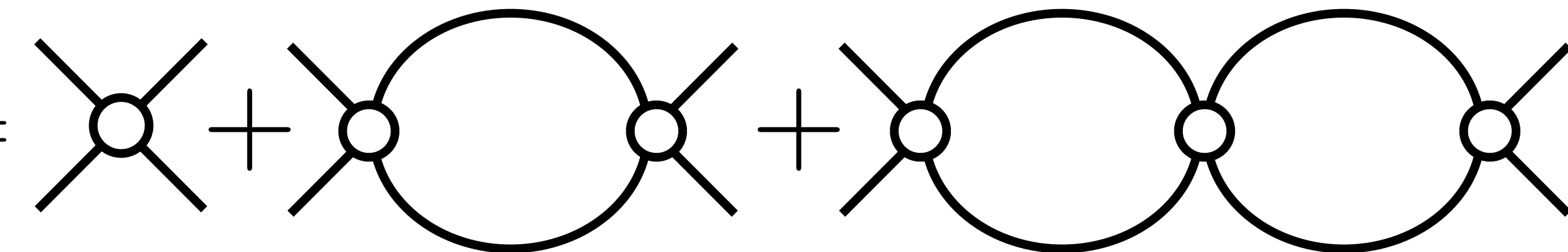
$$\begin{aligned} \text{one-loop} &= \int \frac{d^4k}{(2\pi)^4} [iB(k, P)]^2 \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(P - k)^2 - m^2 + i\epsilon} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{[iB(k, P)]^2}{(2\omega_k)^2} \pi\delta(E - 2\omega_k) + \text{“PV integral”} \\ &= [iB_{on}] \rho [iB_{on}] + \text{“PV integral”} \\ &= \text{cut diagram} + \text{PV diagram} \end{aligned}$$

$$\rho \equiv \frac{p}{8\pi E} \sim \sqrt{s - s_{th}}$$

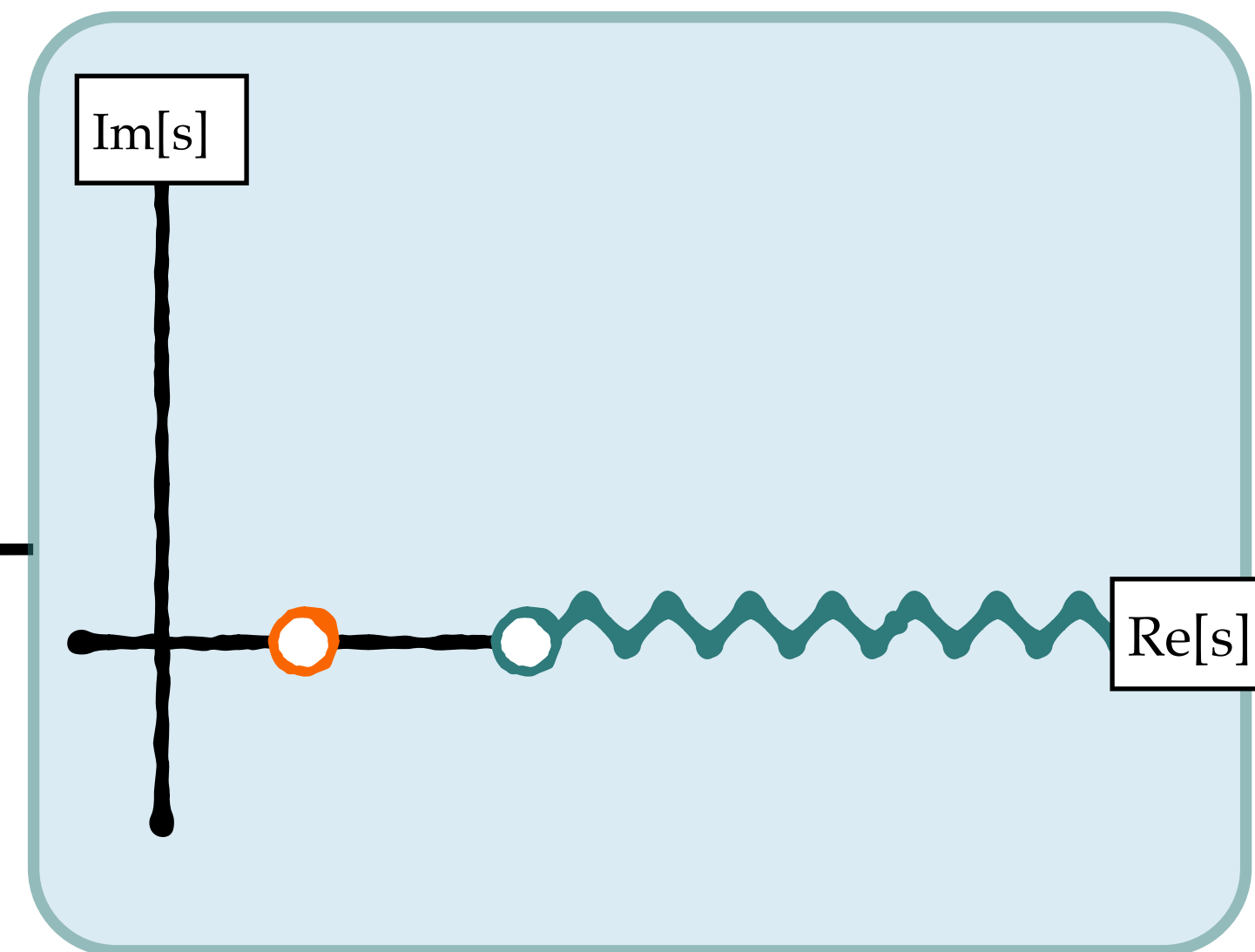
square root singularity.

Two-hadron scattering

Unitarity using all orders perturbation theory:

$$i\mathcal{M} = \text{tree} + \text{one-loop} + \text{two-loop} + \dots$$


2^N sheets for N open channels



$$\rho \equiv \frac{p}{8\pi E} \sim \sqrt{s - s_{th}}$$

square root singularity.

Two-hadron scattering

Unitarity using all orders perturbation theory:

$$i\mathcal{M} = \text{tree} + \text{one-loop} + \text{two-loop} + \dots$$

$$= \text{cut tree} + \dots$$

$$= \underbrace{\text{cut tree}}_{K\text{-matrix}} \left\{ \text{tree} + \text{one-loop PV} + \dots \right\}$$

Two-hadron scattering

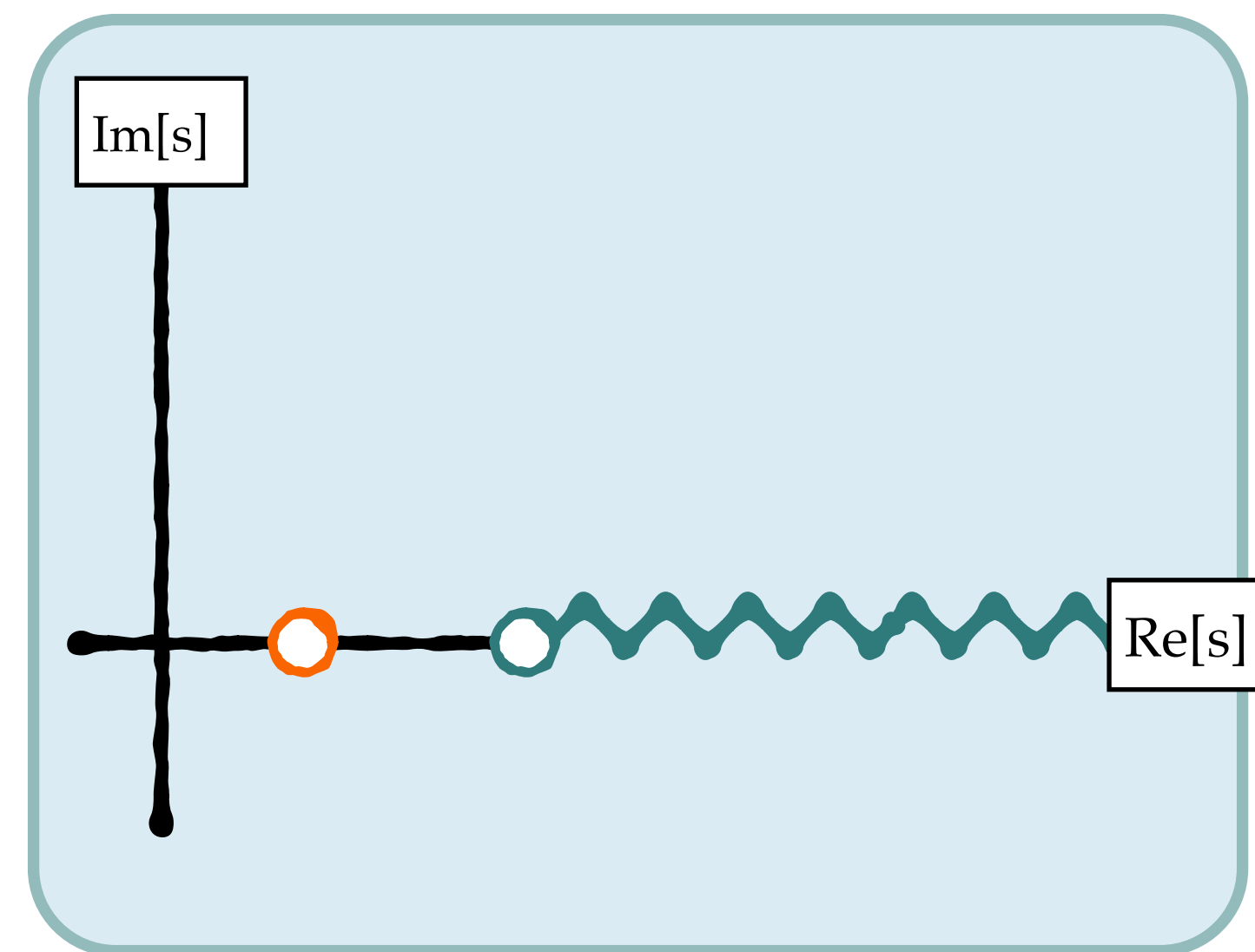
Unitarity using all orders perturbation theory:

$$i\mathcal{M} = \text{tree} + \text{one-loop} + \text{two-loop} + \dots$$

$$= \text{tree} + \text{tree} \circlearrowleft \text{tree} + \dots$$

$$\rho \equiv \frac{p}{8\pi E} \sim \sqrt{s - s_{th}}$$

square root singularity.



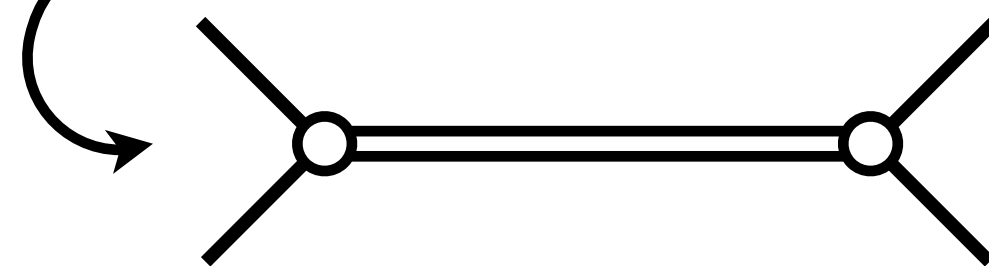
Two-hadron scattering

Unitarity using all orders perturbation theory:

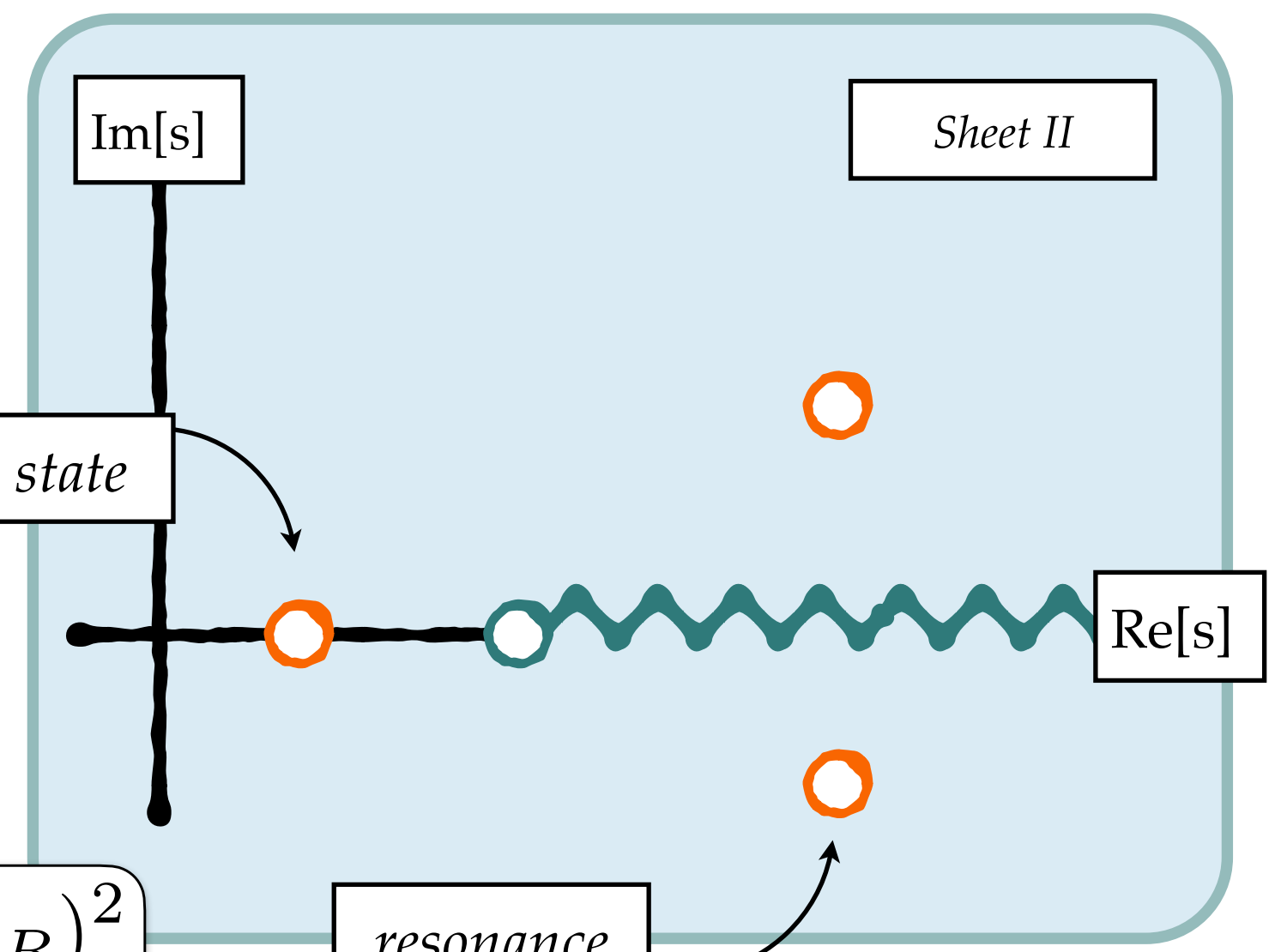
$$i\mathcal{M} = \text{tree} + \text{one-loop} + \text{two-loop} + \dots$$

$$= \text{tree} + \text{one-loop with cut} + \text{two-loop with cut} + \dots$$

$$= \frac{i}{\mathcal{K}^{-1} - i\rho}$$



bound state



$$s_R = (E_R - \frac{i}{2}\Gamma_R)^2$$

resonance

Two-hadron scattering

Unitarity using all orders perturbation theory:

$$\begin{aligned}
 i\mathcal{M} &= \text{[tree]} + \text{[one loop]} + \text{[two loops]} + \dots \\
 &= \text{[tree with } i \text{]} + \text{[one loop with } i \text{ and } \infty \text{]} + \text{[two loops with } i \text{ and } \infty \text{]} + \dots \\
 &= \frac{i}{\mathcal{K}^{-1} - i\rho}
 \end{aligned}$$

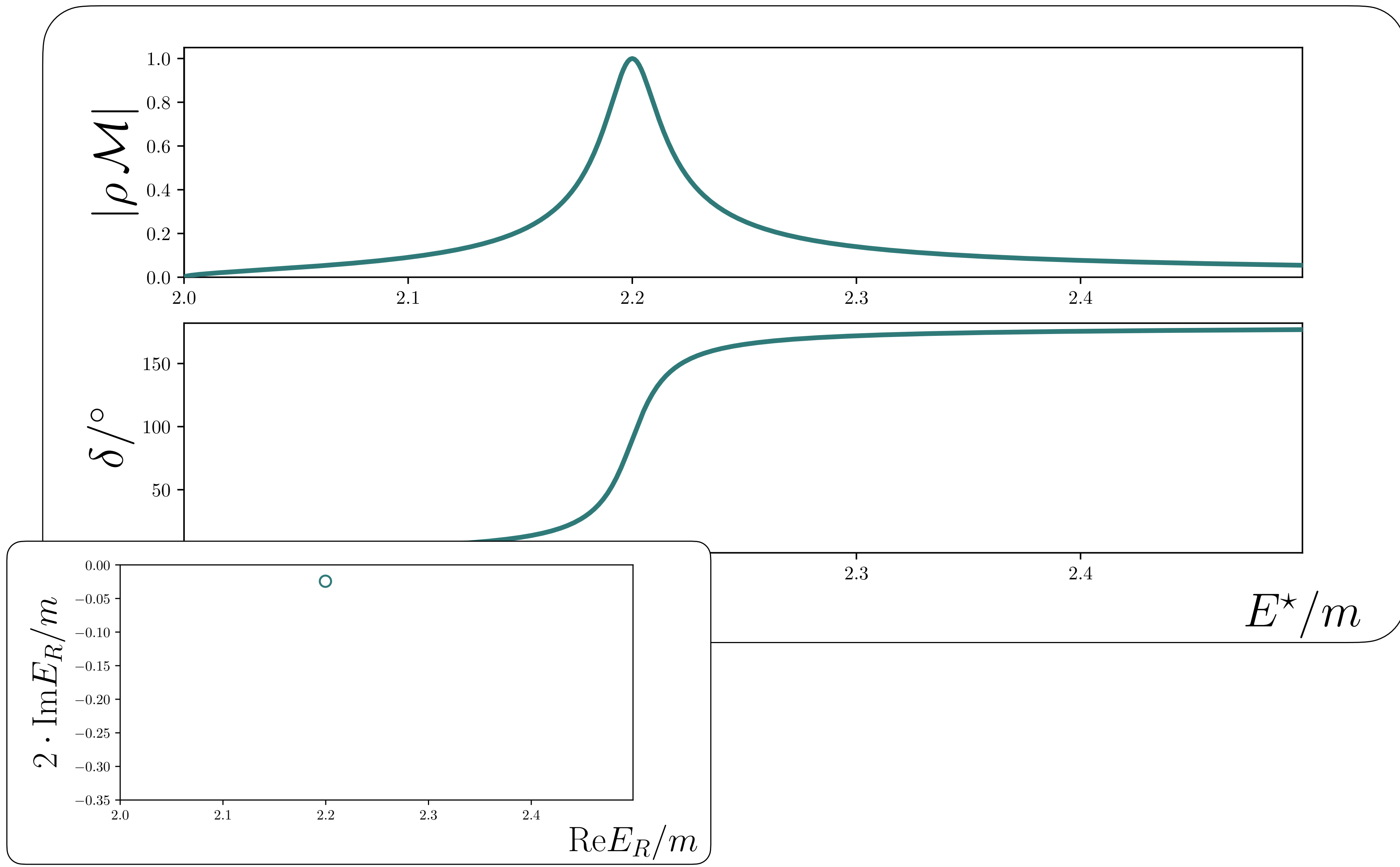
Equating this to the elastic S matrix...

$$S = e^{2i\delta} = 1 + 2i\rho\mathcal{M}$$

$$\begin{aligned}
 \mathcal{K}^{-1} &= \rho \cot \delta \\
 \mathcal{M} &= \frac{\sin \delta}{\rho} e^{i\delta}
 \end{aligned}$$

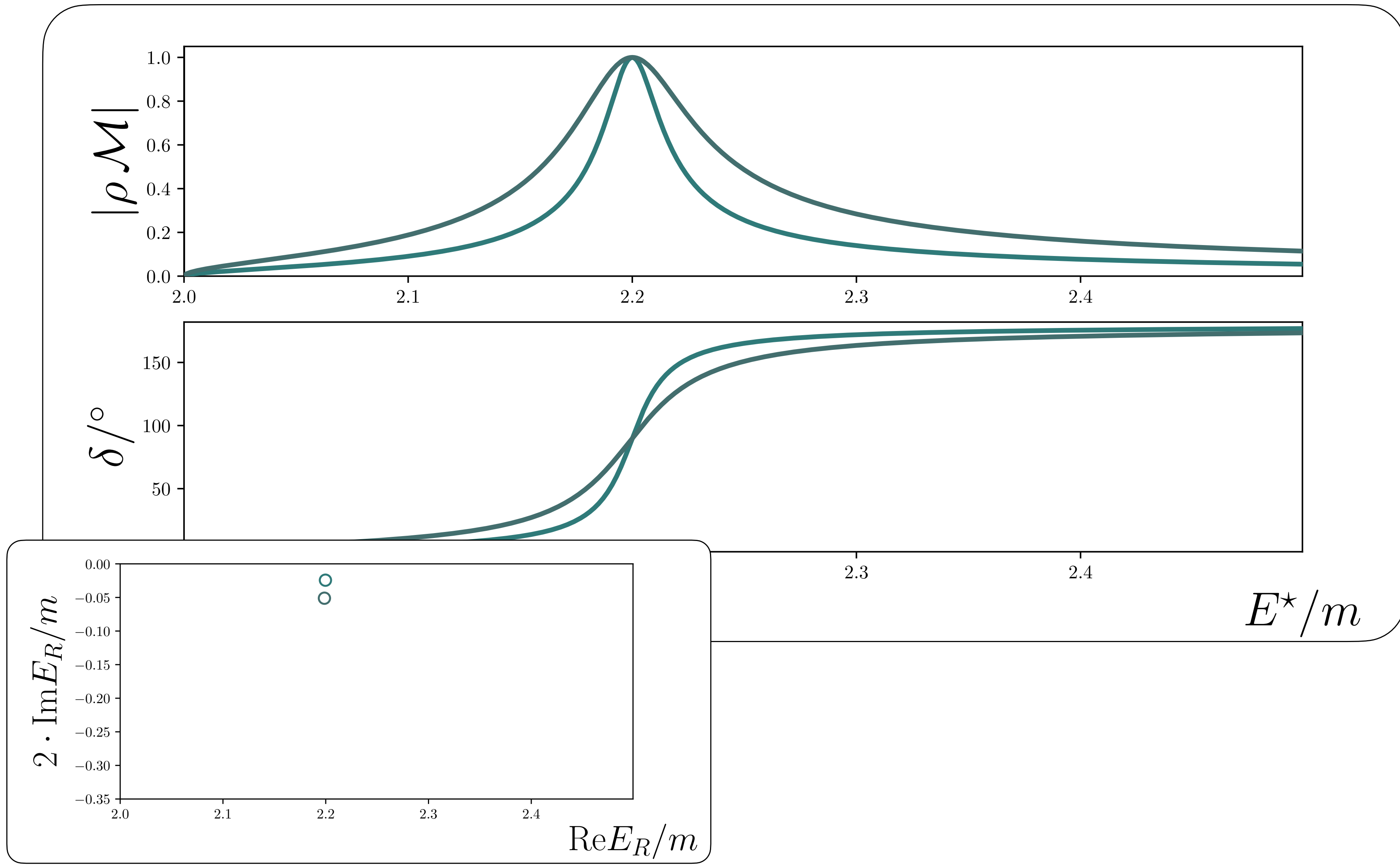
Resonances

To build some intuition: $\mathcal{M} = \frac{\sin \delta}{\rho} e^{i\delta}$



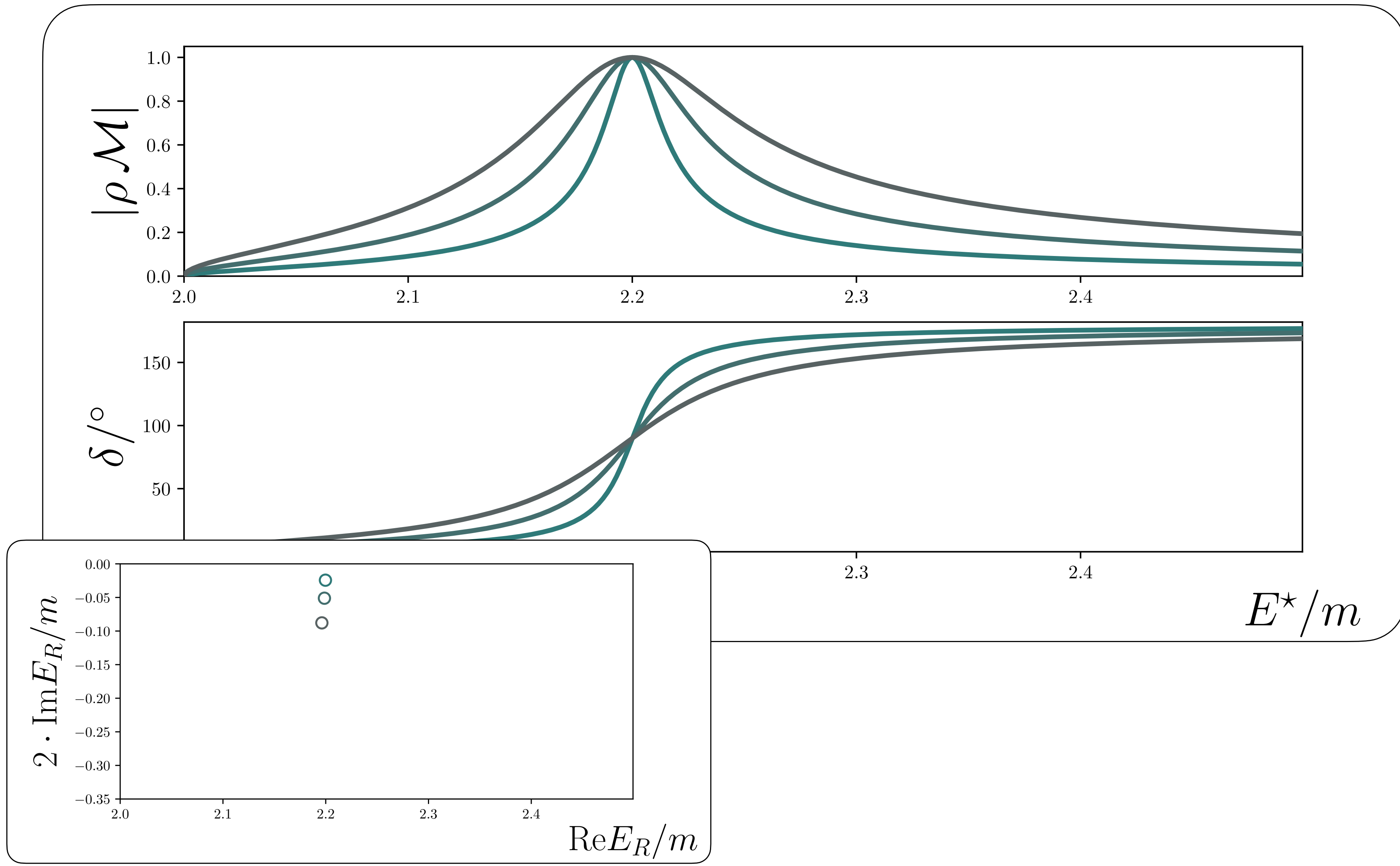
Resonances

To build some intuition: $\mathcal{M} = \frac{\sin \delta}{\rho} e^{i\delta}$



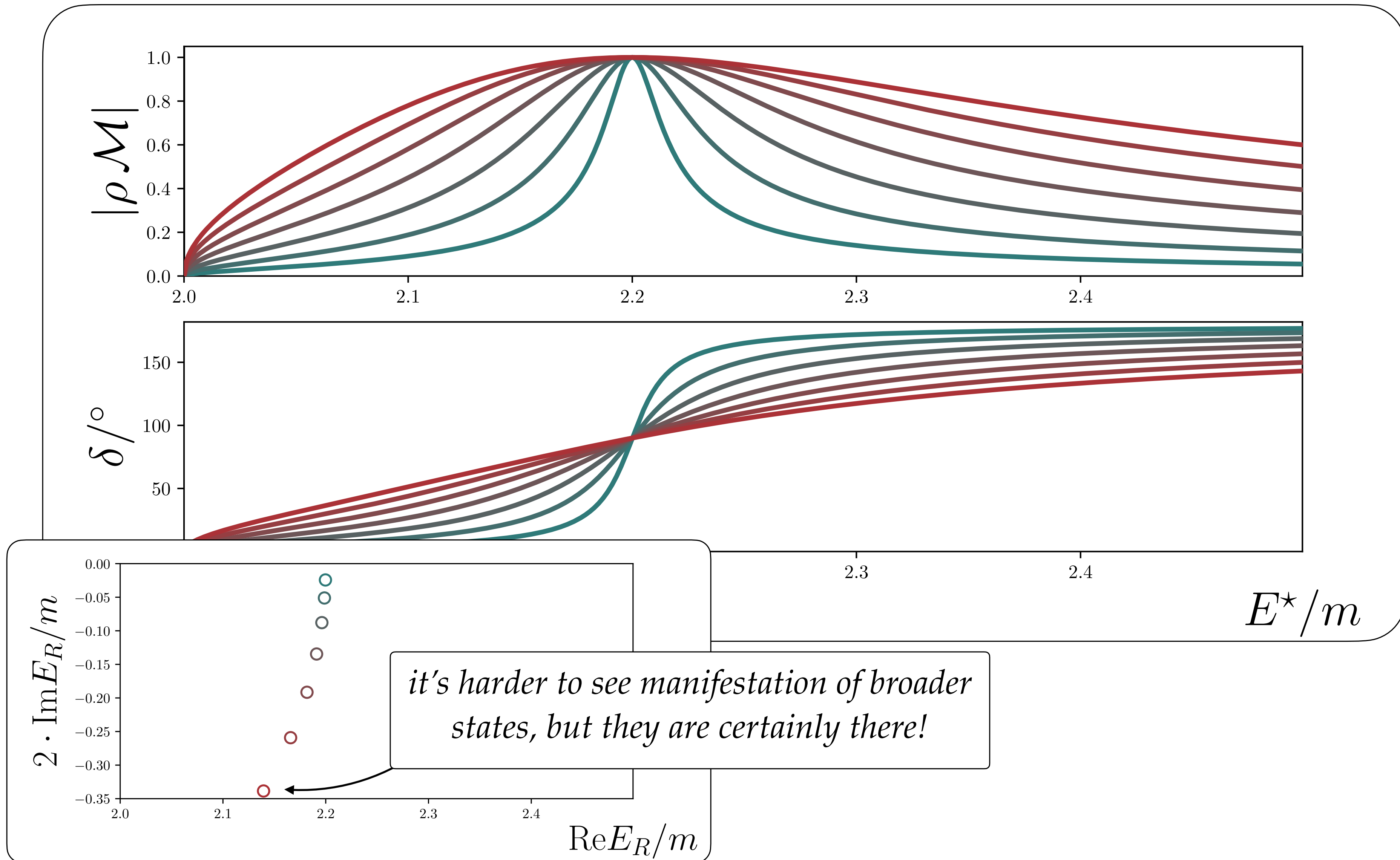
Resonances

To build some intuition: $\mathcal{M} = \frac{\sin \delta}{\rho} e^{i\delta}$



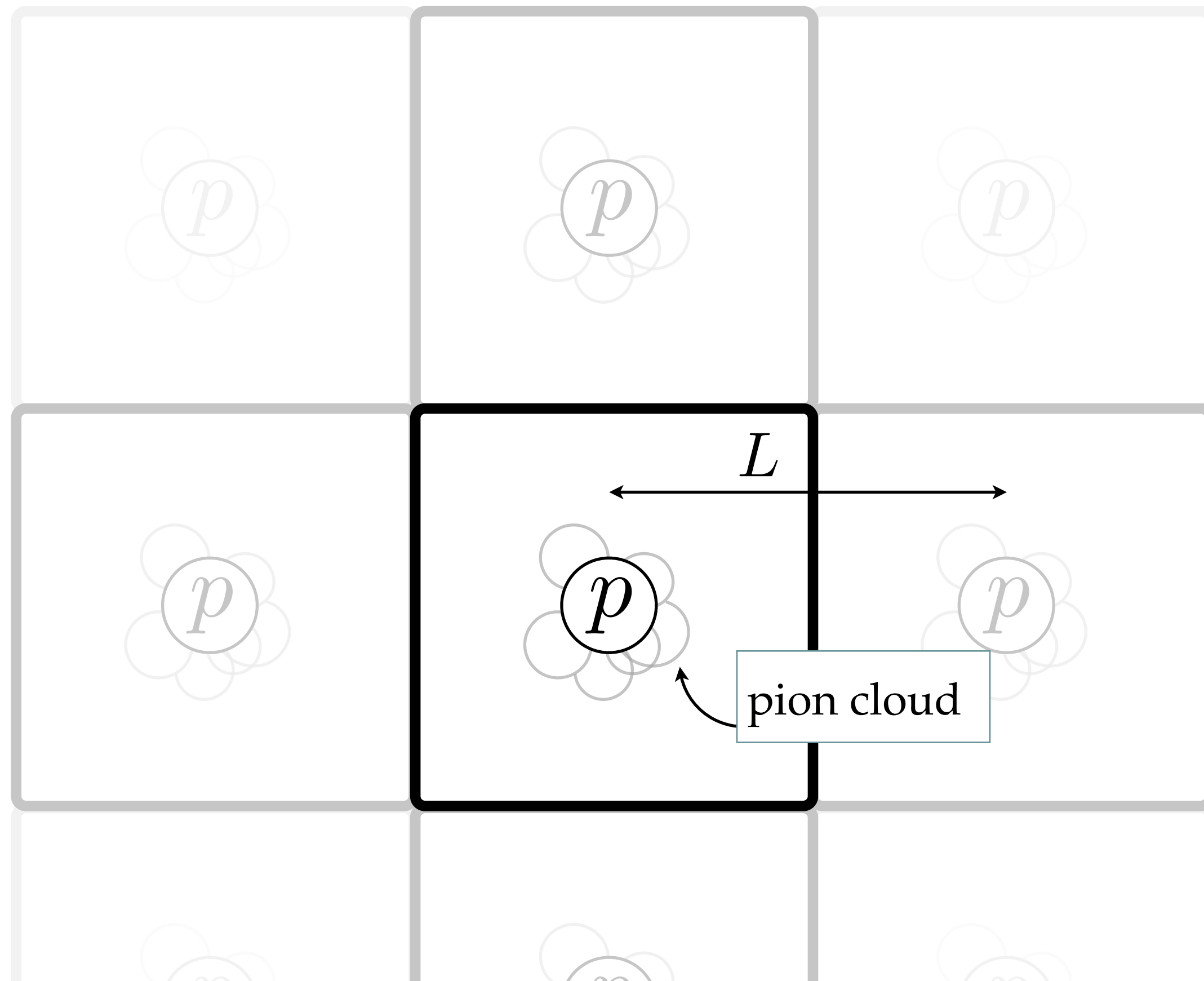
Resonances

To build some intuition: $\mathcal{M} = \frac{\sin \delta}{\rho} e^{i\delta}$



One particle in finite volume

- 📌 Stable hadron size $\sim \mathcal{O}(1/m_\pi)$
- 📌 If $L \gg m_\pi^{-1}$, finite-volume errors are suppressed

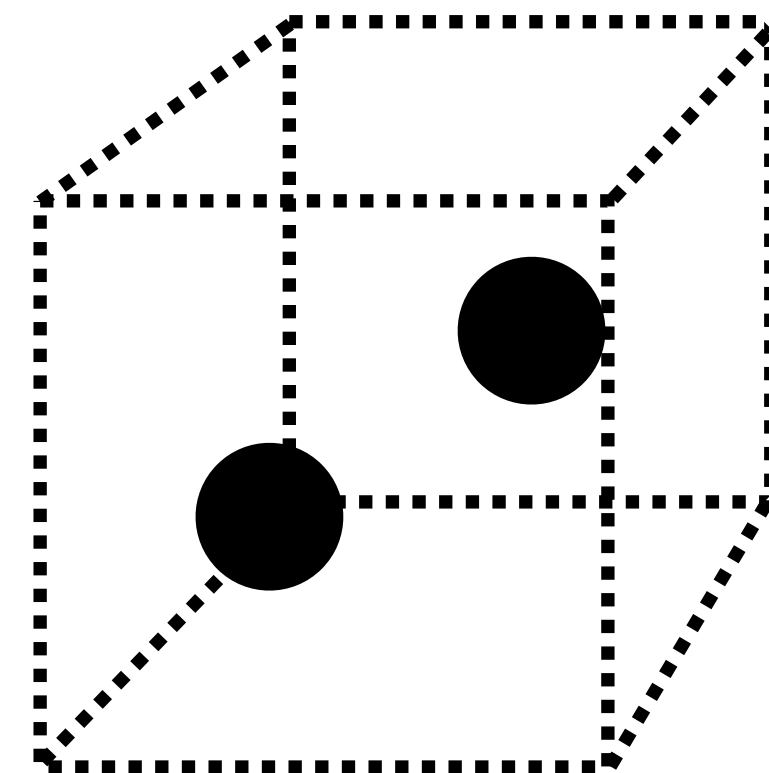
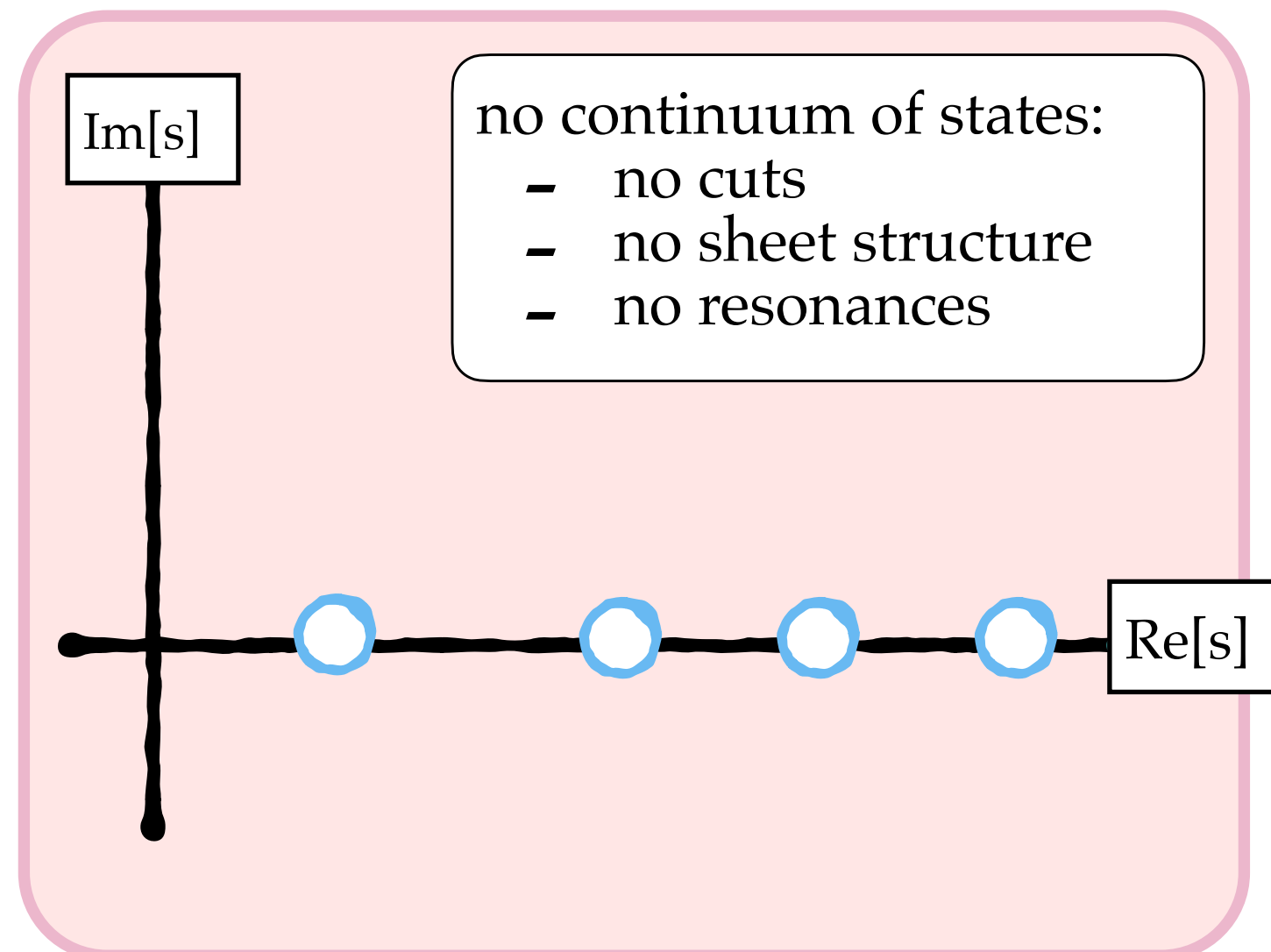


$$m_h(L) = m_h(\infty) + \mathcal{O}(e^{-m_\pi L})$$

Two particle in finite volume

Consider the finite-volume two-particle correlator ($E \sim 2m$):

$$C_L^{2pt.}(P) = \text{[Diagram 1]} + \text{[Diagram 2]} + \dots$$



Two particle in finite volume

Consider the finite-volume two-particle correlator ($E \sim 2m$):

$$\begin{aligned} C_L^{2pt.}(P) &= \text{[Diagram 1]} + \text{[Diagram 2]} + \dots \\ &= C_\infty(P) + \dots \end{aligned}$$

The diagrams represent Feynman diagrams for two-particle correlators. The first diagram is a single circle with two external legs, labeled V . The second diagram is two such circles connected at a vertex, also labeled V . Ellipses indicate higher-order terms in the expansion.

Two particle in finite volume

Consider the finite-volume two-particle correlator ($E \sim 2m$):

$$\begin{aligned}
 C_L^{2pt.}(P) &= \text{[Diagram: circle with two external lines and label } V \text{]} + \text{[Diagram: two circles with two external lines and label } V \text{]} + \dots \\
 &= C_\infty(P) + \text{[Diagram: dashed circle with two external lines and label } V - \infty \text{]} + \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{[Diagram: dashed circle with two external lines and label } V - \infty \text{]} &= (iB) \left(\left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int \frac{d^3k}{(2\pi)^3} \right] \frac{1}{(2\omega_k)^2} \frac{i}{E - 2\omega_k + i\epsilon} \right) (iB) \\
 &\equiv [iB] iF [iB]
 \end{aligned}$$

F replaces ρ

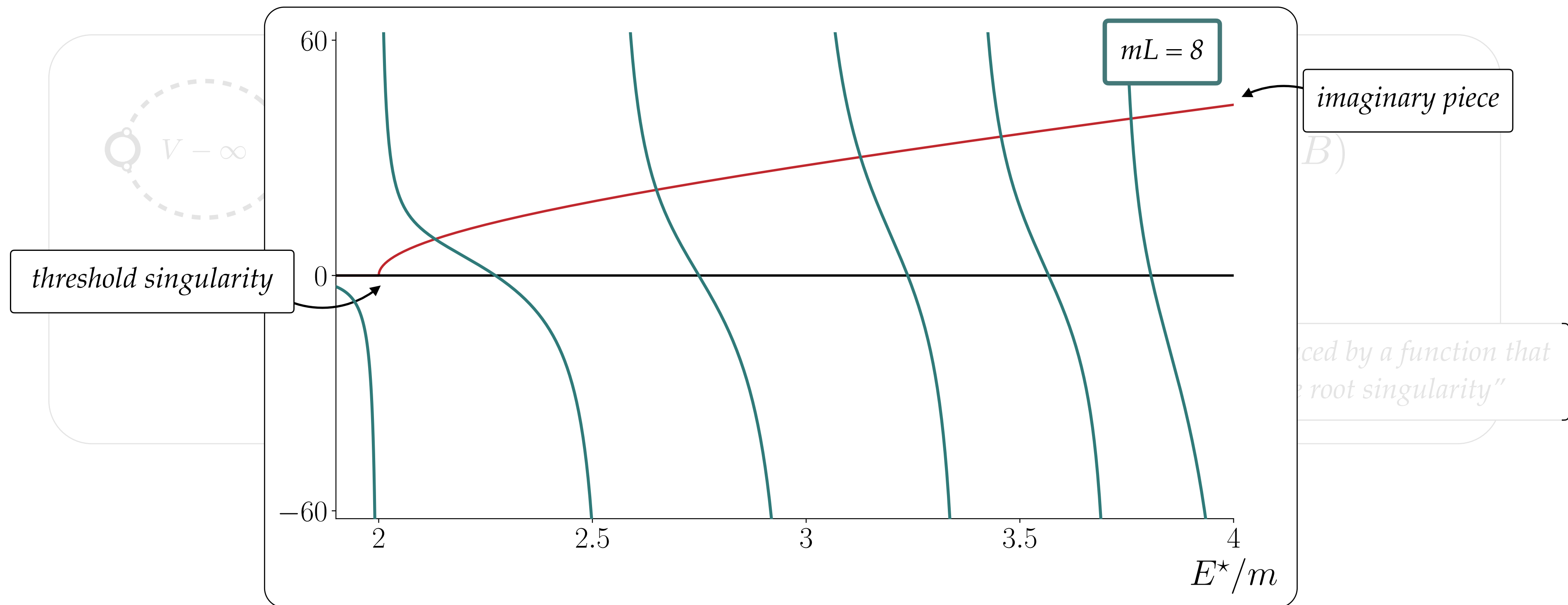
a simple square root singularity is replaced by a function that has both simple poles and the square root singularity

Two particle in finite volume

Consider the finite-volume two-particle correlator ($E \sim 2m$):

$$C_L^{2pt.}(P) = \text{[diagram: circle with V and two external lines]} + \text{[diagram: two circles with V and four external lines]} + \dots$$

$$= C_\infty(P) + \text{[diagram: dashed circle with V-\infty and two external lines]} + \dots$$



Two particle in finite volume

Consider the finite-volume two-particle correlator ($E \sim 2m$):

$$\begin{aligned} C_L^{2pt.}(P) &= \text{[diagram: a circle with two external legs and a vertex labeled } V] + \text{[diagram: two circles with two external legs and vertices labeled } V] + \dots \\ &= C_\infty(P) + \text{[diagram: a dashed circle with two external legs and vertices labeled } V - \infty] + \text{[diagram: two dashed circles with two external legs and vertices labeled } V - \infty] + \dots \\ &= \text{“smooth”} + A \frac{i}{F^{-1} + \mathcal{M}} B^\dagger \end{aligned}$$

poles satisfy: $\det[F^{-1}(P, L) + \mathcal{M}(P)] = 0$

• Lüscher (1986, 1991)

• Rummukainen & Gottlieb (1995)

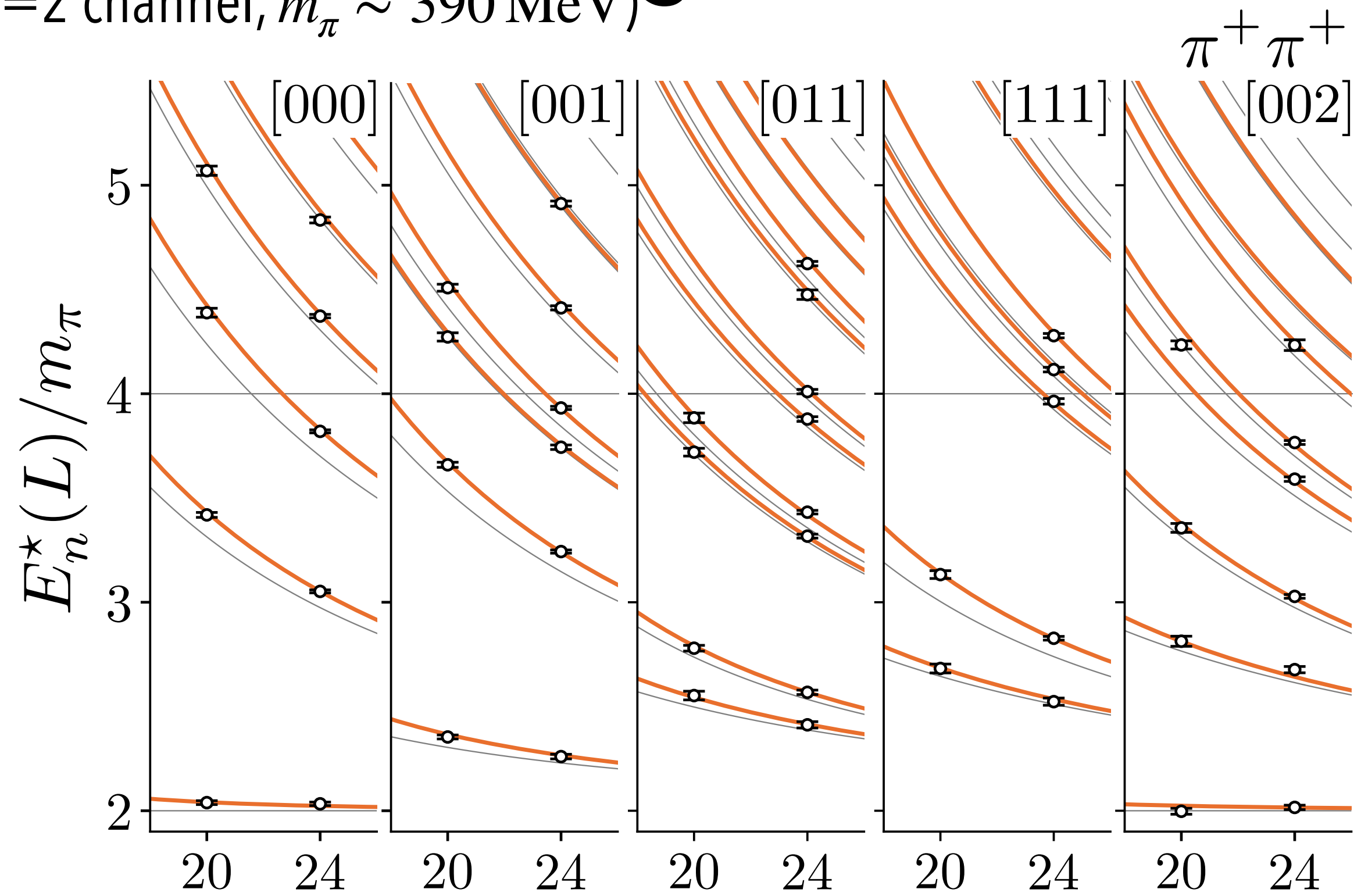
• Kim, Sachrajda, & Sharpe / Christ, Kim & Yamazaki (2005)

• Feng, Li, & Liu (2004); Hansen & Sharpe / RB & Davoudi (2012)

• RB (2014)

$\pi\pi$ scattering

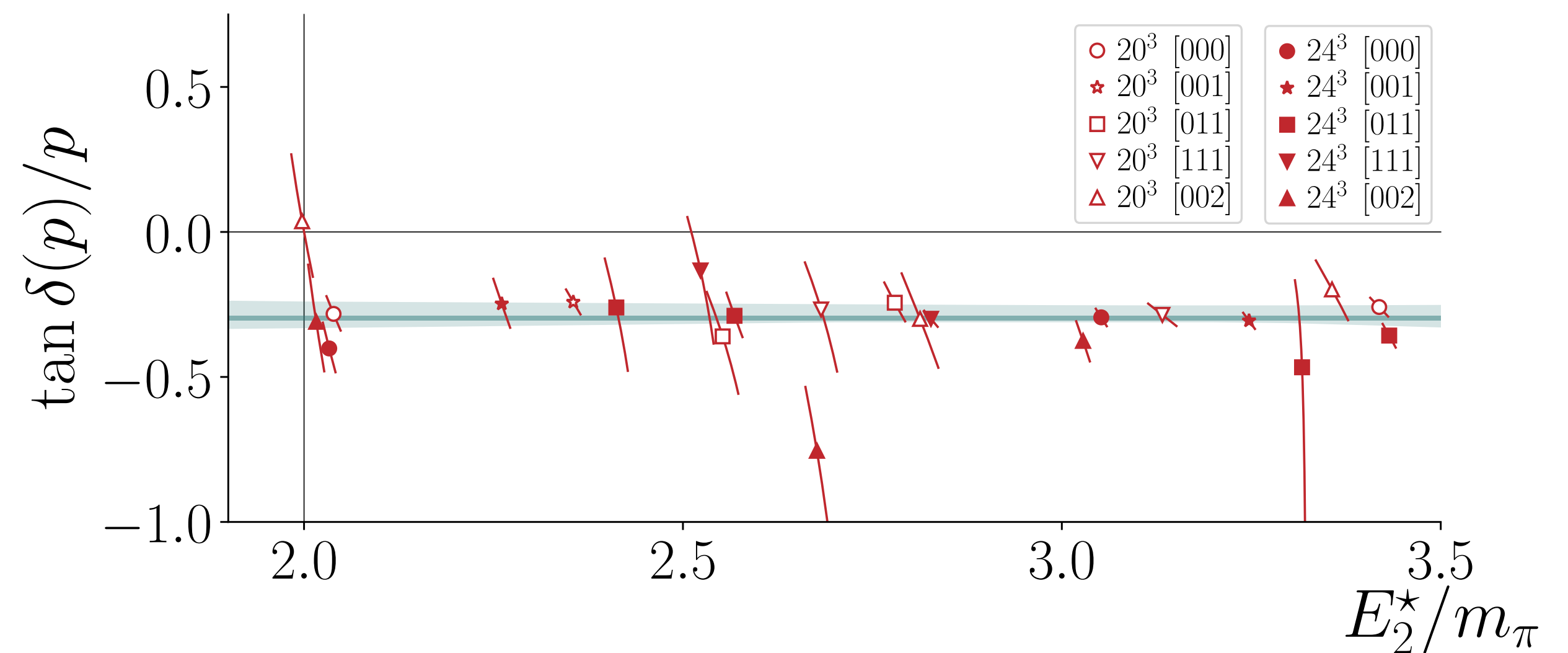
($l=2$ channel, $m_\pi \sim 390$ MeV)



$\pi^+ \pi^+$

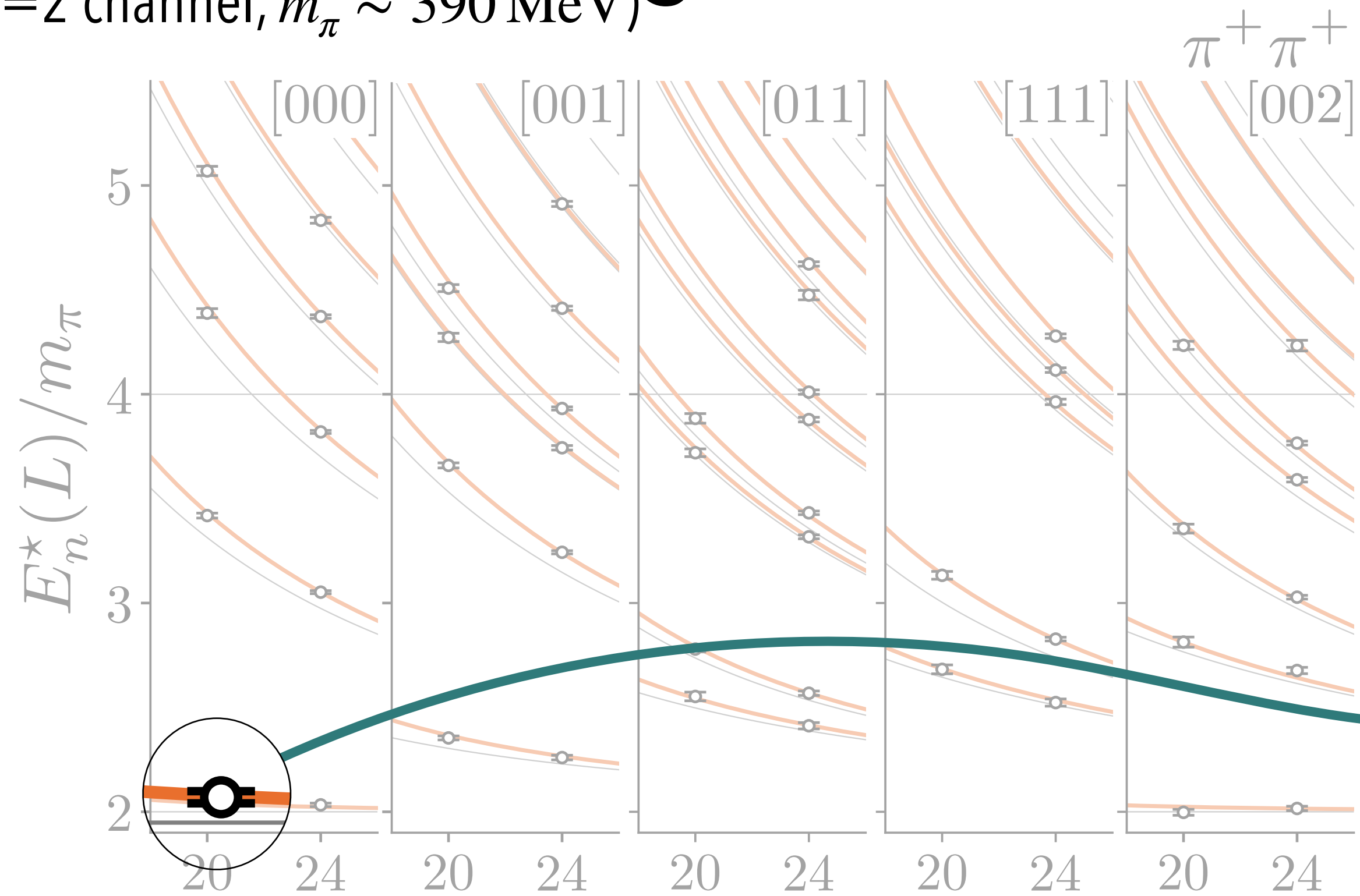
$$\det[F^{-1}(P, L) + \mathcal{M}(P)] = 0$$

$$\mathcal{M} \sim \frac{1}{p \cot \delta - ip}$$



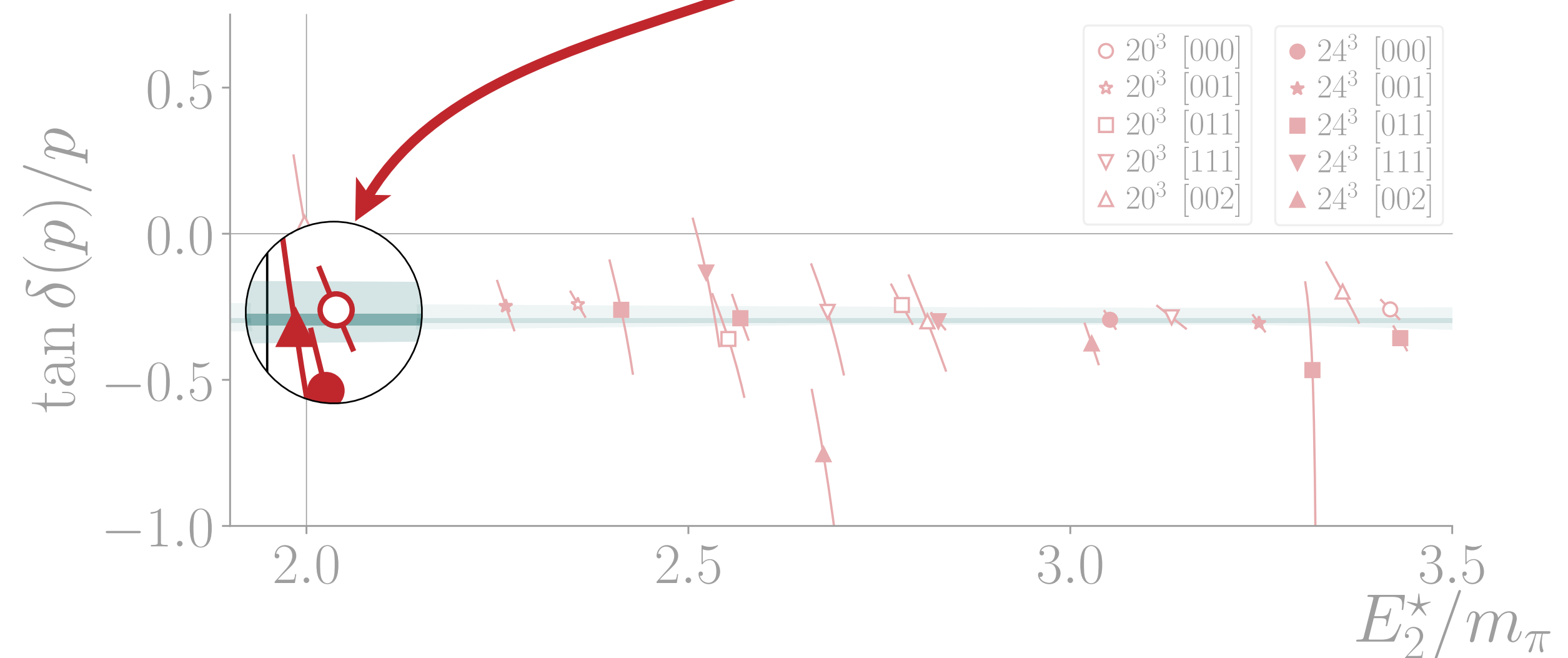
$\pi\pi$ scattering

($l=2$ channel, $m_\pi \sim 390$ MeV)



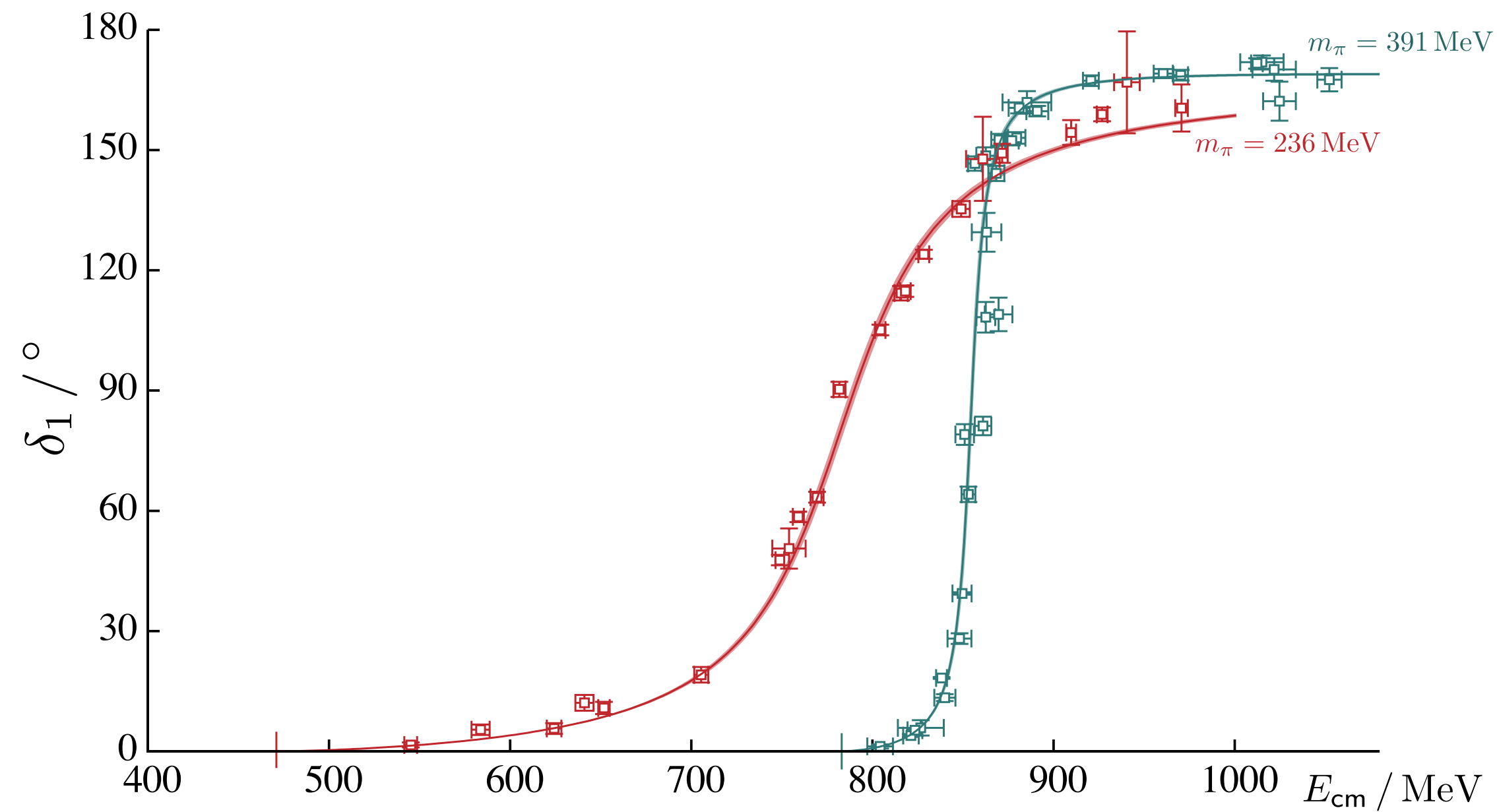
$$\det[F^{-1}(P, L) + \mathcal{M}(P)] = 0$$

$$\mathcal{M} \sim \frac{1}{p \cot \delta - ip}$$

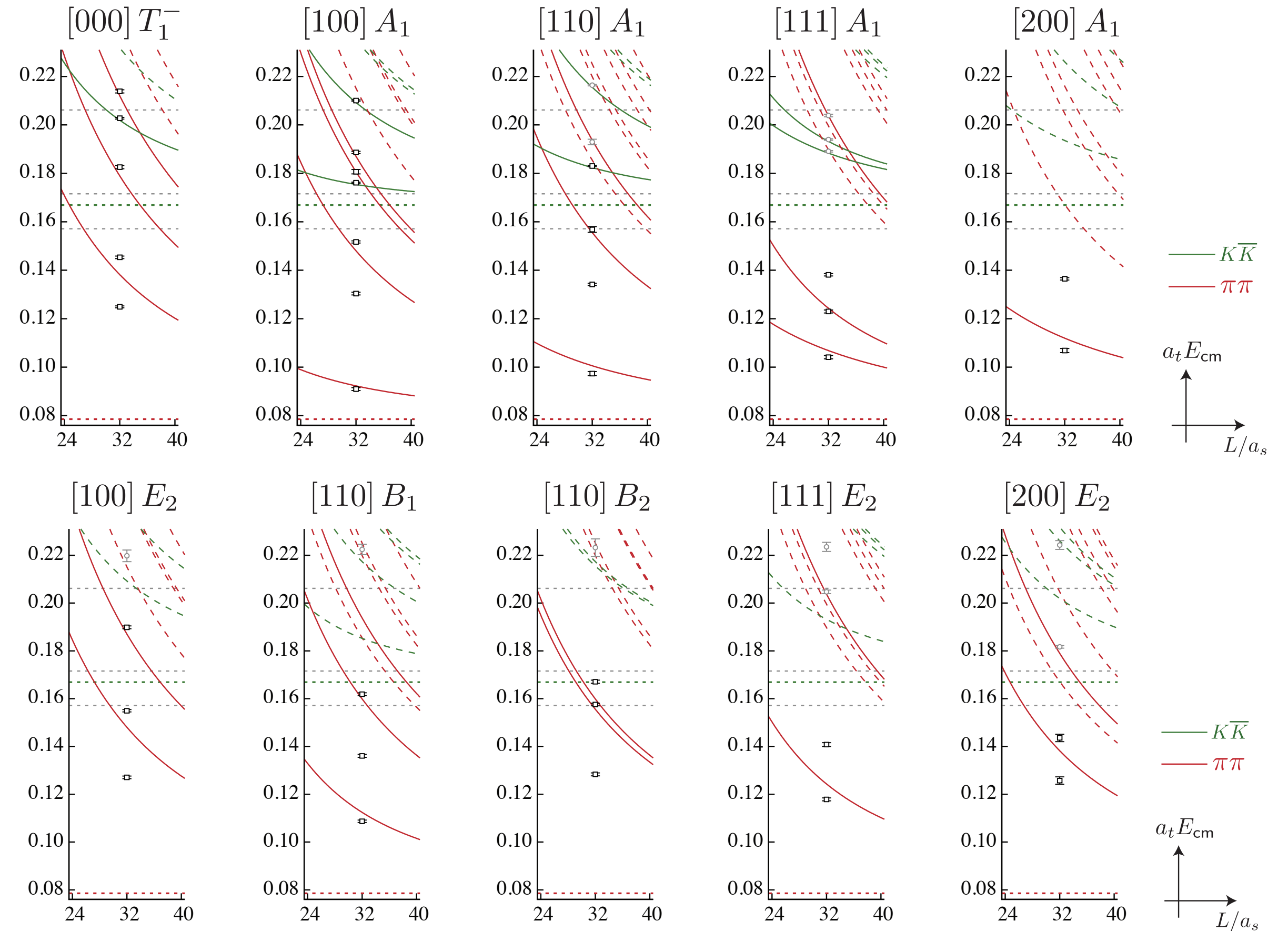


$\pi\pi$ scattering

($l=1$ channel)

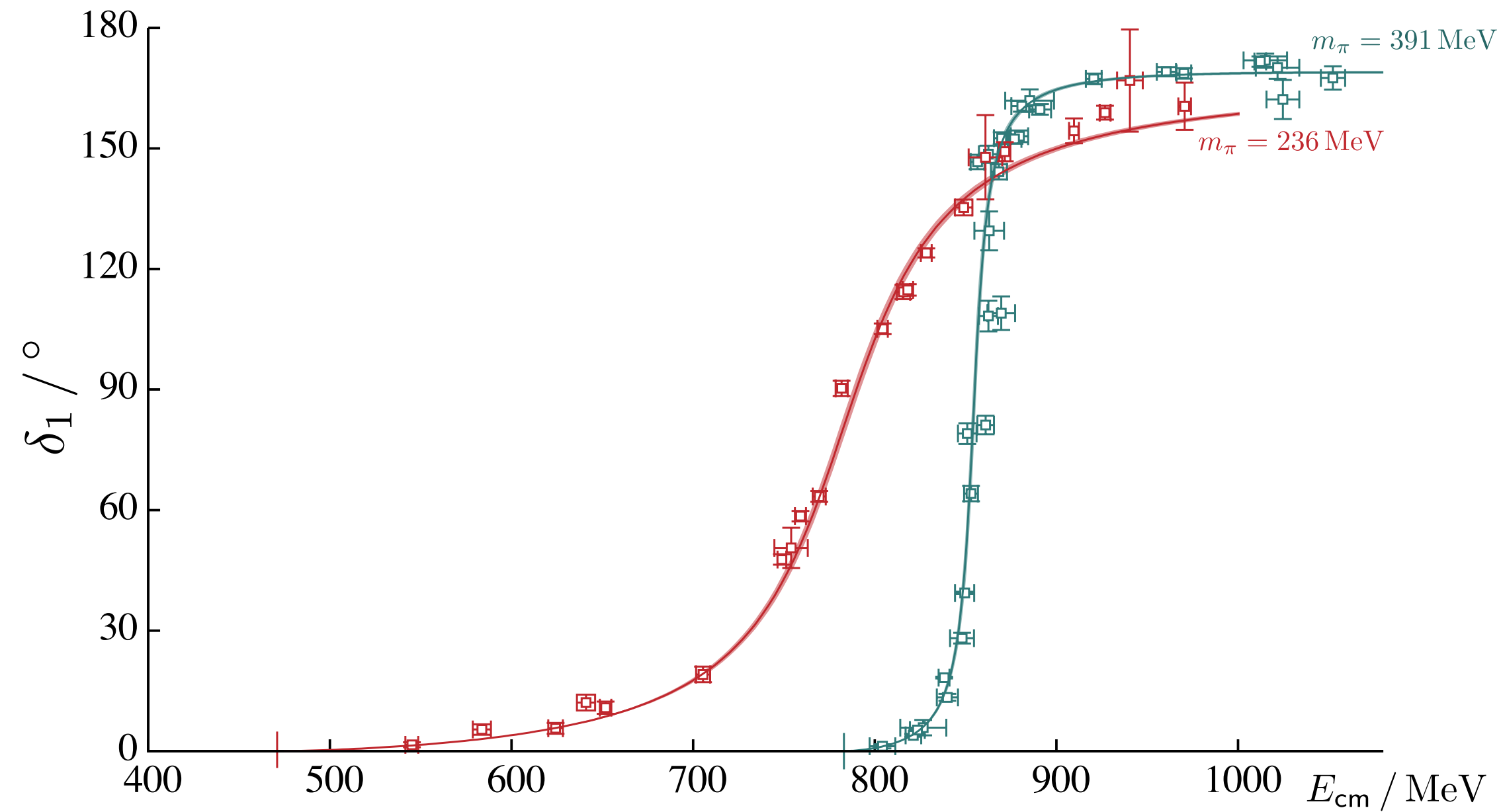


$$\mathcal{M} \sim \frac{1}{p \cot \delta - ip}$$

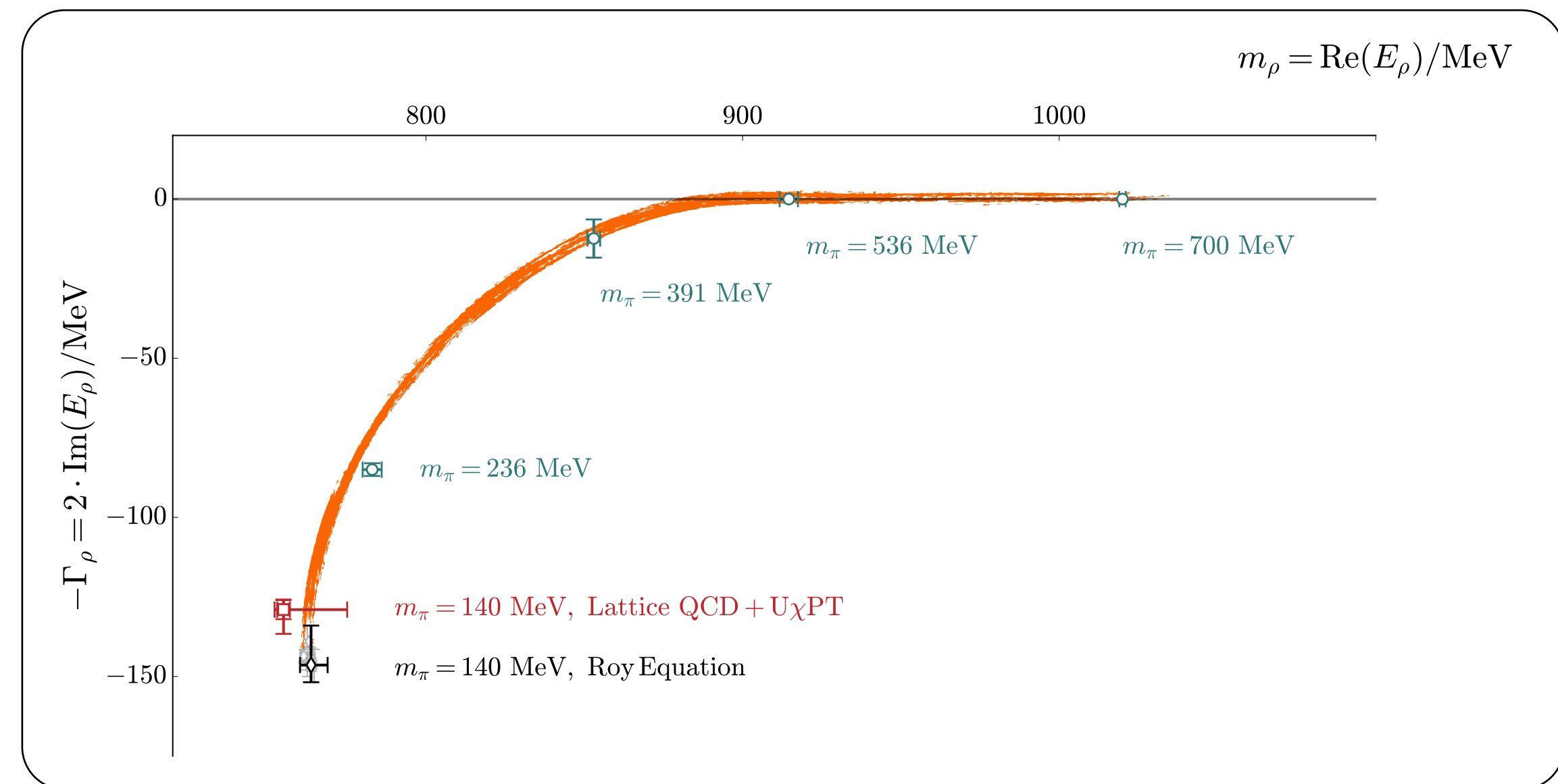
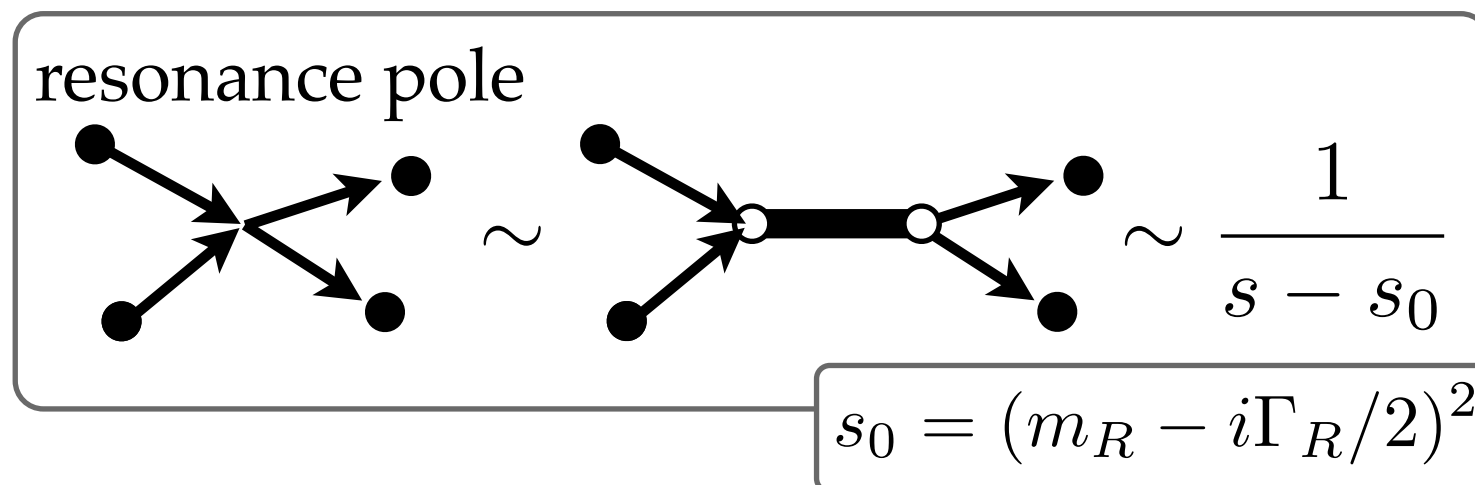


$\pi\pi$ scattering

($l=1$ channel)

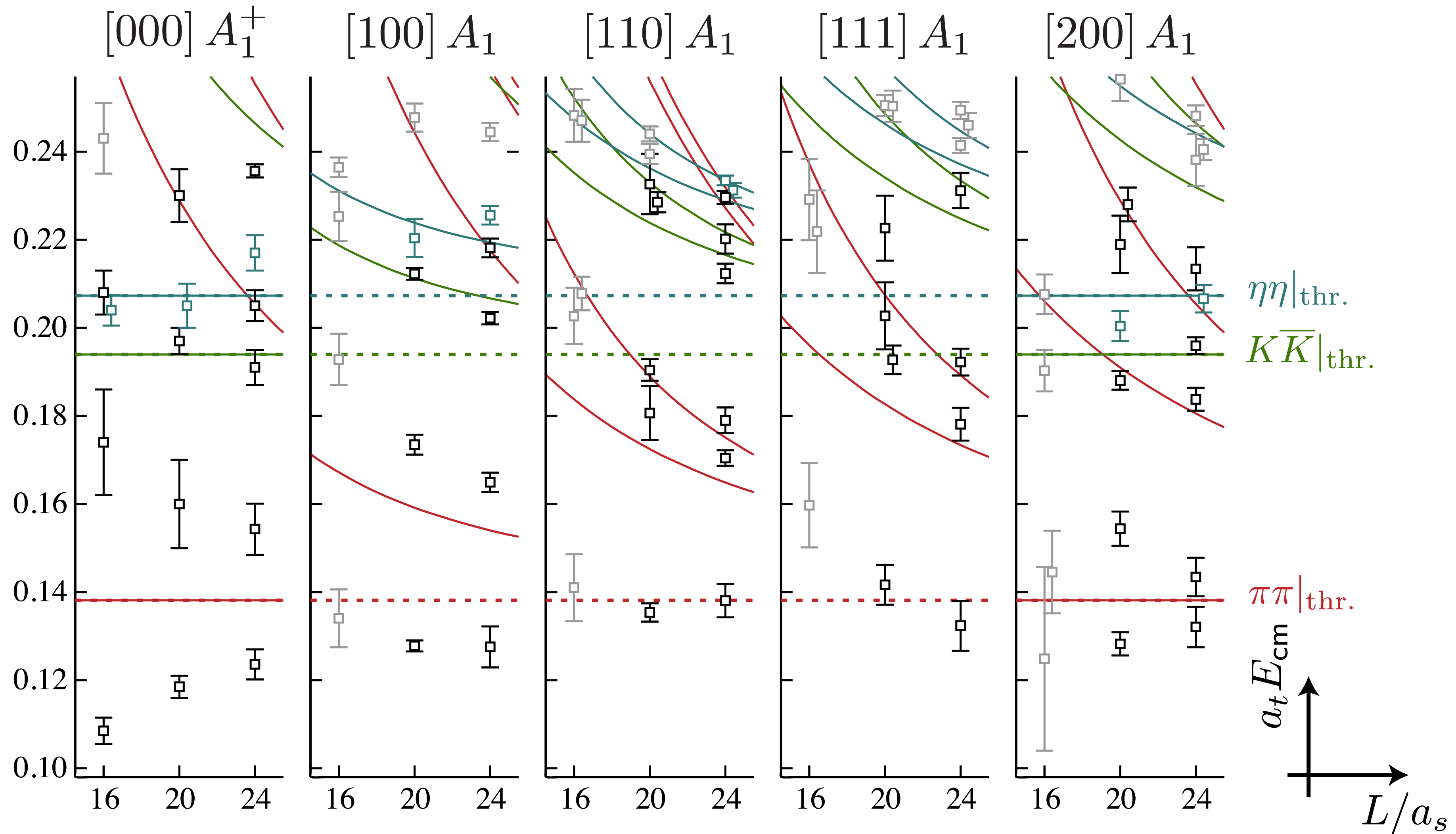


$$\mathcal{M} \sim \frac{1}{p \cot \delta - ip}$$



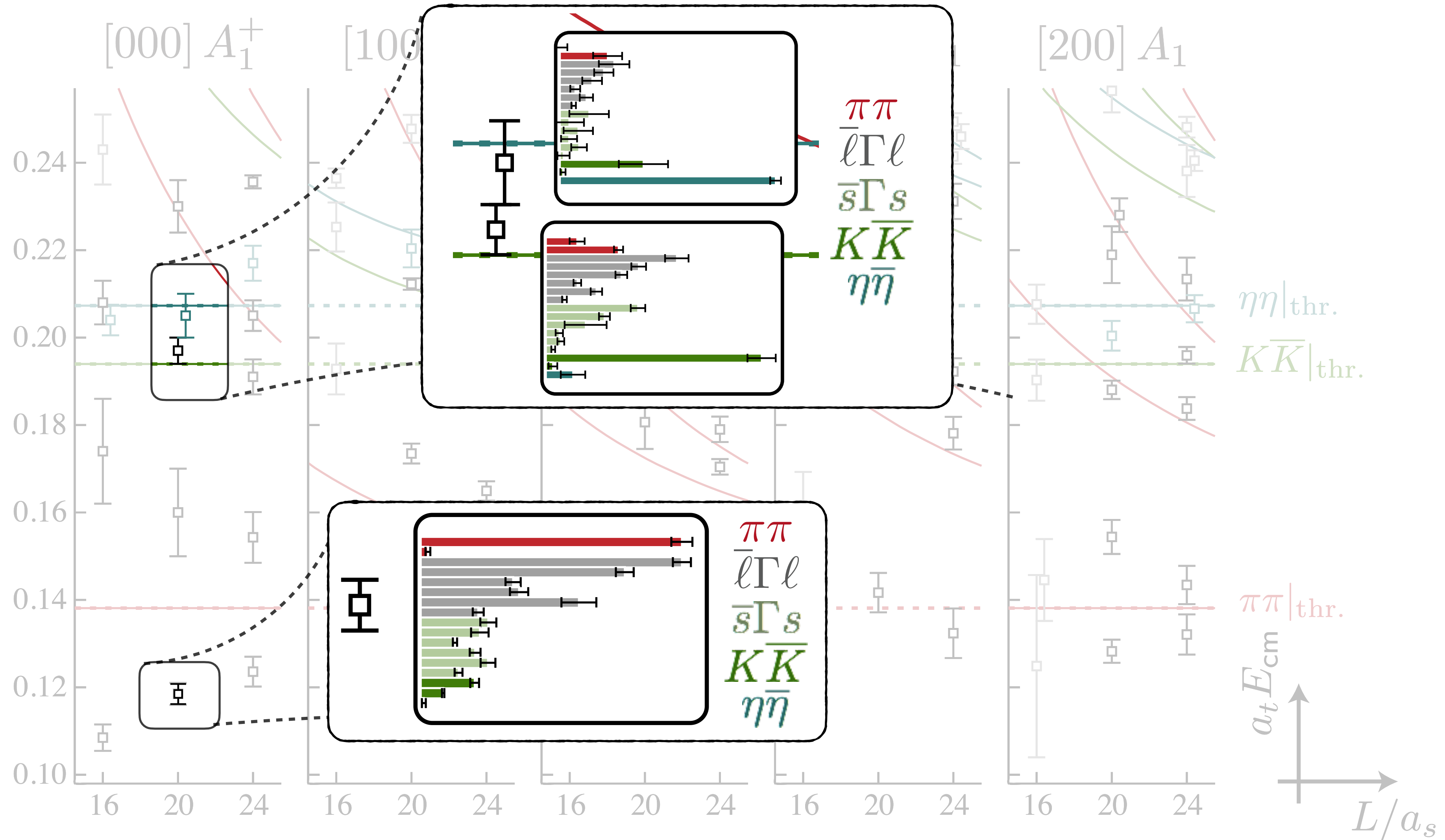
The vacuum channel in a finite volume

($l=0$ channel, $m_\pi \sim 390$ MeV)



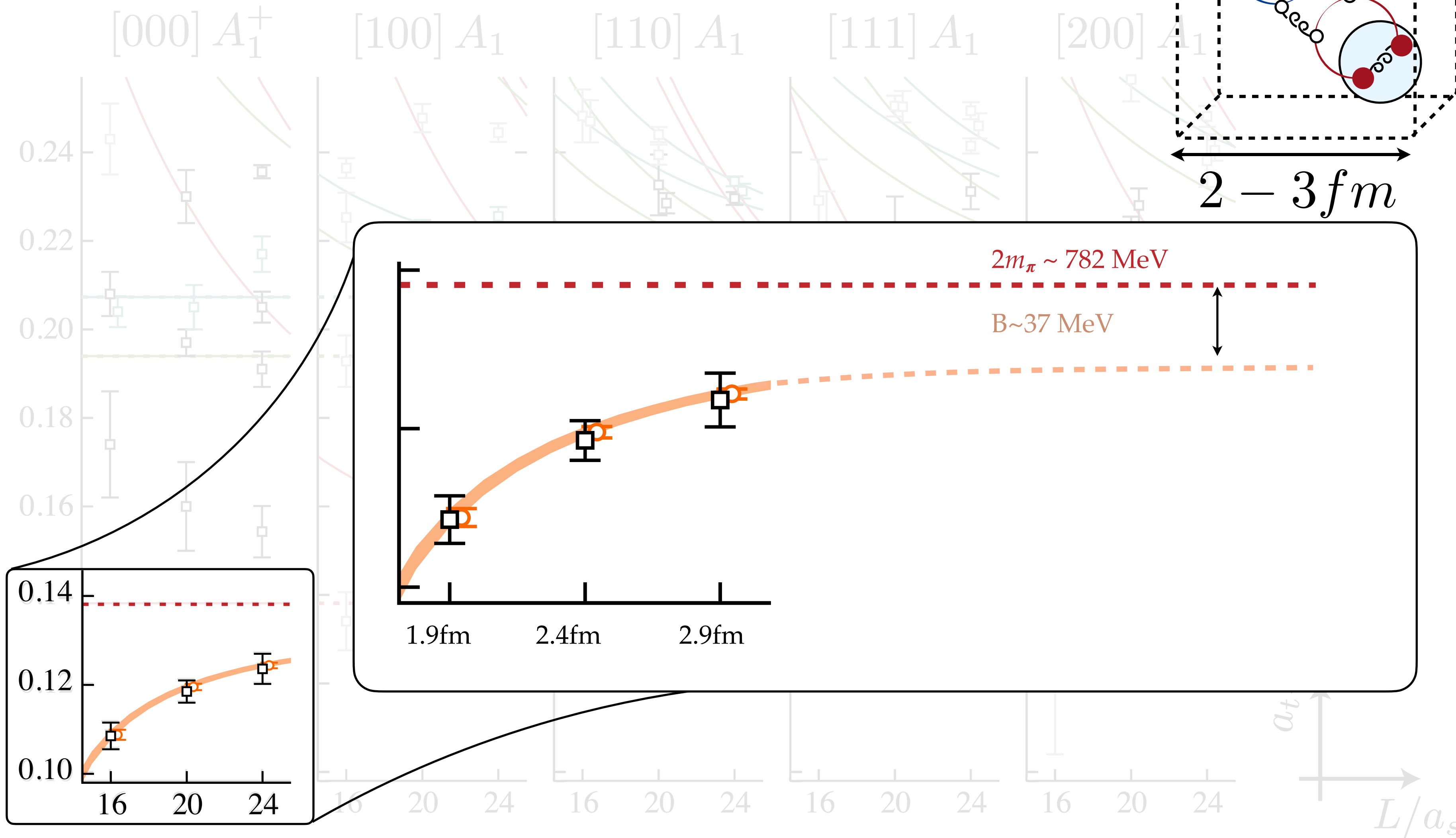
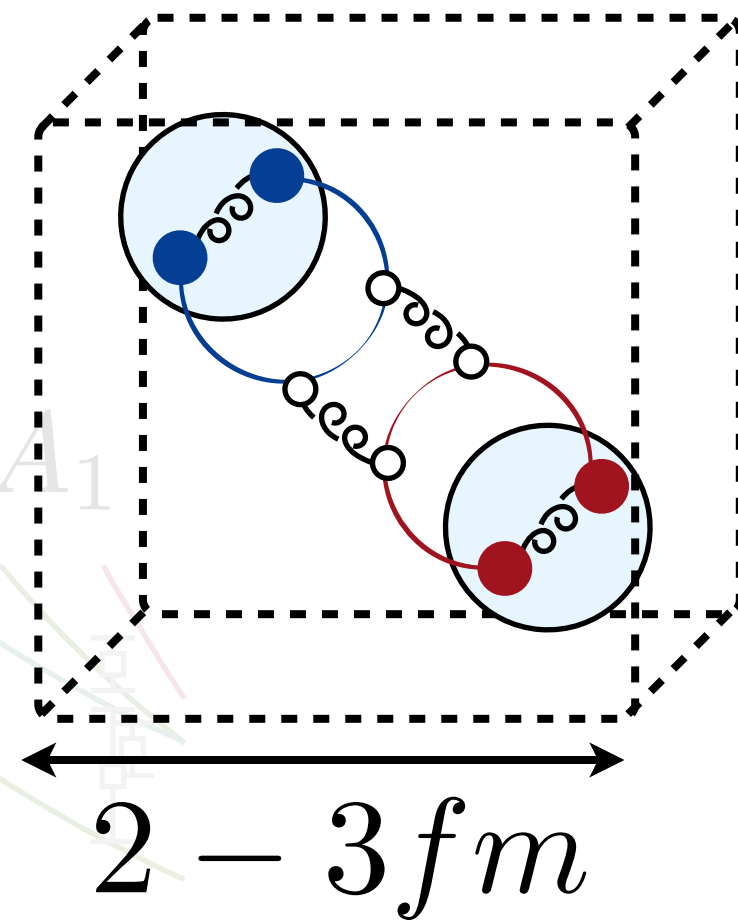
The vacuum channel in a finite volume

($l=0$ channel, $m_\pi \sim 390$ MeV)



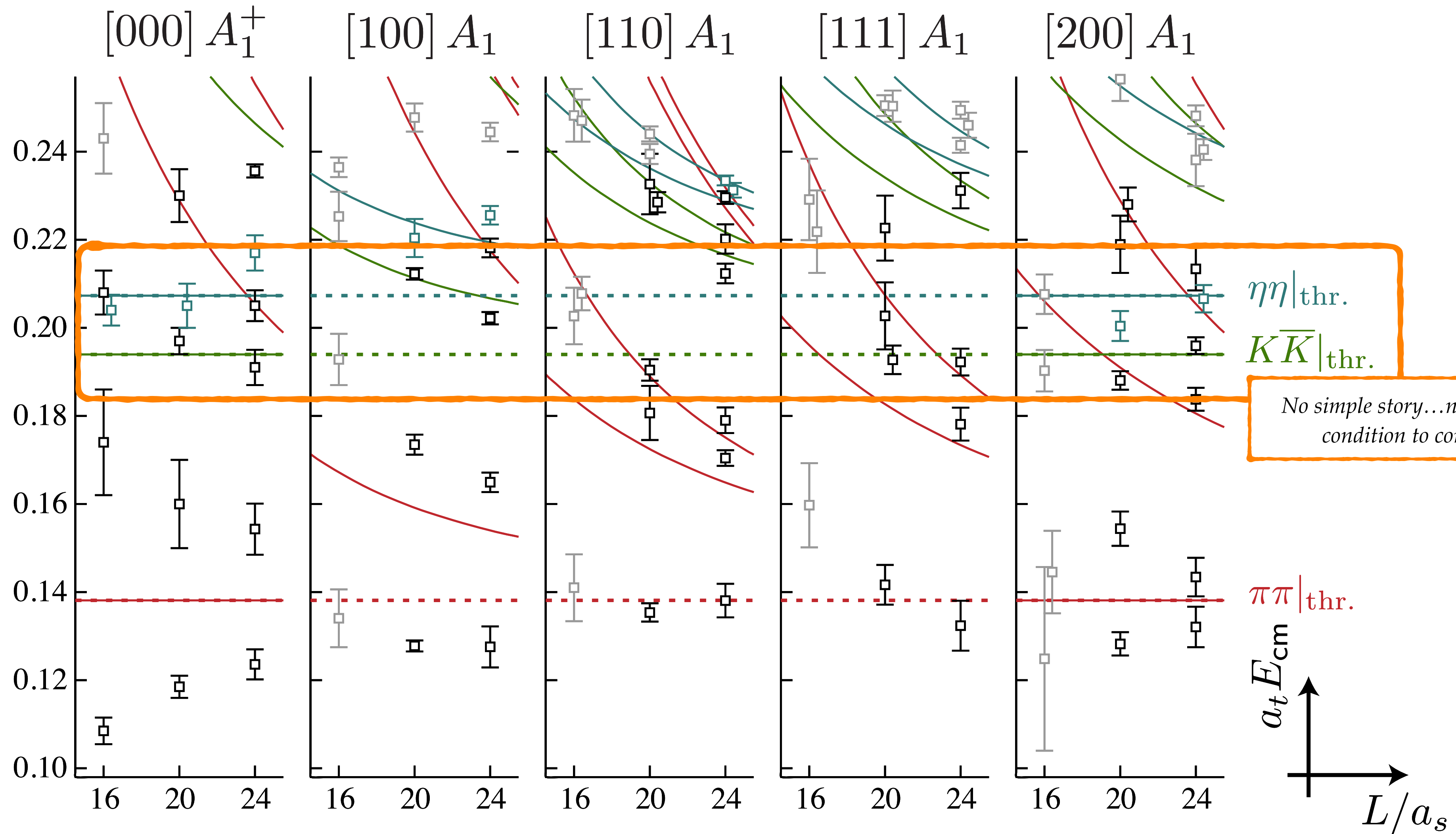
The vacuum channel in a finite volume

($l=0$ channel, $m_\pi \sim 390$ MeV)



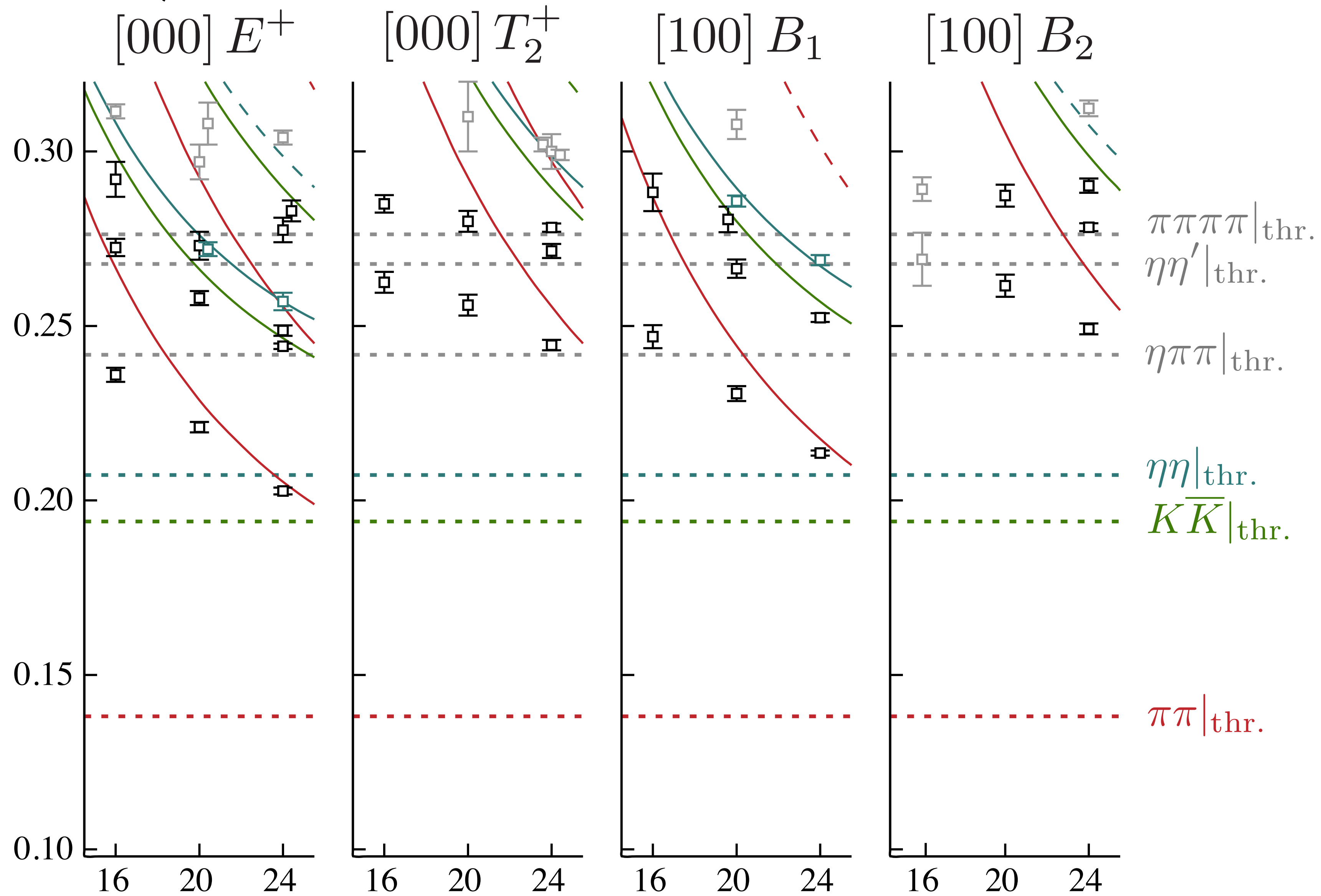
The vacuum channel in a finite volume

($l=0$ channel, $m_\pi \sim 390$ MeV)



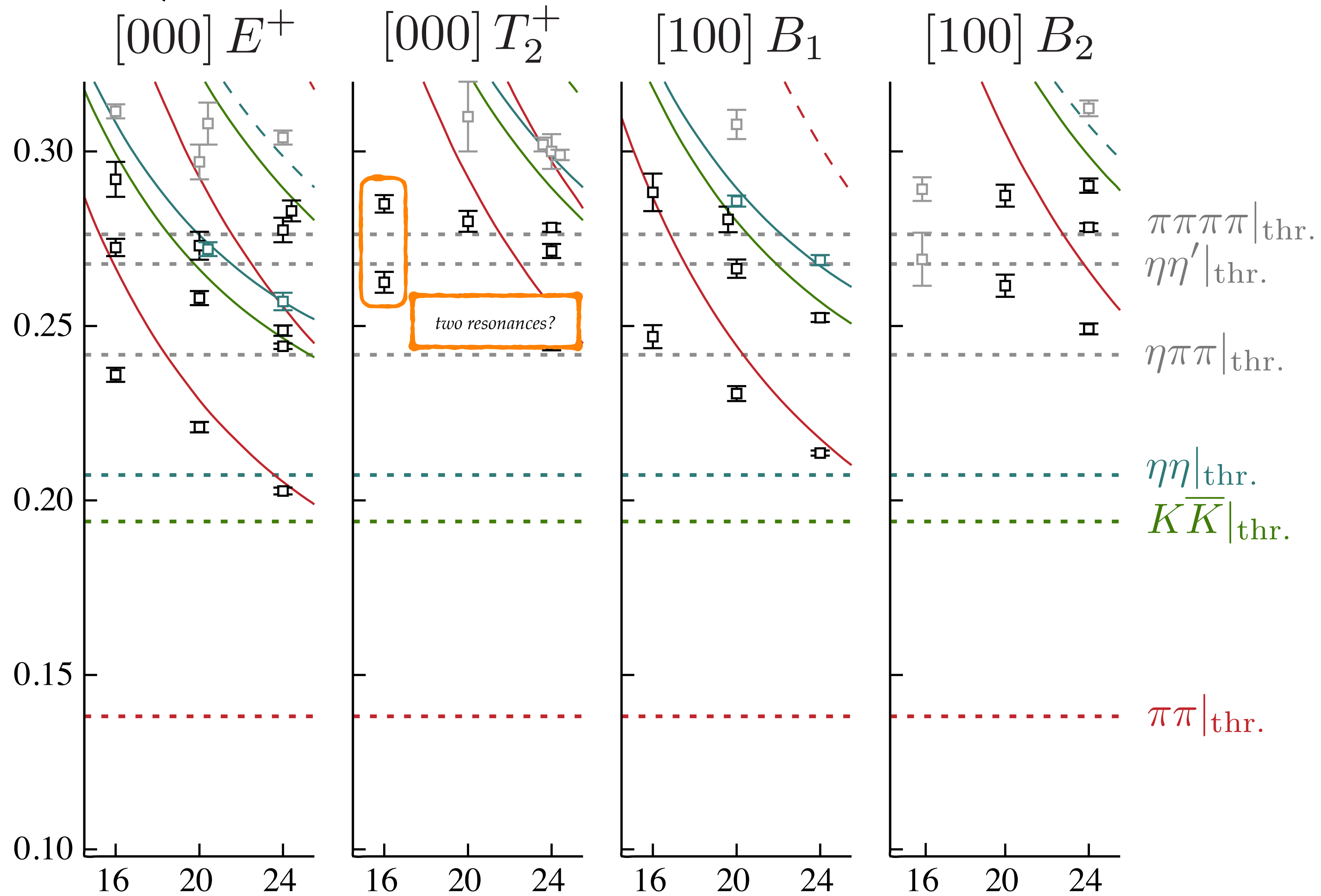
Isoscalar spectra // D-wave dominant

($l=0$ channel, $m_\pi \sim 390$ MeV)



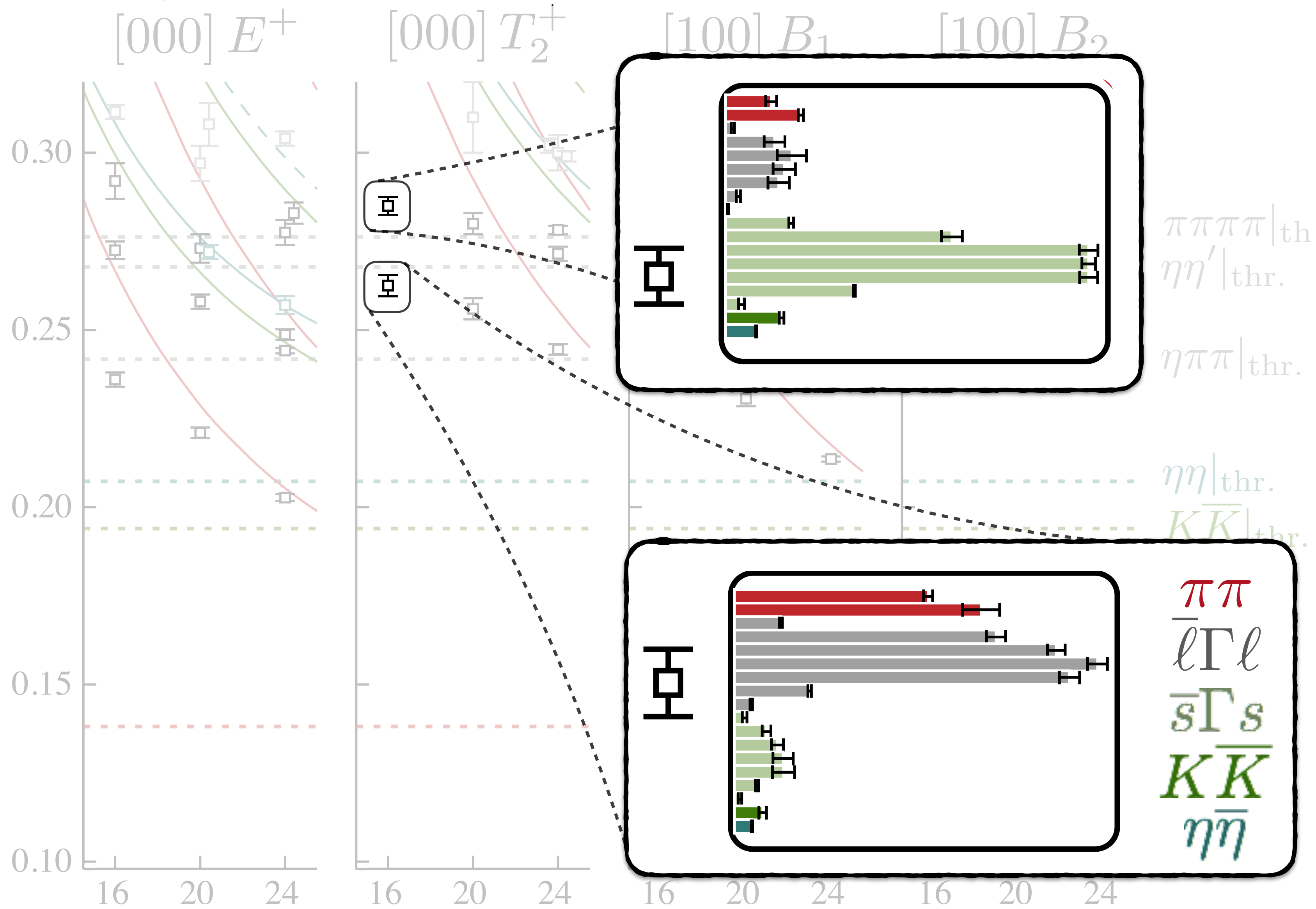
Isoscalar spectra // D-wave dominant

($l=0$ channel, $m_\pi \sim 390$ MeV)



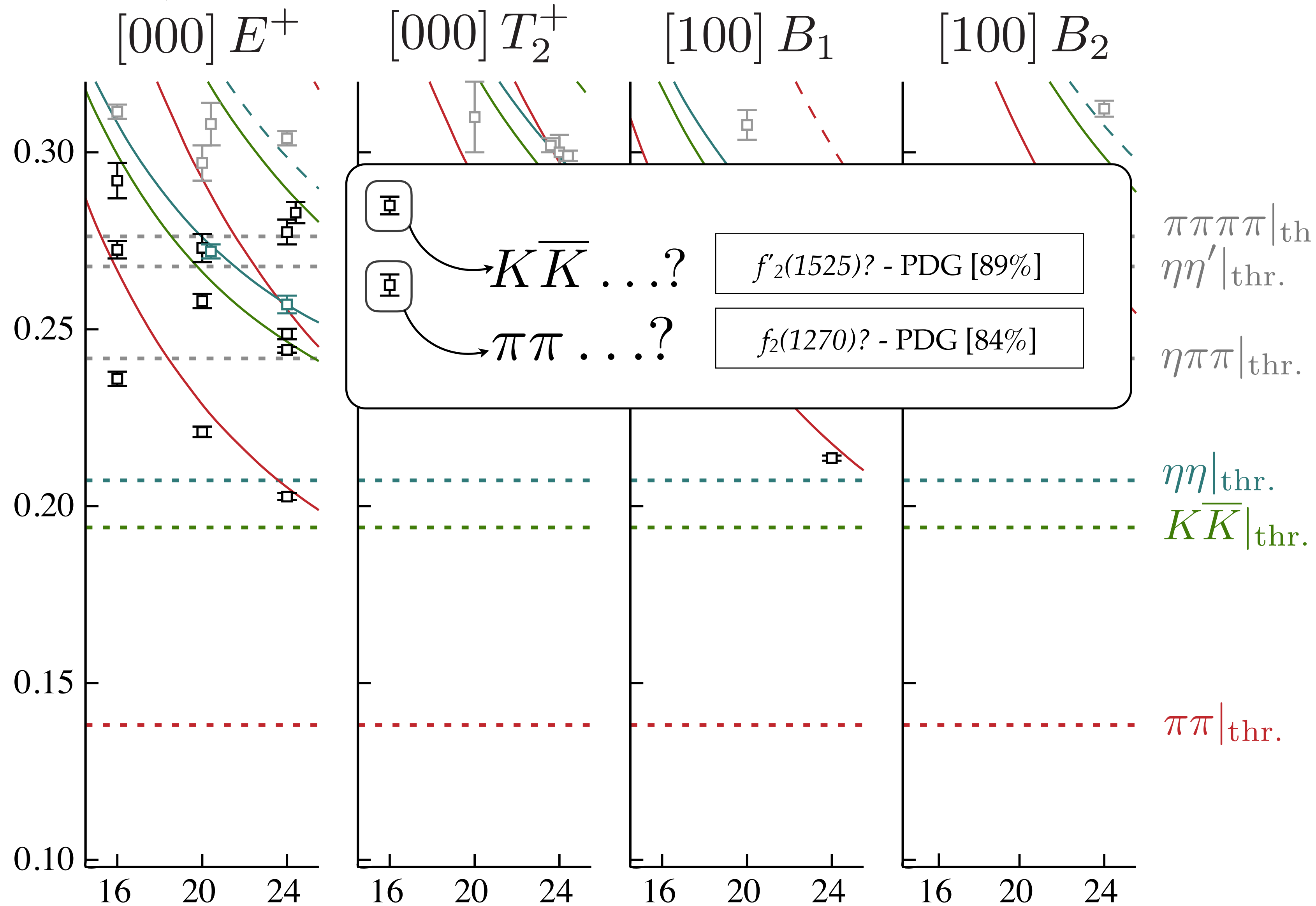
Isoscalar spectra // D-wave dominant

($l=0$ channel, $m_\pi \sim 390$ MeV)



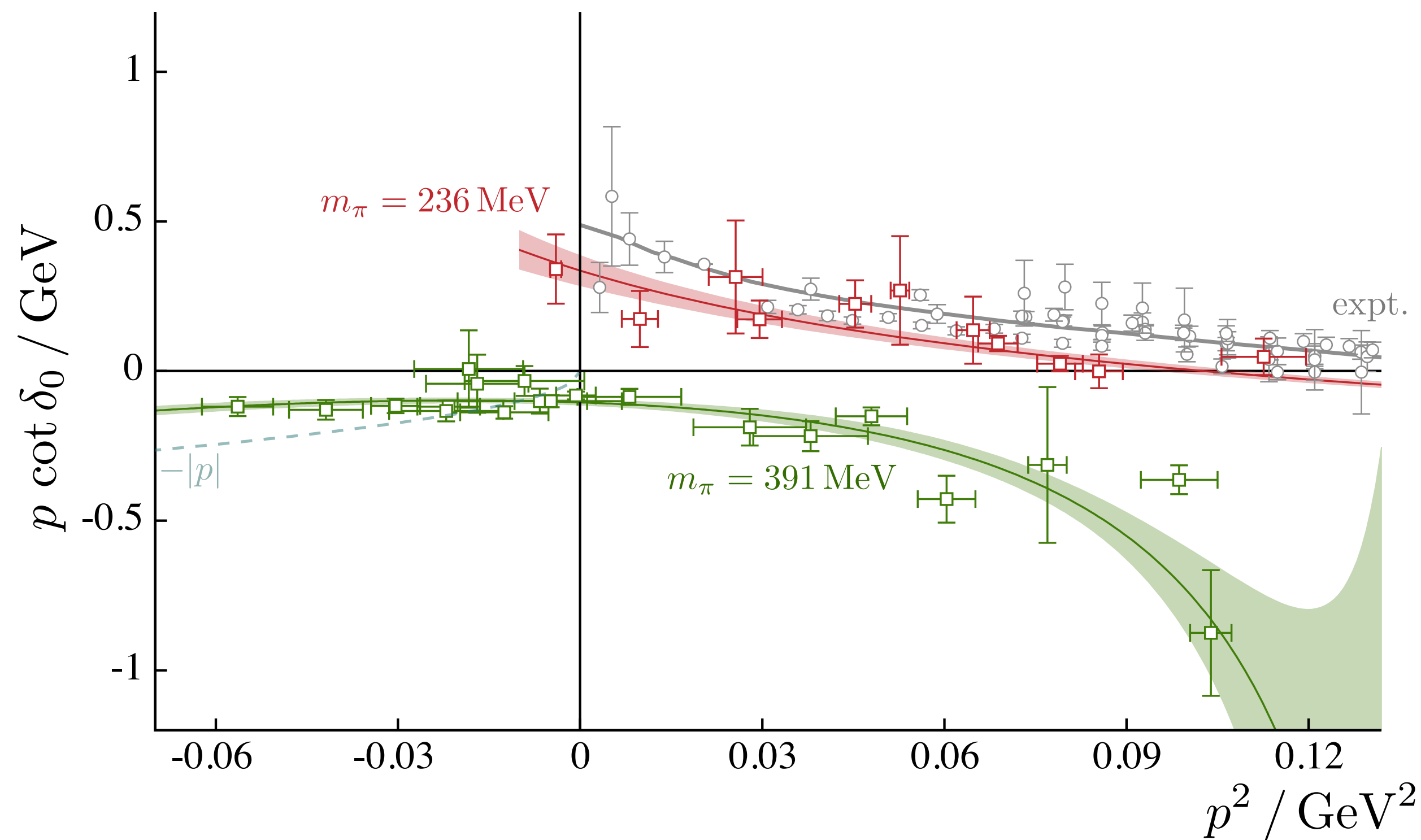
Isoscalar spectra // D-wave dominant

($l=0$ channel, $m_\pi \sim 390$ MeV)

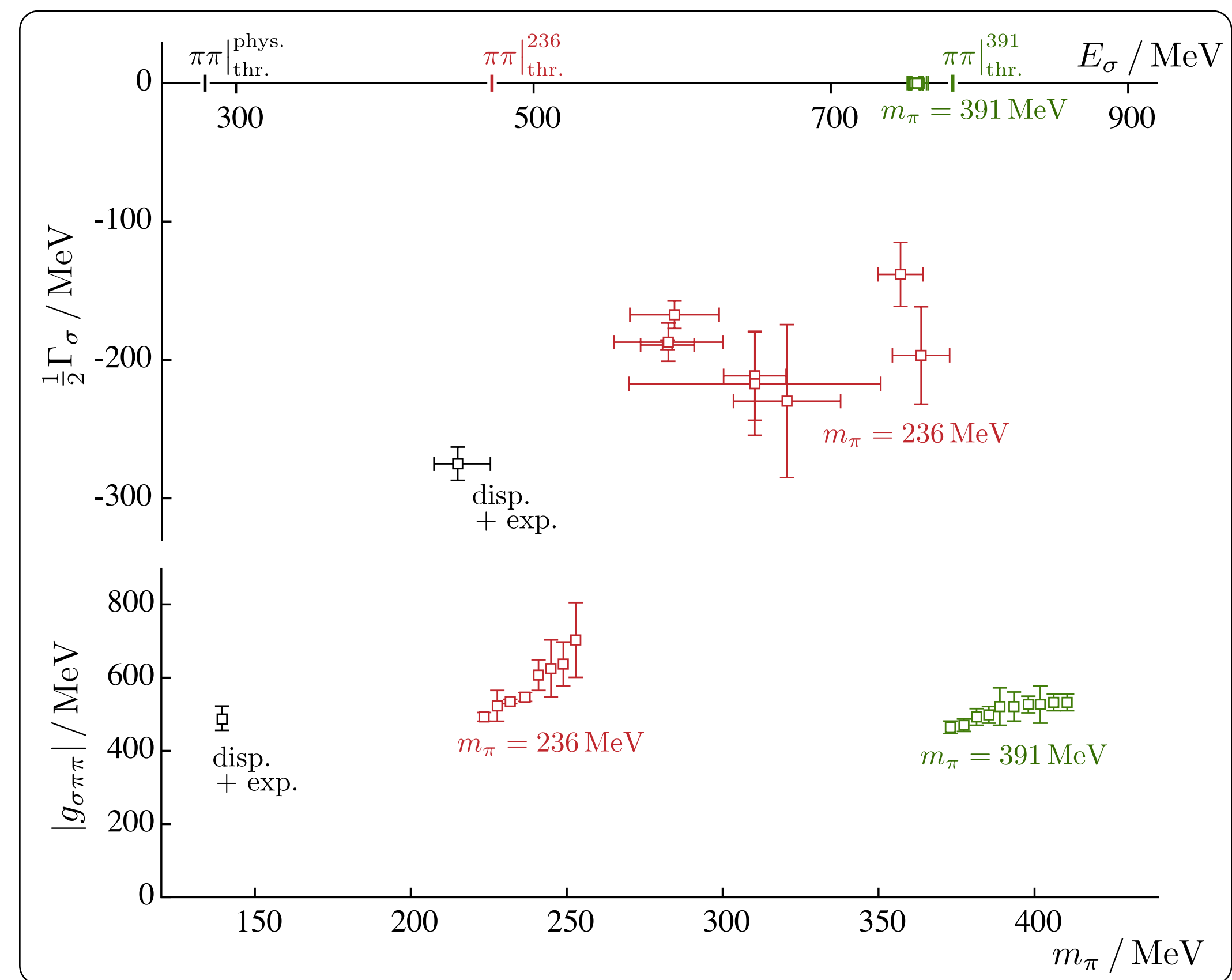
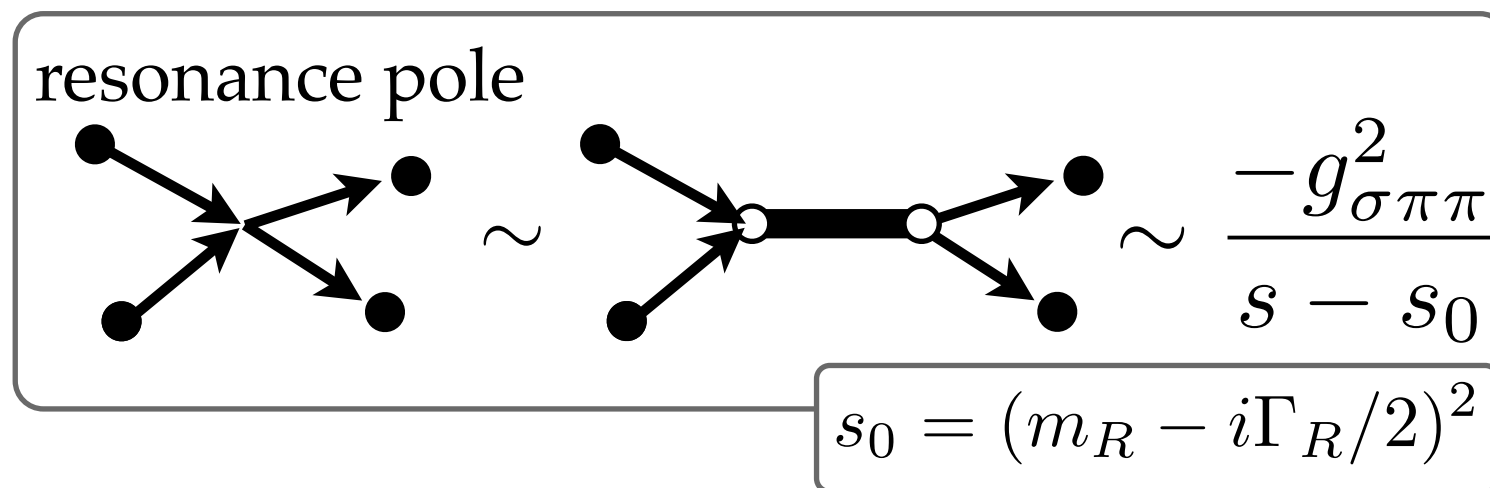


$\pi\pi$ scattering

($l=0$ // S-wave channel)



$$\mathcal{M} \sim \frac{1}{p \cot \delta - ip}$$



Outline

First half:

Elastic two-body scattering

Second half:

Two-body, coupled-channel scattering

Three-body scattering

Coupled-channels analysis

📌 Above $2m_K$, there is not a one-to-one correspondence

$$\det \begin{bmatrix} F_{\pi\pi}^{-1} + \mathcal{M}_{\pi\pi,\pi\pi} & \mathcal{M}_{\pi\pi,K\bar{K}} \\ \mathcal{M}_{\pi\pi,K\bar{K}} & F_{K\bar{K}}^{-1} + \mathcal{M}_{K\bar{K},K\bar{K}} \end{bmatrix} = 0$$

📌 In general, must constrain $(1/2) [N^2 + N]$ functions of energy

📌 Need that many energy levels at the same energy

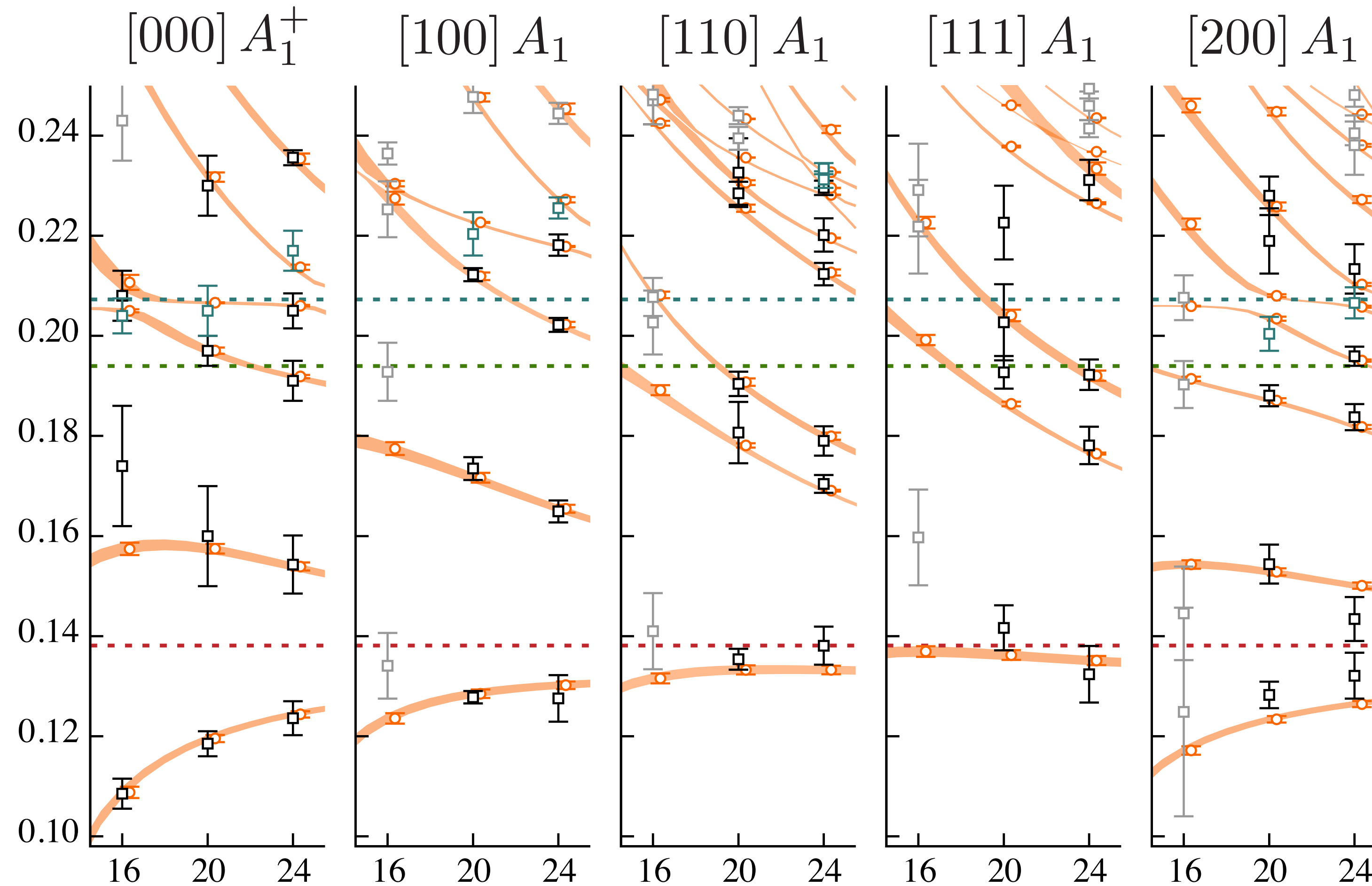
📌 Alternatively, parametrize scattering amplitude and do a global fit

Coupled-channels analysis

📌 S-wave above $2m_\pi$, $2m_K$, and $2m_\eta$

📌 Ansatz $\mathbf{K}^{-1}(s) = \begin{pmatrix} a + bs & c + ds & e \\ c + ds & f & g \\ e & g & h \end{pmatrix}$

$$\chi^2/N_{\text{dof}} = \frac{44.0}{57 - 8} = 0.90$$

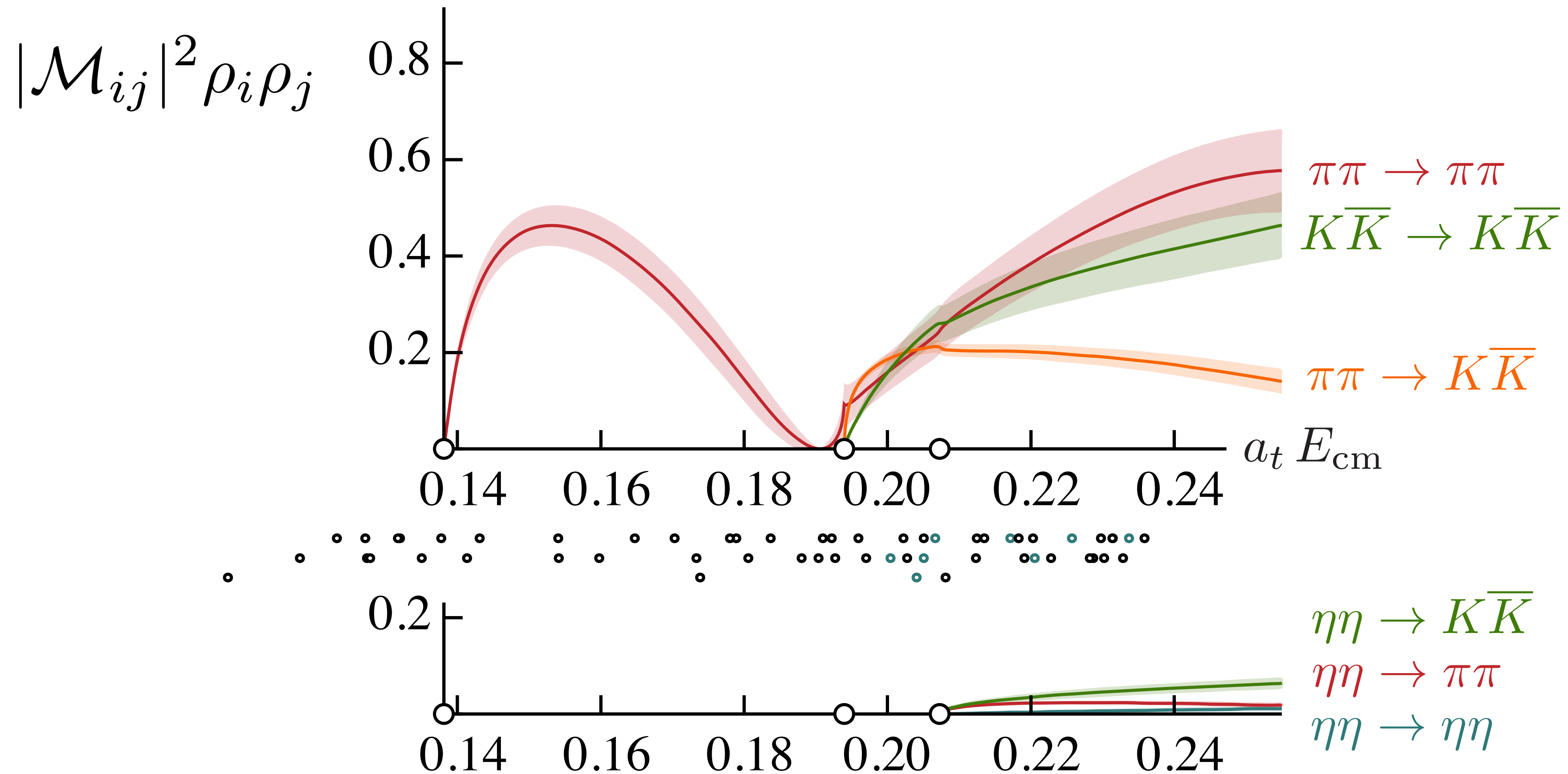


57 energy levels

Coupled-channels analysis

📌 S-wave above $2m_\pi$, $2m_K$, and $2m_\eta$

📌 Ansatz $\mathbf{K}^{-1}(s) = \begin{pmatrix} a + bs & c + ds & e \\ c + ds & f & g \\ e & g & h \end{pmatrix}$

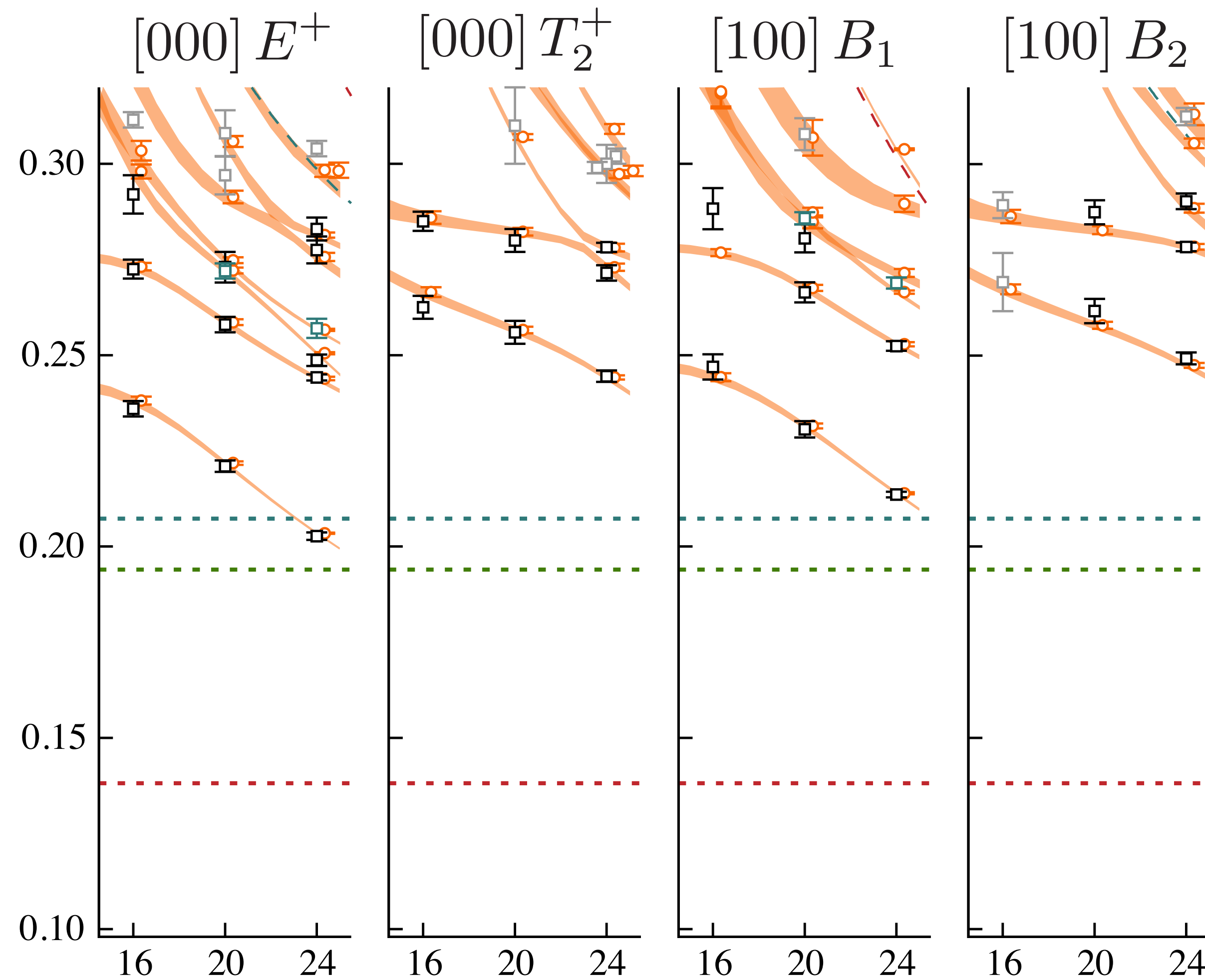


Coupled-channels analysis

📌 D-wave above $2m_\pi$, $2m_K$, and $2m_\eta$

📌 Ansatz
$$K_{ij}(s) = \frac{g_i^{(1)} g_j^{(1)}}{m_1^2 - s} + \frac{g_i^{(2)} g_j^{(2)}}{m_2^2 - s} + \gamma_{ij} \quad \begin{array}{l} \gamma_{\eta\eta} \neq 0 \\ \gamma_{ij} = 0 \text{ otherwise} \end{array}$$

$$\chi^2/N_{\text{dof}} = \frac{28.9}{34 - 9} = 1.15$$

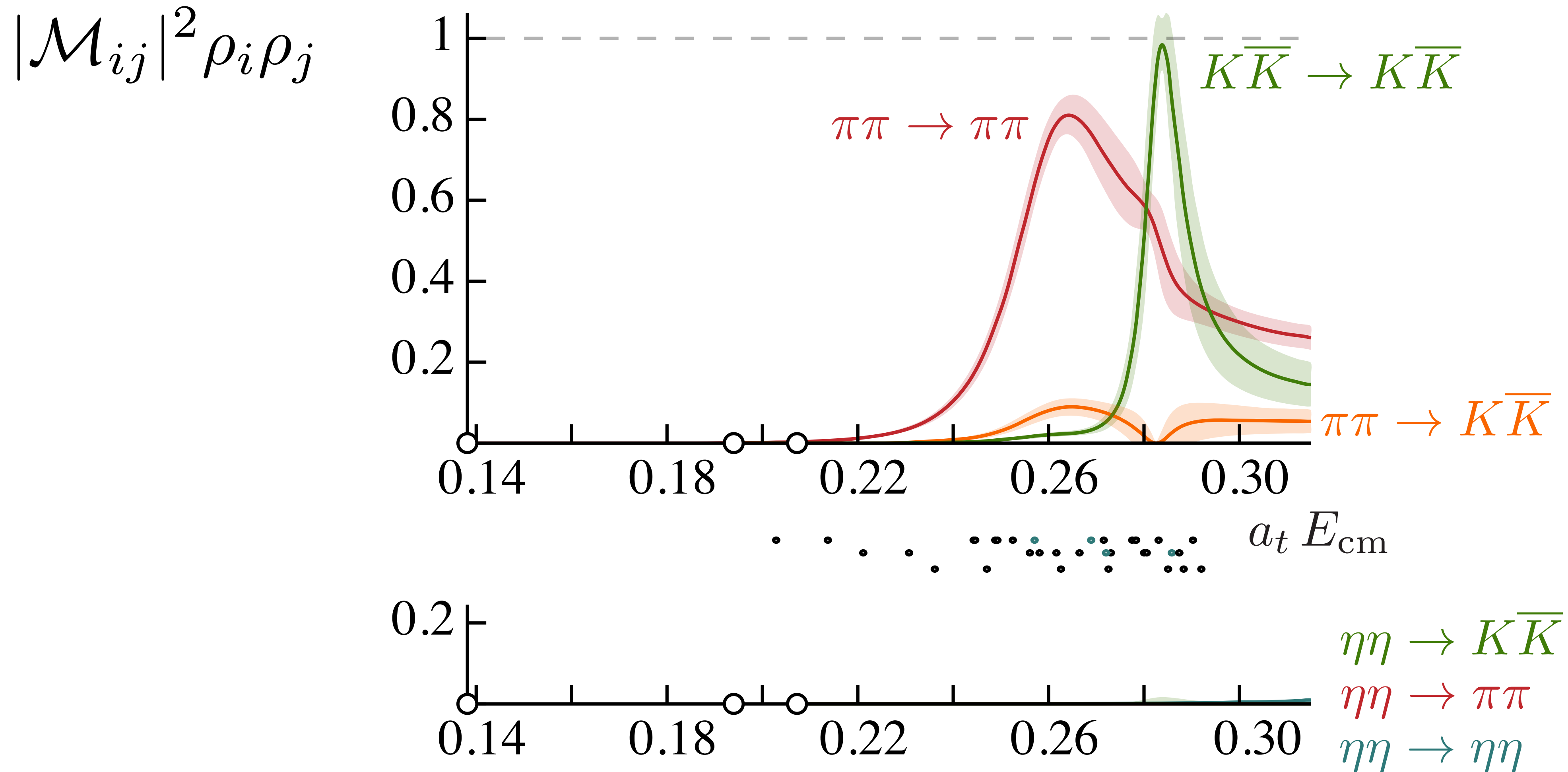


34 energy levels

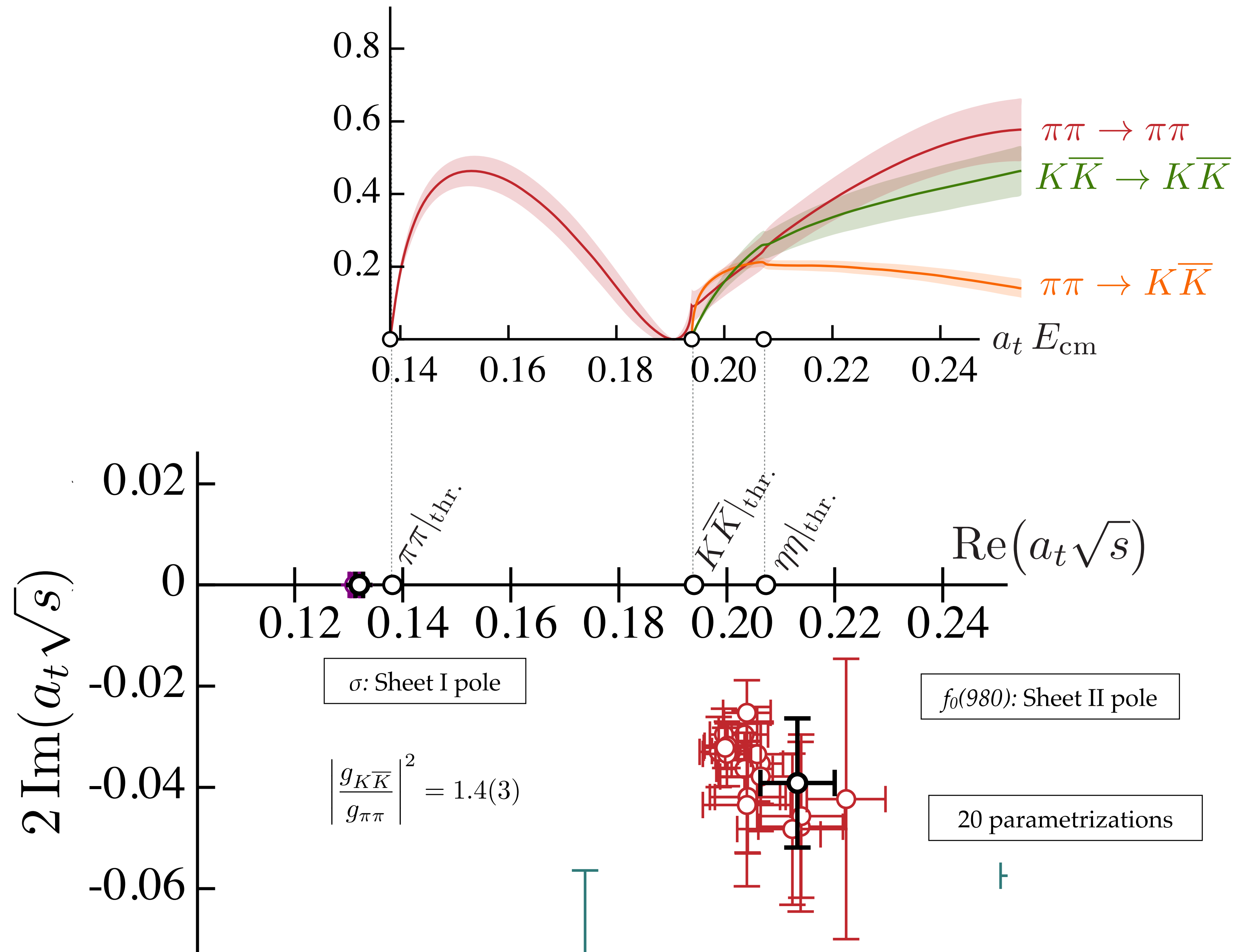
Coupled-channels analysis

📌 D-wave above $2m_\pi$, $2m_K$, and $2m_\eta$

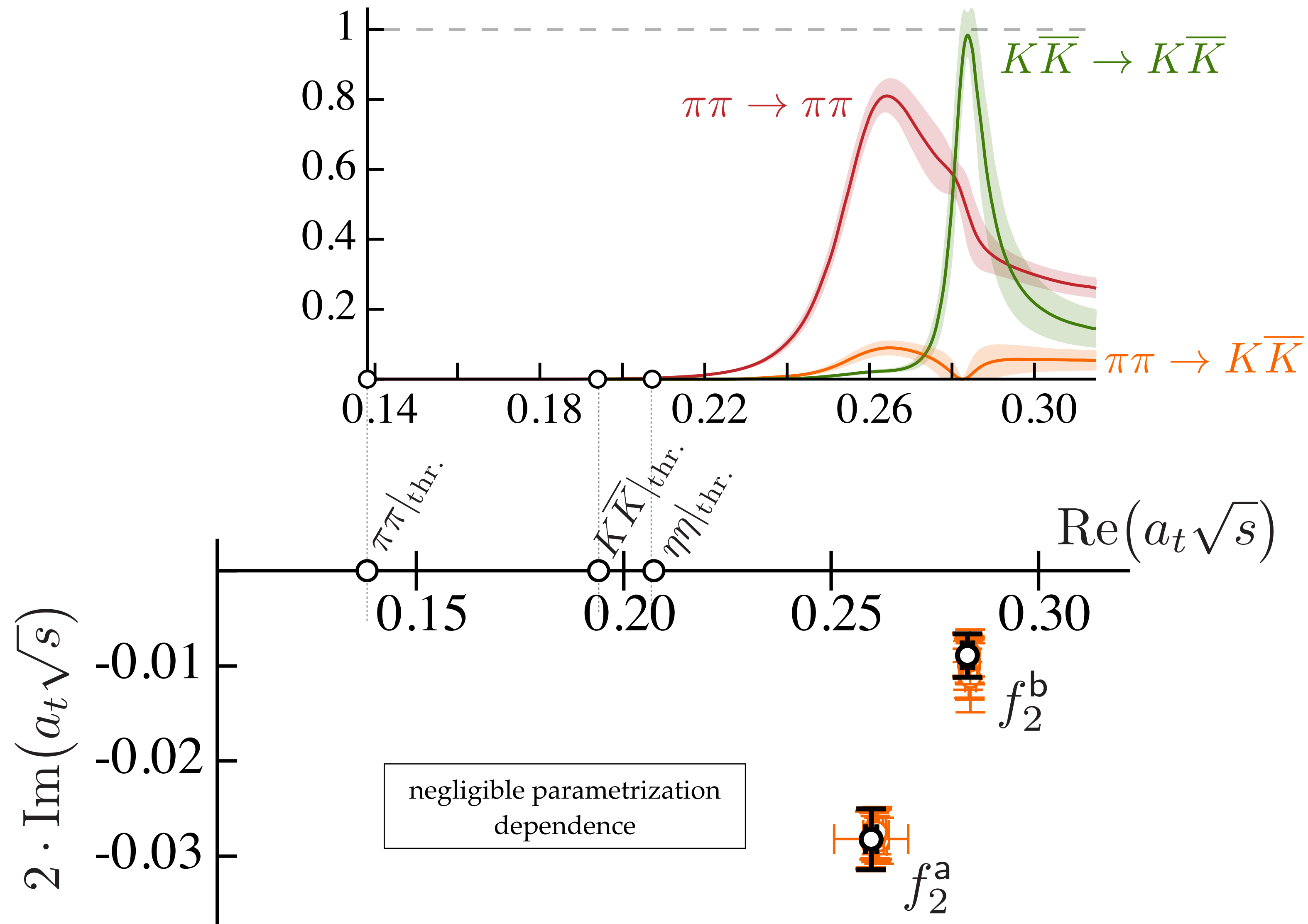
📌 Ansatz
$$K_{ij}(s) = \frac{g_i^{(1)} g_j^{(1)}}{m_1^2 - s} + \frac{g_i^{(2)} g_j^{(2)}}{m_2^2 - s} + \gamma_{ij} \quad \begin{array}{l} \gamma_{\eta\eta} \neq 0 \\ \gamma_{ij} = 0 \text{ otherwise} \end{array}$$



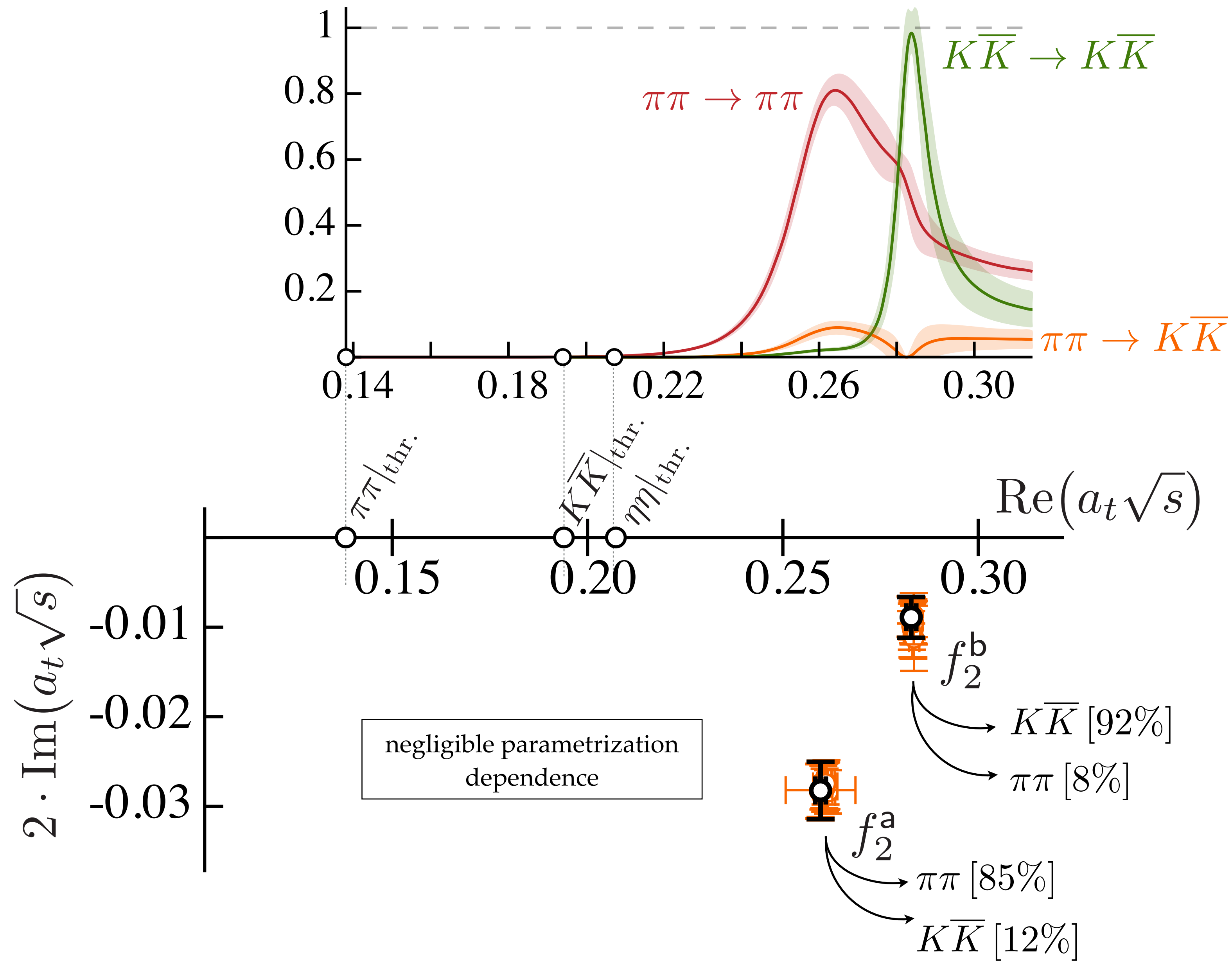
Scalar poles: σ and $f_0(980)$



Tensor poles: the f_2 's



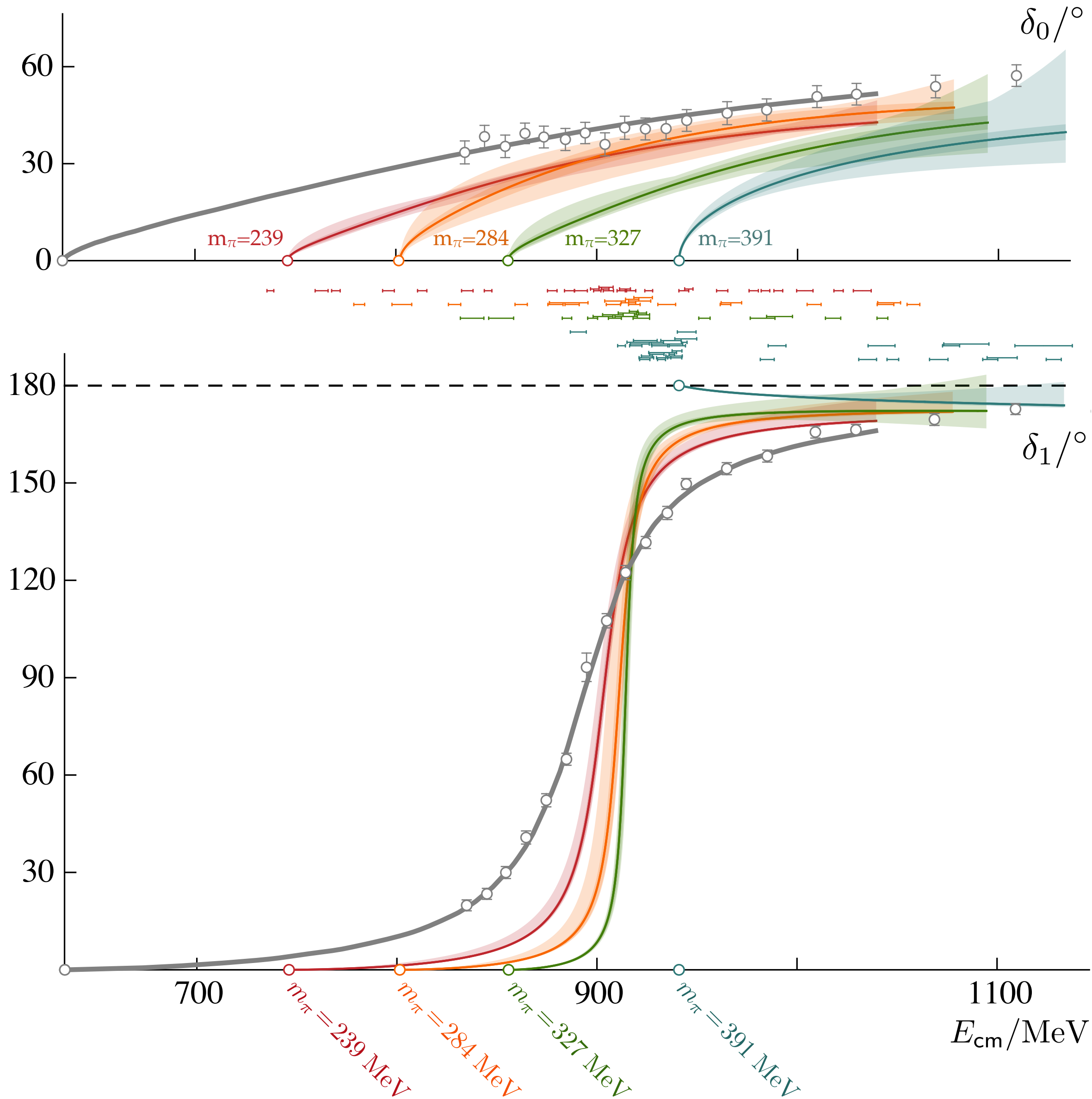
Tensor poles: the f_2 's



πK scattering

($l=1/2$ channel)

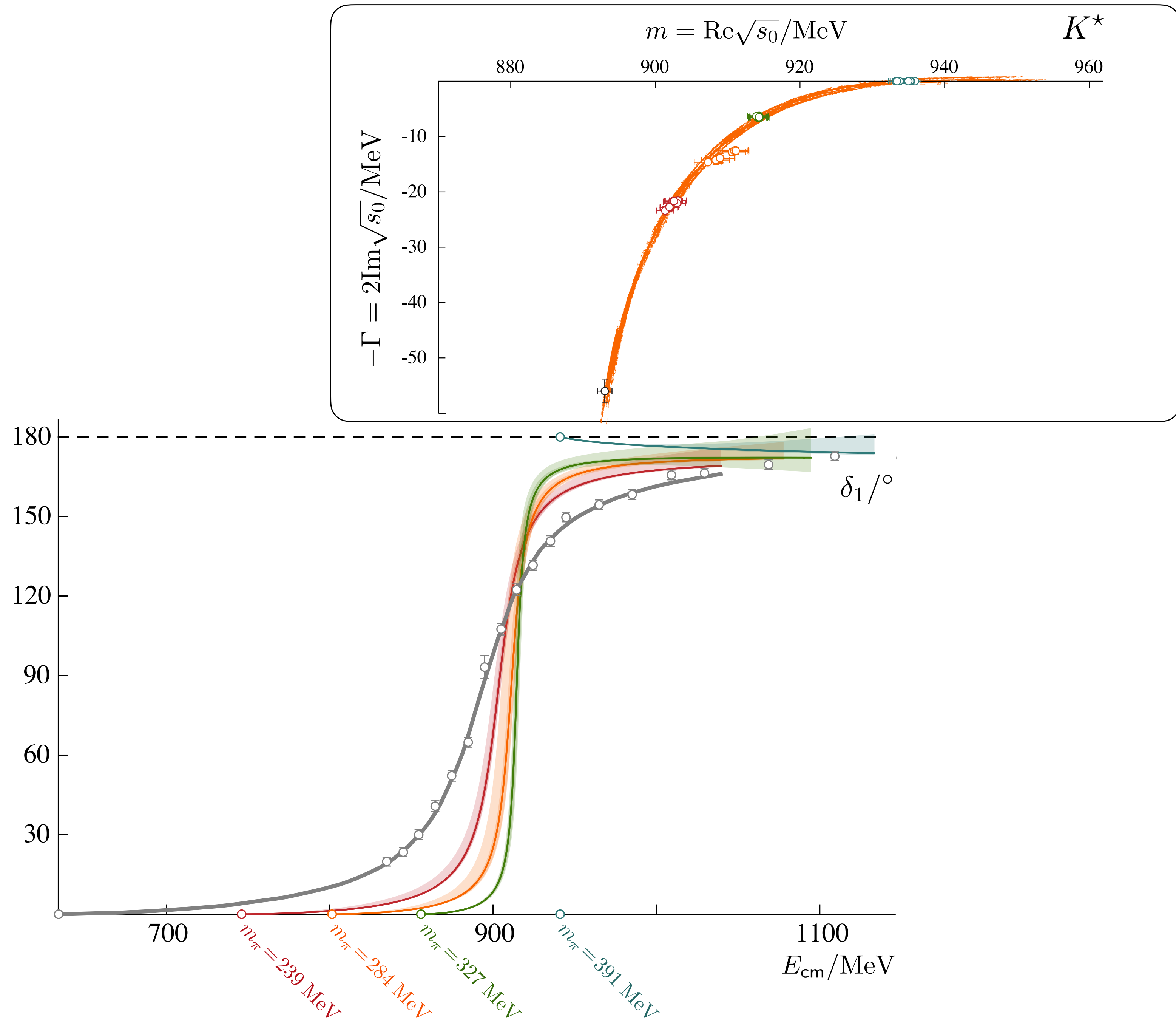
$$\mathcal{M} \sim \frac{1}{p \cot \delta - ip}$$



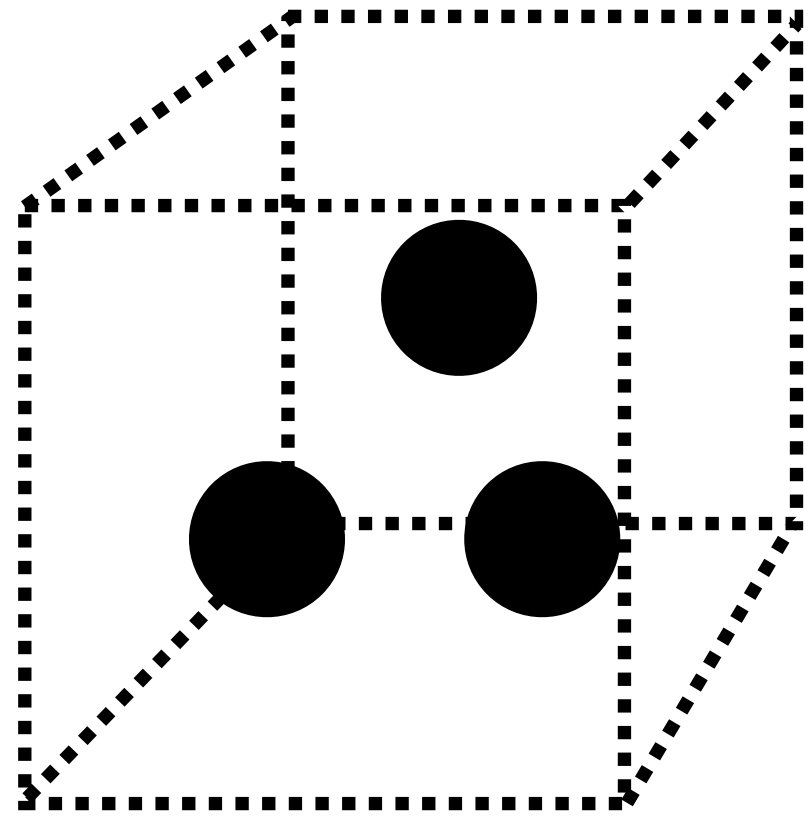
πK scattering

($l=1/2$ channel)

$$\mathcal{M} \sim \frac{1}{p \cot \delta - ip}$$



Three-hadron systems

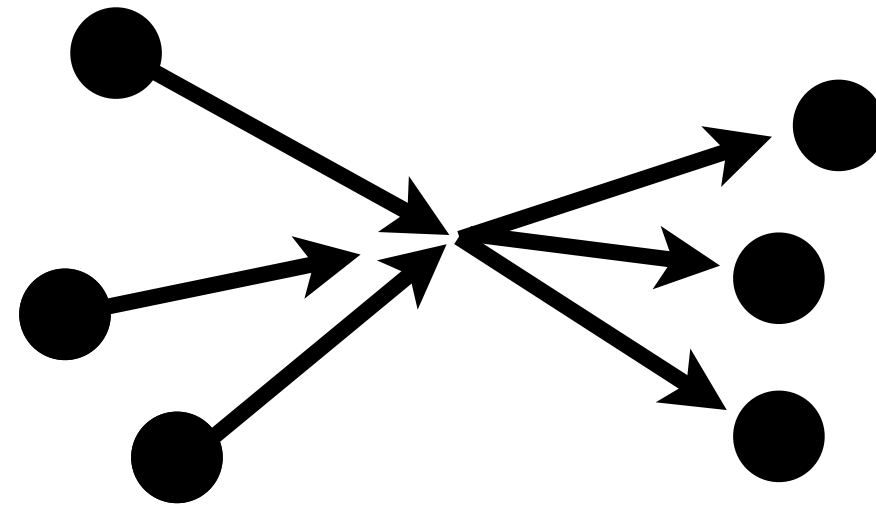


finite-volume
spectroscopy

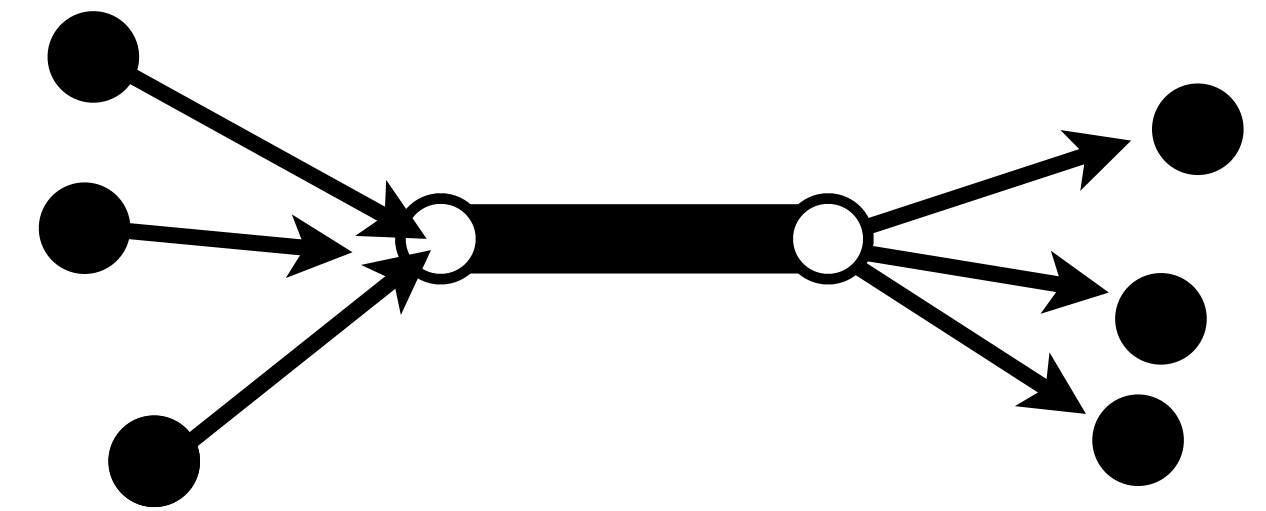
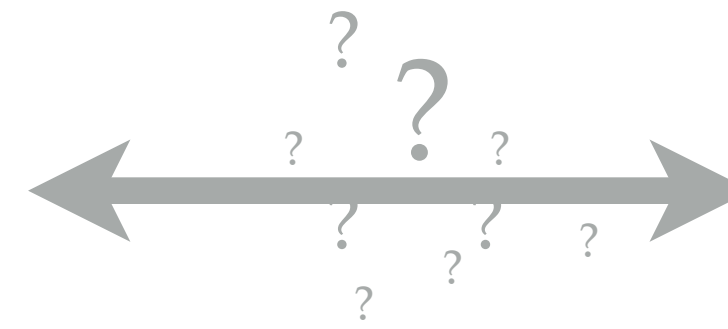
$$\det [F_3(E_L, L) + \mathcal{K}_{\text{df},3}^{-1}(E_L)] = 0$$



Hansen & Sharpe ('14, '15)
Mai & Döring ('17)
RB, Hansen & Sharpe ('18)
Hansen, Romero-Lopez & Sharpe ('20)
Blanton & Sharpe ('20)
Jackura et al. ('20)



infinite-volume
scattering amplitudes

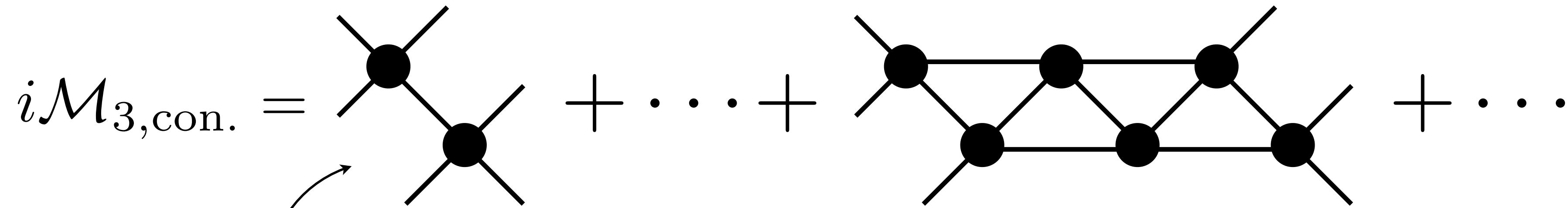


bound state and
resonance poles

Three hadrons in an infinite volume

The three-body scattering amplitude using all orders perturbation theory.

Sum over all connected 3-to-3 diagrams...



$$G \sim \frac{1}{(P - p - k)^2 - m^2}$$

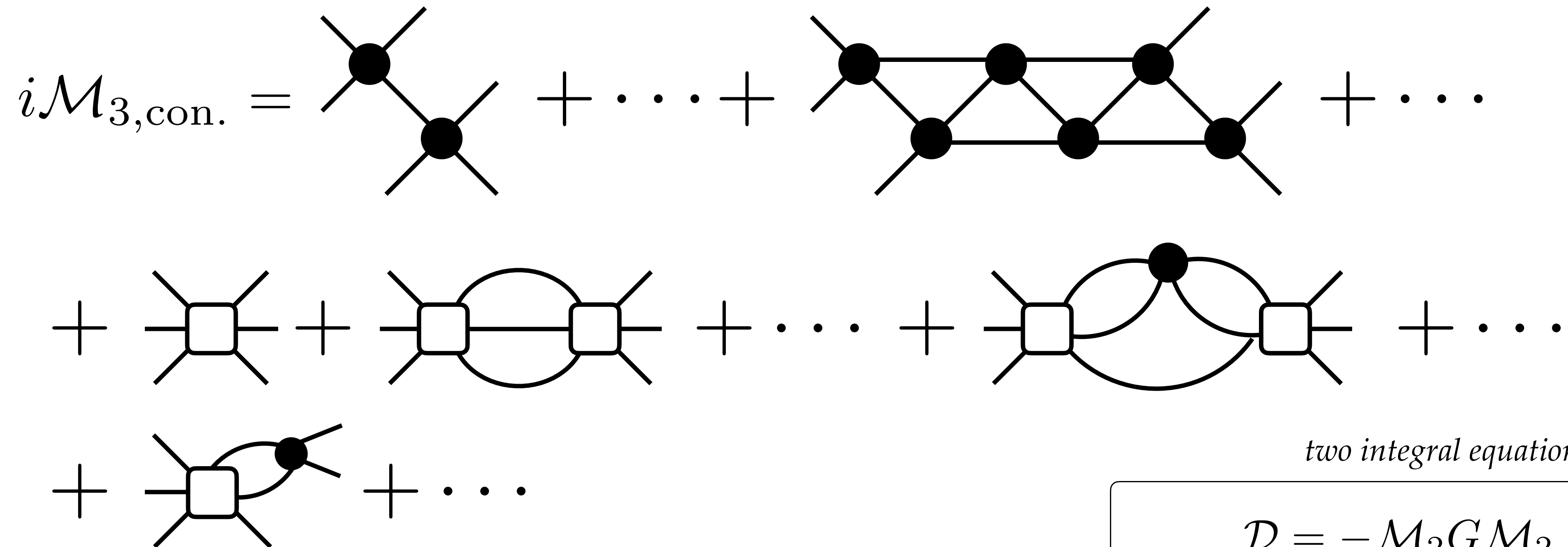
satisfies an integral equations

$$i\mathcal{D} = i\mathcal{M}_2 iG i\mathcal{M}_2 + \int i\mathcal{M}_2 iG i\mathcal{D}$$

Three hadrons in an infinite volume

The three-body scattering amplitude using all orders perturbation theory.

Sum over all connected 3-to-3 diagrams...



two integral equations and you're done!

$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \int \mathcal{M}_2 G \mathcal{D}$$

$$\mathcal{L} = \frac{1}{3} + \mathcal{M}_2 \rho - \mathcal{D} \rho$$

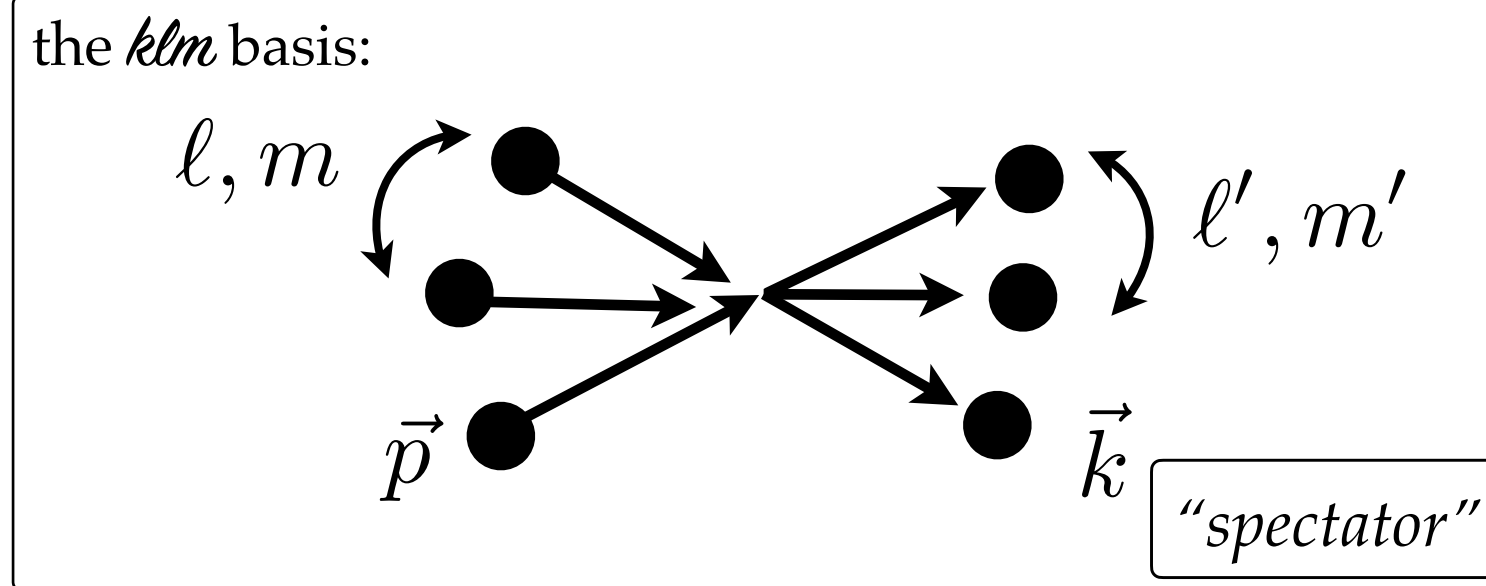
$$\mathcal{T} = \mathcal{K}_{\text{df},3} - \int \mathcal{K}_{\text{df},3} \rho \mathcal{L} \mathcal{T}$$

$$\mathcal{M}_{3,\text{con.}} = \mathcal{S} \{ \mathcal{D} + \mathcal{L} \mathcal{T} \mathcal{L}^T \}$$

Three hadrons in a finite volume

Finite-volume correlator...

$$C_L^{2pt.}(P) = \text{[Diagram: A circle with a horizontal line through its center. The top and bottom halves are labeled 'V'. The circle is circled in green.] } + \dots$$



not a natural basis in a infinite volume...that's ok!

More degrees of freedom.
In infinite volume = (6 particles with 3 unfixed momenta)
- (10 Poincare generators) = 8 d.o.f.
The klm basis has $1 + 4 + 6 = 11$ d.o.f. ...also ok!

Three hadrons in a finite volume

Finite-volume correlator...

$$C_L^{2pt.}(P) = \text{[Diagram: A circle with two vertices on a horizontal line, labeled V above and V below, with a teal highlight]} + \dots$$

$$\text{[Diagram: A circle with two vertices on a horizontal line, labeled V above and V below]} = \text{[Diagram: A dashed circle with two vertices on a horizontal line, labeled V-\infty above and V below]} + \text{[Diagram: A circle with two vertices on a horizontal line, no labels]}$$

$$F_{k\ell m; p\ell' m'} = \delta_{kp} \overset{\text{cut-off}}{H(\vec{k})} \left[\frac{1}{L^3} \sum_{\vec{a}}^{\text{UV}} - \int_a^{\text{UV}} \right] \frac{4\pi Y_{\ell m}(\hat{a}) Y_{\ell' m'}^*(\hat{a})}{4\omega_a \omega_{ka} (E - \omega_k - \omega_a - \omega_{ka})}$$

Three hadrons in a finite volume

Finite-volume correlator...

$$C_L^{2pt.}(P) = \text{[Diagram 1]} + \text{[Diagram 2]} + \dots + \text{[Diagram 3]} + \text{[Diagram 4]} + \dots$$

The diagrams represent terms in a perturbative expansion of the finite-volume correlator. Diagram 1 is a bubble with two external legs and two internal vertices labeled V . Diagram 2 is a chain of two such bubbles. Diagram 3 is a diagram with three vertices and three internal lines, all labeled V . Diagram 4 is a more complex diagram with four vertices and four internal lines, all labeled V , and is highlighted with a green oval. A callout box labeled "exchange cuts" shows a diagram with two vertices and two external legs, with a dashed vertical line representing a cut, and the expression $\sim i\mathcal{K}_2 iG i\mathcal{K}_2$.

Three hadrons in a finite volume

Finite-volume correlator...

$$C_L^{2pt.}(P) = \begin{array}{c} \text{Diagram 1} + \text{Diagram 2} + \dots \\ \text{Diagram 3} + \text{Diagram 4} + \dots \end{array}$$

The diagrams are:

- Diagram 1: A circle with two external vertices on the left and right, and two internal vertices at the top and bottom. Both internal edges are labeled V .
- Diagram 2: Two Diagram 1 circles connected at their right vertex to the left vertex of the second circle.
- Diagram 3: A circle with two external vertices on the left and right, and one internal vertex at the top. Three edges are labeled V : the top edge, the left edge, and the bottom edge.
- Diagram 4: A circle with two external vertices on the left and right, and two internal vertices at the top and bottom. Four edges are labeled V : the top edge, the left edge, the bottom edge, and the right edge.

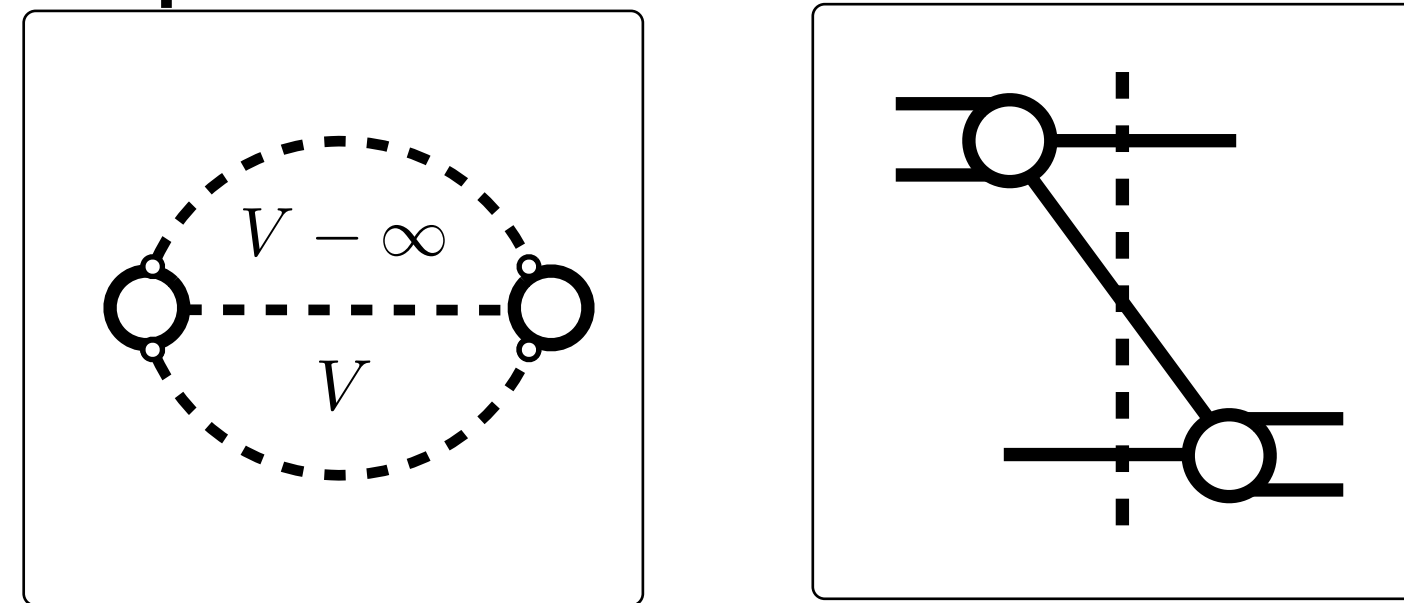
after a lot of massaging...

$$\det \left[F_3(E_L, L) + \mathcal{K}_{\text{df},3}^{-1}(E_L) \right] = 0$$

Isotropic approximation

Consider the case where two-body system is an S-wave

$$[F_3^s]_{kp} = \frac{1}{L^3} \left[\frac{F^s}{3} - F^s \frac{1}{1/(2\omega\mathcal{K}_2^s) + F^s + G^s} F^s \right]_{kp}$$



Furthermore, consider the case where the K-matrix does not depend on the spectator momentum. Then the quantization condition reduces to:

$$F_3^{\text{iso}}(E, \vec{P}, L) = -1/\mathcal{K}_{\text{df},3}^{\text{iso}}(E^*)$$

Where:

$$F_3^{\text{iso}}(E, L) = \sum_{k,p} [F_3^s]_{kp}$$

Solving integral equations

Assuming $\mathcal{K}_{\text{df}} = 0$ and partial wave-projecting onto $\ell = 0$ sector,

$$\mathcal{D}_s^{(u,u)}(p, k) = -\mathcal{M}_2(E_{2,p}^*)G_s(p, k, \epsilon)\mathcal{M}_2(E_{2,k}^*) - \mathcal{M}_2(E_{2,p}^*) \int_0^{k_{\text{max}}} \frac{k'^2 dk'}{(2\pi)^2 \omega_{k'}} G_s(p, k', \epsilon) \mathcal{D}_s^{(u,u)}(k', k),$$

- Discretized momenta: meshes, splines,...
- Soften or isolate possible singularities: non-zero epsilon, integrate out poles analytically, ...
- Write as a matrix form:

$$\mathbf{D}(N, \epsilon) = -\mathbf{M} \cdot \mathbf{G}(\epsilon) \cdot \mathbf{M} - \mathbf{M} \cdot \mathbf{G}(\epsilon) \cdot \mathbf{P} \cdot \mathbf{D}(N, \epsilon)$$

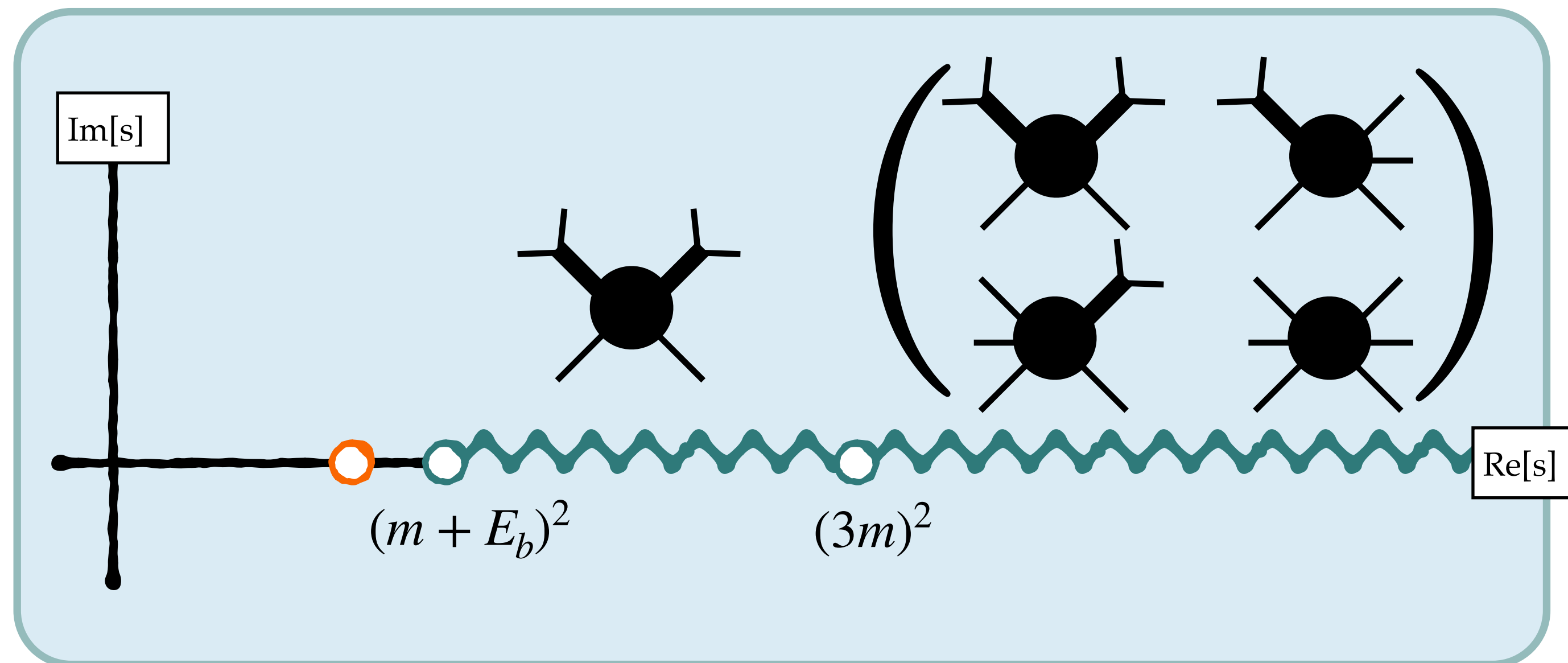
- Recover the exact results when taking $\epsilon = \eta/N \rightarrow 0$.

Consistency checks for toy model

Consider a toy theory with a two-body bound state

□ Simple case $q \cot \delta = -\frac{1}{a}$

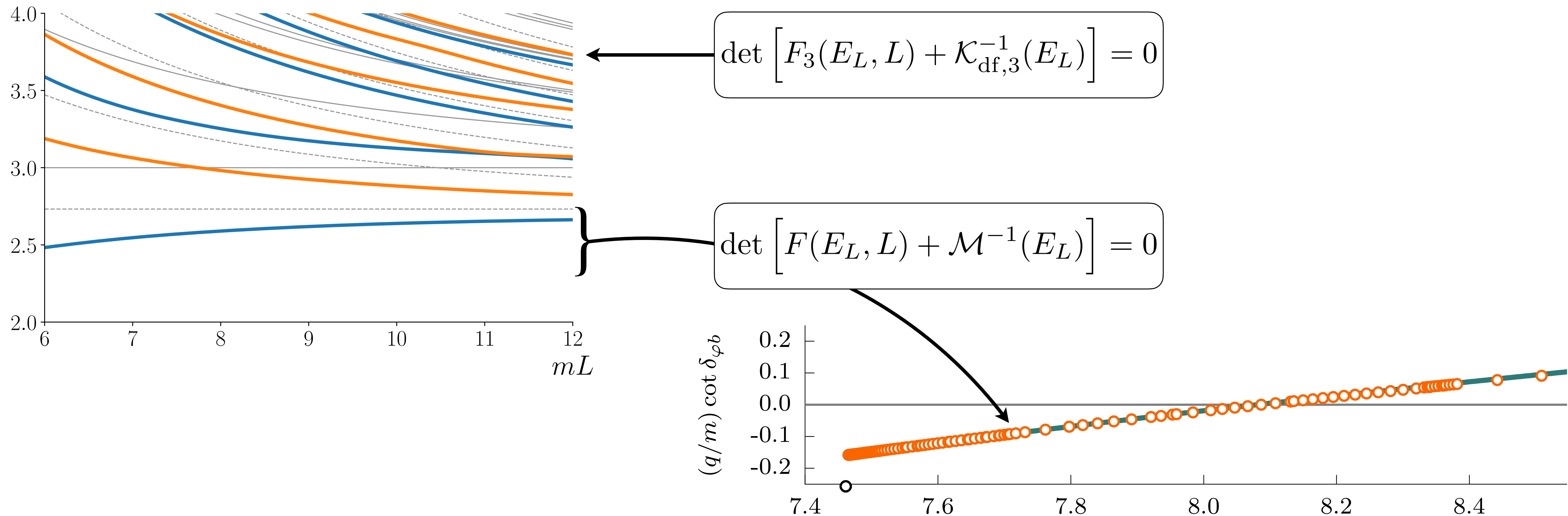
□ if $a > 0$, there is a bound state with binding energy $E_b = 2\sqrt{m^2 - \frac{1}{a^2}}$



Consistency checks for toy model

Alternatively to solving the integral equations, one can instead:

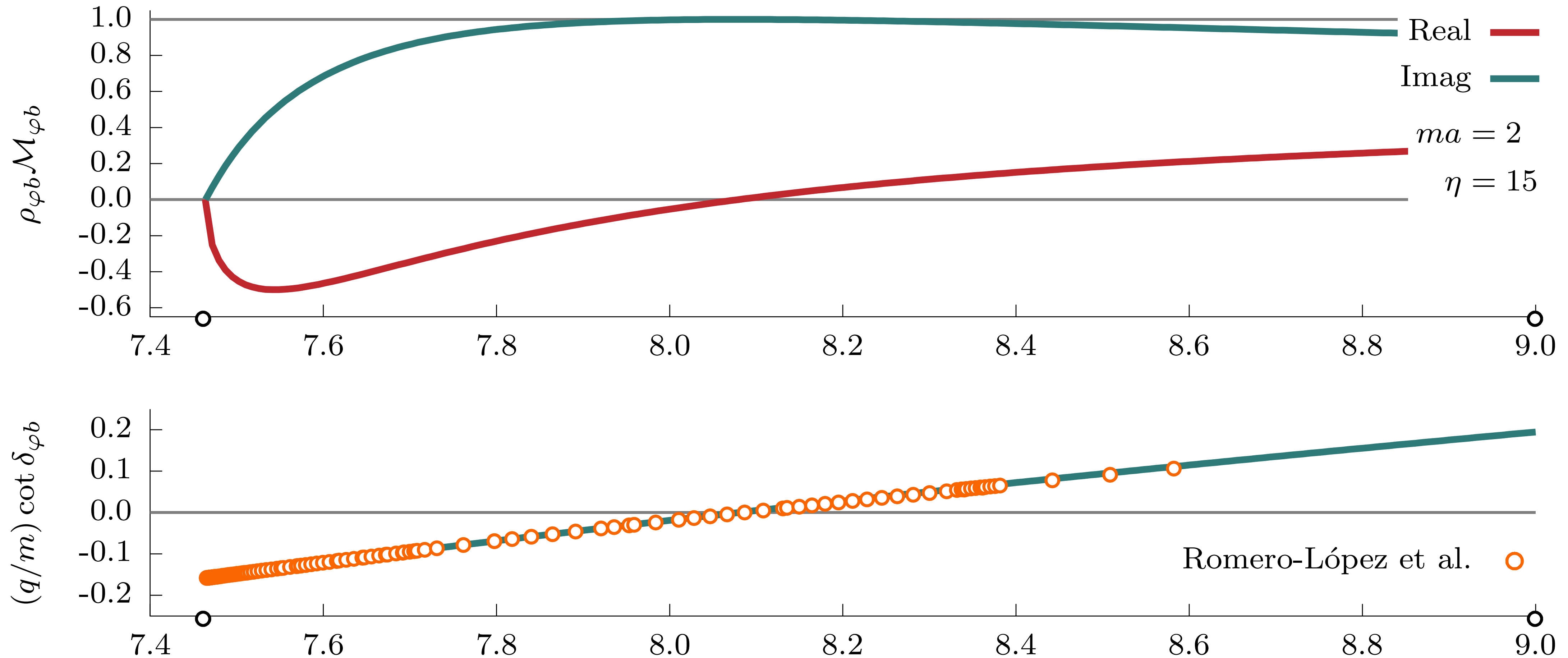
- Fix $ma = 2$ and $\mathcal{K}_{\text{df}} = 0$
- obtain E_L using 3-body quantization by Hansen & Sharpe (2014),
- obtain $q \cot \delta_{\varphi b}$ using the two-body Lüscher formalism.



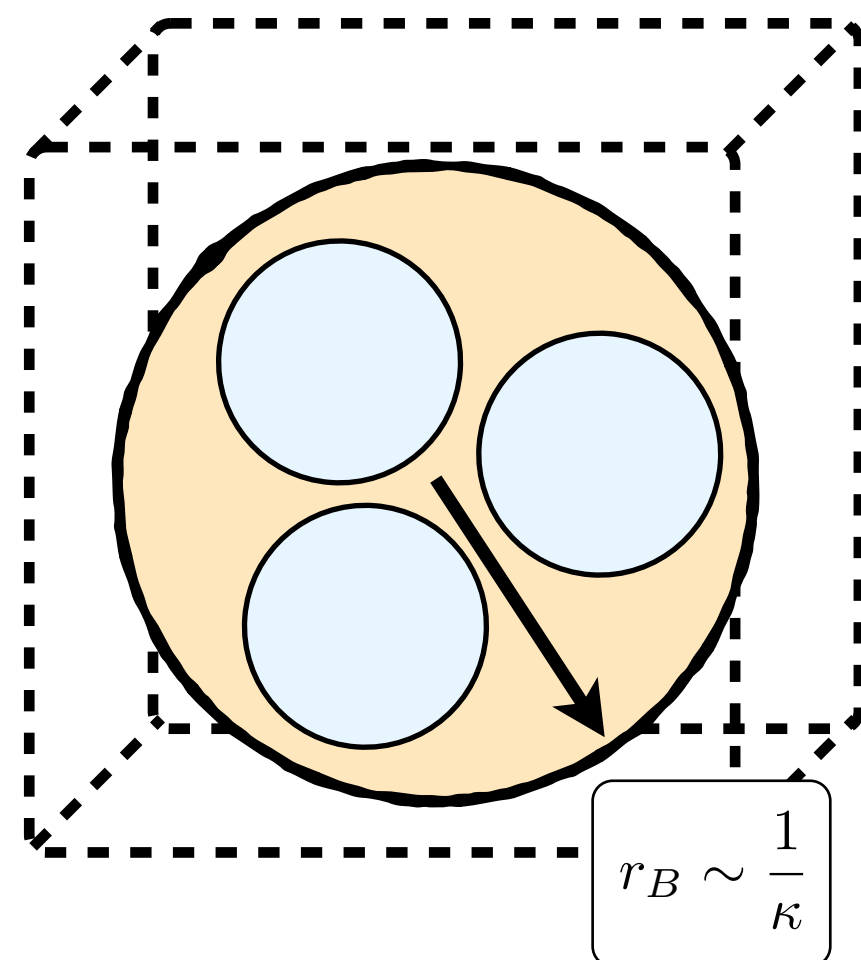
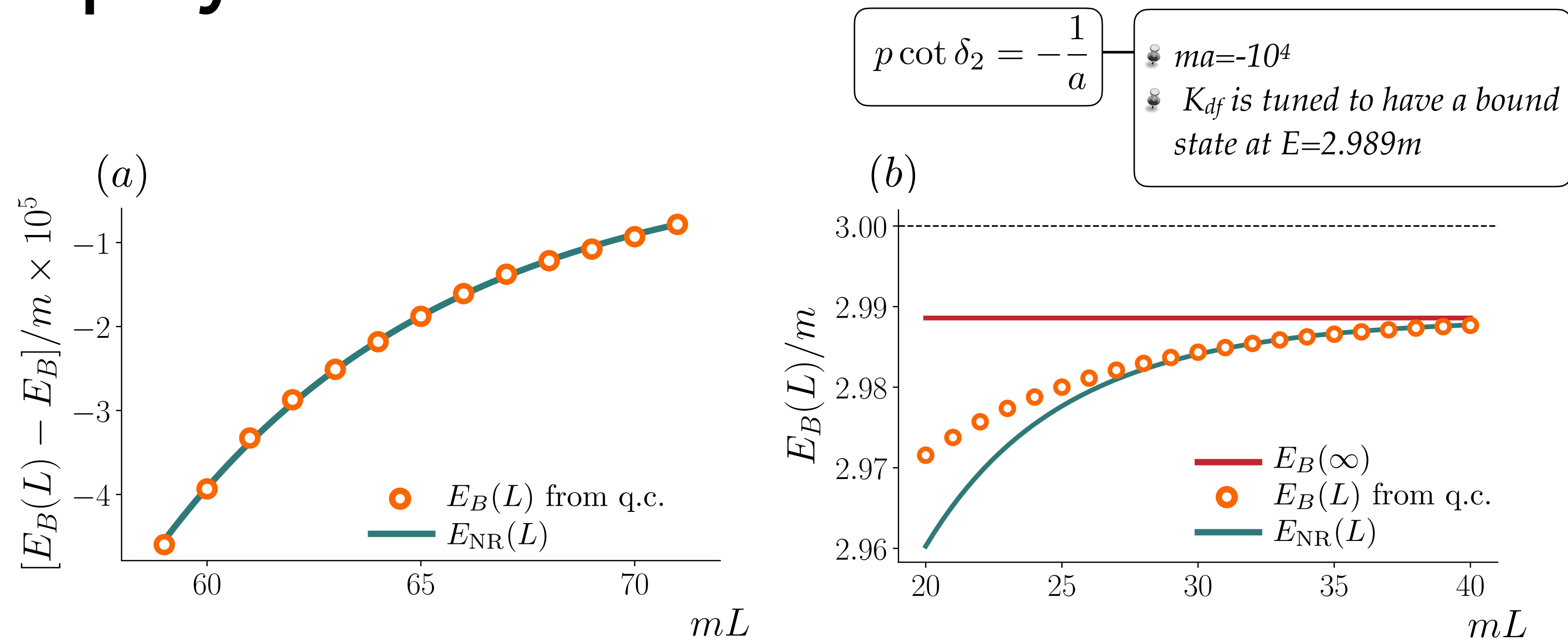
Consistency checks for toy model

One can also solve the integral equations to find agreement!

Two independent methods reproducing same exact amplitudes!



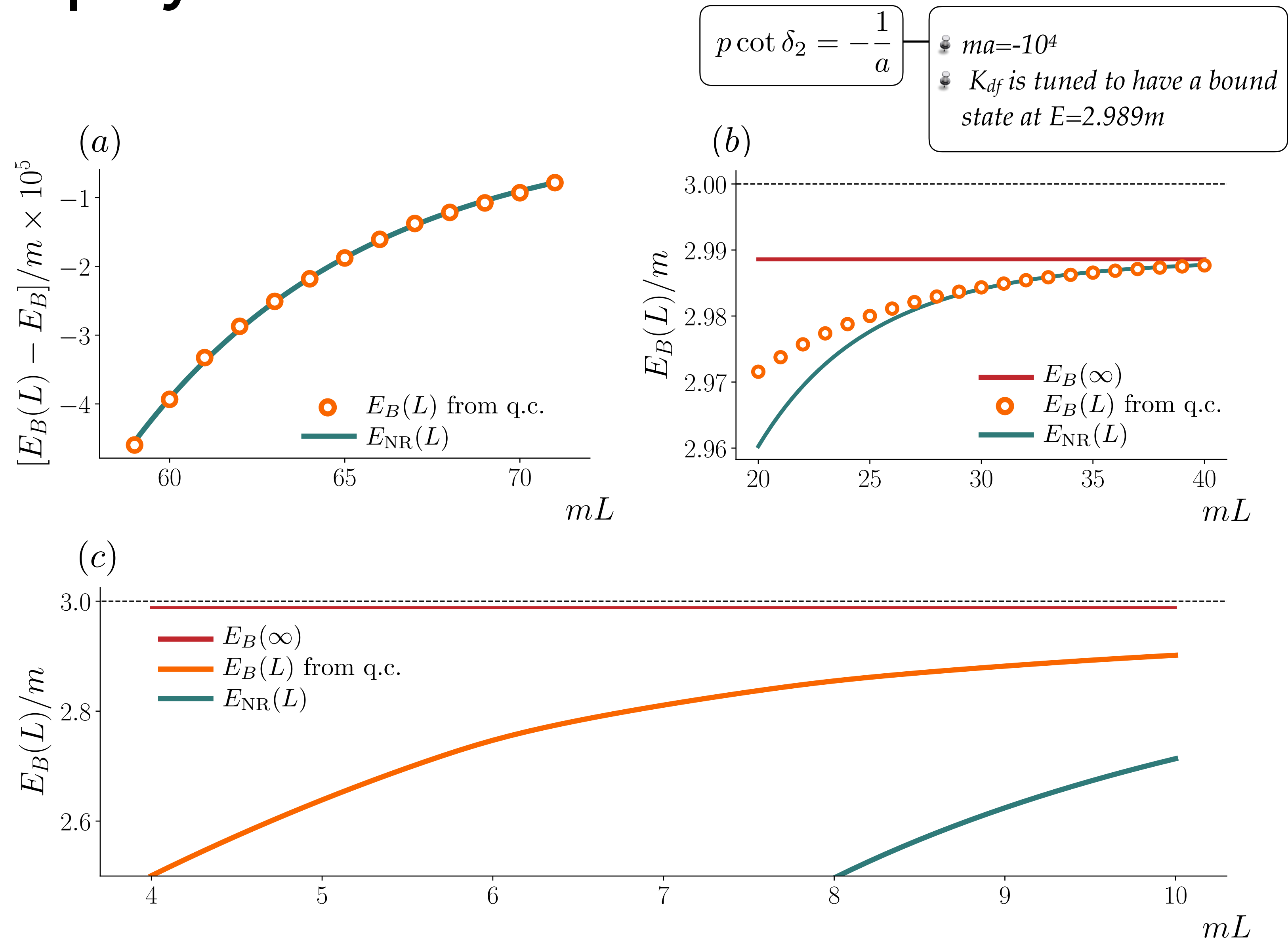
Efimov-like physics



$$E_B(L) = 3 - \kappa^2 - 96.35|A|^2 \kappa^2 \frac{e^{-2\kappa L/\sqrt{3}}}{(\kappa L)^{3/2}}$$

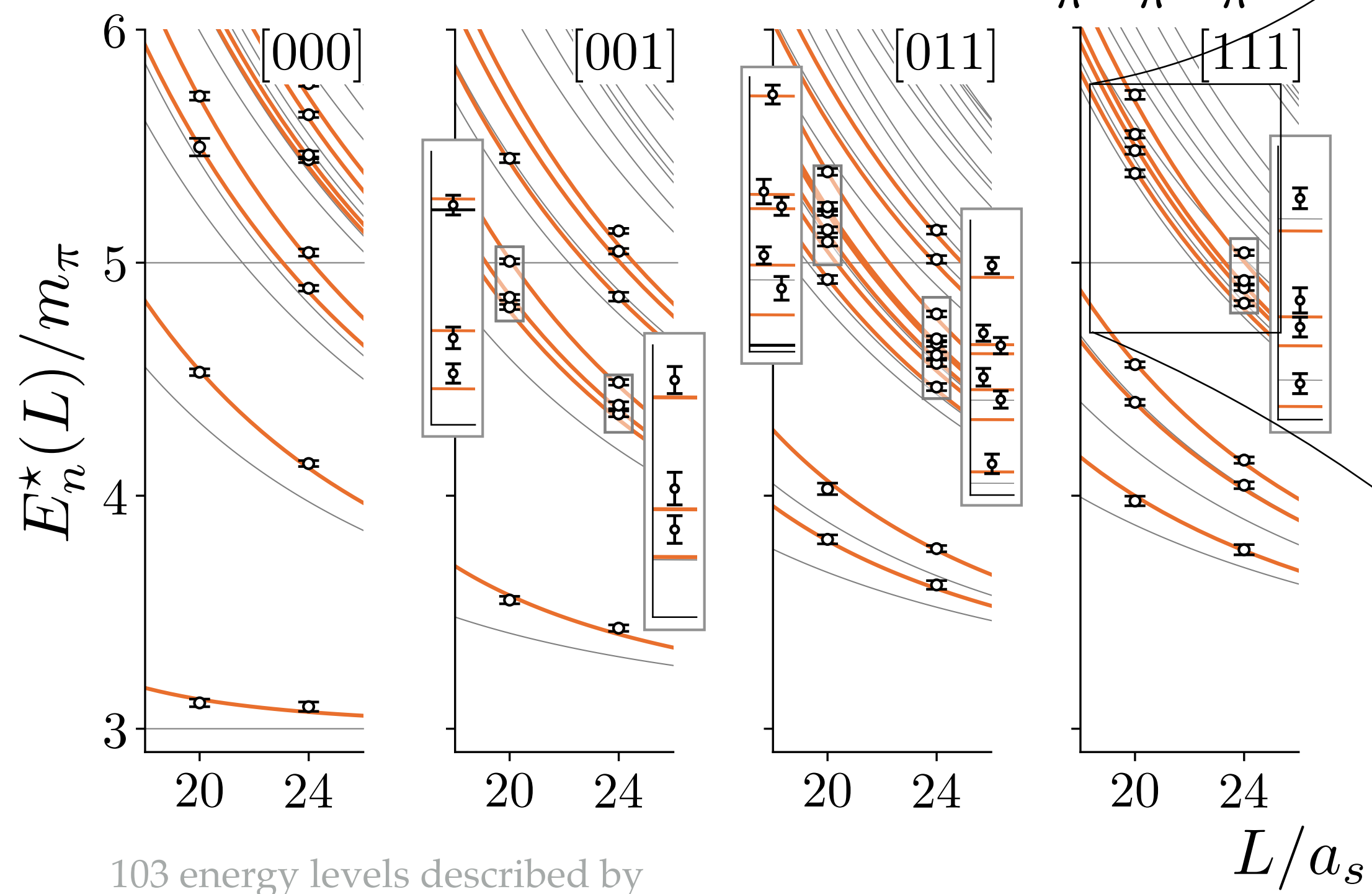
Meißner, Ríos, & Rusetsky (2015)

Efimov-like physics

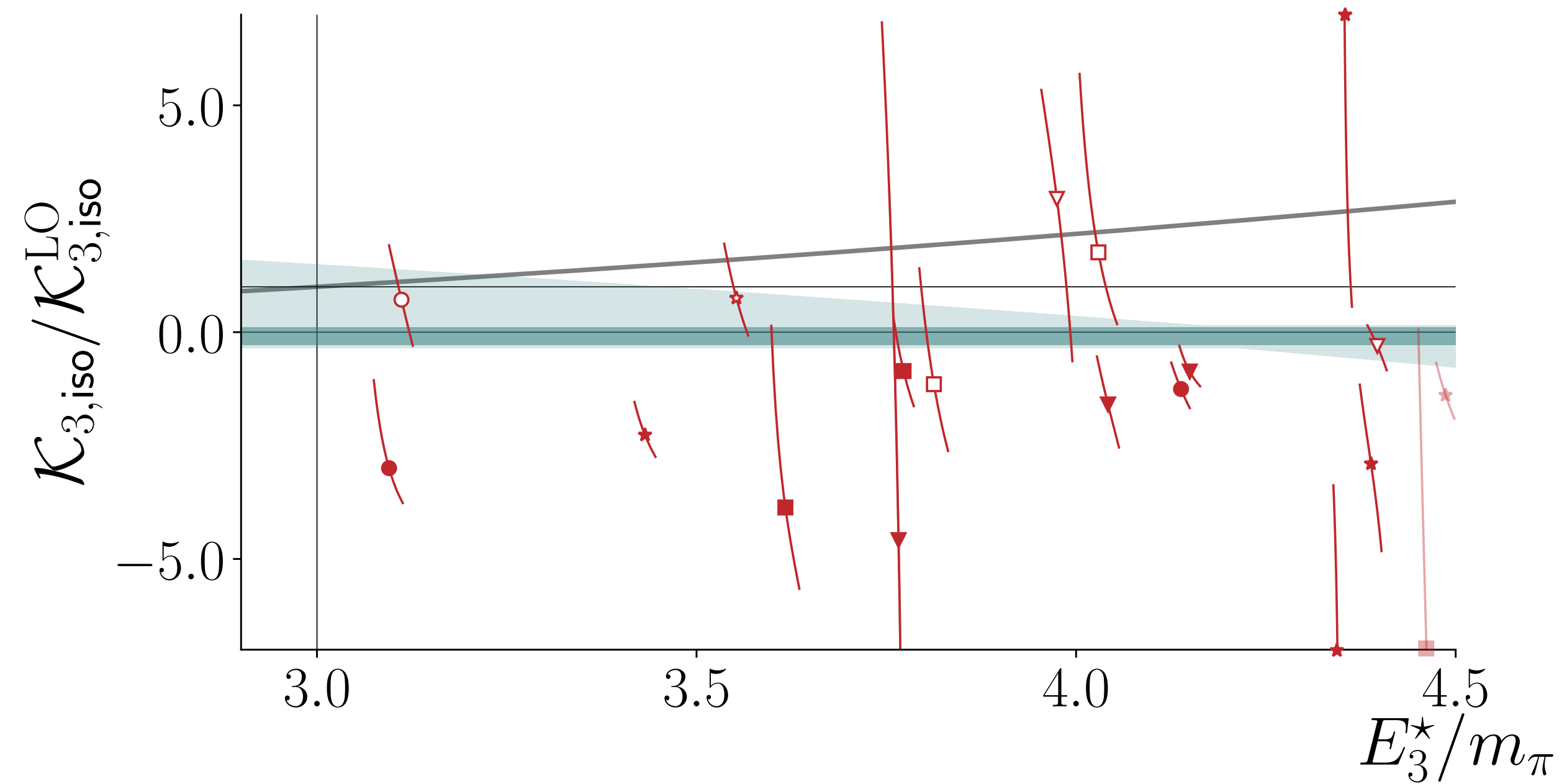


$\pi\pi\pi$

($l=3$ channel, $m_\pi \sim 390$ MeV)



103 energy levels described by three numbers: m_π , $a_{\pi\pi}$, $\mathcal{K}_{3,\text{iso}}$



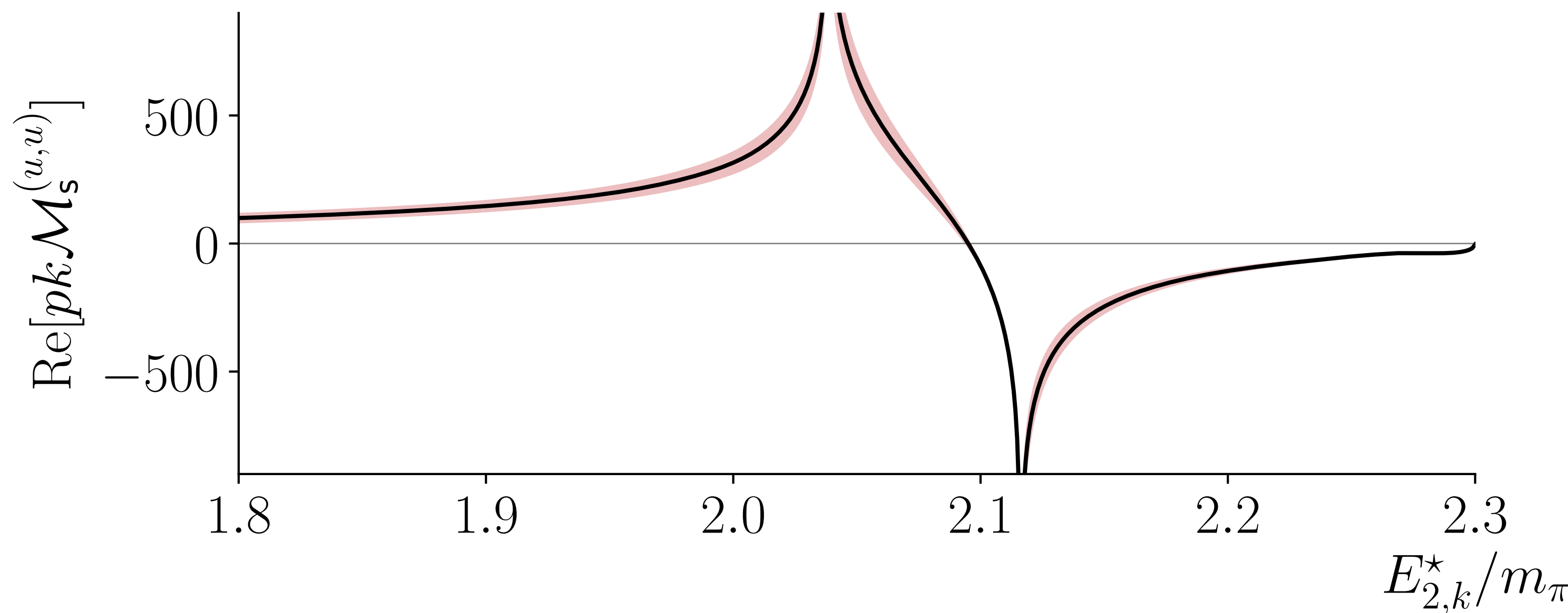
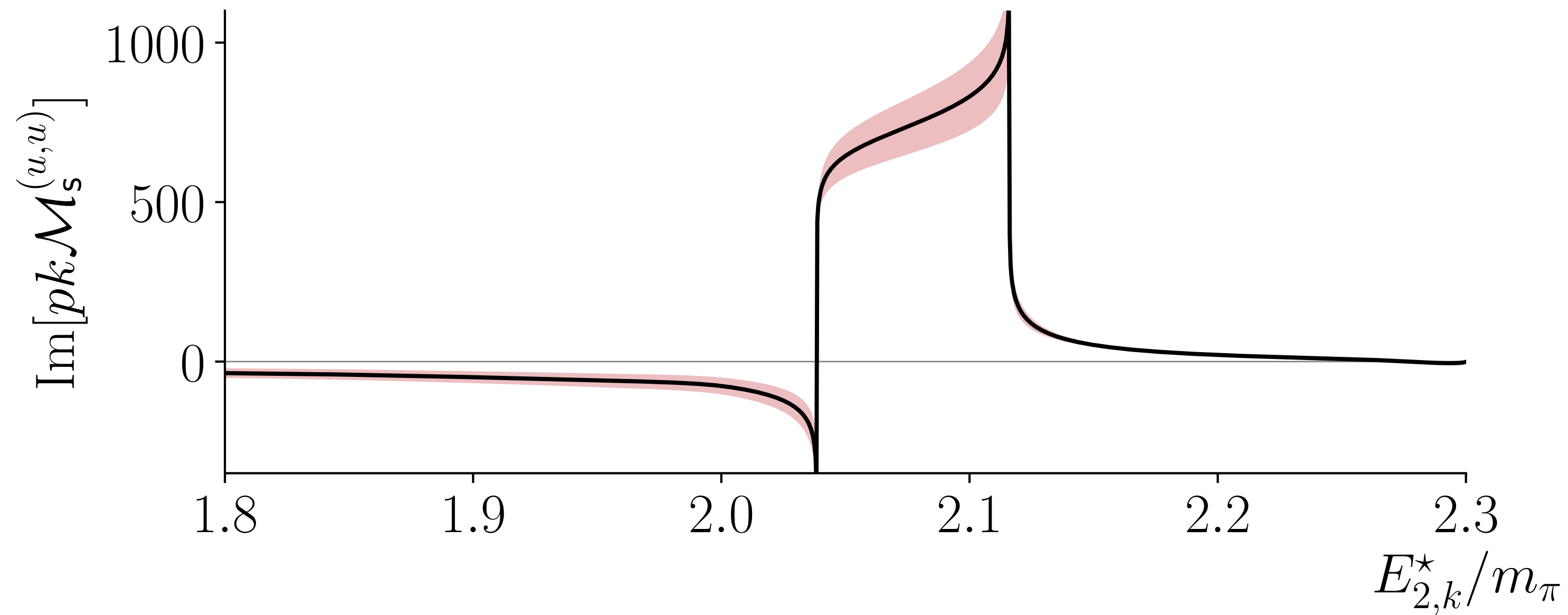
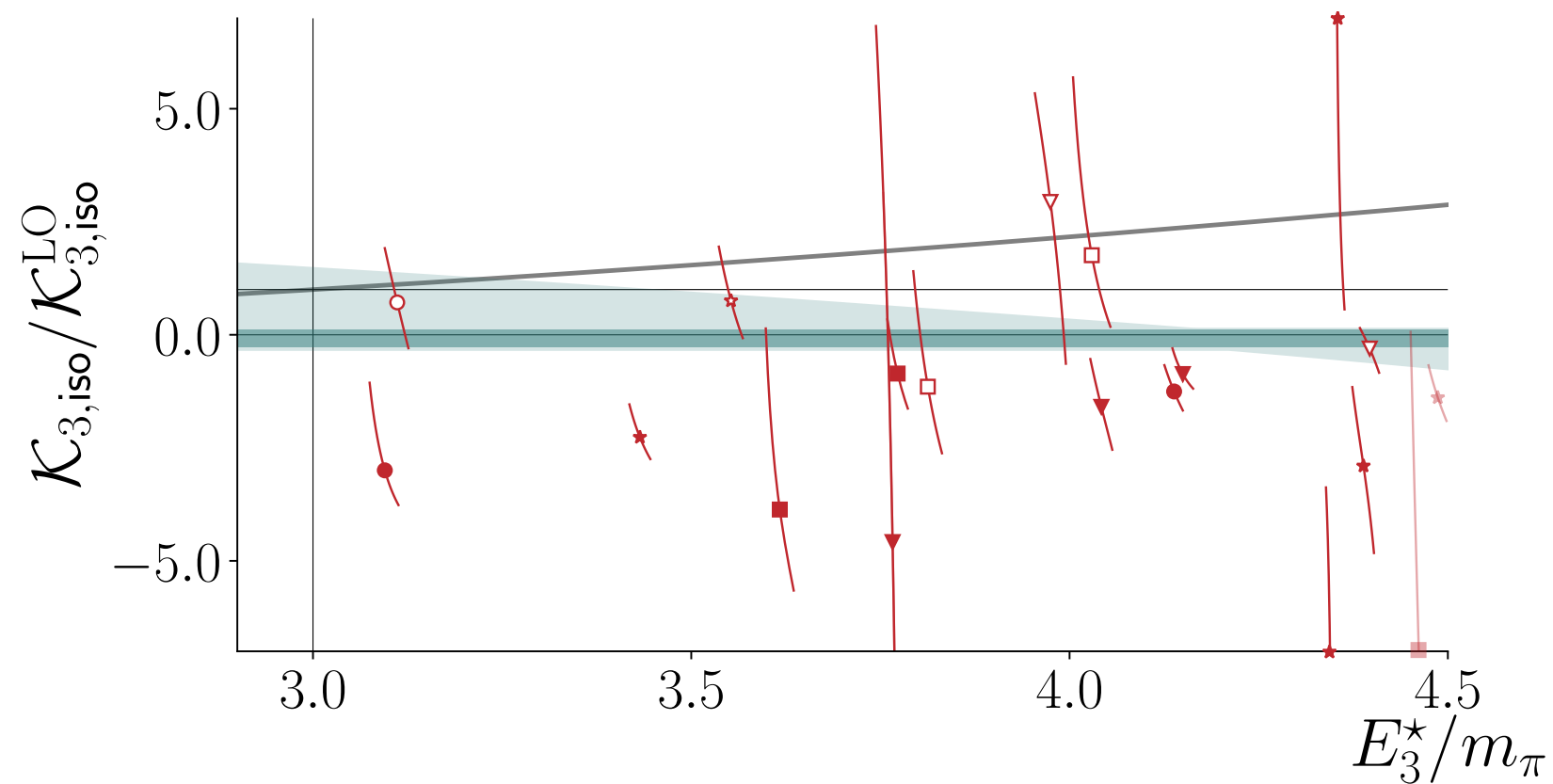
$m_\pi \sim 390$ MeV

Hansen, RB, Edwards, Thomas, & Wilson (2020)

$\pi\pi\pi$ scattering

($l=3$ channel, $m_\pi \sim 390$ MeV)

first 3body scattering amplitude from the lattice QCD!



$m_\pi \sim 390$ MeV

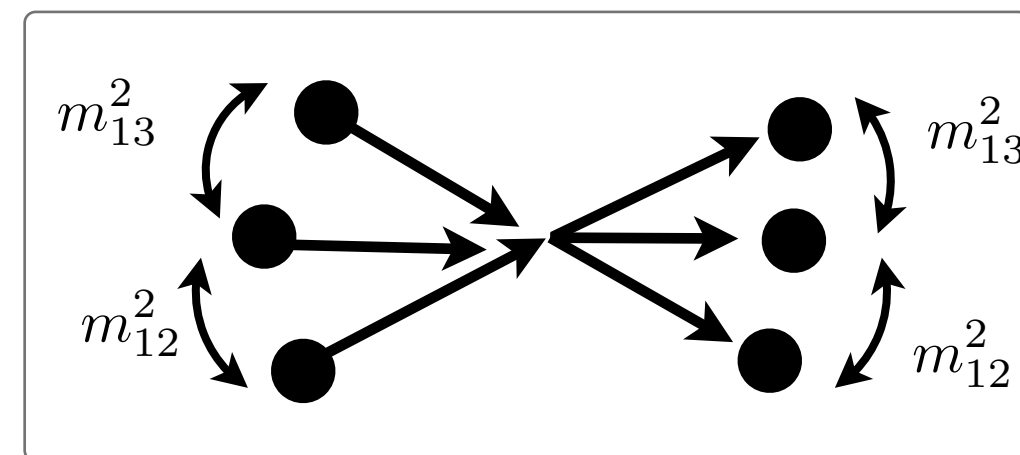
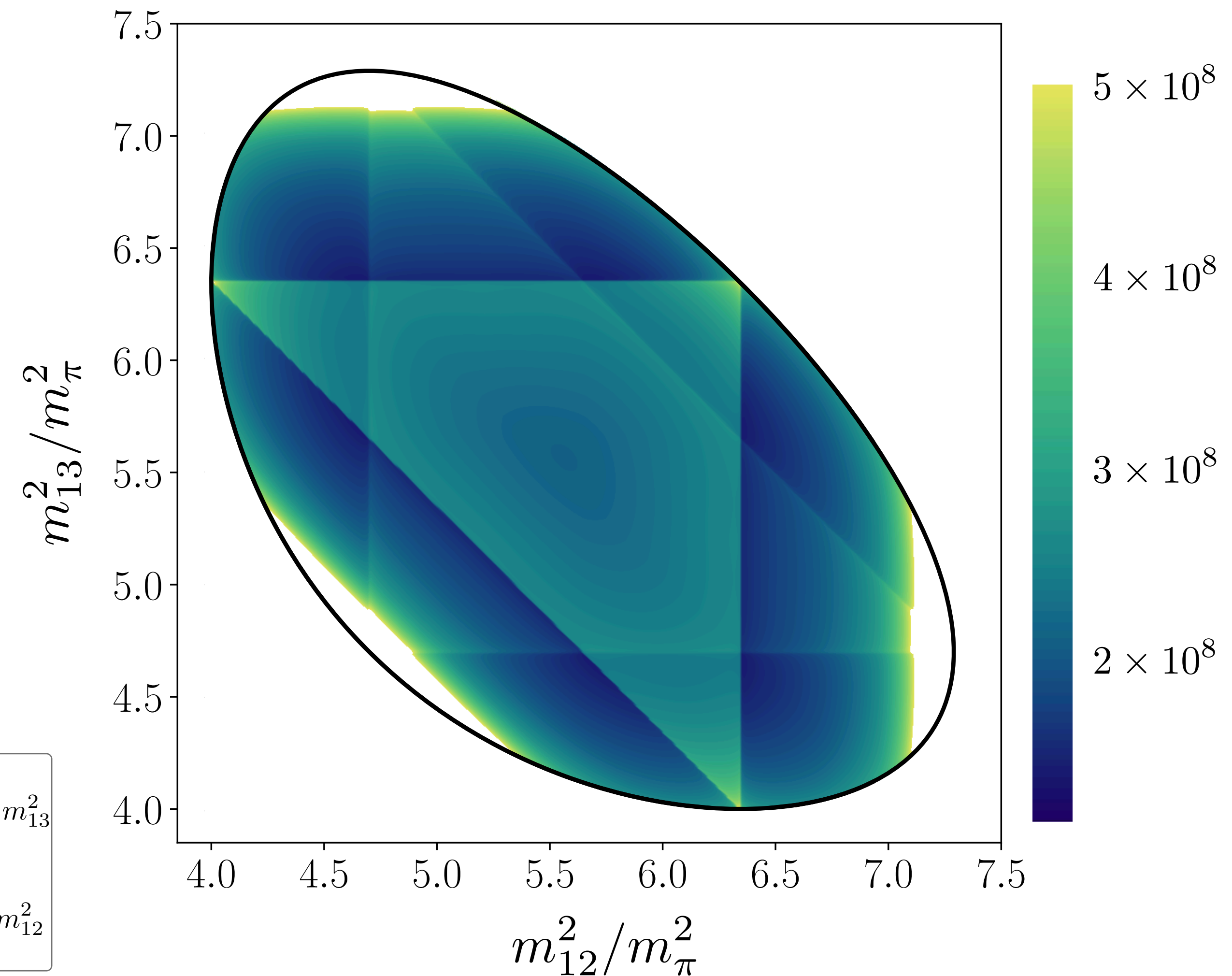
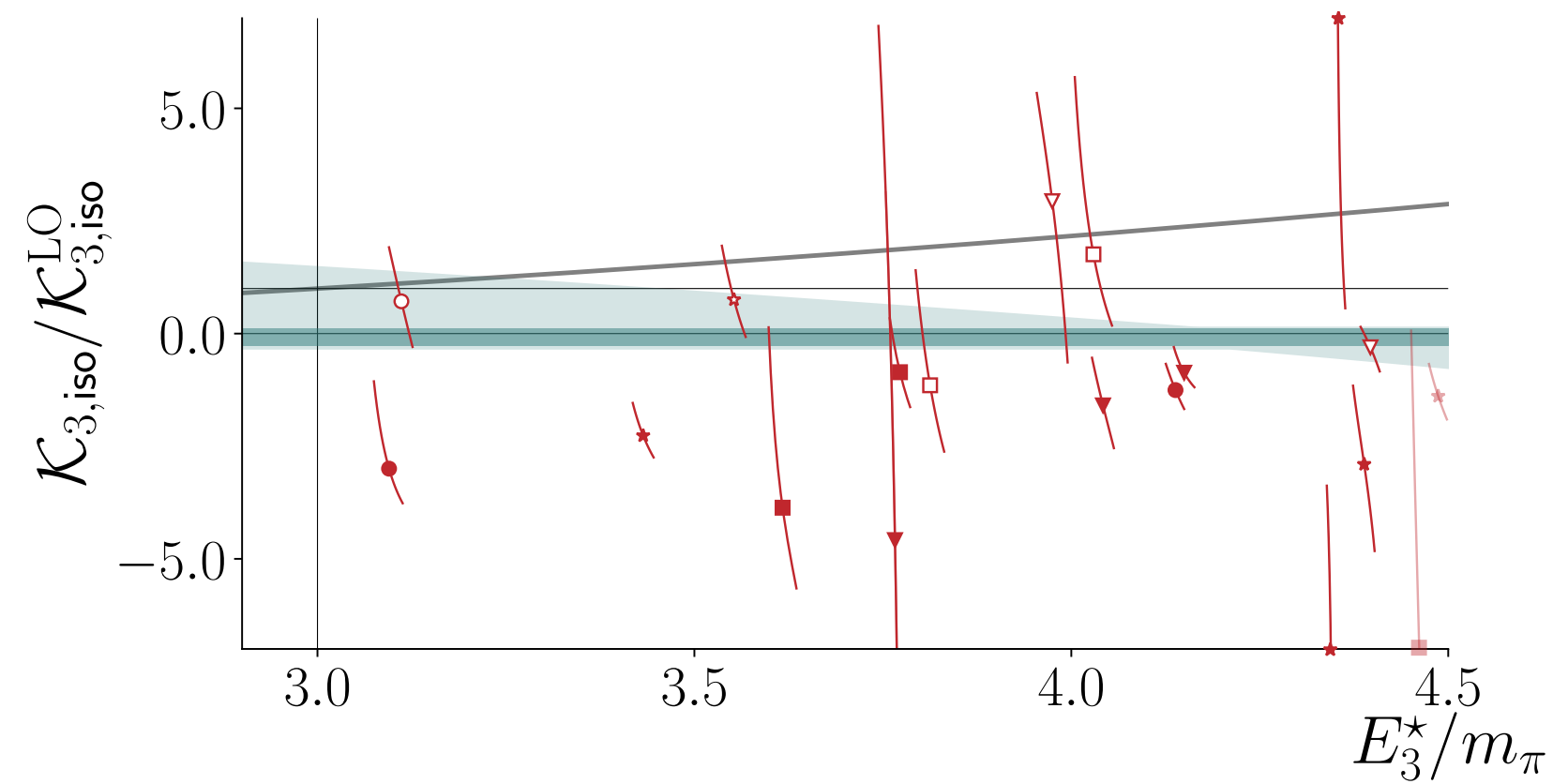
Hansen, RB, Edwards, Thomas, & Wilson (2020)

$$\mathcal{D}_s^{(u,u)}(p, k) = -\mathcal{M}_2(E_{2,p}^*)G_s(p, k, \epsilon)\mathcal{M}_2(E_{2,k}^*) - \mathcal{M}_2(E_{2,p}^*) \int_0^{k_{\max}} \frac{k'^2 dk'}{(2\pi)^2 \omega_{k'}} G_s(p, k', \epsilon) \mathcal{D}_s^{(u,u)}(k', k),$$

$\pi\pi\pi$ scattering

($l=3$ channel, $m_\pi \sim 390$ MeV)

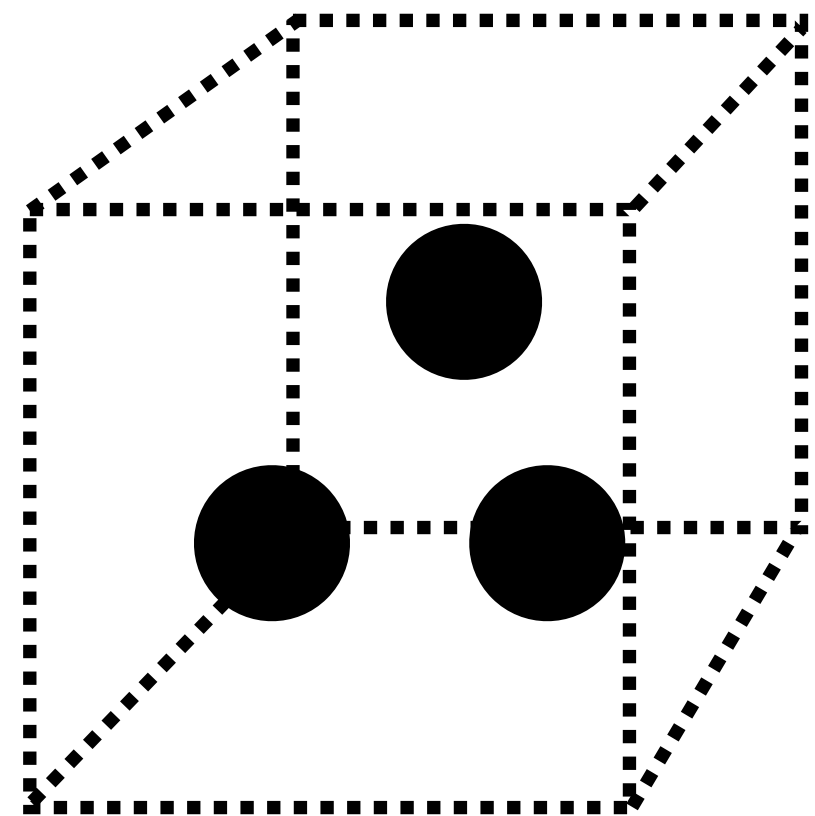
first 3body scattering amplitude from the lattice QCD!



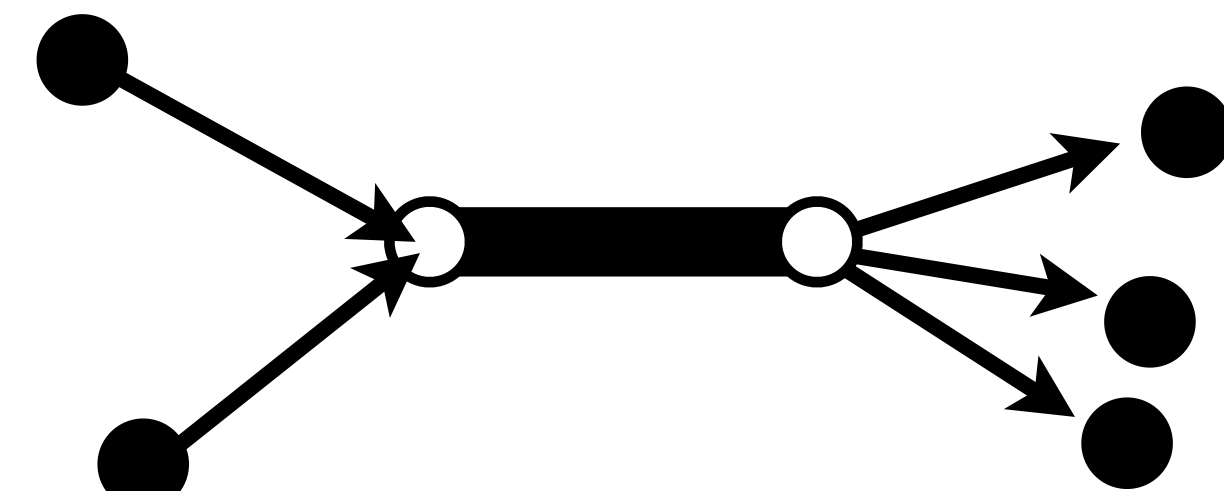
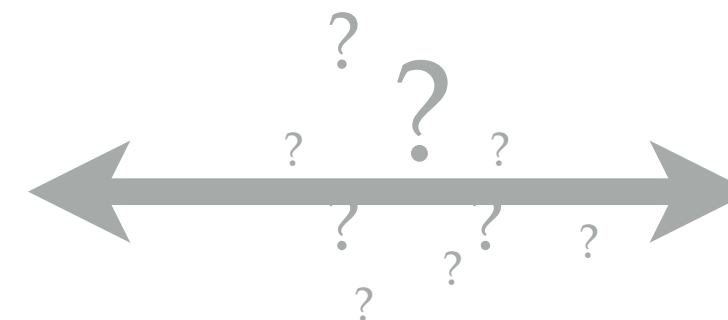
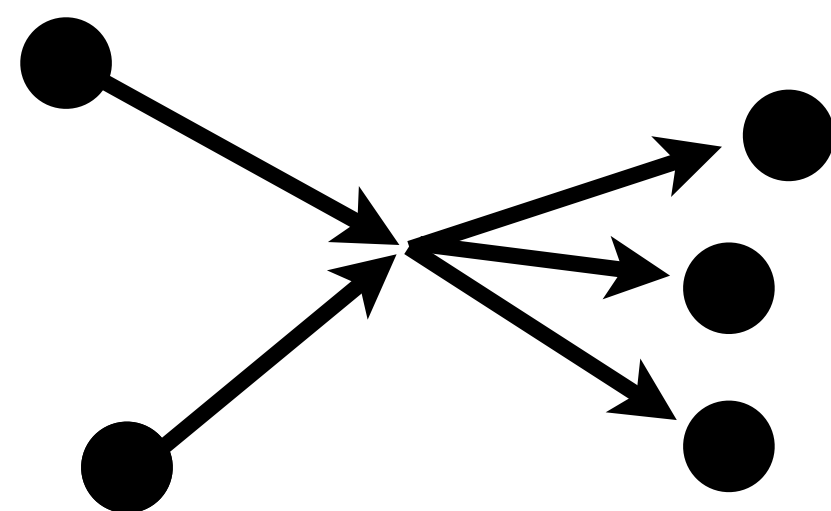
$m_\pi \sim 390$ MeV

Hansen, RB, Edwards, Thomas, & Wilson (2020)

We are just getting started!



Hansen & Sharpe ('14, '15)
 Mai & Döring ('17)
 RB, Hansen & Sharpe ('18)
 Hansen, Romero-Lopez & Sharpe ('20)
 Blanton & Sharpe ('20)
 Jackura et al. ('20)



Unitarity in 3Body systems

FV formalism for spinless states

coupled 2/3B systems

analytic and numerical checks

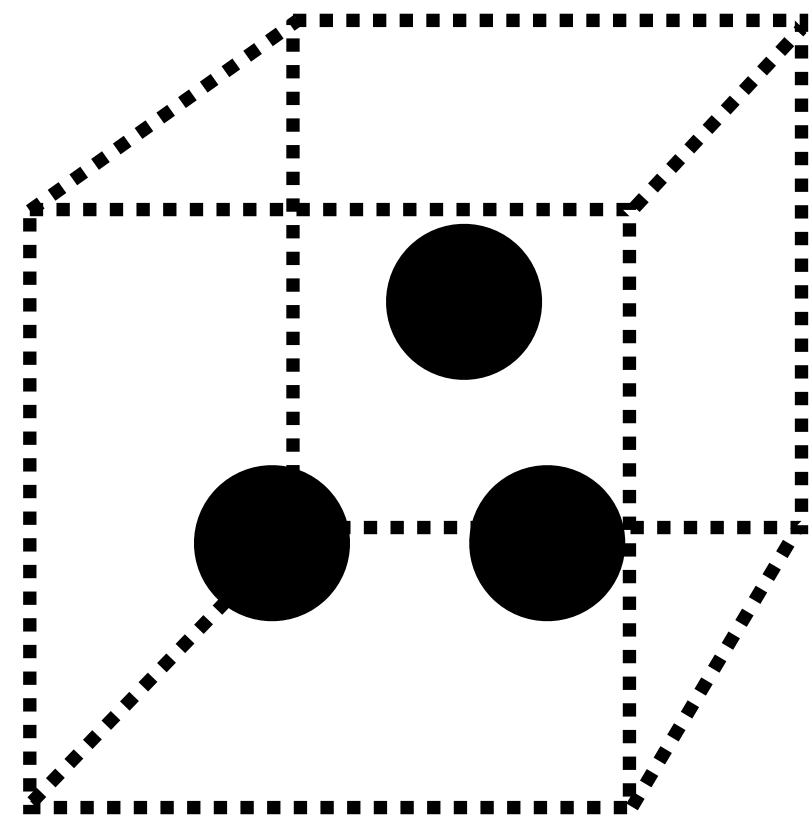
integral equations

Actual calculations

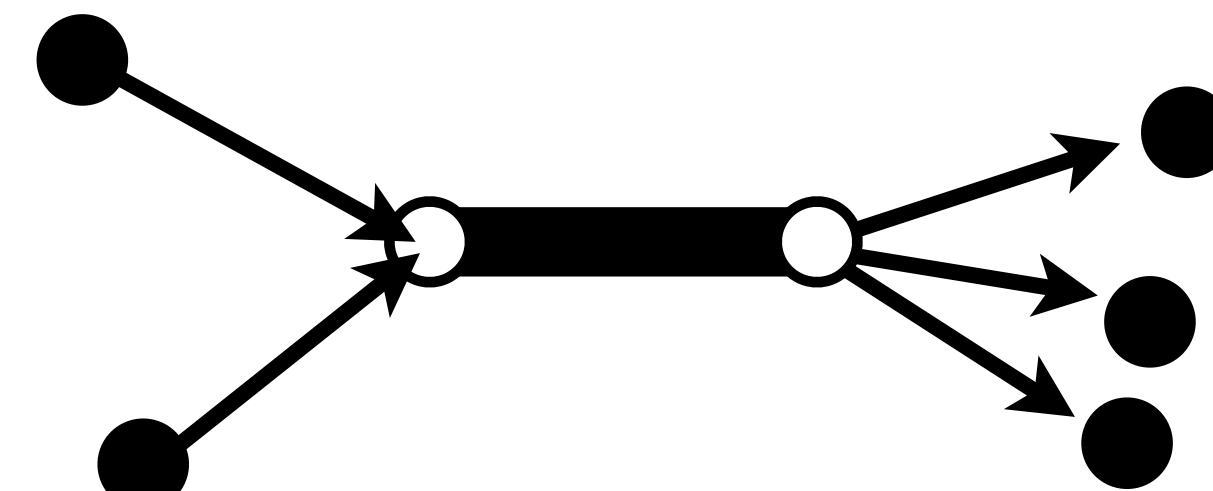
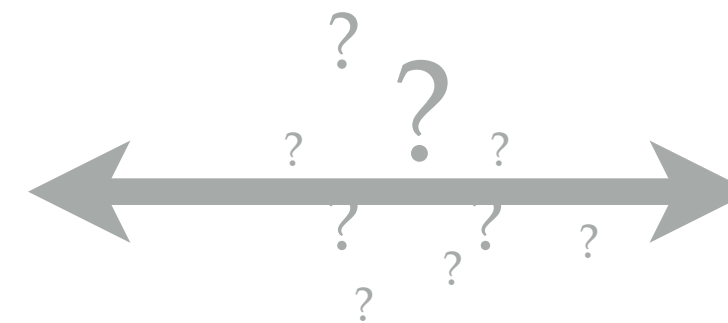
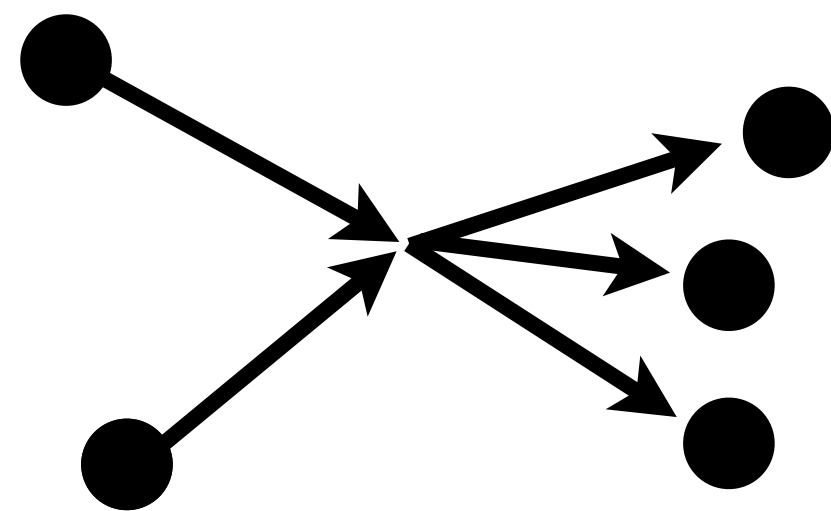
Spin, multichannels

Analytic continuation

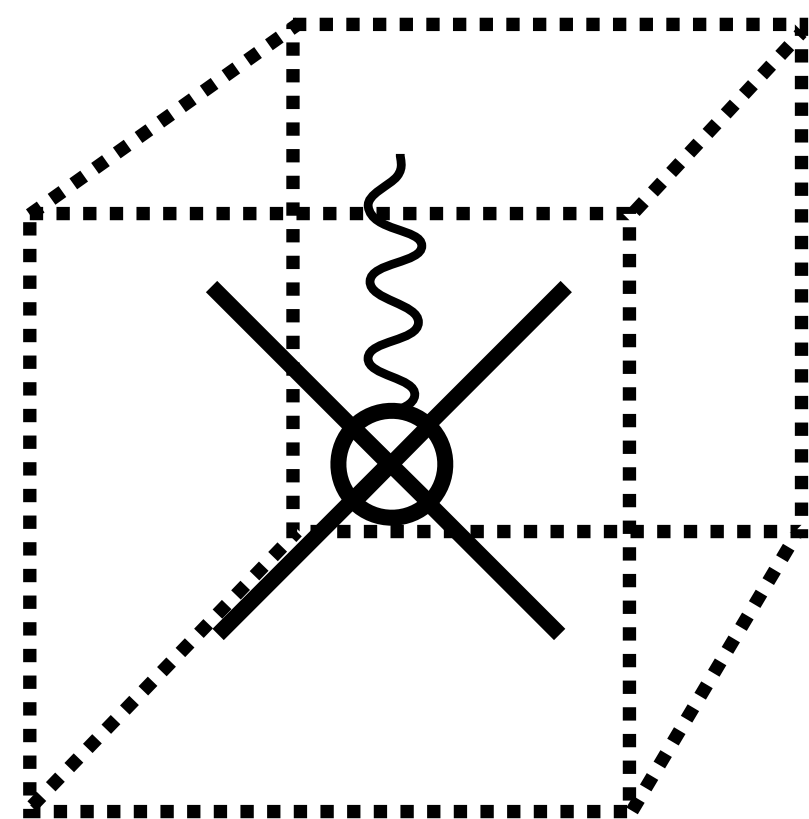
We are just getting started!



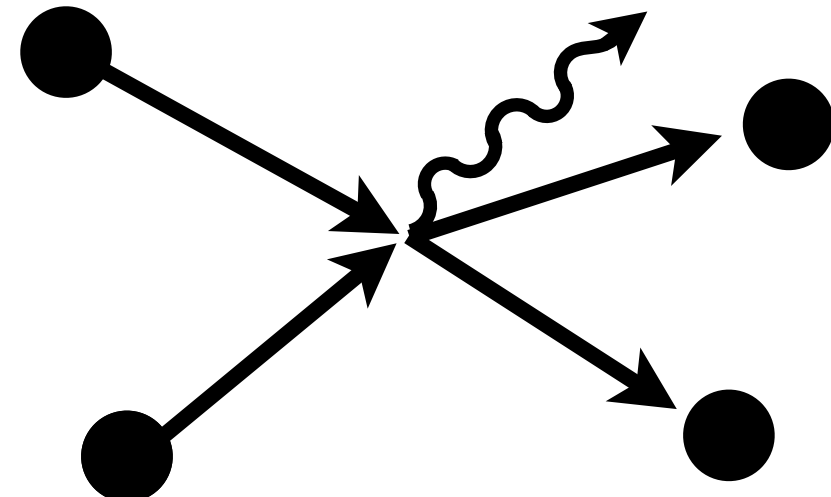
Hansen & Sharpe ('14, '15)
Mai & Döring ('17)
RB, Hansen & Sharpe ('18)
Hansen, Romero-Lopez & Sharpe ('20)
Blanton & Sharpe ('20)
Jackura et al. ('20)



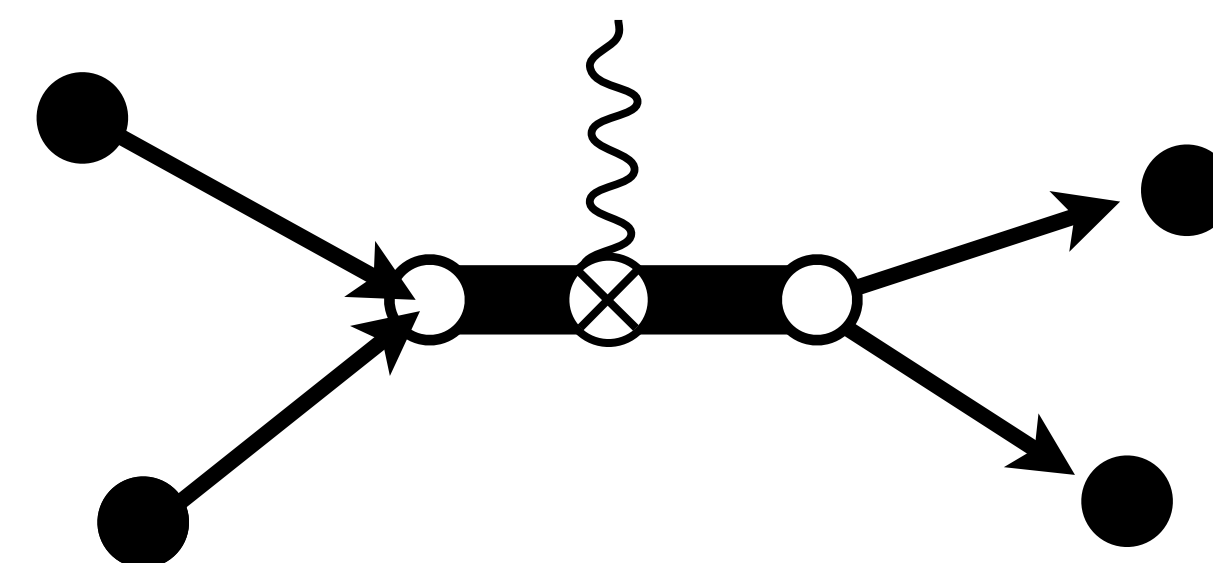
Structural information



RB & Hansen ('15)
Baroni, RB, Jackura, Hansen
& Ortega-Gama ('18)



RB, Jackura, Ortega-Gama,
Sherman ('21)



A review/introduction

slightly out of date

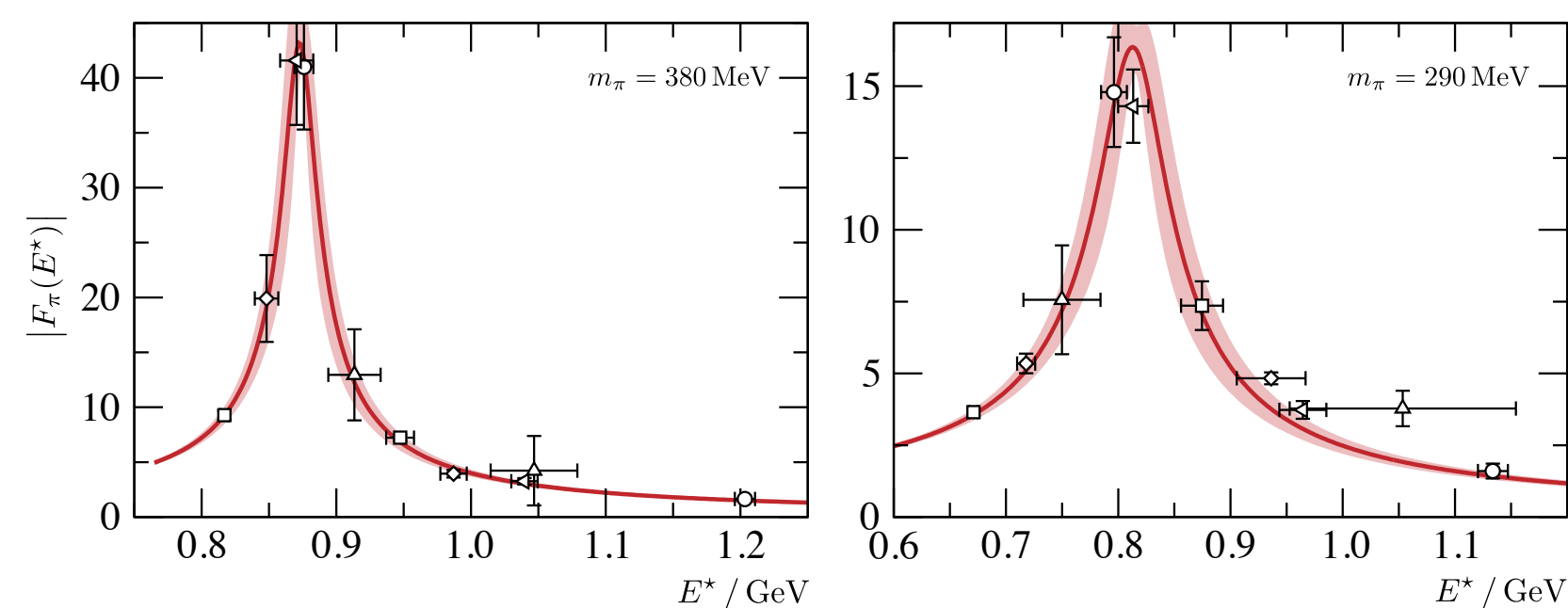
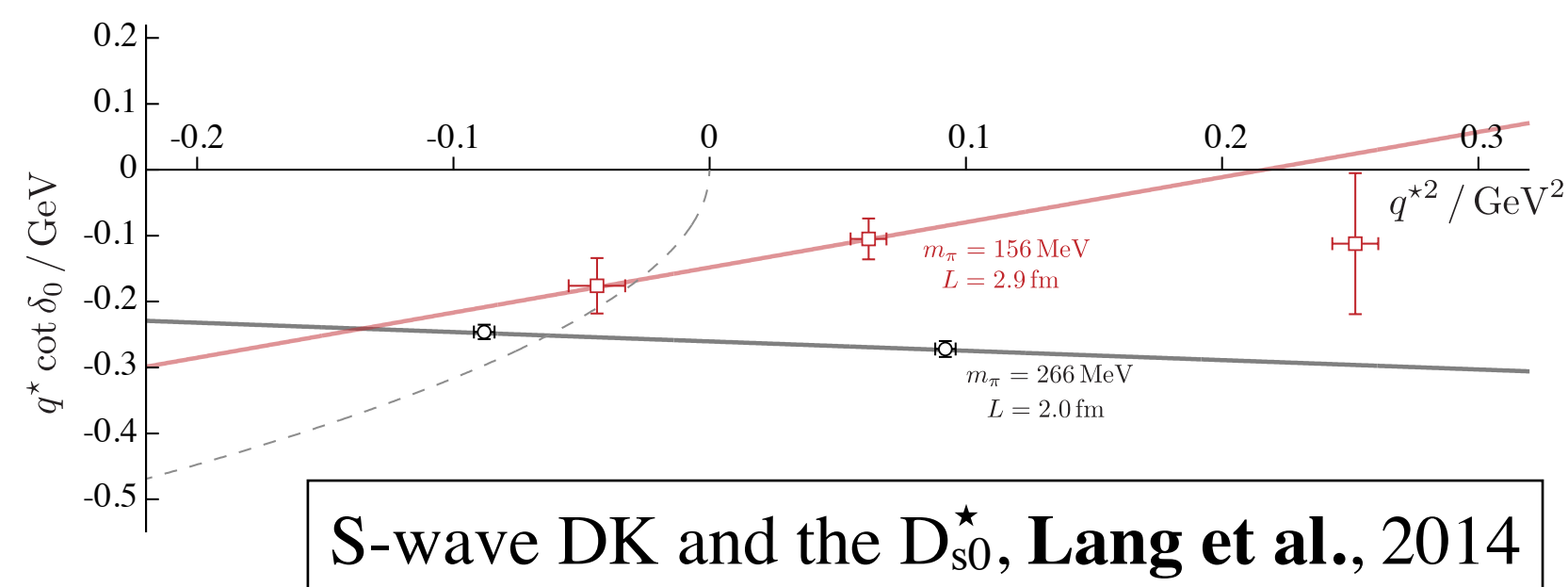
Scattering processes and resonances from lattice QCD

Raúl A. Briceño,^{1,*} Jozef J. Dudek,^{1,2,†} and Ross D. Young^{3,‡}

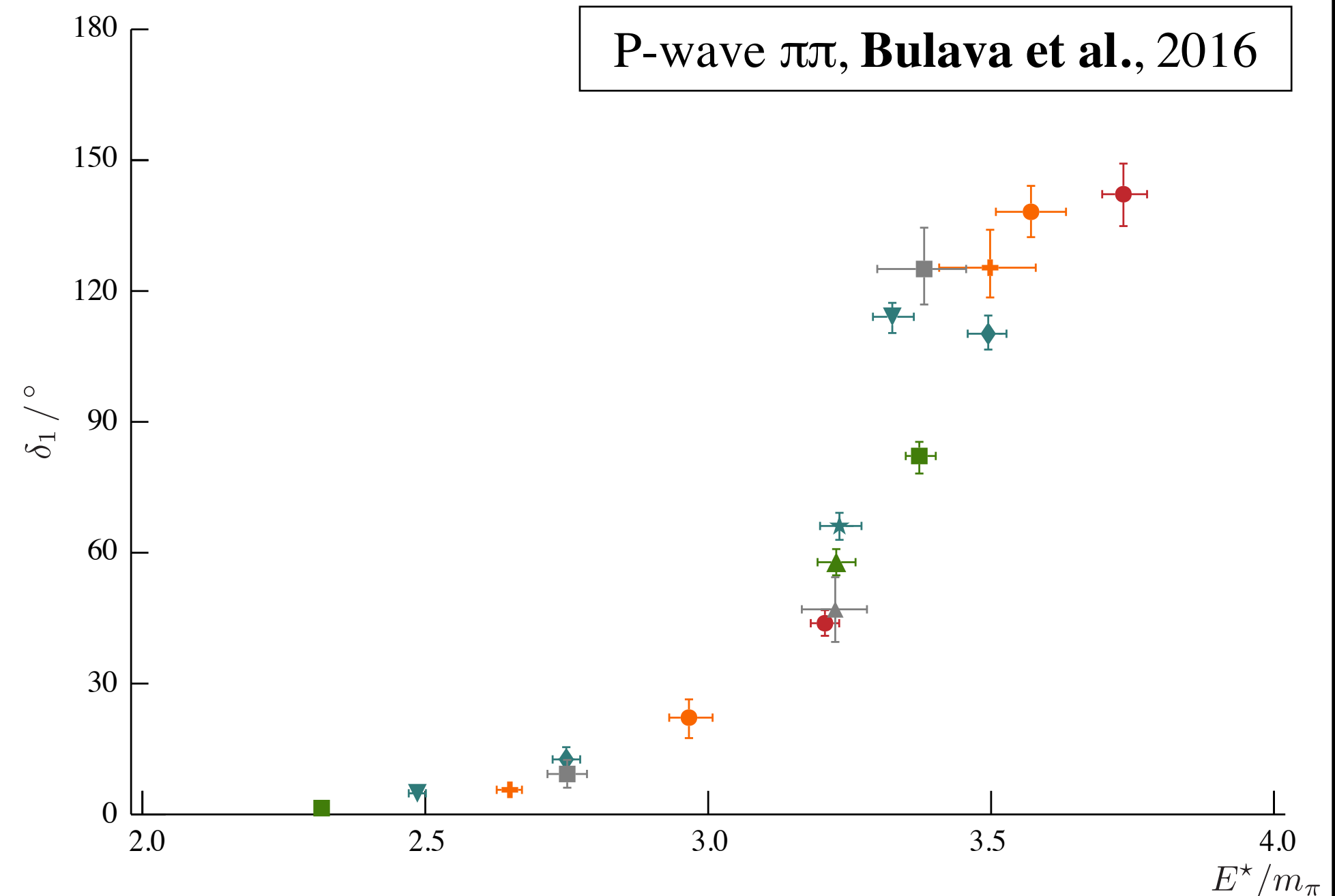
¹ Thomas Jefferson National Accelerator Facility, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA

² Department of Physics, College of William and Mary, Williamsburg, Virginia 23187, USA

³ Special Research Center for the Subatomic Structure of Matter (CSSM), Department of Physics, University of Adelaide, Adelaide 5005, Australia



γ^* -to- $\pi\pi$, Feng et al., 2015



HadSpec+

Jefferson Lab



Edwards

Old Dominion University / Jefferson Lab



Jackura

DAMTP, University of Cambridge



Wilson



Thomas

MIT



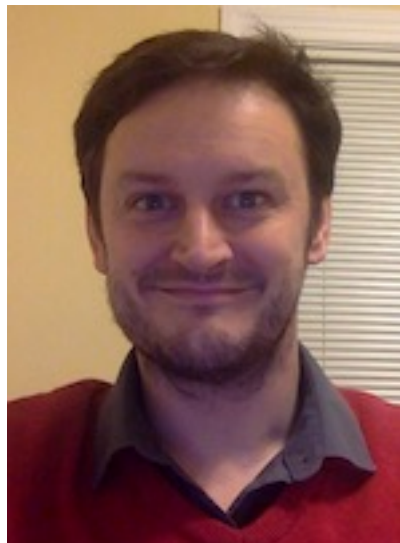
Romero-Lopez

Old Dominion University



Islam

William and Mary / Jefferson Lab



Dudek

Edinburgh



Hansen

University of Washington



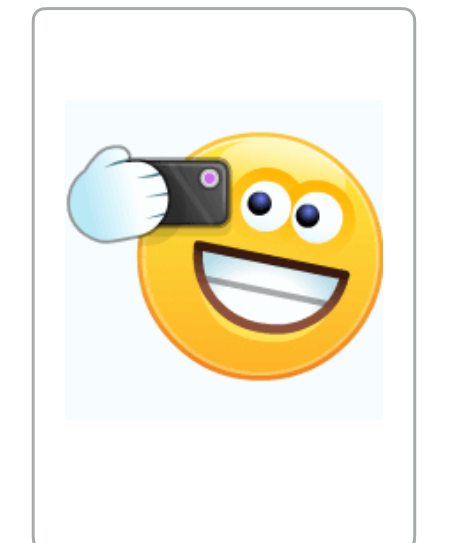
Sharpe

Indiana University



Dawid

University of Maryland



Blanton

back-up slides

Weinberg compositeness criterion for the σ

• The σ is a bound state, so we can apply Weinberg's criterion

$$|\sigma\rangle_{391} \sim \sqrt{Z} \left(\text{diagram 1} + \text{diagram 2} + \dots \right) + \sqrt{1-Z} \text{diagram 3} \text{diagram 4}$$

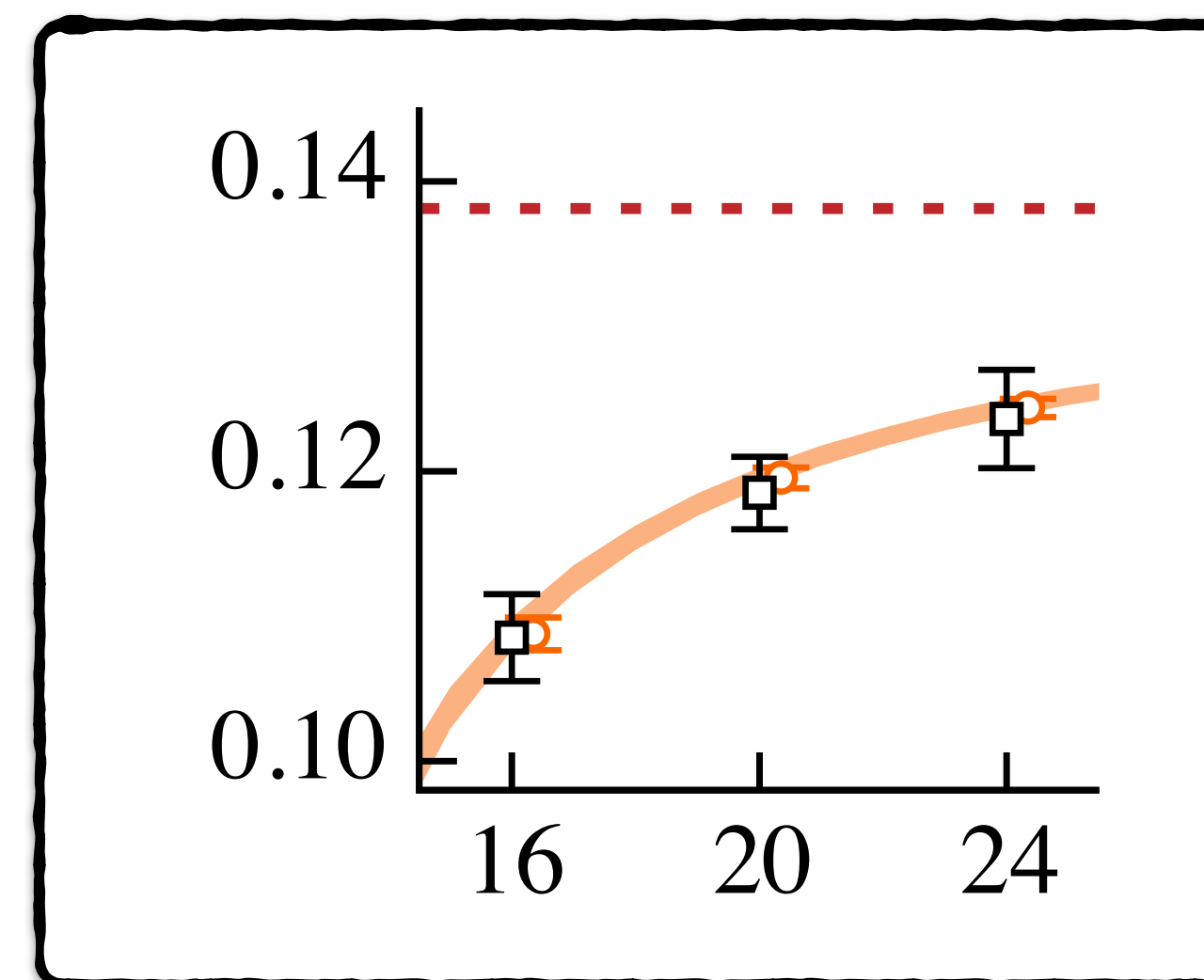
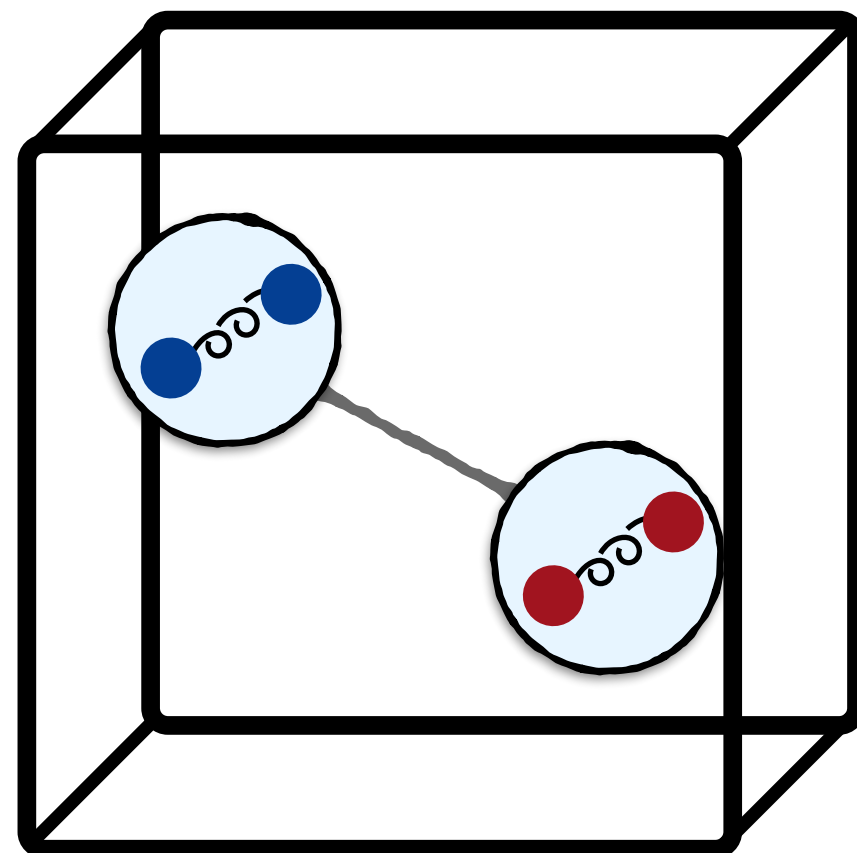
• Can relate Z to scattering information

$$a = -2 \frac{1-Z}{2-Z} \frac{1}{\sqrt{m_\pi B_\sigma}},$$

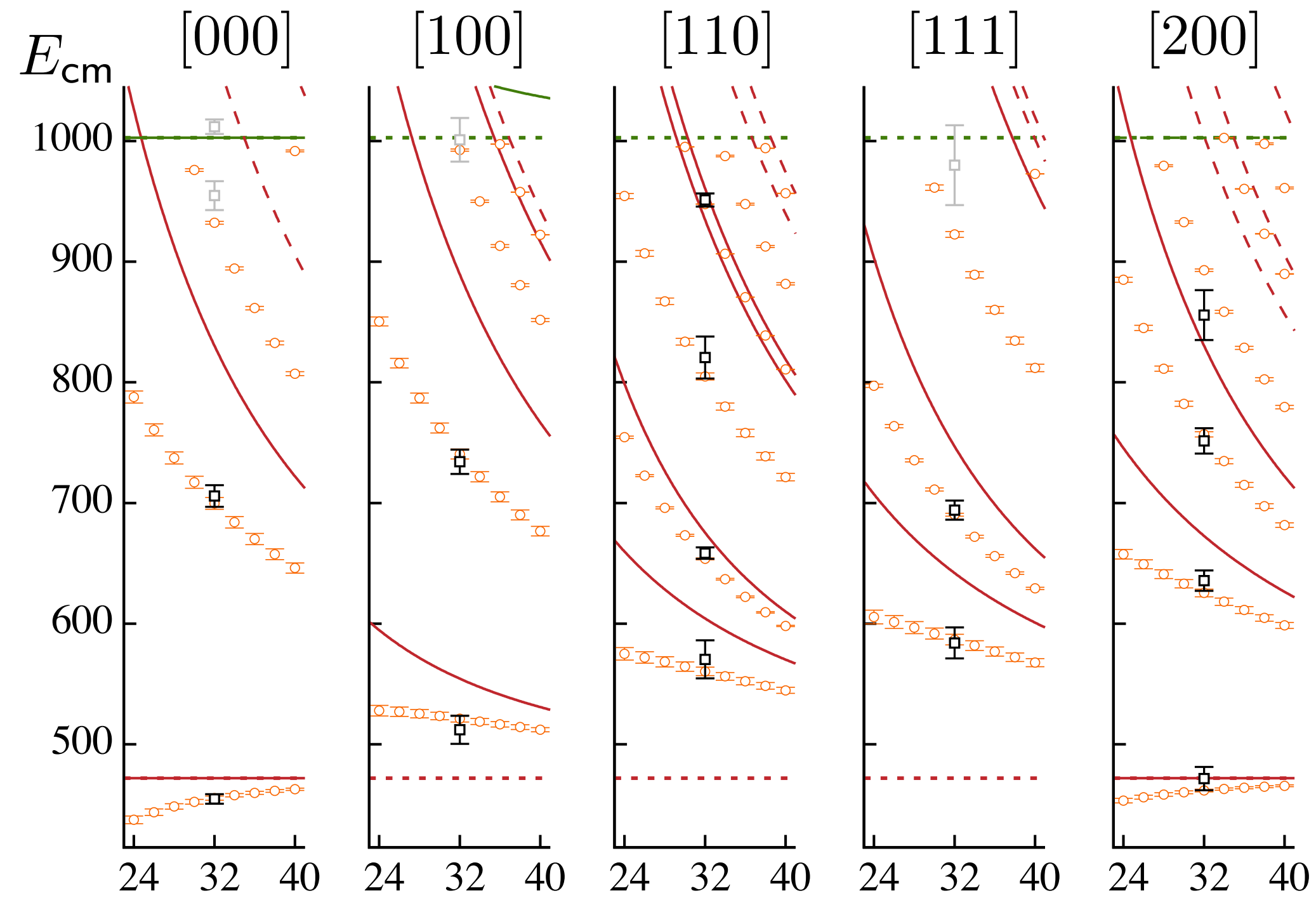
$$r = -\frac{Z}{1-Z} \frac{1}{\sqrt{m_\pi B_\sigma}}$$

• To obtain: $Z \sim 0.3(1)$

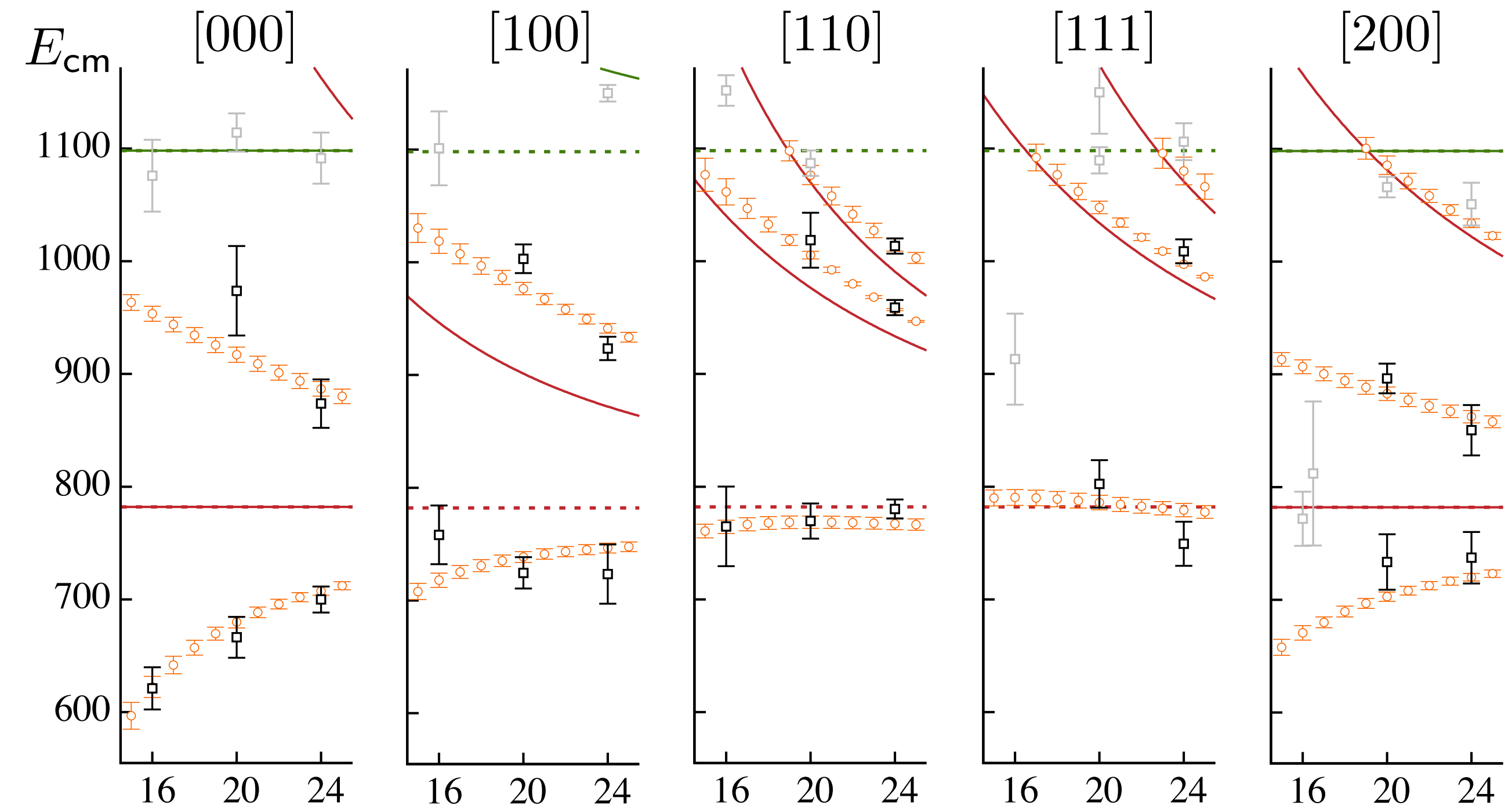
• Consistent with the large FV effects



Isoscalar, elastic spectrum fits

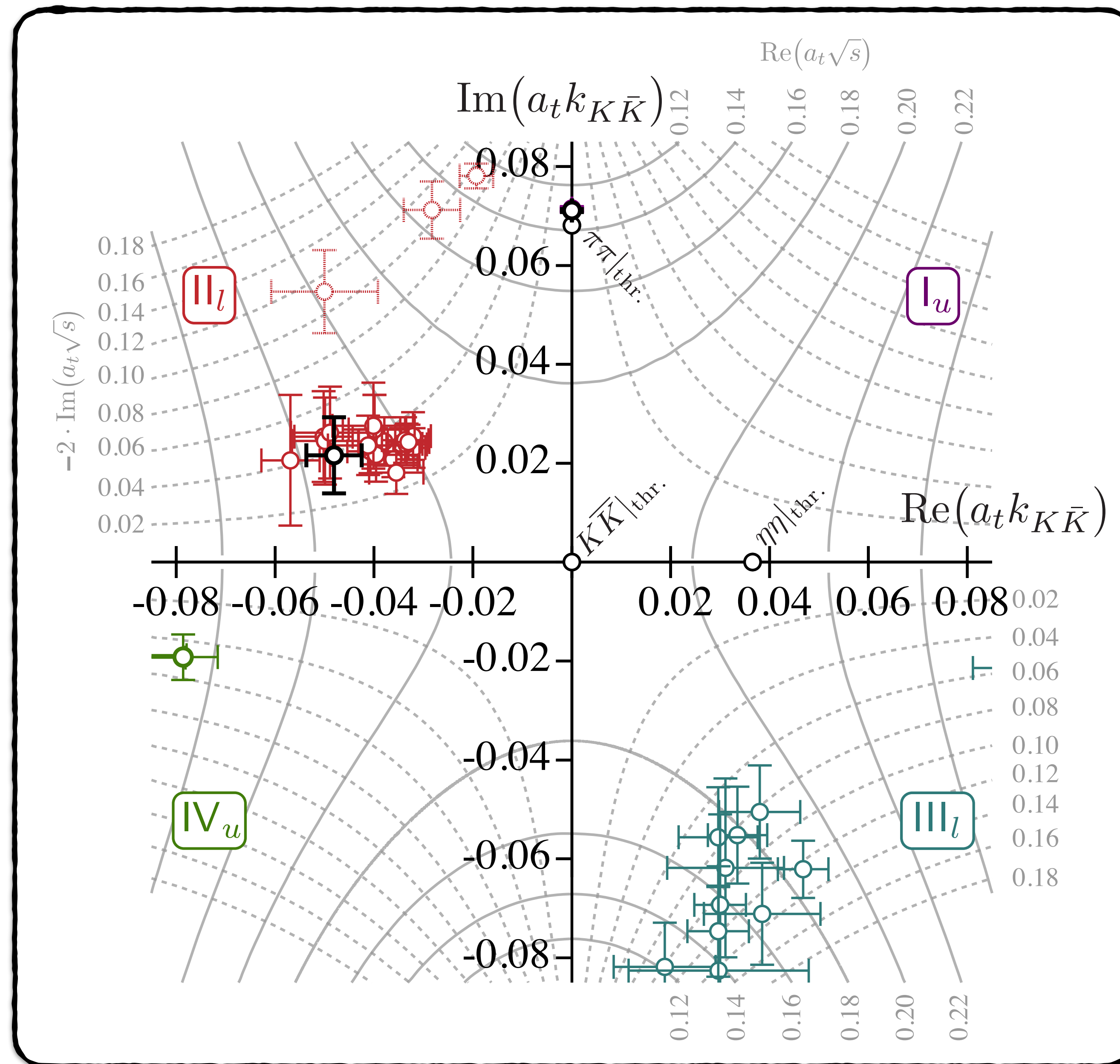


$m_\pi \sim 260$ MeV

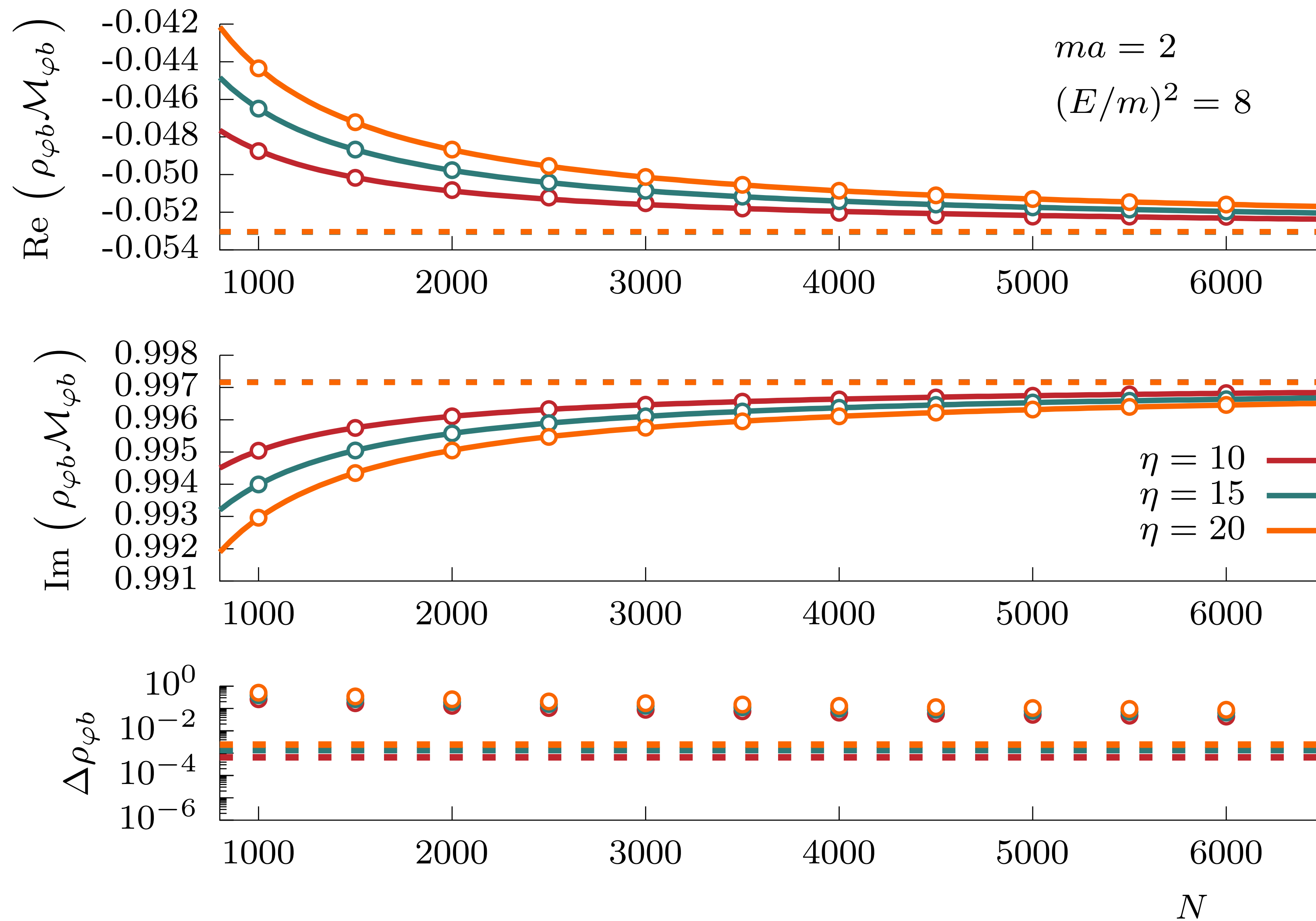


$m_\pi \sim 390$ MeV

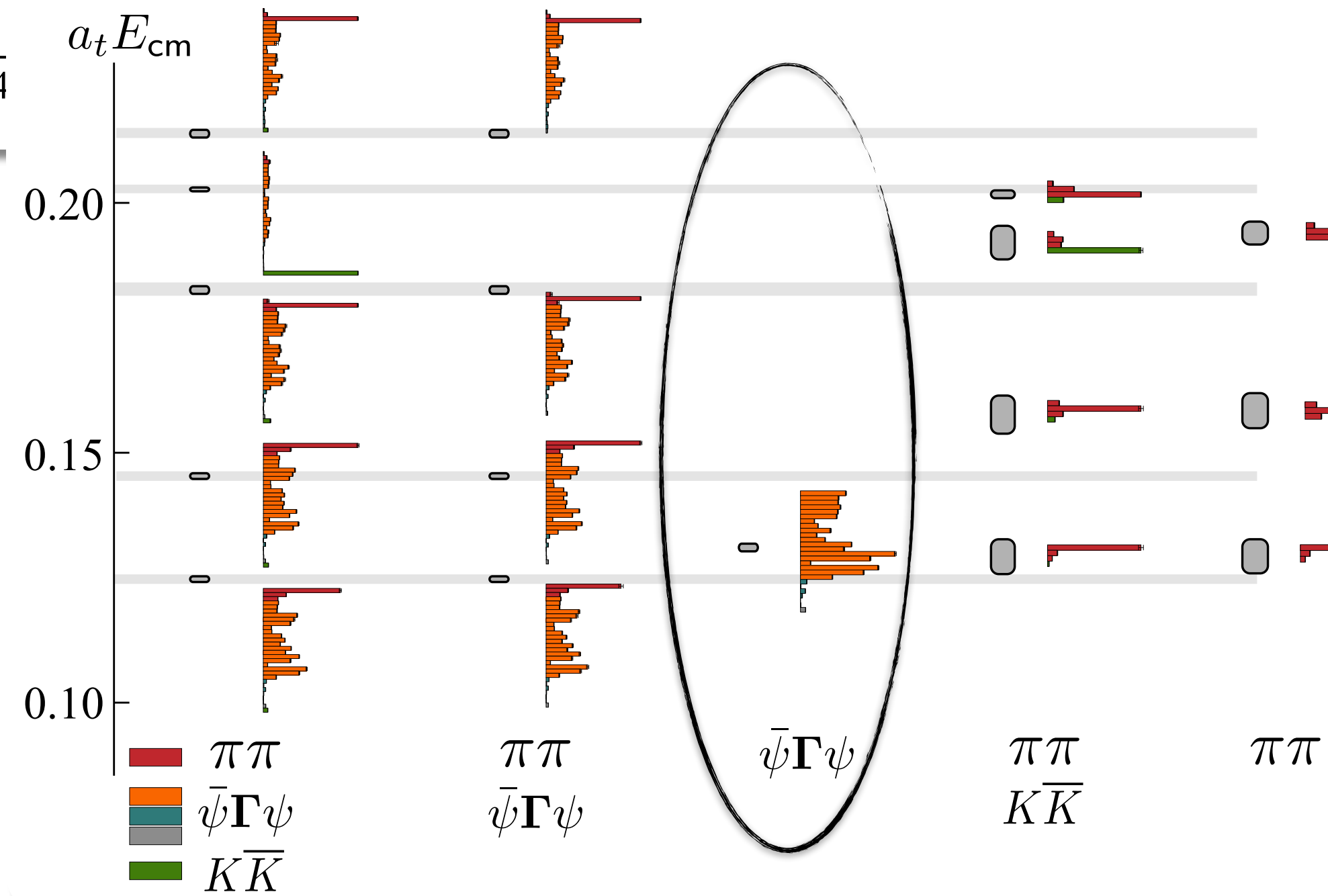
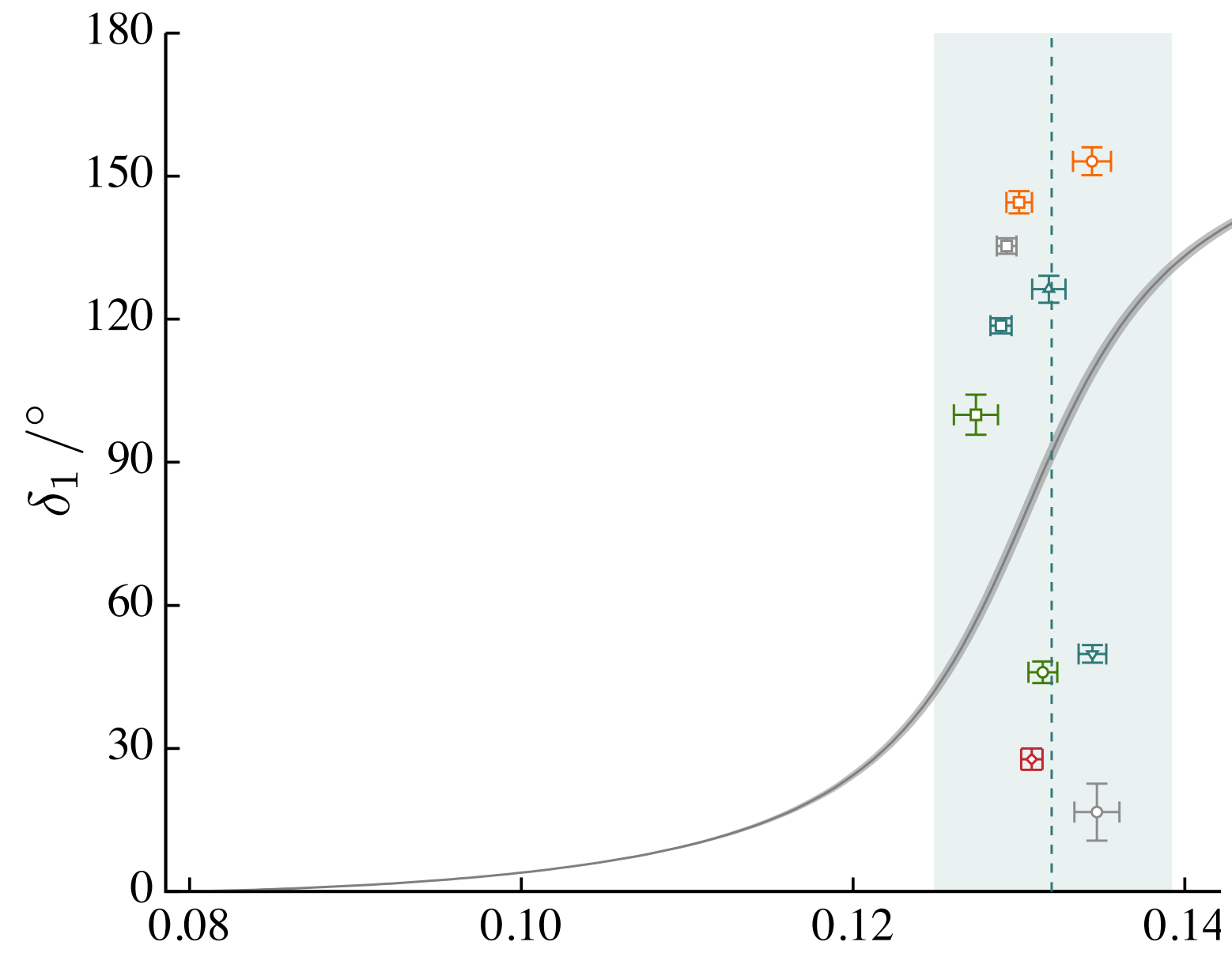
Complex momentum plane



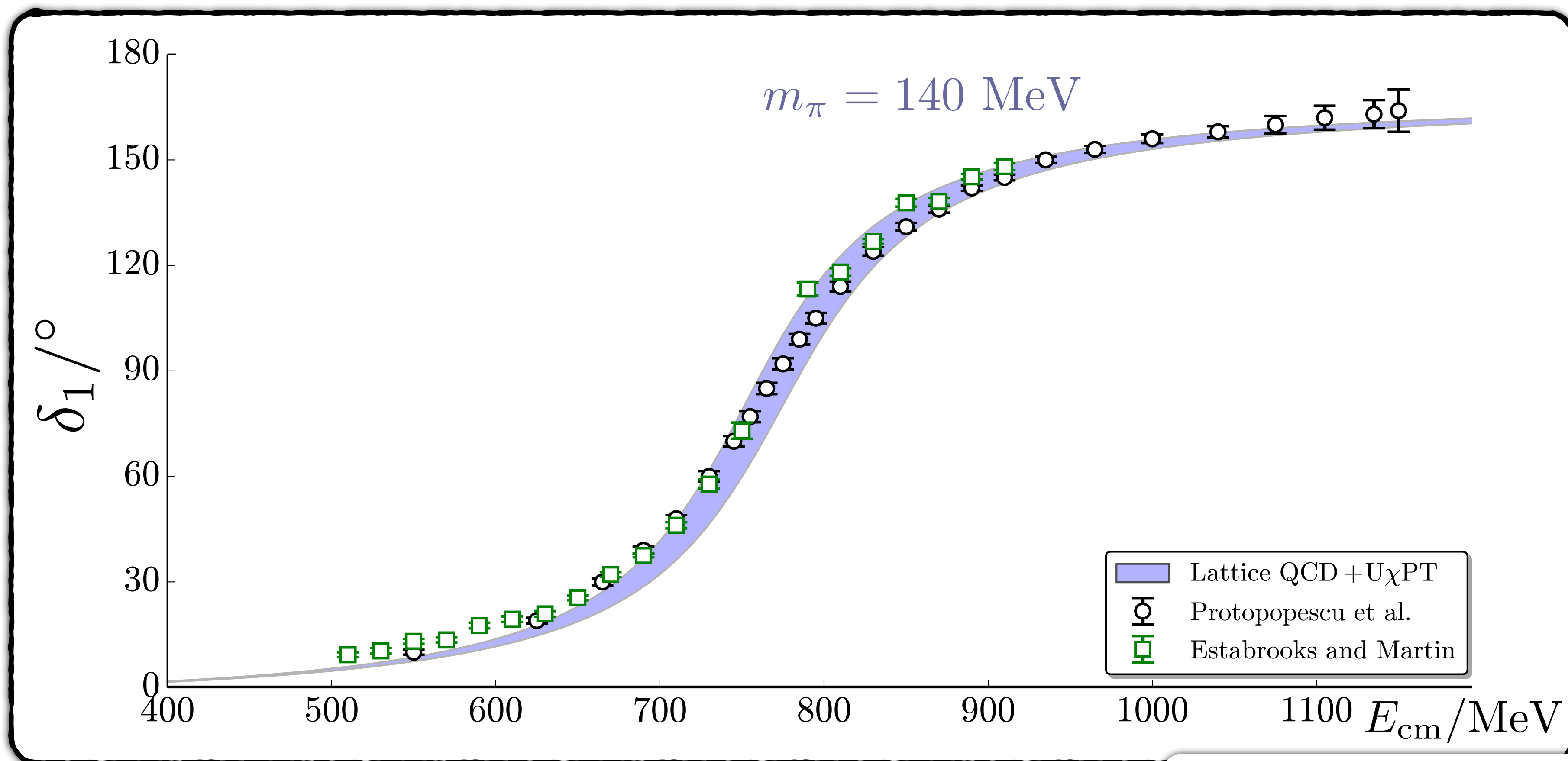
Convergence tests



The incorrect answer



Comparison with experiment



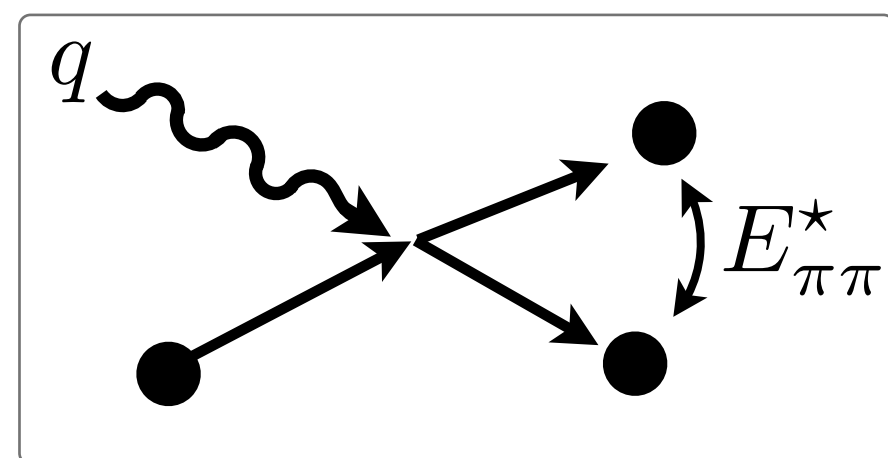
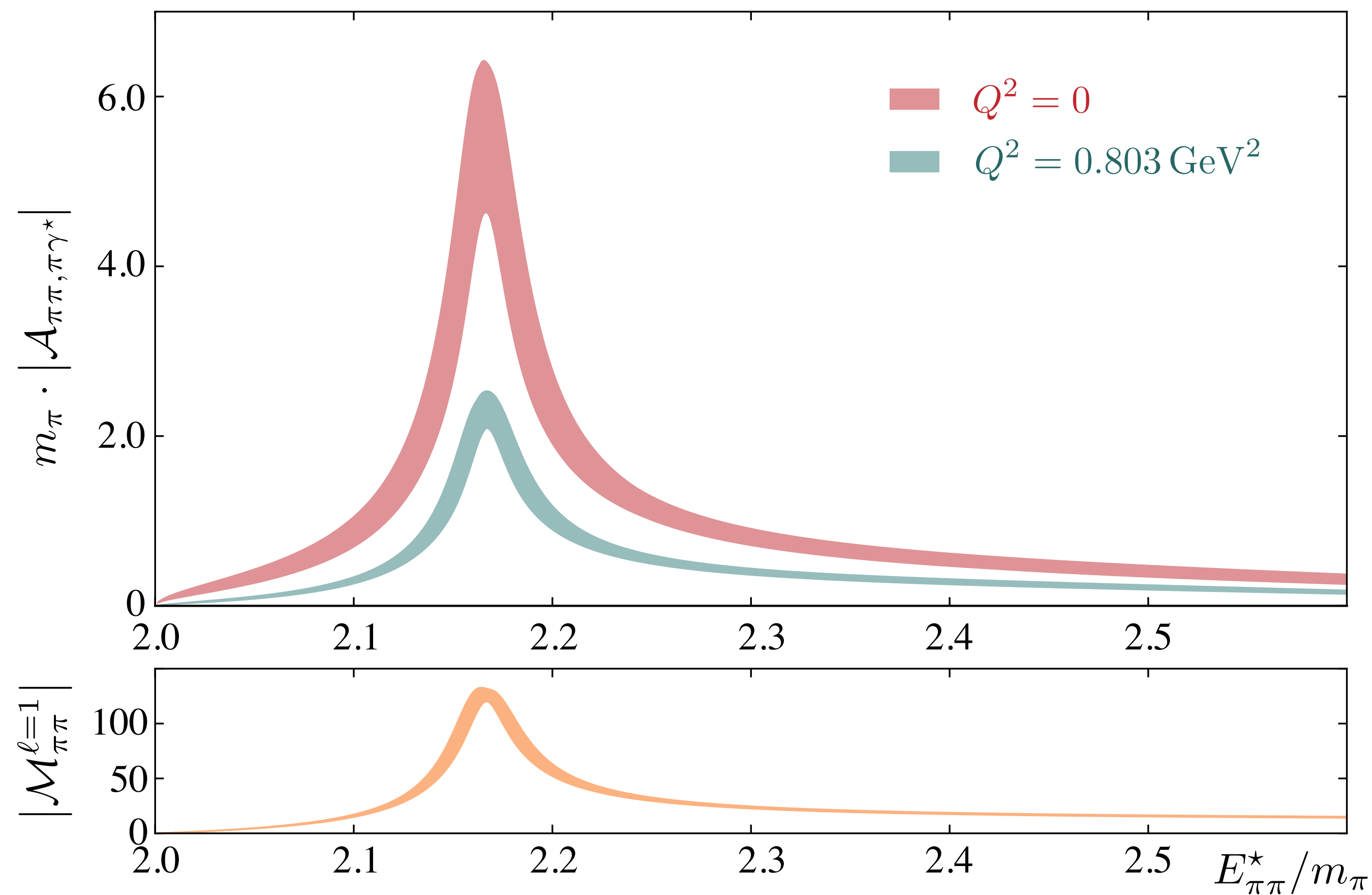
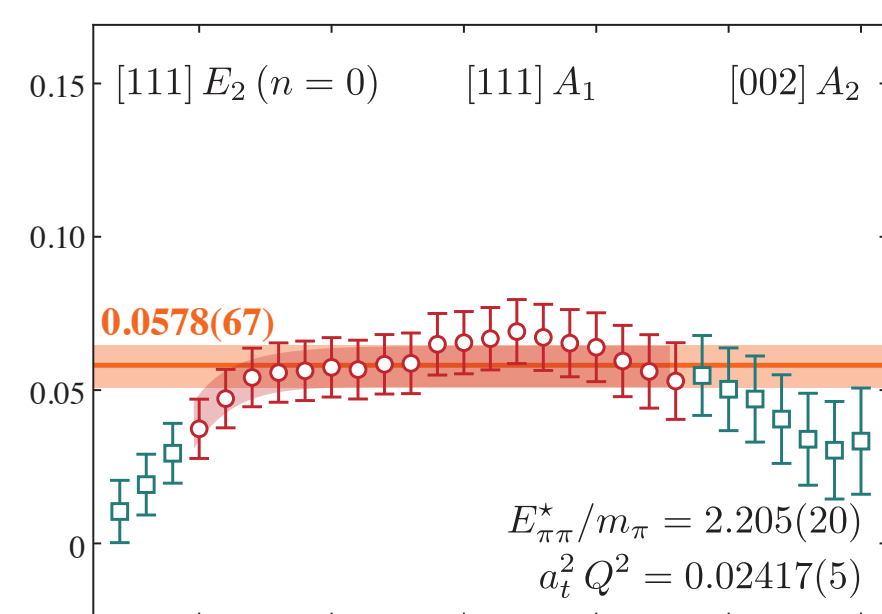
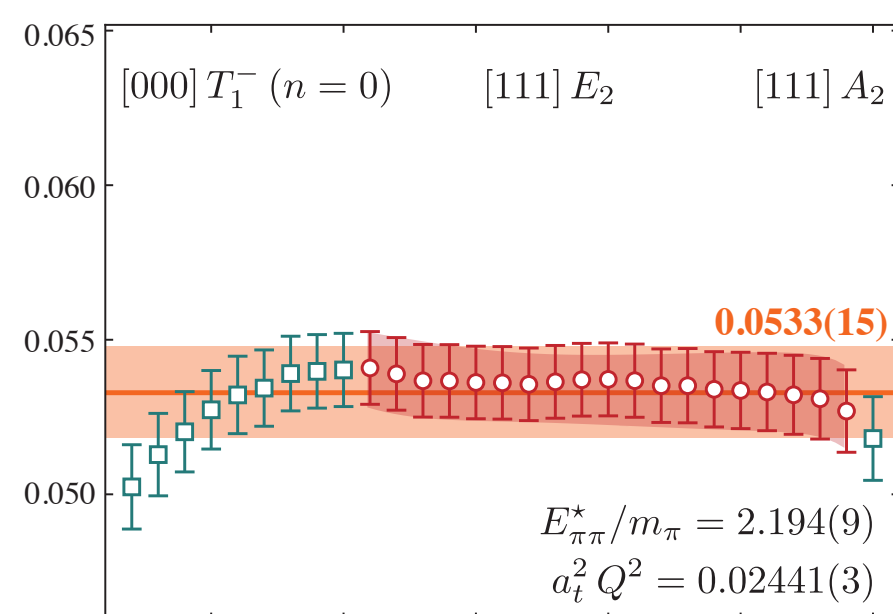
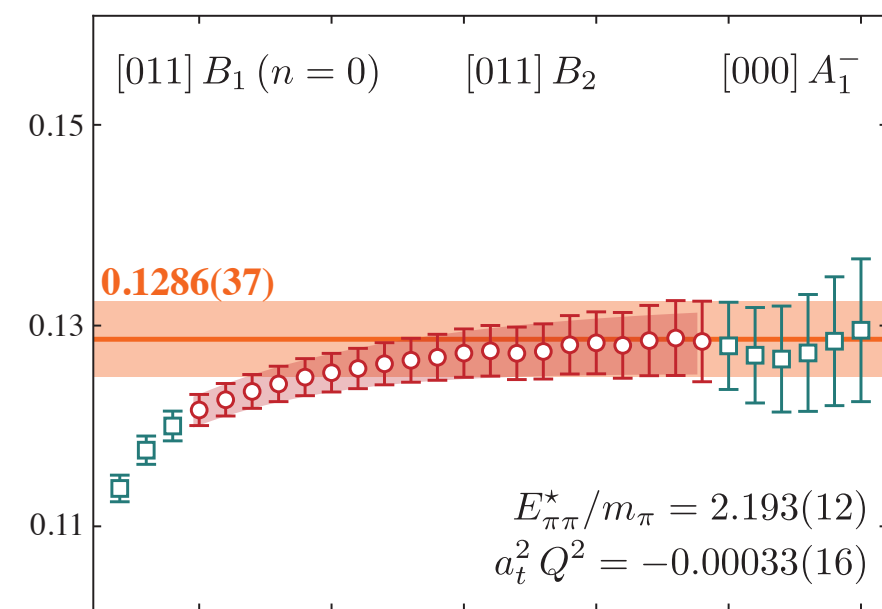
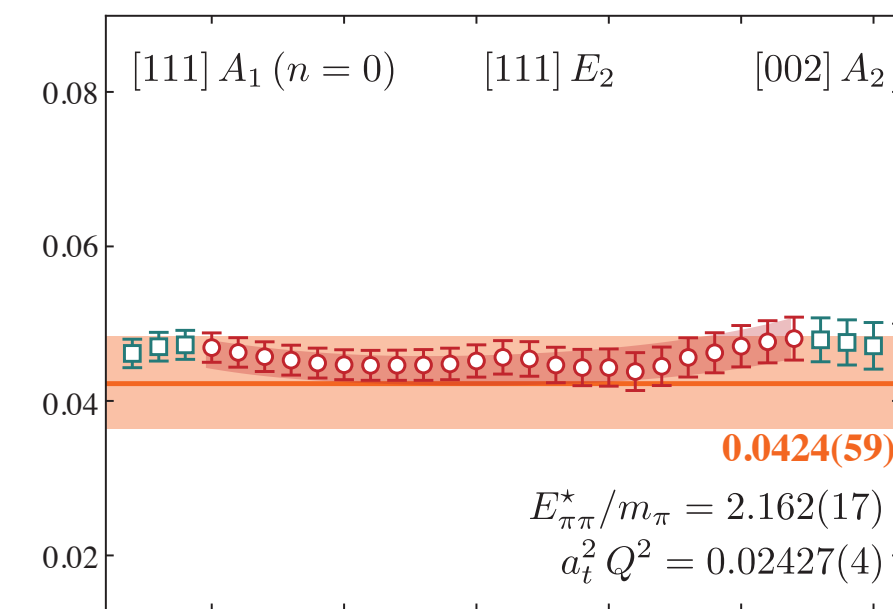
Bolton, RB & Wilson (2016)

• $U\chi PT$ in a nut-shell:

• enforce unitarity exactly: $\mathcal{M}_{U\chi PT}^{-1} \equiv \text{Re} \left(\mathcal{M}_{\chi PT}^{-1} \right) - i\rho$

• treat dynamics perturbatively: $\text{Re} \left(\mathcal{M}_{\chi PT}^{-1} \right) = \text{Re} \left(\frac{1}{\mathcal{M}_2 + \mathcal{M}_4 + \dots} \right) \approx \frac{1}{\mathcal{M}_2} - \frac{\text{Re}(\mathcal{M}_4)}{\mathcal{M}_2^2}$

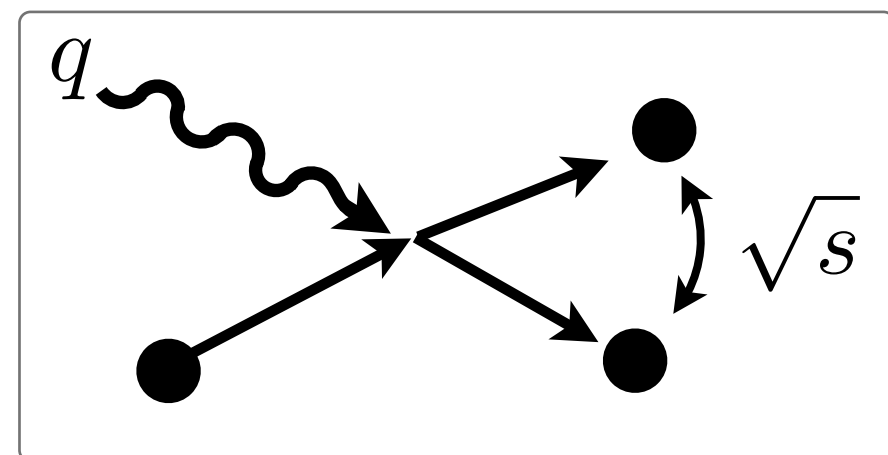
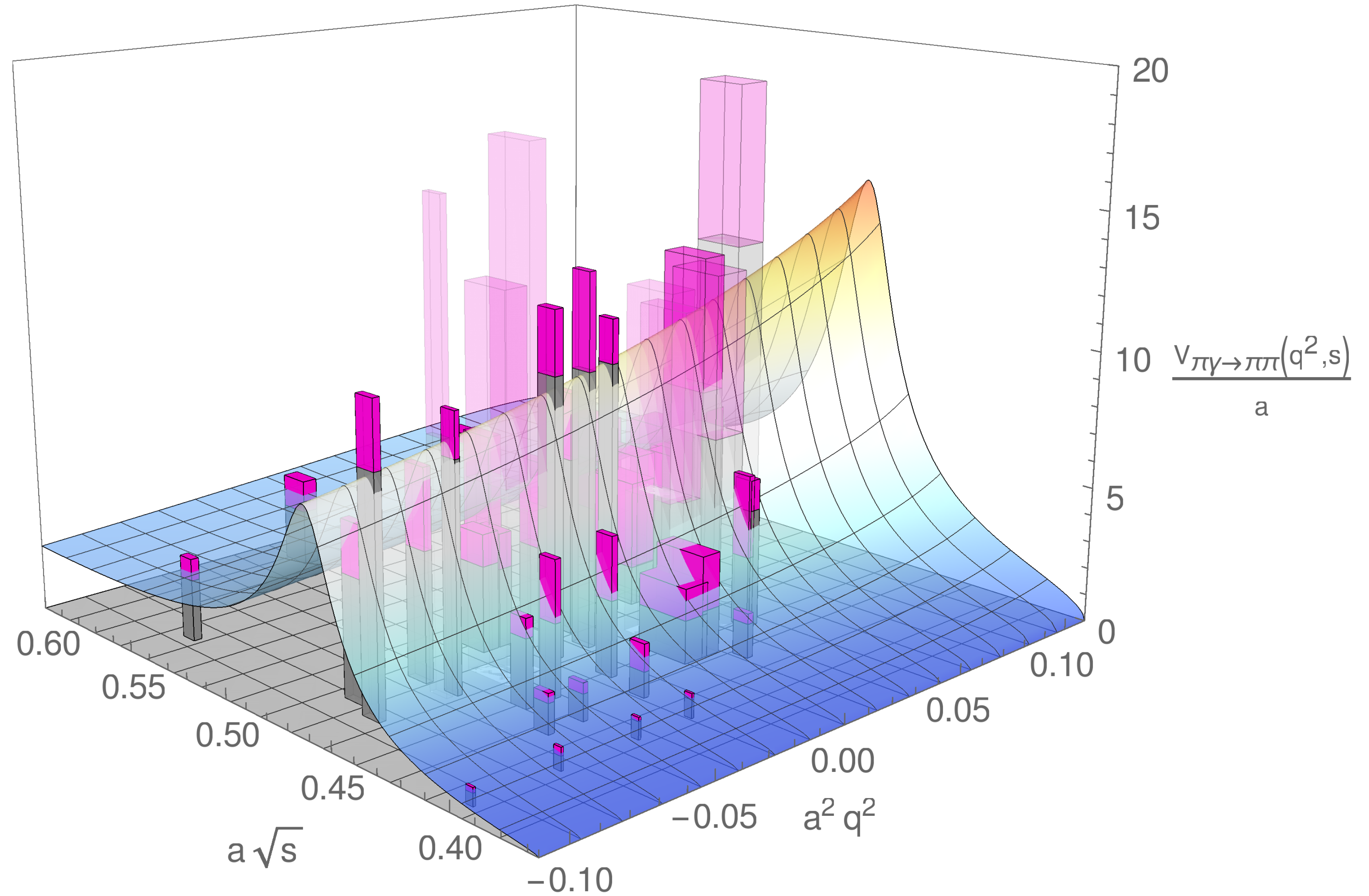
$\pi\gamma \Rightarrow \pi\pi$



$m_\pi \sim 390 \text{ MeV}$

RB, Dudek, Edwards, Shultz, & Thomas (2015)

$\pi\gamma \Rightarrow \pi\pi$



$m_\pi \sim 320 \text{ MeV}$

Alexandrou, Leskovec, Meinel, Negele,
Paul, Petschlies, Pochinsky, Rendon, Syritsyn(2018)