Local boundary constraints on free edges of unstretchable sheets

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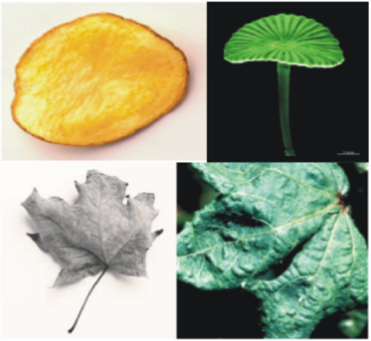
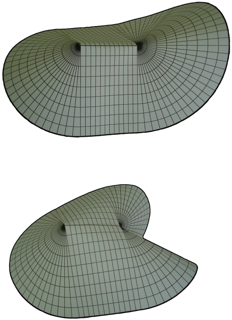
June 2021

Elastic sheets



J. Dervaux, M. Ben Amar, *PRL* 2008



J. Guven *et al.*, *EPJE* 2013

soft growing tissues & defects in paper

Surface model

- Use a continuum description and model the elastic sheet as a two-dimensional surface.

- Its energy is given as a surface integral over a scalar density $\mathcal{H}$:

$$H = \int_\Sigma \mathrm{d}A \, \mathcal{H} \, .$$

- Elastic sheets offer resistance to bending and tangential strain.

$$\boxed{H_{\text{elastic sheet}} = \int_\Sigma \mathrm{d}A \, (\mathcal{H}_{\text{bend}} + \mathcal{H}_{\text{stretching}})}$$

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Surface model

- Bending:

$$\boxed{\mathcal{H}_{\text{bend}} = \frac{1}{2} k K^2 + \bar{k} K_{\text{G}}}$$

Helfrich (1973)

k: bending modulus
$\bar{k}$: saddle splay modulus

Surface model

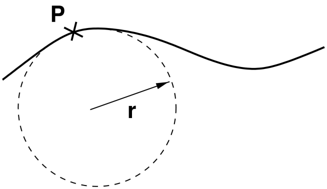
- Bending:

$$\boxed{\mathcal{H}_{\text{bend}} = \frac{1}{2}K^2 + k_G K_{\text{G}}}$$

Helfrich (1973)

Scaling: $k_G = k/\bar{k}$

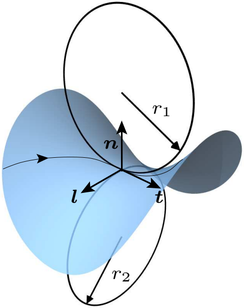
- Curvature in one dimension: $c = \dfrac{1}{r}$



Definition of K and K_{G}

In two dimensions:

$$H_{\text{bend}} = \int \mathrm{d}A \left(\frac{1}{2}K^2 + k_G K_{\mathsf{G}}\right)$$



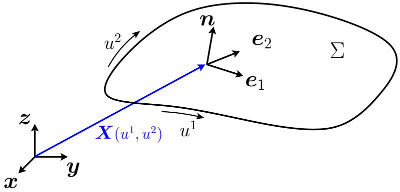
$$K = c_1 + c_2 = \frac{1}{r_1} + \frac{1}{r_2}$$

$$K_{\mathsf{G}} = c_1 \cdot c_2 = \frac{1}{r_1} \cdot \frac{1}{r_2}$$

Differential geometry

- Consider 2D surface Σ, which is described locally by its position $\boldsymbol{X}(u^1, u^2) \in \mathbb{R}^3$, where the u^a are a suitable set of local coordinates on the surface.
- Basis:

$$\boldsymbol{e}_a = \frac{\partial \boldsymbol{X}}{\partial u^a}; \quad \boldsymbol{n} = \frac{\boldsymbol{e}_1 \times \boldsymbol{e}_2}{|\boldsymbol{e}_1 \times \boldsymbol{e}_2|}.$$



- metric and extrinsic curvature tensor ($a, b \in \{1, 2\}$):

$$g_{ab} = \boldsymbol{e}_a \cdot \boldsymbol{e}_b \; ; \qquad K_{ab} = \boldsymbol{e}_a \cdot \frac{\partial \boldsymbol{n}}{\partial u^b} = \boldsymbol{e}_a \cdot \nabla_b \boldsymbol{n} \; .$$

- Note that: $K = K_{ab} g^{ab}$ and $K_G = \frac{1}{2}(K^2 - K_{ab}K^{ab})$.

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Surface model

- Elastic sheets offer resistance to tangential strain.

$$H_{\text{elastic sheet}} = \int_\Sigma \mathrm{d}A \, (\underbrace{\mathcal{H}_{\text{bend}}}_{\propto h^3} + \underbrace{\mathcal{H}_{\text{stretching}}}_{\propto h})$$

h: thickness

- IDEA for $h \to 0$: fix the metric of the surface to $g_{ab}^{(0)}$ via a Lagrange multiplier T^{ab}.

$$H = \int_\Sigma \mathrm{d}A \, \mathcal{H}_{\text{bend}} - \frac{1}{2} \int_\Sigma \mathrm{d}A \; T^{ab}(g_{ab} - g_{ab}^{(0)})$$

J. Guven, M. M. Müller, *J. Phys. A* **41**, 055203 (2008).

Shape equation

- Variation of the surface vector function: $\boldsymbol{X} \to \boldsymbol{X} + \delta \boldsymbol{X} \Rightarrow \delta H$
- Ground state in equilibrium $\Rightarrow \delta H = 0$

$$\delta H = \int_\Sigma \mathrm{d}A \, (\nabla_a \boldsymbol{f}^a) \cdot \delta \boldsymbol{X} + \delta H_{\text{boundary}} = 0 \, . \qquad (\boldsymbol{f}^a\text{: stress tensor})$$

Shape equation

- Variation of the surface vector function: $\boldsymbol{X} \rightarrow \boldsymbol{X} + \delta\boldsymbol{X} \Rightarrow \delta H$
- Decomposition: $\delta\boldsymbol{X} = \Psi^a \boldsymbol{e}_a + \Phi \boldsymbol{n}$

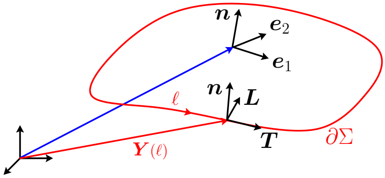
$$\delta H = \int_\Sigma \mathrm{d}A \, \left(\mathcal{E}_0 - K_{ab}T^{ab}\right)\Phi + \left(\nabla_a T^{ab}\right)\Psi_b + \delta H_{\text{boundary}} = 0 \, .$$

$$\Rightarrow \boxed{-\Delta K - K\left(\frac{K^2}{2} - 2K_{\mathrm{G}}\right) - K_{ab}T^{ab} = 0 \quad \text{and} \quad \nabla_a T^{ab} = 0 \, .}$$

- This set of nonlinear partial differential equations has to be solved to determine the shape of the sheet.

- But: What about the boundary conditions?

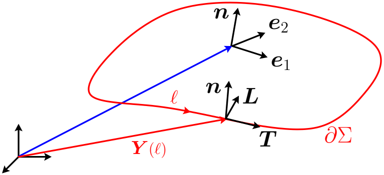
Darboux frame adapted to the boundary



- Darboux frame adapted to the boundary $\boldsymbol{Y}(\ell)$ with tangent $\boldsymbol{T} = T^a \boldsymbol{e}_a$ and conormal $\boldsymbol{L} = \boldsymbol{n} \times \boldsymbol{T} = L^a \boldsymbol{e}_a$
- Motion of the frame along the boundary ($\dot{\boldsymbol{T}} = \partial_\ell \boldsymbol{T}, \ldots$):

$$\begin{pmatrix} \dot{\boldsymbol{T}} \\ \dot{\boldsymbol{L}} \\ \dot{\boldsymbol{n}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & -\kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{L} \\ \boldsymbol{n} \end{pmatrix}$$

Darboux frame adapted to the boundary



- Darboux frame adapted to the boundary $\boldsymbol{Y}(\ell)$ with tangent $\boldsymbol{T} = T^a \boldsymbol{e}_a$ and conormal $\boldsymbol{L} = \boldsymbol{n} \times \boldsymbol{T} = L^a \boldsymbol{e}_a$
- Geodesic curvature $\kappa_g = \dot{\boldsymbol{T}} \cdot \boldsymbol{L} = L_a T^b \nabla_a T^a$

  Normal curvature $\kappa_n = \dot{\boldsymbol{n}} \cdot \boldsymbol{T} = K_{ab} T^a T^b = K_{\parallel}$

  Geodesic torsion $\tau_g = \dot{\boldsymbol{L}} \cdot \boldsymbol{n} = -K_{ab} L^a T^b = -K_{\parallel\perp}$

The boundary integral

- Variation of the energy:

$$\delta H = \int_\Sigma \mathrm{d}A \, \left(\nabla_a \boldsymbol{f}^a\right) \cdot \delta \boldsymbol{X} + \delta H_{\text{boundary}} \; .$$

The boundary integral

- Variation of the energy due to surface boundary changes:

$$\delta H_{\text{boundary}} = -\oint_{\partial\Sigma} \mathrm{d}\ell \, L_a \left(\boldsymbol{f}^a \cdot \delta \boldsymbol{X} - \mathcal{H}^{ab} \boldsymbol{e}_b \cdot \delta \boldsymbol{n} \right) \, ,$$

where

$$\begin{aligned} \boldsymbol{f}^a &= K(K^{ab} - \frac{1}{2} K g^{ab} + T^{ab}) \, \boldsymbol{e}_b - \nabla^a K \, \boldsymbol{n} \quad \text{and} \\ \mathcal{H}^{ab} &= K g^{ab} + k_G (K g^{ab} - K^{ab}) \, . \end{aligned}$$

- Decomposition w.r.t the Darboux frame:

$$\delta \boldsymbol{X} = \Psi_{\|} \, \boldsymbol{T} + \Psi_{\perp} \, \boldsymbol{L} + \Phi \boldsymbol{n} \, .$$

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The boundary integral

- In equilibrium ($\nabla_\perp = L^a \nabla_a$):

$$\delta H_{\text{boundary}} = -\oint_{\partial\Sigma} \mathrm{d}\ell \Big[T^{\mathcal{D}}_{\perp\|}\, \Psi_\| + (T^{\mathcal{D}}_{\perp} - \mathcal{H}_{\text{bend}})\Psi_\perp$$
$$- (\nabla_\perp K + k_G \dot{\tau}_g)\,\Phi + (K + k_G \kappa_n)\nabla_\perp \Phi \Big] = 0 \,,$$

- Naïvely, $\Psi_\|$, $\Psi_\perp$, Φ and $\nabla_\perp \Phi$ can be varied independently on *free* boundaries.

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The boundary integral

- In equilibrium ($\nabla_\perp = L^a \nabla_a$):

$$\delta H_{\text{boundary}} = -\oint_{\partial\Sigma} \mathrm{d}\ell \Big[T^{\mathcal{D}}_{\perp\|} \Psi_{\|} + (T^{\mathcal{D}}_{\perp} - \mathcal{H}_{\text{bend}}) \Psi_\perp - (\nabla_\perp K + k_G \dot{\tau}_g)\Phi + (K + k_G \kappa_n) \nabla_\perp \Phi \Big] = 0 \,,$$

- Naïvely, $\Psi_{\|}$, $\Psi_\perp$, Φ and $\nabla_\perp \Phi$ can be varied independently on *free* boundaries.
- However, consider the balance of torques captured by

$$K + k_G \kappa_n = 0 \,.$$

The boundary integral

- In equilibrium ($\nabla_\perp = L^a \nabla_a$):

$$\delta H_{\text{boundary}} = -\oint_{\partial\Sigma} d\ell \Big[T^{\mathcal{D}}_{\perp\|}\, \Psi_\| + (\, T^{\mathcal{D}}_\perp - \mathcal{H}_{\text{bend}})\Psi_\perp - (\nabla_\perp K + k_G \dot{\tau}_g)\, \Phi + (K + k_G \kappa_n)\nabla_\perp \Phi \Big] = 0 \,,$$

- Naïvely, $\Psi_\|$, $\Psi_\perp$, Φ and $\nabla_\perp \Phi$ can be varied independently on *free* boundaries.
- For a flat sheet:

$$K = 0 \,.$$

This does not make sense!

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The boundary integral on free edges

- The three components of $\delta \boldsymbol{X}$ on free boundaries are *not* independent!
- Proposition: Fix the intrinsic geometry of the boundary curve itself, *i.e.*, local arc-length ℓ and geodesic curvature κ_g.
- Using Lagrange multipliers $\mathcal{T}$ and Λ:

$$H^{\text{tot}} = \int_\Sigma \mathrm{d}A\, \mathcal{H}_{\text{bend}} - \frac{1}{2}\int_\Sigma \mathrm{d}A\; T^{ab}(g_{ab} - g_{ab}^{(0)}) + \oint_{\partial\Sigma} \mathrm{d}\nu\, \mathcal{T}\,(\sqrt{G} - \sqrt{G^{(0)}}) + \oint_{\partial\Sigma} \mathrm{d}\nu\, \Lambda\,(\sqrt{G}\kappa_g - \sqrt{G^{(0)}}\kappa_g^{(0)})$$

finally yields

The boundary integral on free edges

$$
\begin{aligned}
\delta H^{\text{tot}}_{\text{boundary}} \quad = \quad & - \oint_{\partial\Sigma} \mathrm{d}\ell \left[T^{\mathcal{D}}_{\perp\|} + \dot{\mathcal{T}} + \dot{\Lambda}\kappa_g \right] \Psi_{\|} \\
& - \oint_{\partial\Sigma} \mathrm{d}\ell \left[T^{\mathcal{D}}_{\perp} - \frac{1}{2}K^2 + \kappa_g \mathcal{T} - \ddot{\Lambda} - (\Lambda + k_G)K_G \right] \Psi_{\perp} \\
& + \oint_{\partial\Sigma} \mathrm{d}\ell \left[\nabla_{\perp} K + \kappa_n \mathcal{T} + 2\dot{\Lambda}\tau_g + (\Lambda + k_G)\dot{\tau}_g \right] \Phi \\
& + \oint_{\partial\Sigma} \mathrm{d}\ell \left[K + (\Lambda + k_G)\kappa_n) \right] \nabla_{\perp}\Phi
\end{aligned}
$$

J. Guven, M. M. Müller & P. Vzquez-Montejo, Math. Mech. Solids 24, 4051 (2019).

The boundary integral on free edges

$$
\begin{aligned}
\delta H^{\text{tot}}_{\text{boundary}} \;=\; & -\oint_{\partial\Sigma} d\ell \left[T^{\mathcal{D}}_{\perp\parallel} + \dot{\mathcal{T}} + \dot{\Lambda}\kappa_g \right] \Psi_\parallel \\
& -\oint_{\partial\Sigma} d\ell \left[T^{\mathcal{D}}_{\perp} - \frac{1}{2}K^2 + \kappa_g \mathcal{T} - \ddot{\Lambda} - (\Lambda + k_G)K_G \right] \Psi_\perp \\
& +\oint_{\partial\Sigma} d\ell \left[\nabla_\perp K + \kappa_n \mathcal{T} + 2\dot{\Lambda}\tau_g + (\Lambda + k_G)\dot{\tau}_g \right] \Phi \\
& +\oint_{\partial\Sigma} d\ell \left[K + (\Lambda + k_G)\kappa_n \right] \nabla_\perp \Phi
\end{aligned}
$$

Terms from bending and the constraints on surface isometry, arc-length, and geodesic curvature

Boundary conditions on free edges

- With $\Lambda \to \Lambda + k_G$, we obtain the boundary conditions:

$$T^{\mathcal{D}}_{\perp\|} + \dot{\mathcal{T}} + \dot{\Lambda}\kappa_g = 0$$

$$T^{\mathcal{D}}_{\perp} - \frac{1}{2}K^2 + \kappa_g \mathcal{T} - \ddot{\Lambda} - \Lambda K_G = 0$$

$$\nabla_\perp K + \kappa_n \mathcal{T} + 2\dot{\Lambda}\tau_g + \Lambda\dot{\tau}_g = 0$$

$$K + \Lambda\,\kappa_n = 0$$

Solve the shape equations subject to these boundary conditions!

Boundary conditions on free edges

- With $\Lambda \to \Lambda + k_G$, we obtain the boundary conditions:

$$T^{\mathcal{D}}_{\perp\|} + \dot{\mathcal{T}} + \dot{\Lambda}\kappa_g = 0 \tag{1}$$

$$T^{\mathcal{D}}_{\perp} - \frac{1}{2}K^2 + \kappa_g \mathcal{T} - \ddot{\Lambda} - \Lambda K_G = 0 \tag{2}$$

$$\nabla_\perp K + \kappa_n \mathcal{T} + 2\dot{\Lambda}\tau_g + \Lambda\dot{\tau}_g = 0 \tag{3}$$

$$K + \Lambda\,\kappa_n = 0 \tag{4}$$

- For a boundary with $\kappa_n \neq 0$, we obtain $\Lambda = -\frac{K}{\kappa_n}$ from (4).
- Eq. (3) $\Rightarrow \mathcal{T}$ in terms of the boundary geometry

  $\Rightarrow T^{\mathcal{D}}_{\perp}$ and $T^{\mathcal{D}}_{\perp\|}$ *via* (1) and (2).

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Conical geometry

Position vector:

$$\boldsymbol{X}(r,s) = r\,\boldsymbol{u}(s)$$

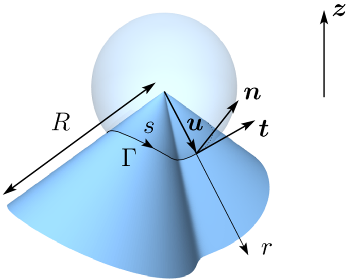
Curvature $(a,b) \in \{r,s\}$:

$$K_{ab} = r\begin{pmatrix} 0 & 0 \\ 0 & \kappa(s) \end{pmatrix}$$

with $\kappa(s) = -\boldsymbol{n}\cdot\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = -\boldsymbol{n}\cdot\boldsymbol{t}'$.

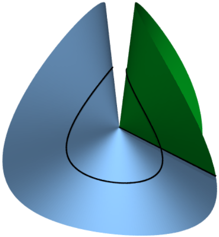
$$\boxed{\kappa'' + \frac{\kappa^3}{2} + (1 + r^2 T_{\parallel})\,\kappa = 0}$$

with $T_{\parallel} = t^a t^b T_{ab} = -C_{\parallel}(s)/r^2$.



The conical defect

- For a circular outer boundary ($r = R$), the energy is rotationally invariant $\Rightarrow C_{\|}$ is constant.
- No need to solve the other shape equation $\nabla_a T^{ab} = 0$ to determine the equilibrium geometry!



M. M. Müller, M. Ben Amar & J. Guven, *Phys. Rev. Lett.* **101**, 156104 (2008).
N. Stoop *et al.*, *Phys. Rev. Lett.* **105**, 068101 (2010).
J. Guven, M. M. Müller & P. Vzquez-Montejo, *J. Phys. A* **45**, 015203 (2012).

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Including the boundary conditions

- Let's do it now! Solution of $\nabla_a T^{ab} = 0$:

$$
\begin{aligned}
T_{\parallel}(s,r) &:= t^a t^a T_{ab} = -\frac{C_{\parallel}}{r^2} \\
T_{\perp\parallel}(s,r) &:= l^a t^b T_{ab} = -\frac{1}{r^2}\left(C'_{\parallel}\ln r + C_{\parallel\perp}\right) \\
T_{\perp}(s,r) &:= l^a l^b T_{ab} = \frac{1}{r^2}\left[C''_{\parallel}(\ln r + 1) + C_{\parallel} + C'_{\parallel\perp}\right] + \frac{C_{\perp}}{r}
\end{aligned}
$$

Including the boundary conditions

- Solution for a circular outer boundary ($C_\parallel = \text{const.}$)

$$\begin{aligned} T_\parallel(r) &= -\frac{C_\parallel}{r^2} \\ T_{\perp\parallel}(s,r) &= -\frac{1}{r^2} C_{\parallel\perp}(s) \\ T_\perp(s,r) &= \frac{1}{r^2}\left[C_\parallel + C_{\parallel\perp}(s)'\right] + \frac{C_\perp(s)}{r} \end{aligned}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} + \dot{\mathcal{T}} + \dot{\Lambda}\kappa_g = 0$$

$$T^{\mathcal{D}}_{\perp} - \frac{1}{2}K^2 + \kappa_g \mathcal{T} - \ddot{\Lambda} - \Lambda K_G = 0$$

$$\nabla_\perp K + \kappa_n \mathcal{T} + 2\dot{\Lambda}\tau_g + \Lambda\dot{\tau}_g = 0$$

$$K + \Lambda\,\kappa_n = 0$$

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Including the boundary conditions

- Solution for a circular outer boundary ($C_{\parallel} = \text{const.}$)

$$T_{\parallel}(r) = -\frac{C_{\parallel}}{r^2}$$

$$T_{\perp\parallel}(s,r) = -\frac{1}{r^2} C_{\parallel\perp}(s)$$

$$T_{\perp}(s,r) = \frac{1}{r^2}\left[C_{\parallel} + C_{\parallel\perp}(s)'\right] + \frac{C_{\perp}(s)}{r}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} + \dot{\mathcal{T}} + \dot{\Lambda}\kappa_g = 0$$

$$T^{\mathcal{D}}_{\perp} - \frac{1}{2}K^2 + \kappa_g \mathcal{T} - \ddot{\Lambda} - \Lambda K_G = 0$$

$$\nabla_{\perp} K + K\mathcal{T} + 2\dot{\Lambda}\tau_g + \Lambda\dot{\tau}_g = 0$$

$$K + \Lambda K = 0$$

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Including the boundary conditions

- Solution for a circular outer boundary ($C_\parallel = \text{const.}$)

$$\begin{aligned} T_\parallel(r) &= -\frac{C_\parallel}{r^2} \\ T_{\perp\parallel}(s,r) &= -\frac{1}{r^2} C_{\parallel\perp}(s) \\ T_\perp(s,r) &= \frac{1}{r^2}\left[C_\parallel + C_{\parallel\perp}(s)'\right] + \frac{C_\perp(s)}{r} \end{aligned}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} + \dot{\mathcal{T}} + \dot{\Lambda}\kappa_g = 0$$

$$T^{\mathcal{D}}_{\perp} - \frac{1}{2}K^2 + \kappa_g \mathcal{T} - \ddot{\Lambda} - \Lambda K_G = 0$$

$$\nabla_\perp K + K\mathcal{T} + 2\dot{\Lambda}\tau_g + \Lambda\dot{\tau}_g = 0$$

$$\Lambda = -1$$

Including the boundary conditions

- Solution for a circular outer boundary ($C_{\|} = \text{const.}$)

$$\begin{aligned} T_{\|}(r) &= -\frac{C_{\|}}{r^2} \\ T_{\perp\|}(s,r) &= -\frac{1}{r^2} C_{\|\perp}(s) \\ T_{\perp}(s,r) &= \frac{1}{r^2}\left[C_{\|} + C_{\|\perp}(s)'\right] + \frac{C_{\perp}(s)}{r} \end{aligned}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\|} + \dot{\mathcal{T}} = 0$$

$$T^{\mathcal{D}}_{\perp} - \frac{1}{2}K^2 + \kappa_g \mathcal{T} = 0$$

$$\nabla_{\perp} K + K\mathcal{T} = 0$$

$$\Lambda = -1$$

Including the boundary conditions

- Solution for a circular outer boundary ($C_\parallel = \text{const.}$)

$$
\begin{aligned}
T_\parallel(r) &= -\frac{C_\parallel}{r^2} \\
T_{\perp\parallel}(s,r) &= -\frac{1}{r^2} C_{\parallel\perp}(s) \\
T_\perp(s,r) &= \frac{1}{r^2}\left[C_\parallel + C_{\parallel\perp}(s)'\right] + \frac{C_\perp(s)}{r}
\end{aligned}
$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} + \dot{\mathcal{T}} = 0$$

$$T^{\mathcal{D}}_{\perp} - \frac{\kappa^2}{2R^2} + \kappa_g \mathcal{T} = 0$$

$$-\partial_r\left(\frac{\kappa}{r}\right)\Big|_{r=R} + \frac{\kappa}{R}\mathcal{T} = 0$$

$$\Lambda = -1$$

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Including the boundary conditions

- Solution for a circular outer boundary ($C_{\|} = \text{const.}$)

$$
\begin{aligned}
T_{\|}(r) &= -\frac{C_{\|}}{r^2} \\
T_{\perp\|}(s,r) &= -\frac{1}{r^2} C_{\|\perp}(s) \\
T_{\perp}(s,r) &= \frac{1}{r^2}\left[C_{\|} + C_{\|\perp}(s)'\right] + \frac{C_{\perp}(s)}{r}
\end{aligned}
$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\|} + \dot{\mathcal{T}} = 0$$

$$T^{\mathcal{D}}_{\perp} - \frac{\kappa^2}{2R^2} + \kappa_g \mathcal{T} = 0$$

$$\mathcal{T} = -\frac{1}{R}$$

$$\Lambda = -1$$

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Including the boundary conditions

- Solution for a circular outer boundary ($C_\parallel = \text{const.}$)

$$\begin{aligned} T_\parallel(r) &= -\frac{C_\parallel}{r^2} \\ T_{\perp\parallel}(s,r) &= -\frac{1}{r^2} C_{\parallel\perp}(s) \\ T_\perp(s,r) &= \frac{1}{r^2}\left[C_\parallel + C_{\parallel\perp}(s)'\right] + \frac{C_\perp(s)}{r} \end{aligned}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} = 0$$

$$T^{\mathcal{D}}_{\perp} - \frac{\kappa^2}{2R^2} - \frac{1}{R^2} = 0$$

$$\mathcal{T} = -\frac{1}{R}$$

$$\Lambda = -1$$

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Including the boundary conditions

- Solution for a circular outer boundary ($C_\parallel = \text{const.}$)

$$\begin{aligned} T_\parallel(r) &= -\frac{C_\parallel}{r^2} \\ T_{\perp\parallel}(s,r) &= -\frac{1}{r^2} C_{\parallel\perp}(s) \\ T_\perp(s,r) &= \frac{1}{r^2}\left[C_\parallel + C_{\parallel\perp}(s)'\right] + \frac{C_\perp(s)}{r} \end{aligned}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} = 0$$

$$T^{\mathcal{D}}_{\perp} = \frac{1}{R^2}\left(\frac{\kappa^2}{2} + 1\right)$$

$$\mathcal{T} = -\frac{1}{R}$$

$$\Lambda = -1$$

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Including the boundary conditions

- Solution for a circular outer boundary ($C_{\parallel} = \text{const.}$)

$$\begin{aligned} T_{\parallel}(r) &= -\frac{C_{\parallel}}{r^2} \\ T_{\perp\parallel}(s,r) &= -\frac{1}{r^2} C_{\parallel\perp}(s) \\ T_{\perp}(s,r) &= \frac{1}{r^2}\left[C_{\parallel} + C_{\parallel\perp}(s)'\right] + \frac{C_{\perp}(s)}{r} \end{aligned}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} = 0 = T_{\perp\parallel}(s,R)$$

$$T^{\mathcal{D}}_{\perp} = \frac{1}{R^2}\left(\frac{\kappa^2}{2} + 1\right) = T_{\perp}(s,R)$$

$$\mathcal{T} = -\frac{1}{R}$$

$$\Lambda = -1$$

Including the boundary conditions

- Solution for a circular outer boundary ($C_{\parallel} = \text{const.}$)

$$
\begin{aligned}
T_{\parallel}(r) &= -\frac{C_{\parallel}}{r^2} \\
T_{\perp\parallel}(s,r) &= 0 \\
T_{\perp}(s,r) &= \frac{C_{\parallel}}{r^2} + \frac{C_{\perp}(s)}{r}
\end{aligned}
$$

- Boundary conditions (at $r = R$):

$$T_{\perp\parallel}^{\mathcal{D}} = 0 = T_{\perp\parallel}(s,R)$$

$$T_{\perp}^{\mathcal{D}} = \frac{1}{R^2}\left(\frac{\kappa^2}{2} + 1\right) = T_{\perp}(s,R)$$

$$\mathcal{T} = -\frac{1}{R}$$

$$\Lambda = -1$$

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Including the boundary conditions

- Solution for a circular outer boundary ($C_\parallel = \text{const.}$)

$$\begin{aligned} T_\parallel(r) &= -\frac{C_\parallel}{r^2} \\ T_{\perp\parallel}(s,r) &= 0 \\ T_\perp(s,r) &= \frac{C_\parallel}{r^2} + \frac{C_\perp(s)}{r} \end{aligned}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} = 0 = T_{\perp\parallel}(s,R)$$

$$T^{\mathcal{D}}_\perp = \frac{1}{R^2}\left(\frac{\kappa^2}{2} + 1\right) = \frac{C_\parallel}{R^2} + \frac{C_\perp(s)}{R}$$

$$\mathcal{T} = -\frac{1}{R}$$

$$\Lambda = -1$$

Including the boundary conditions

- Solution for a circular outer boundary ($C_\parallel = \text{const.}$)

$$\begin{aligned} T_\parallel(r) &= -\frac{C_\parallel}{r^2} \\ T_{\perp\parallel}(s,r) &= 0 \\ T_\perp(s,r) &= \frac{C_\parallel}{r^2} + \frac{C_\perp(s)}{r} \end{aligned}$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} = 0 = T_{\perp\parallel}(s,R)$$

$$C_\perp(s) = \frac{1}{R}\left(\frac{\kappa^2}{2} + 1 - C_\parallel\right)$$

$$\mathcal{T} = -\frac{1}{R}$$

$$\Lambda = -1$$

Including the boundary conditions

- Solution for a circular outer boundary ($C_\parallel = \text{const.}$)

$$
\begin{aligned}
T_\parallel(r) &= -\frac{C_\parallel}{r^2} \\
T_{\perp\parallel}(s,r) &= 0 \\
T_\perp(s,r) &= \frac{C_\parallel}{r^2}\left(1-\frac{r}{R}\right)+\frac{1}{rR}\left(\frac{\kappa^2}{2}+1\right)
\end{aligned}
$$

- Boundary conditions (at $r = R$):

$$T^{\mathcal{D}}_{\perp\parallel} = 0 = T_{\perp\parallel}(s,R)$$

$$C_\perp(s) = \frac{1}{R}\left(\frac{\kappa^2}{2}+1-C_\parallel\right)$$

$$\mathcal{T} = -\frac{1}{R}$$

$$\Lambda = -1$$

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Stresses in the bulk

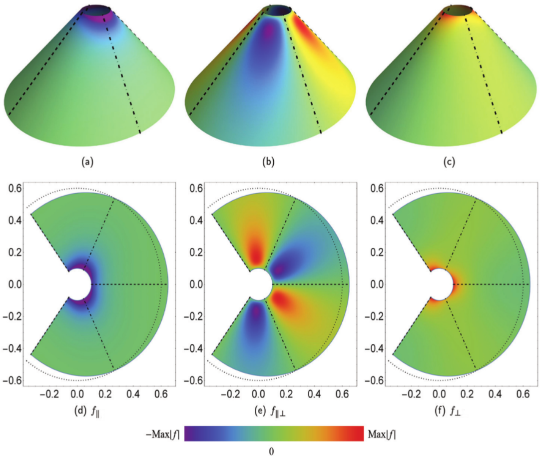
- For the complete tangential stress, one has to add the bending contribution:

$$
\begin{aligned}
f_{\parallel} &= \frac{\kappa^2}{2r^2} - \frac{C_{\parallel}}{r^2} \\
f_{\perp\parallel} &= 0 \\
f_{\perp} &= -\frac{\kappa^2}{2r^2} + \frac{C_{\parallel}}{r^2}\left(1 - \frac{r}{R}\right) + \frac{1}{rR}\left(\frac{\kappa^2}{2} + 1\right)
\end{aligned}
$$

- Example: icecream cone for which $\kappa = \text{const.}$
$\Rightarrow C_{\parallel} = \frac{\kappa^2}{2} + 1$ and thus

$$
f_{\parallel} = -\frac{1}{r^2}\,, \qquad f_{\perp\parallel} = 0\,, \qquad f_{\perp} = \frac{1}{r^2}
$$

Cones with non-circular boundary

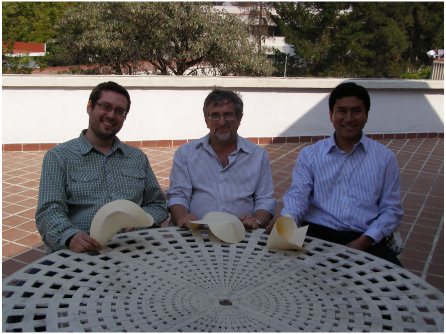


J. Guven, M. M. Müller & P. Vzquez-Montejo, Math. Mech. Solids 24, 4051 (2019).

Conclusions

- Introduction of a framework allowing to analyze the stress distribution in isometric sheets.
- No need to introduce boundary layers!
- The introduction of two local constraints on the boundary curve (arc-length and geodesic curvature) ensures that boundary deformations are consistent with isometry.
- The equilibrium shapes of conical geometries are remarkably insensitive to boundary conditions.
- The framework can easily be extended to other geometries.

Acknowledgements



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Many thanks for your attention!