Diffusion-induced instabilities in soft solid sheets

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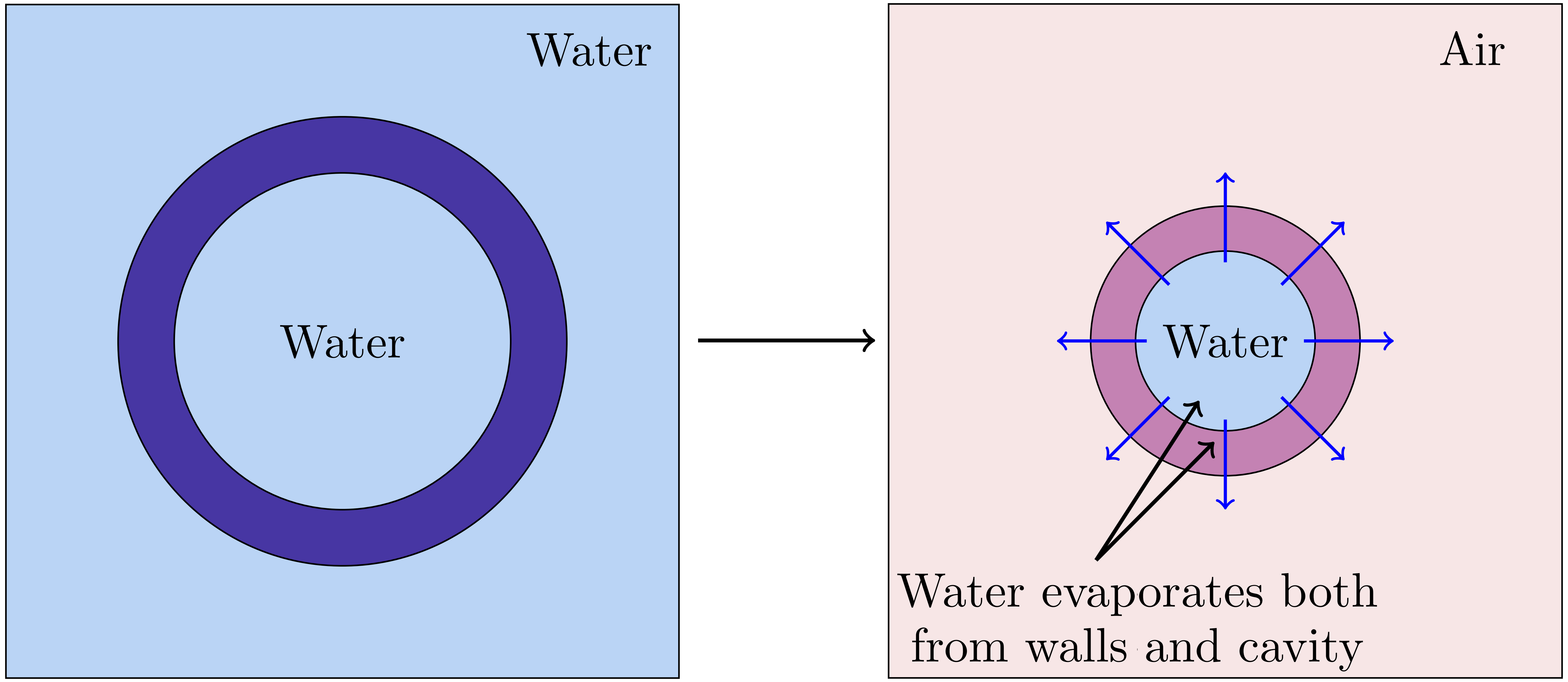
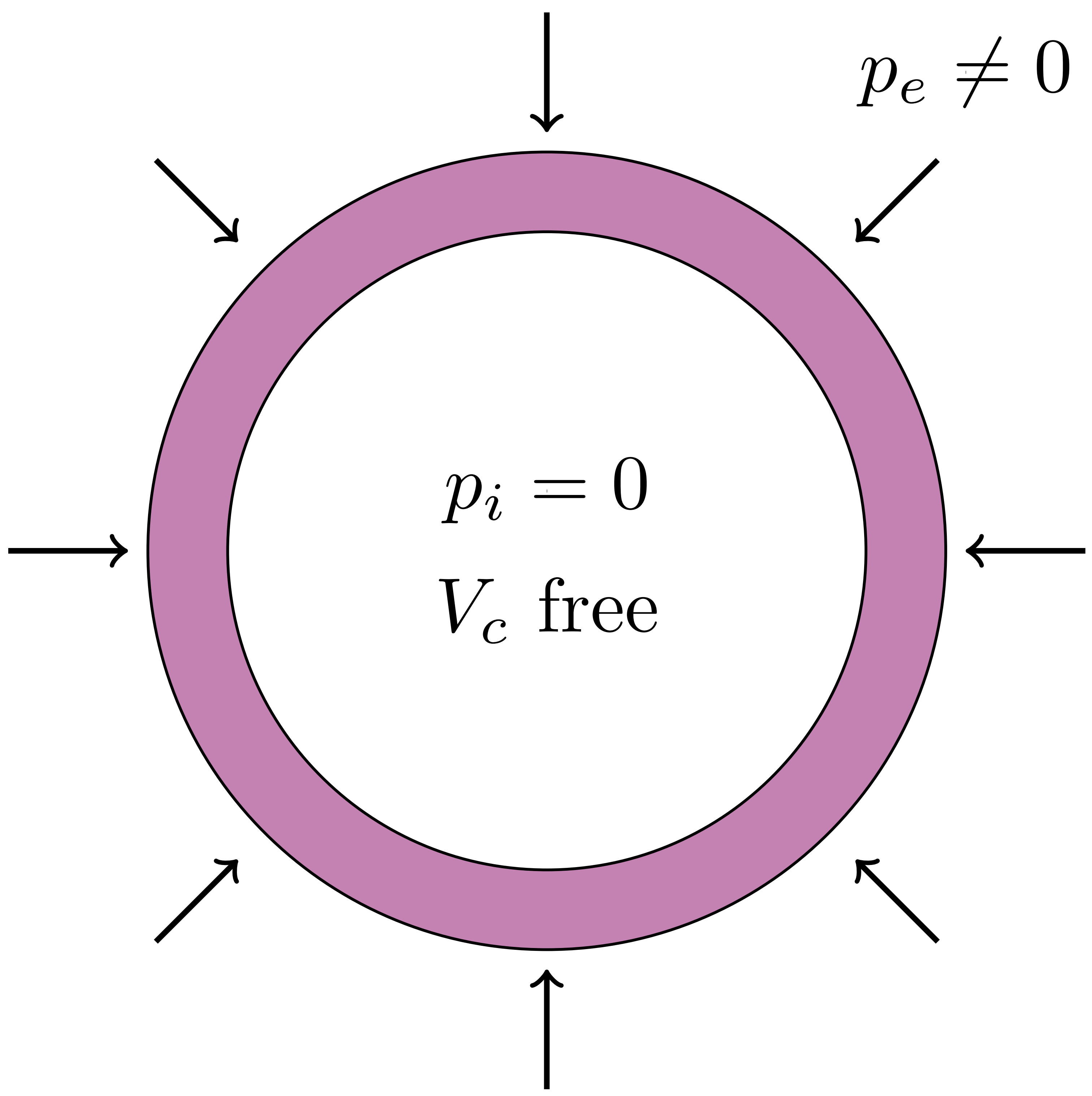
G. Napoli (Universita del Salento) e **S. Turzi** (Politecnico di Milano)

The Physics of Elastic Films: from Biological Membranes to Extreme Mechanics

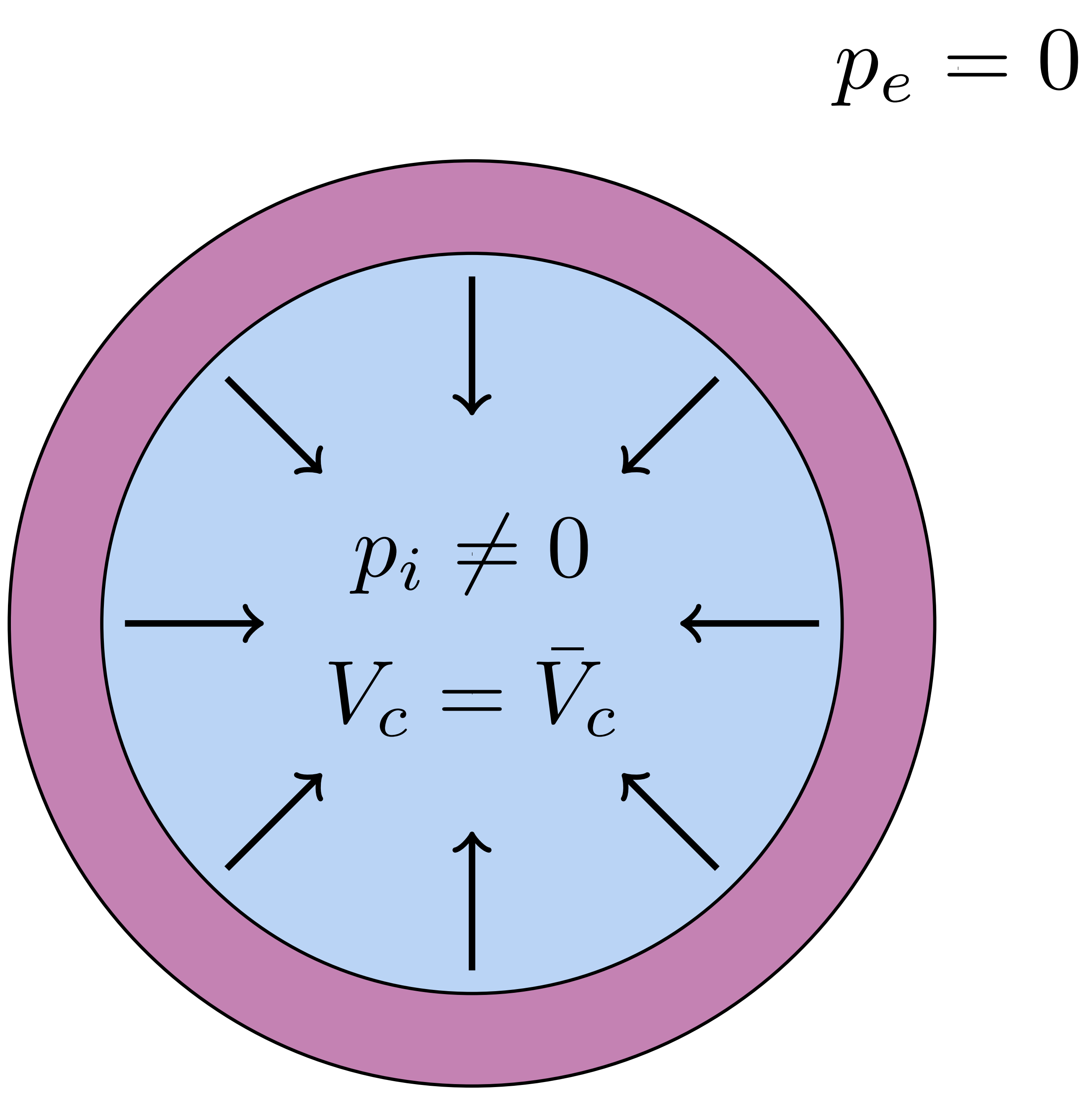
Kavli Institute for Theoretical Physics@University of California - Santa Barbara

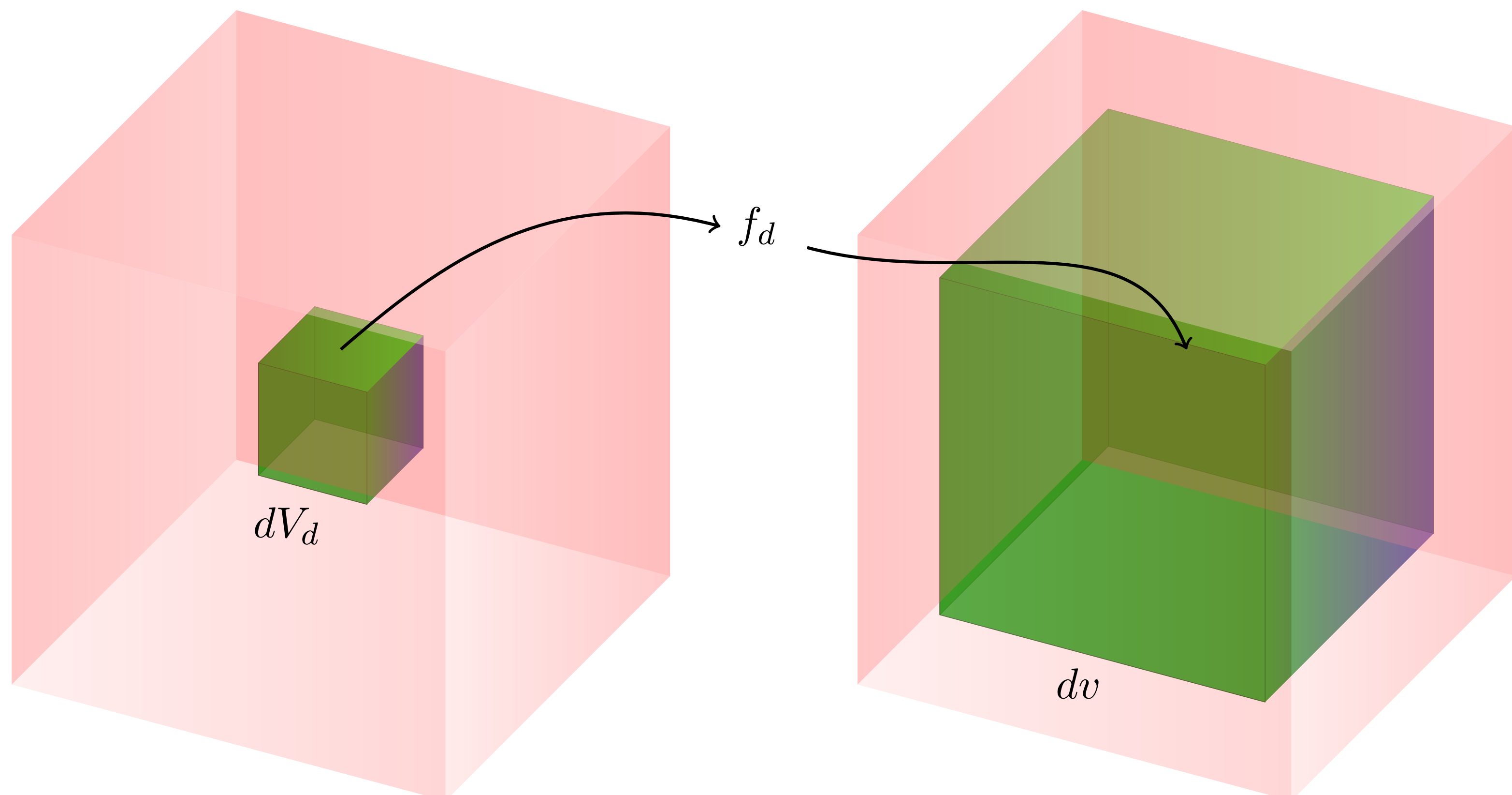


M. Curatolo, G. Napoli, **PN,** S. Turzi: **Dehydration-induced mechanical instabilities in active elastic spherical shells". Submitted, 2021** standard instability



de-hydration instability





displacement field

$$\mathbf{u}_d : \mathcal{B}_d \times \mathcal{T} \to \mathcal{E}\,,$$
$$(X_d, \tau) \mapsto \mathbf{u}(X_d, \tau)\,, [\mathbf{u}] = \mathrm{m}$$

molar liquid concentration

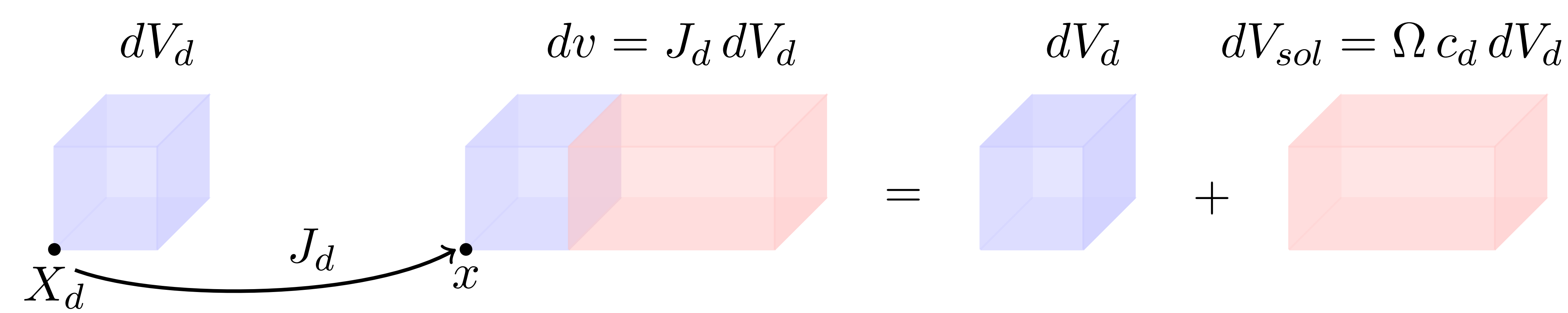
$$c_d : \mathcal{B}_d \times \mathcal{T} \to \mathcal{R}^+\,,$$
$$(X_d, \tau) \mapsto c_d(X_d, \tau)\,, [c_d] = \mathrm{mol/m}^3$$

balance equations

$$\mathrm{div}\,\mathbf{S}_d = \mathbf{0} \quad \mathbf{S}_d = \hat{\mathbf{S}}_d - p\,\mathbf{F}_d^*$$
$$\dot{c} = -\mathrm{div}\,\mathbf{h}_d \quad \mathbf{h}_d = \mathbf{M}\nabla(\hat{\mu} + \Omega\,p)$$
in $\mathcal{B}_d$

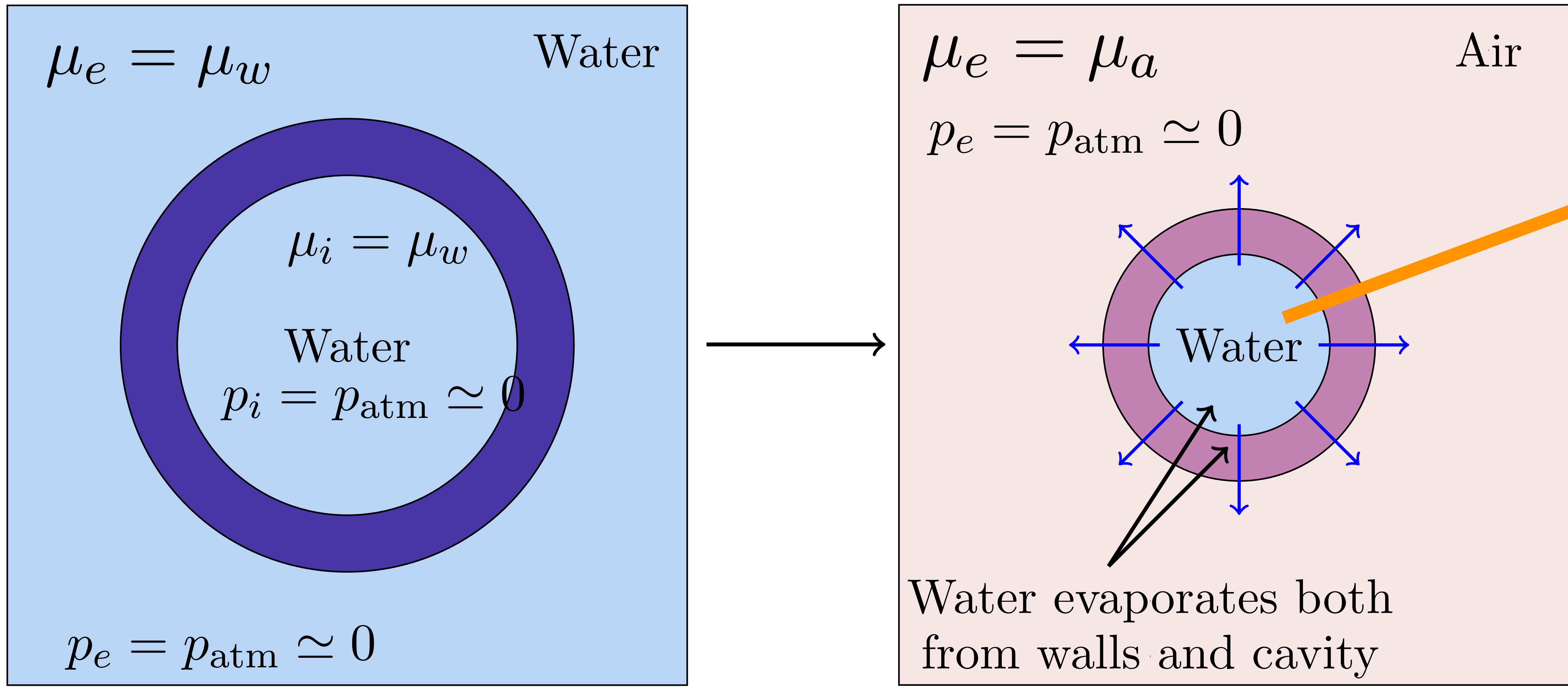
the **constitutive eqns** for stress and chemical potential come from Flory-Rehner thermodynamics

local VC incompressibility effect



$$J_d := \det\mathbf{F}_d = \det(\mathbf{I} + \nabla\mathbf{u}_d) = 1 + \Omega c_d =: \hat{J}_d(c_d)$$

A. Lucantonio, **PN**, L. Teresi: **J Mech Phys Solids 61, 2013; ...;** M. Curatolo, **PN**, L. Teresi: **Soft Matter 14, 2018;** M. Curatolo, **PN**, L. Teresi: **Mech. Soft Mater. 2(13), 2020.**



$$\mu_i = \Omega p_i(t) \quad \text{and} \quad p_i = p_i(t)$$

suction effect

global VC

$$v_c(t) = v_w(t)$$

$$v_w(t) = v_{wo} - v_w^i(t)$$

$$v_{wo} = v_{co} = v_c(0) = \tfrac{1}{3}\int_{\partial\mathcal{C}_d} (X_d + \mathbf{u}_o)\cdot \mathbf{F}_o^\star \mathbf{m}\, dA_d$$

boundary pressure

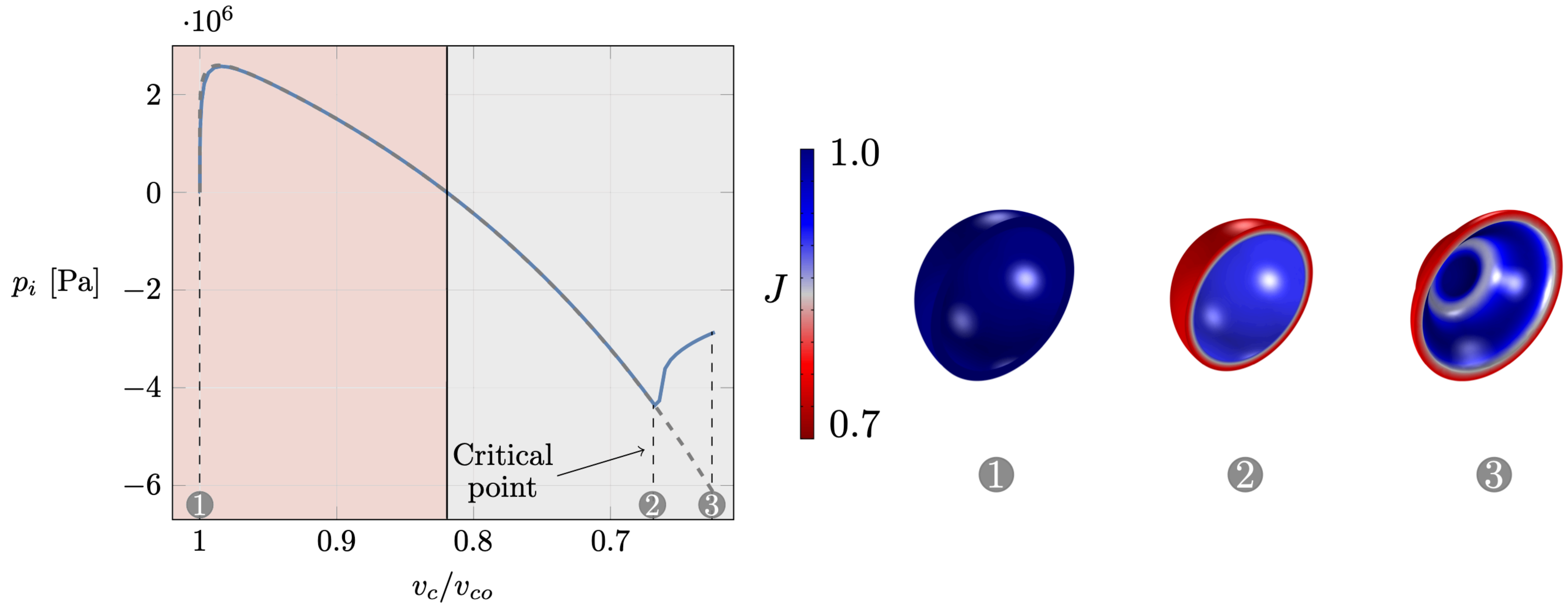
chemical potential of the environment

$$\dot{v}_w^i(t) = \int_{\partial\mathcal{C}_d} q_s\, dA_d = -\int_{\partial\mathcal{C}_d} \mathbf{h}_d\cdot\mathbf{m}\, dA_d\,, \quad v_w^i(0) = 0$$

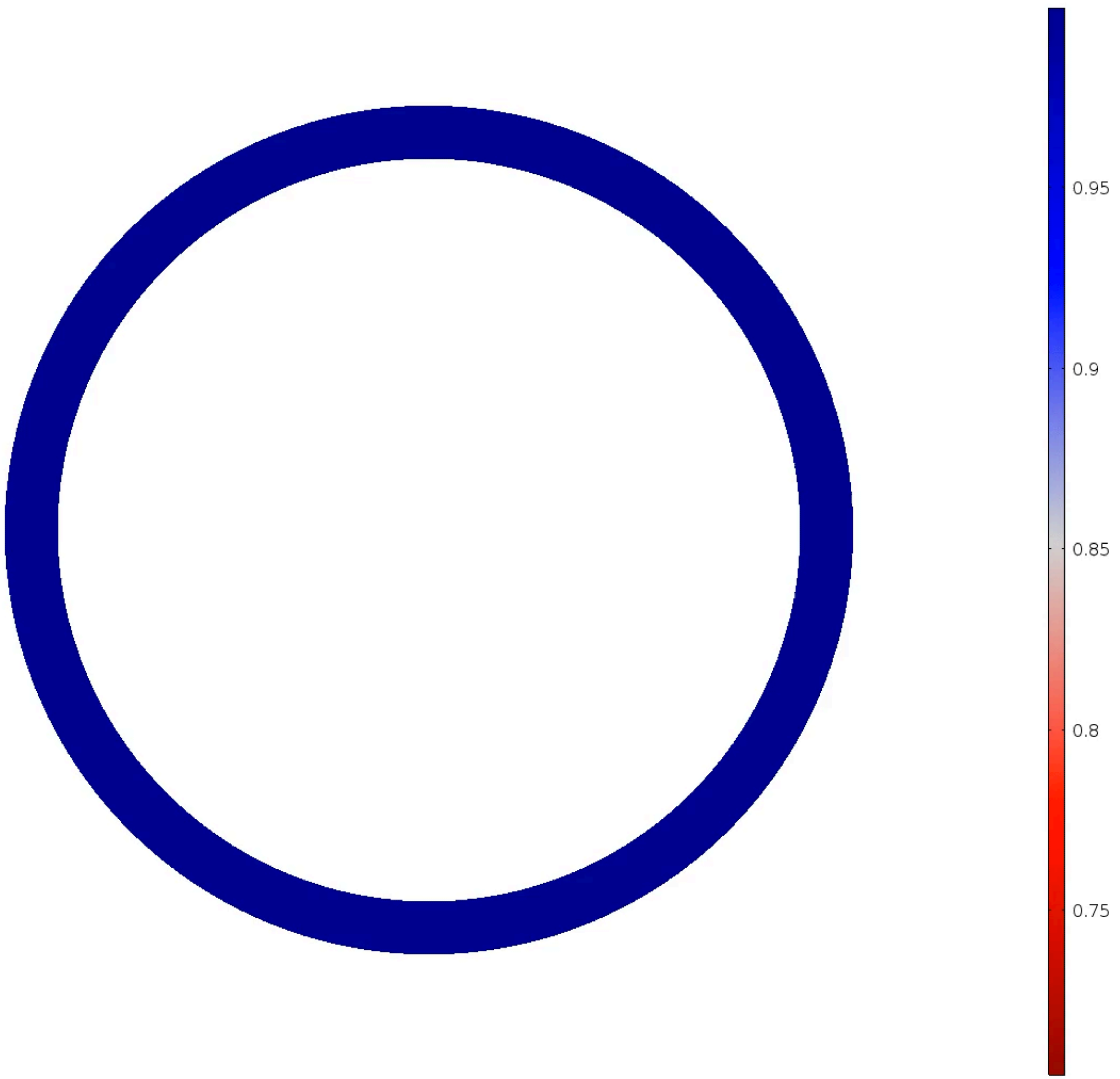
boundary conditions

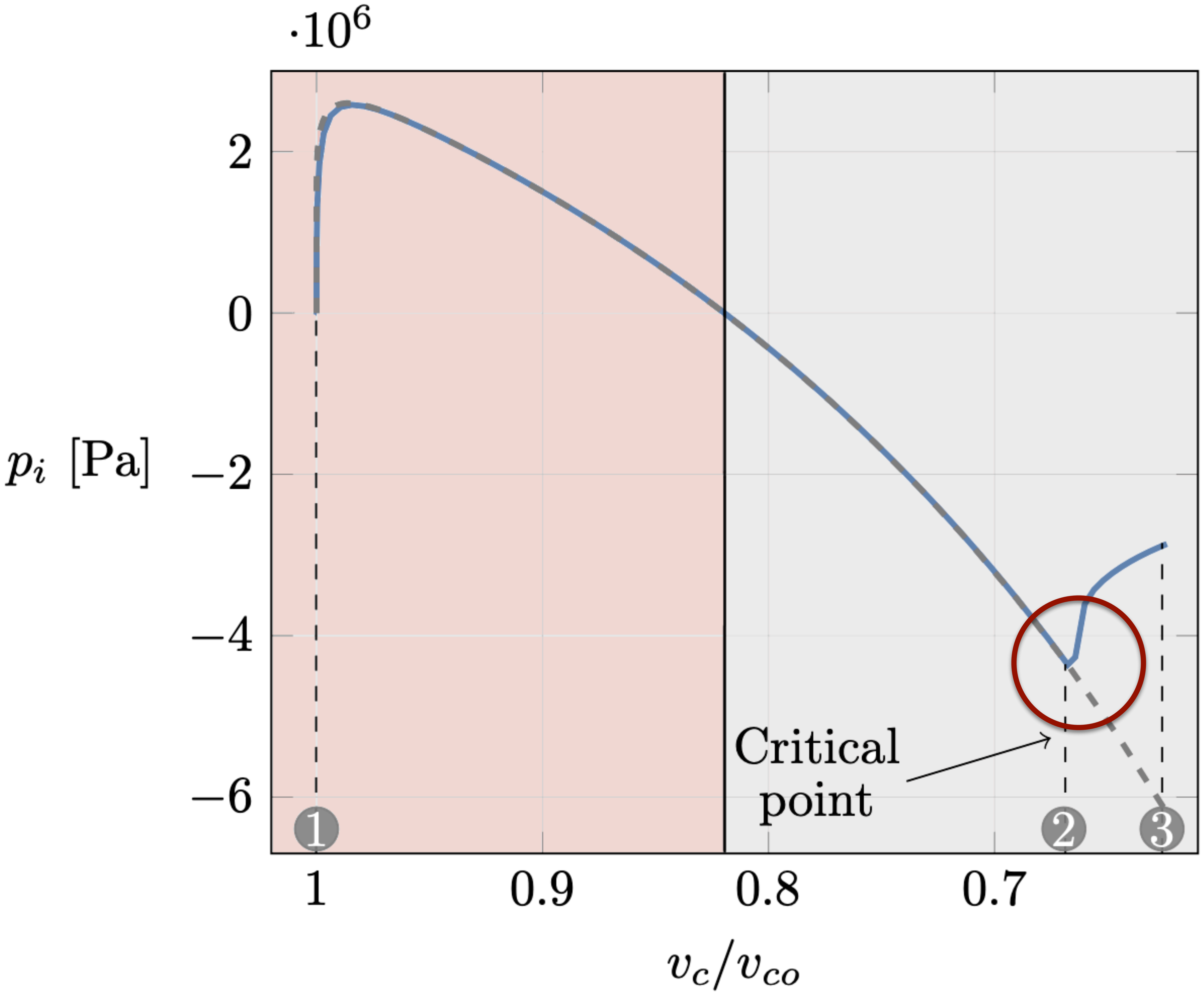
$$\mathbf{S}_d\mathbf{m} = \hat{\mathbf{t}} \quad \text{and} \quad \mu = \mu_e$$

$$v_c(t) = \int_{\mathcal{C}_t} dv = \frac{1}{3}\int_{\partial\mathcal{C}_t} x\cdot\mathbf{n}\, da = \frac{1}{3}\int_{\partial\mathcal{C}_d} (X_d + \mathbf{u}_d)\cdot\mathbf{F}_d^\star\mathbf{m}\, dA_d$$


$\cdot 10^6$
p_i [Pa]
2
0
−2
−4
−6
1
0.9
0.8
0.7
v_c/v_{co}
Critical point
1
2
3
J
1.0
0.7
1
2
3

a first glance at mechanical instabilities/2





* purely mechanical model based on 3D Wesolowski model for the spherical shell (Arch.Mech.Stosow.19(1), 1967)
* liquid diffusion is frozen at the onset of instability (red circle) which occurs instantaneously with respect to the diffusion time
* deformations are purely radial

$$r = r(R)\,, \quad \theta = \Theta\,, \quad \phi = \Phi \quad \text{and} \quad r^2 r' = J_o\, R^2$$

* shell stays spherical until the onset of instability

$$r_c = \left(\frac{3v_c}{4\pi}\right)^{1/3}$$

bulk equations

$$J_0 = 1 + \Omega\, c_0 \qquad r^2 r' = R^2 J_0$$

$$\mathtt{div}\,\mathbf{S}_0 = \mathbf{0} \qquad p_0' R + 2G_d Q_0(-1 + J_0 Q_0^3)^2 - G_d Q_0^4 J_0' R = 0 \qquad Q_0(R) = R/r(R)$$

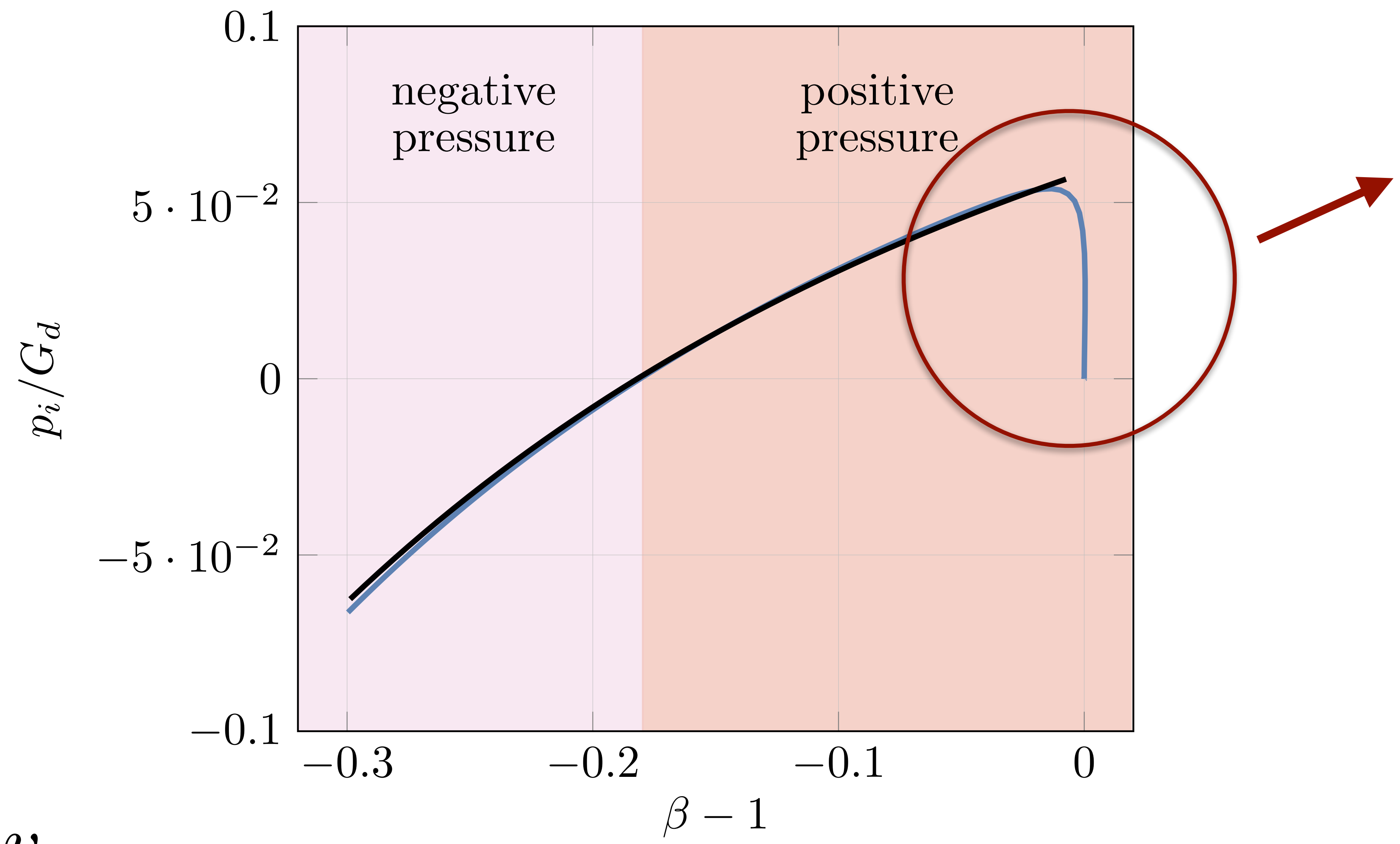
$$\mathtt{div}\,\mathbf{h}_0 = \mathbf{0} \qquad R^2 h_{0_R} = C_0,$$

boundary conditions

$$S_{0_{RR}}(R_d) = 0$$

$$\mu(R_d) = \mu_e$$

$$\mu(R_c) + \Omega Q_0^2(R_c) S_{0_{RR}}(R_c) = 0$$



Where the external chemical potential suddenly drops to its plateau value and the pressure rises at nearly constant cavity volume the static solution cannot reproduce the numerical values

$$\beta := \frac{v_c}{v_{co}}$$

the incremental chemo-mechanical problem

step 1

spherical solution | incremental unknown fields

$$\mathbf{x}(R,\Theta) = r(R)\mathbf{e}_R + \epsilon(u(R,\Theta)\mathbf{e}_R + v(R,\Theta)\mathbf{e}_\Theta)$$

$$p(R,\Theta) = p_0(R) + \epsilon p_1(R,\Theta)$$

$$J_d(R,\Theta) = J_0(R) + \epsilon J_1(R,\Theta)$$

step 2 incremental bulk equations

$$J_1 = J_0 \mathrm{tr}(\mathbf{F}_0^{-1}\mathbf{F} - 1)$$

$$\mathtt{div}\,\mathbf{S}_1 = \mathbf{0} \quad \text{with} \quad \mathbf{S}_1 = -J_0 p_1 \mathbf{F}_0^{-T} - J_1 p_0 \mathbf{F}_0^{-T} + J_0 p_0 \mathbf{F}_0^{-T}\mathbf{F}_1^T\mathbf{F}_0^{-T} + G_d\mathbf{F}_1$$

$$\mathtt{div}\,\mathbf{h}_1 = \mathbf{0} \quad \text{with} \quad \mathbf{h}_1 = -\mathbf{M}_0\nabla\mu_1 - \mathbf{M}_1\nabla\mu_0$$

step 3 incremental boundary conditions

....

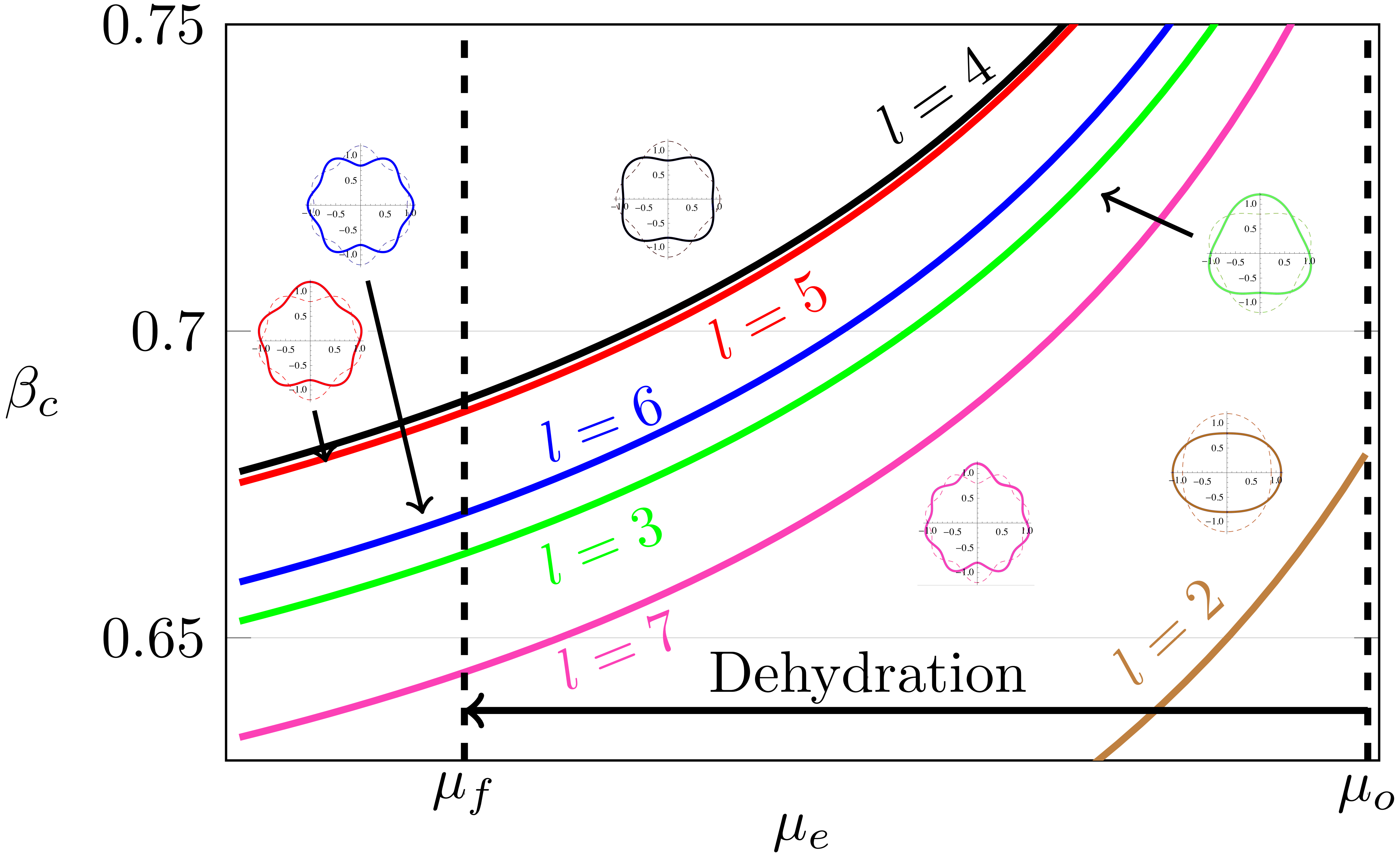
step 4 Ansatz of the solution

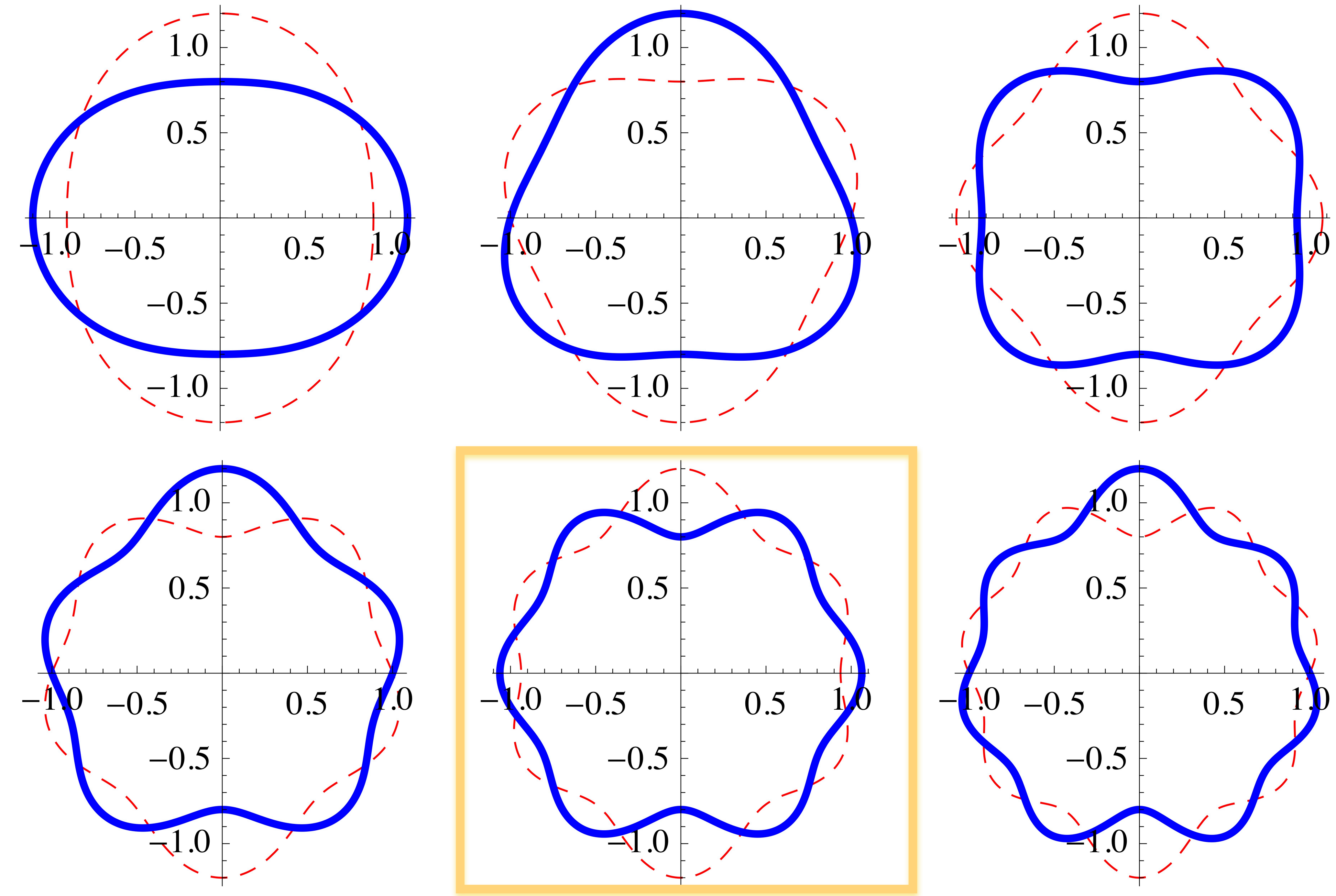
$$u(R,\Theta) = \sum_{l=1}^{\infty}\mathcal{U}_l(R)\mathrm{P}_l(\cos\Theta) \qquad v(R,\Theta) = \sum_{l=1}^{\infty}\mathcal{V}_l(R)\partial_\Theta[\mathrm{P}_l(\cos\Theta)]$$

$$p_1(R,\Theta) = \sum_{l=1}^{\infty}\mathcal{P}_l(R)\mathrm{P}_l(\cos\Theta) \qquad J_1(R,\Theta) = \sum_{l=1}^{\infty}\mathcal{J}_l(R)\mathrm{P}_l(\cos\Theta)$$

- the analysis of the global volume constraint shows that to first order the perturbation of the cavity volume vanishes for any incremental displacement field and the pressure fields of the enclosed liquid remains unchanged up to the first order

a sketch of the solution





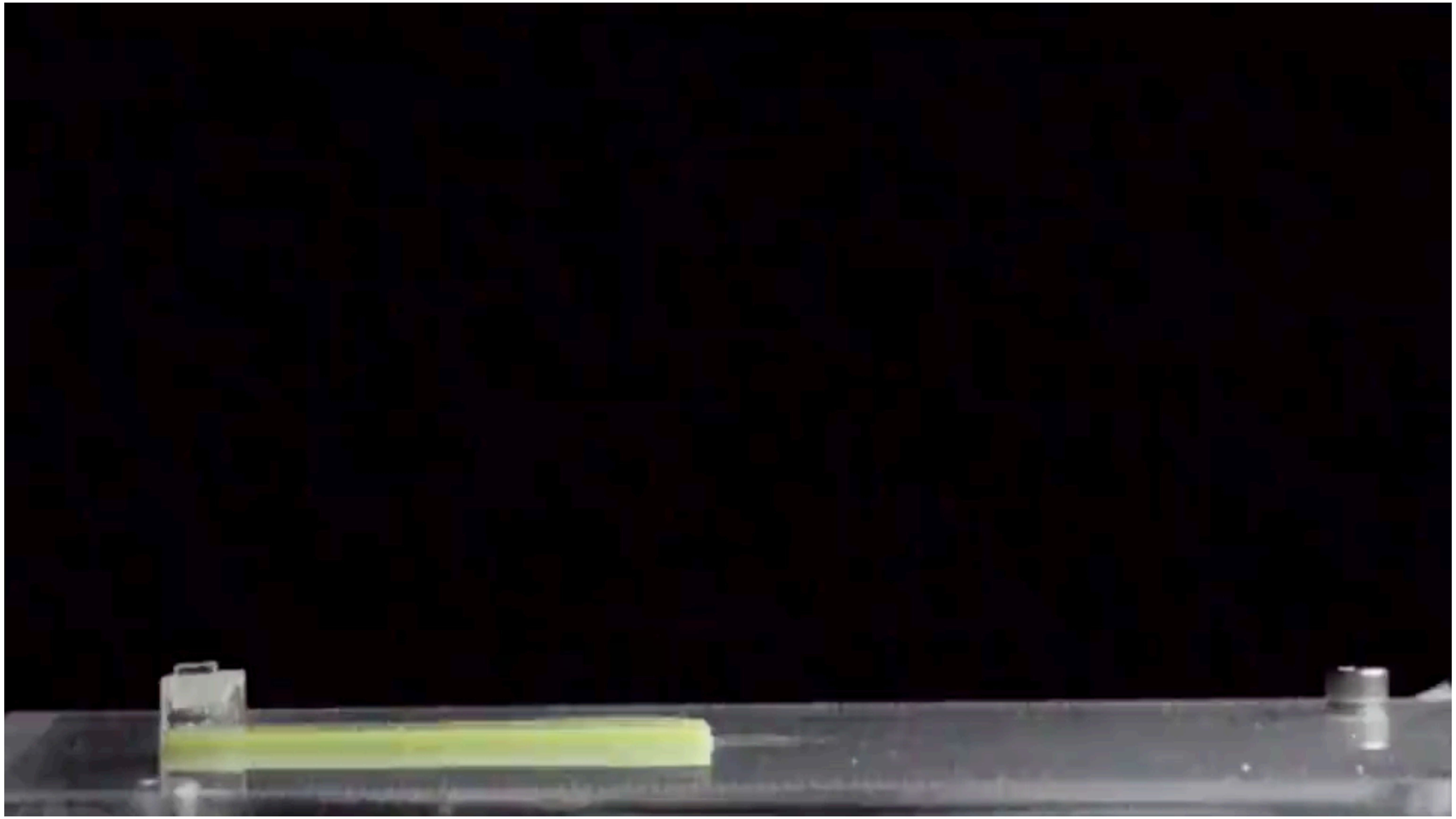
Critical modes and critical profile mode corresponding to the solution of the first order incremental problem

open issues/work in progress

the change of de-hydration across the thickness of the shell delivers a spontaneous curvature and a frustration which may be caught by 2D shell models

occurrence of cavitation instability is also interesting (and here is not considered, by assuming that adhesion energy density is infinite)

D.P. Holmes, J.H. Lee, H.S. Park, M. Pezzulla, **Physical Review E** 102(2), 2020.

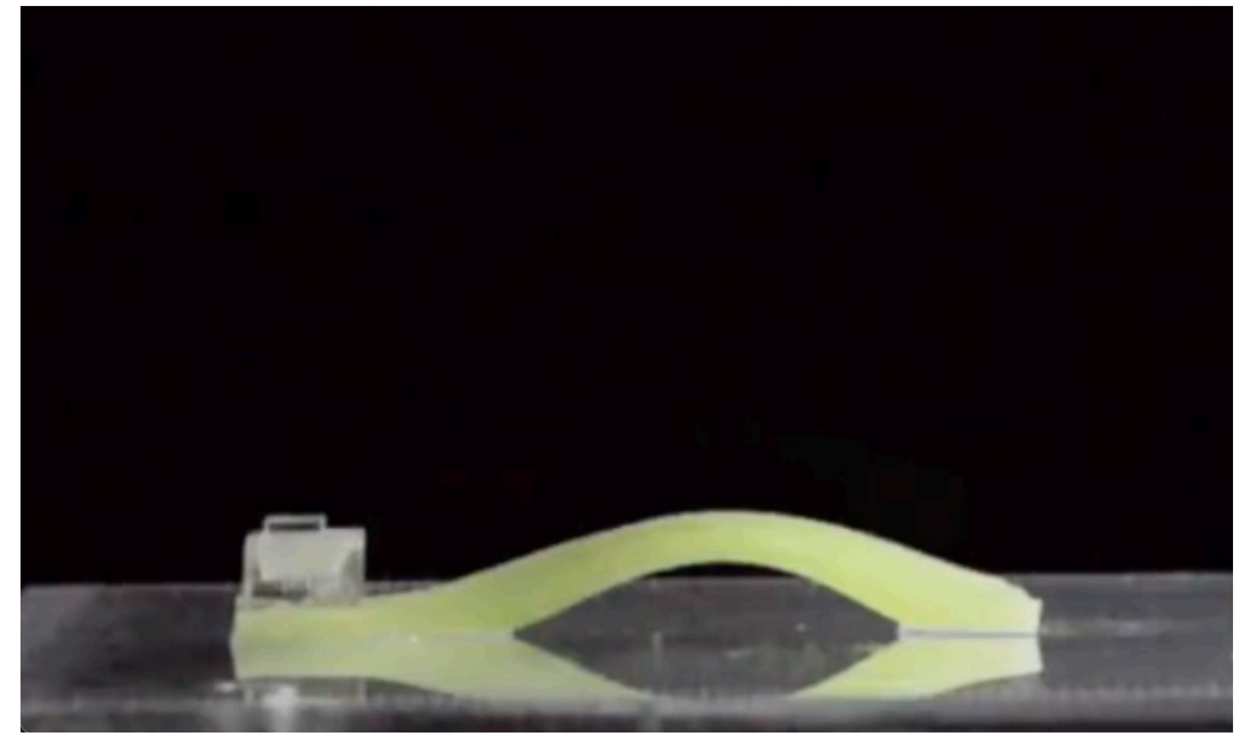


MoSS Lab@Boston University
(under the supervision of D.P. Holmes)

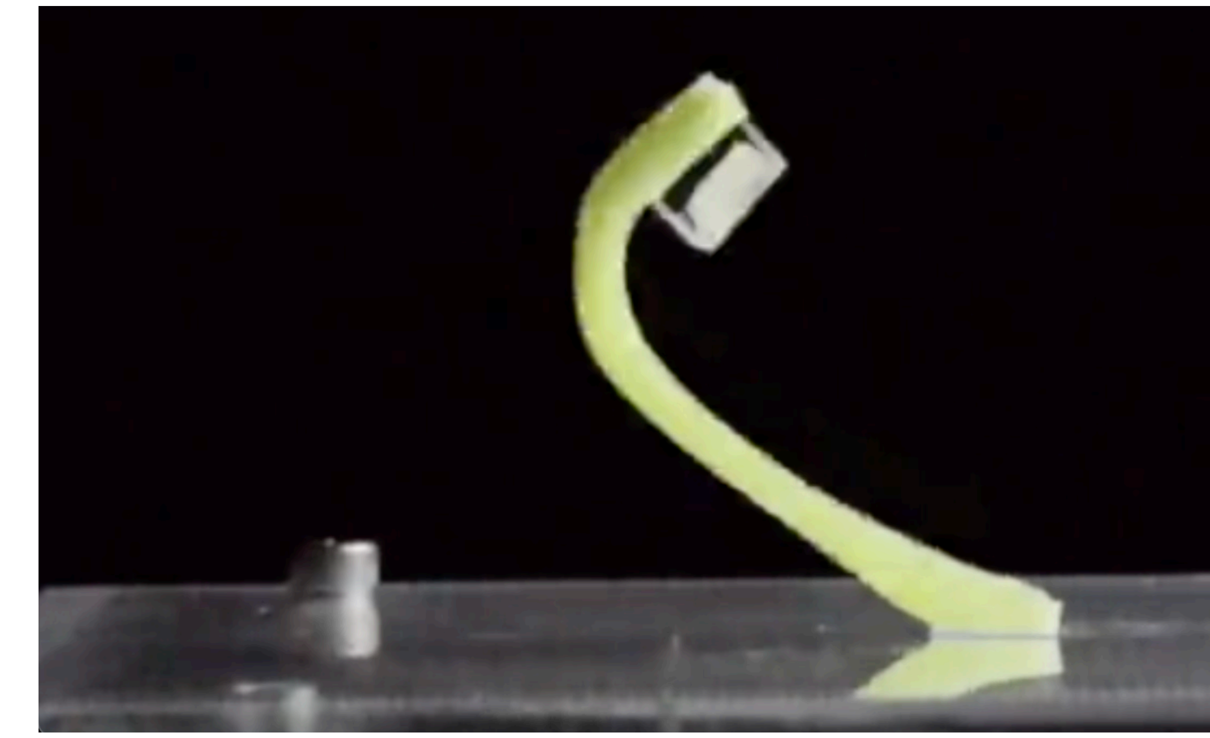
the 3 phases of the bar mechanism

phase 1

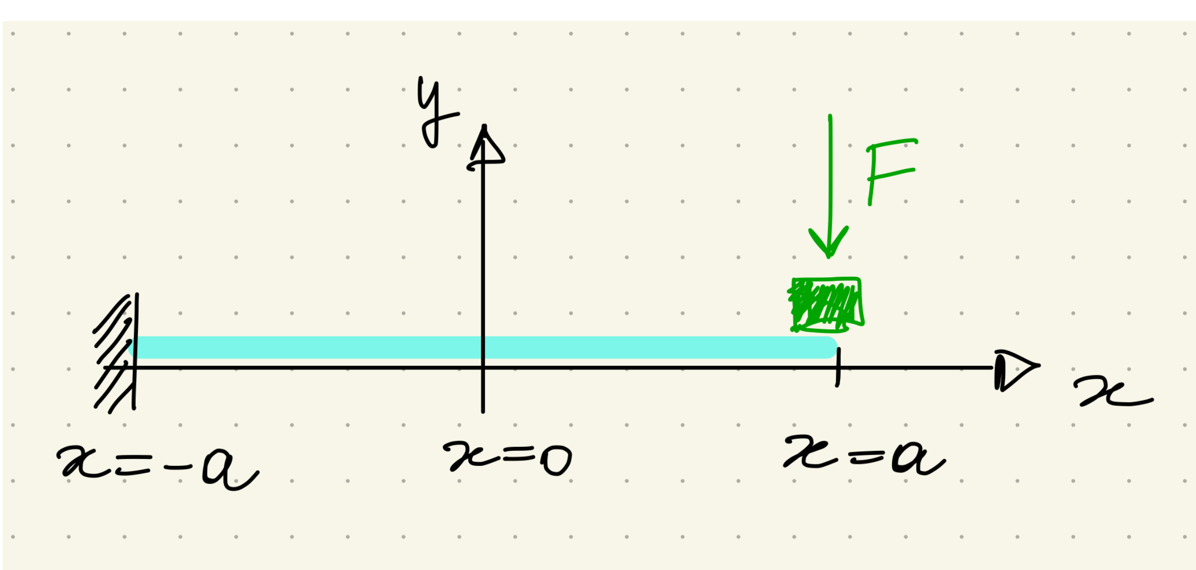
phase 2



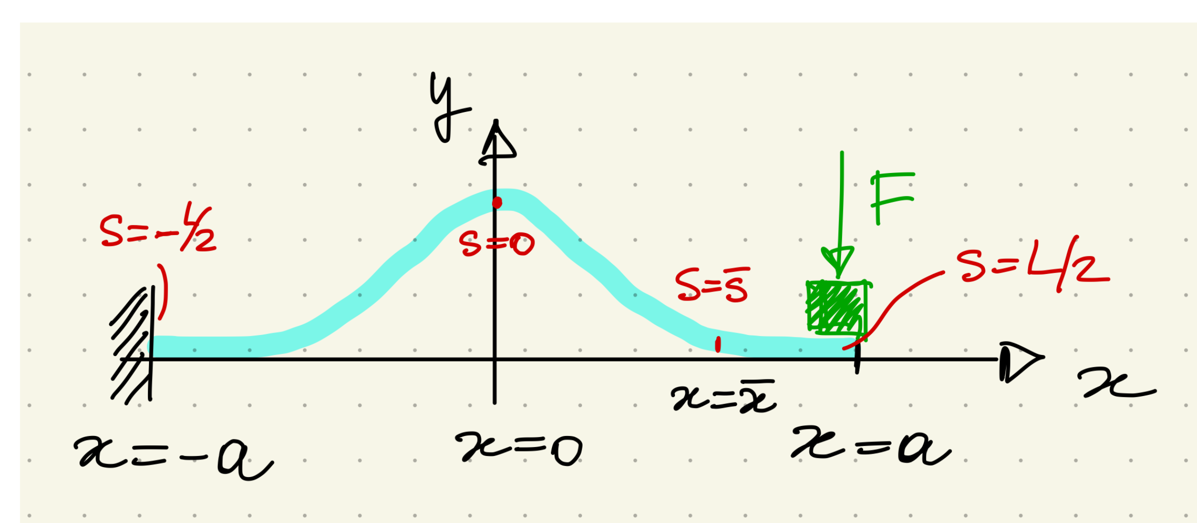
phase 3



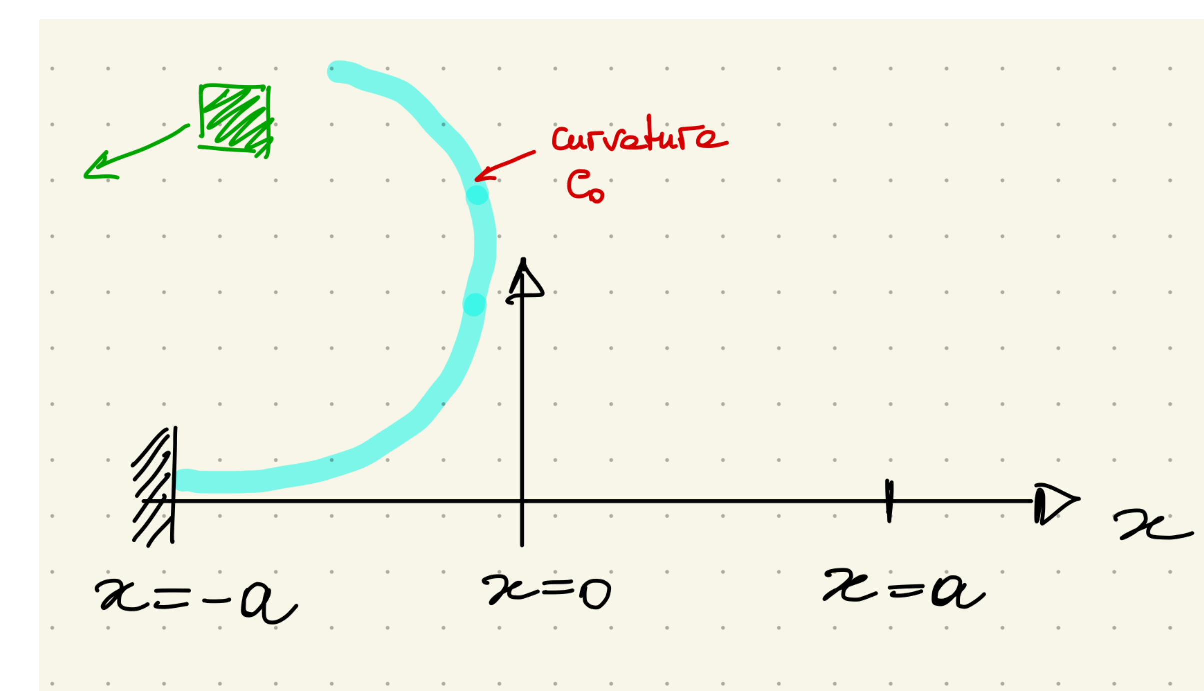
compression phase



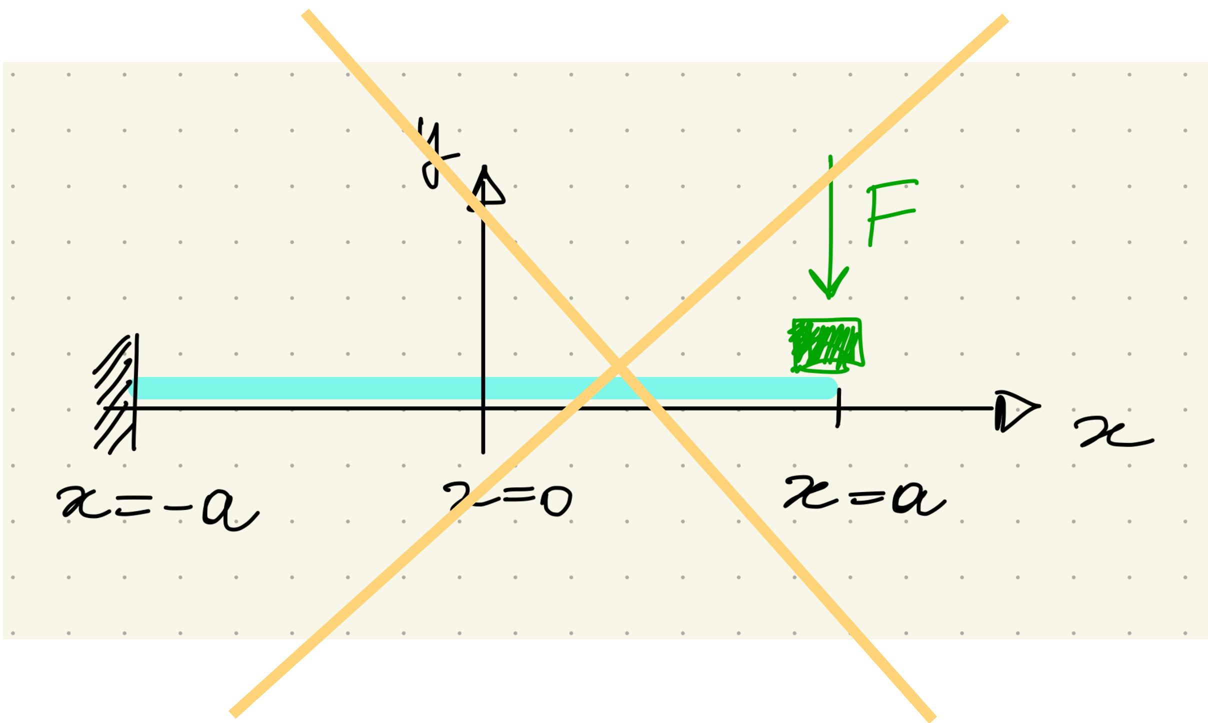
buckling phase



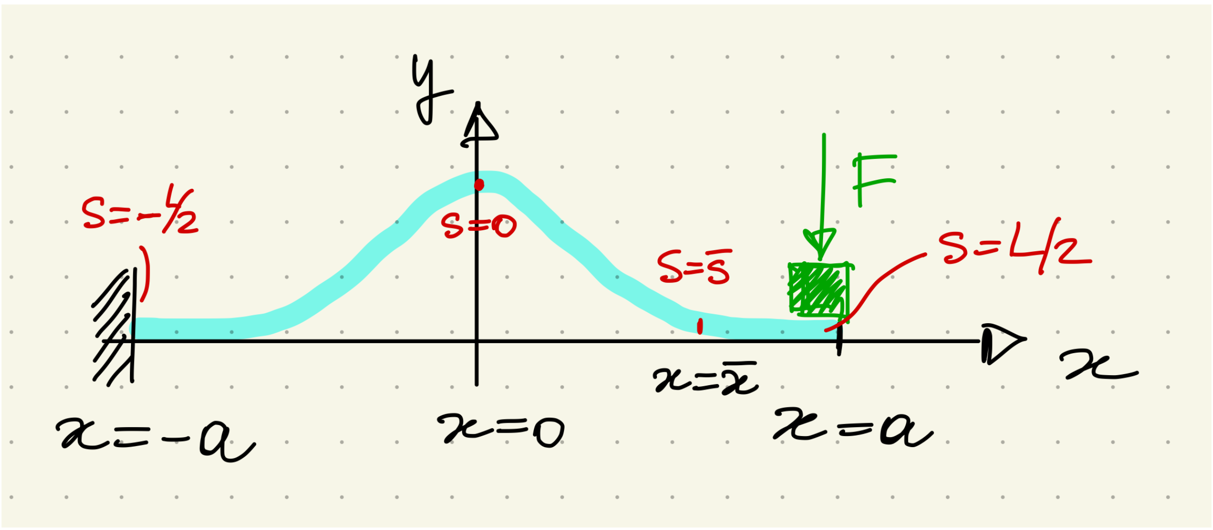
catapult phase



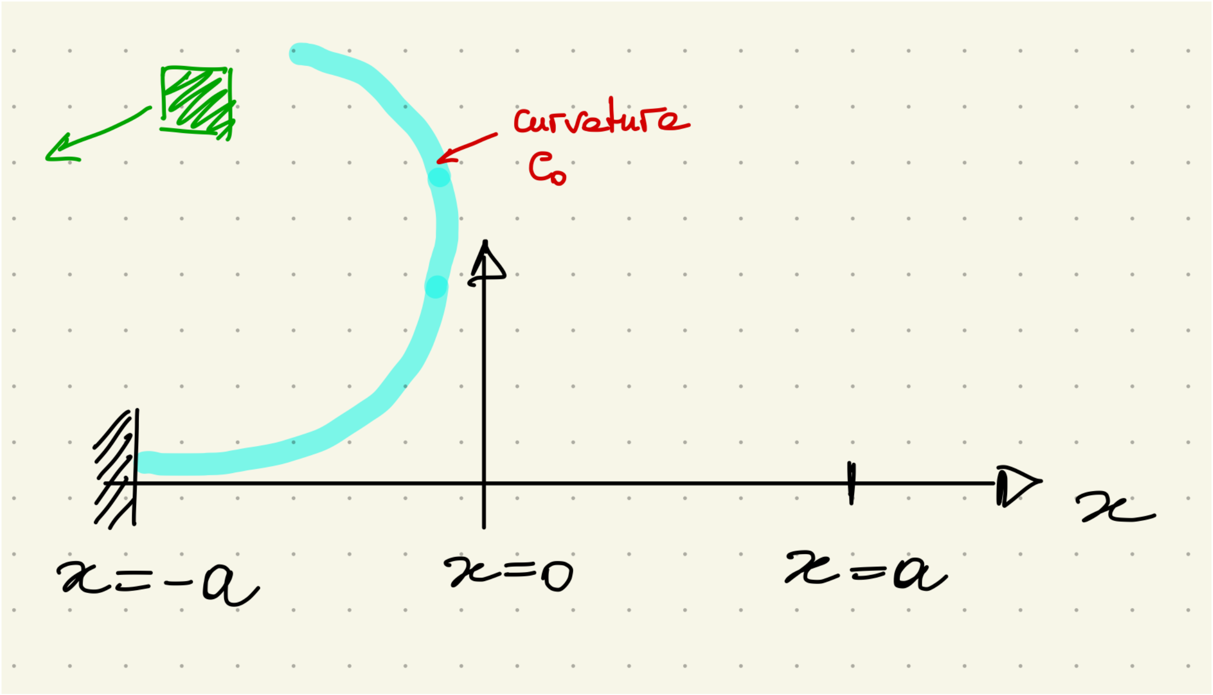
compression phase



buckling phase



catapult phase



$$\mathcal{W}_f = \int_0^{\bar{S}} \kappa(\theta' - c_o)^2 dS$$

energy of the buckled region

$$\mathcal{W}_a = \int_{\bar{S}}^{L/2} (\kappa c_o^2 - 2w) dS$$

energy of the adhered region

$$l_{ec} := \sqrt{\frac{\kappa}{w}}$$

elasto-capillary length

$\theta(0) = 0$ even symmetry

$\theta(\bar{S}) = 0$ continuity

$\kappa[\theta'(\bar{S})]^2 = 2w$ transversality condition

$a = \int_0^{\bar{S}} \cos\theta \, dS + \frac{L}{2} - \bar{S}$ global constraint

T.J.W Wagner, D. Vella, **Soft Matter 9, 2013;** G. Napoli, S. Turzi: **PRSA 471(2183), 2015;** G. Napoli, S. Turzi: **Meccanica 52, 2017.**

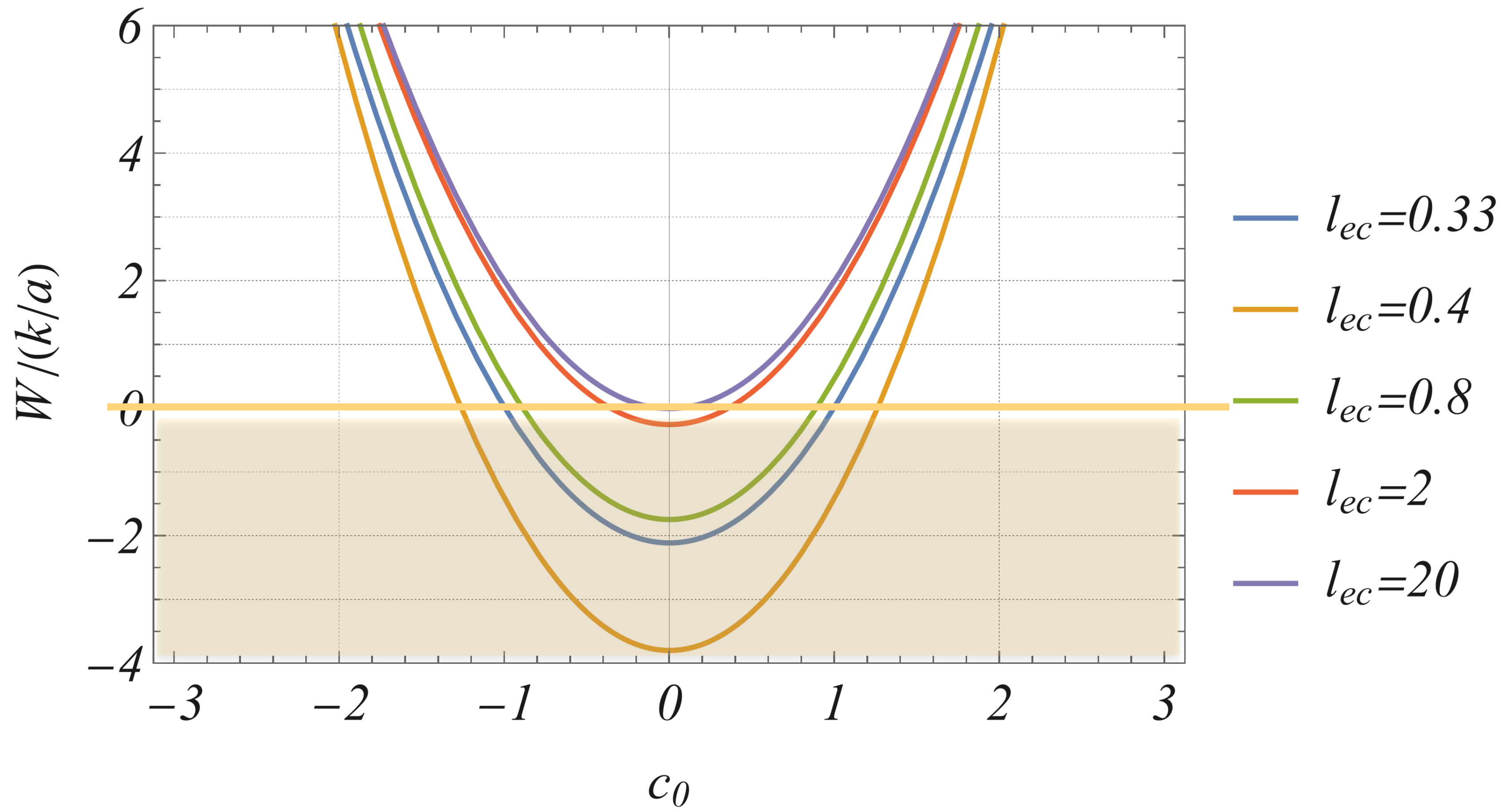
$$W_{\text{buck}} = \frac{2\kappa}{l_{ec}^2}\left[2\bar{x} - a\left(2 + \frac{\epsilon}{1-\cos\theta_0}\right)\right] + 2ac_o^2\kappa(1+\epsilon)$$

$$\epsilon = \frac{L-2a}{2a}$$

$$\frac{W}{\kappa/a} = 2\frac{2a^2}{l_{ec}^2}\left[(l_{ec}c_o)^2(1+\epsilon) - \frac{\epsilon}{1-\cos\theta_0}\right]$$

VS

$$W_{co} = 0$$



- the pattern is not monotonic with lec;
- to each value of lec, it corresponds a value of co which makes the catapult realizable;
- for co=0, we can't have the catapult mechanism as W never goes beyond the 0 energy level

work in progress: relate the change of de-hydration across the thickness of the beam to the spontaneous curvature, analyse the buckled problem and the role of the key parameters, study possible different catapult mechanisms

acknowledgements

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Università del Salento

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Ministero degli Affari Esteri
e della Cooperazione Internazionale

Research project:

Microencapsulation based on intelligent actively-remodeling bio polymer gels for controlled drug release

a joint project with Anne Bernheim
Ben Gurion University at the Negev
(Israel)

Ministero dell'Istruzione, dell'Università e della Ricerca

Research project:

Mathematics of active materials: from mechanobiology to smart devices