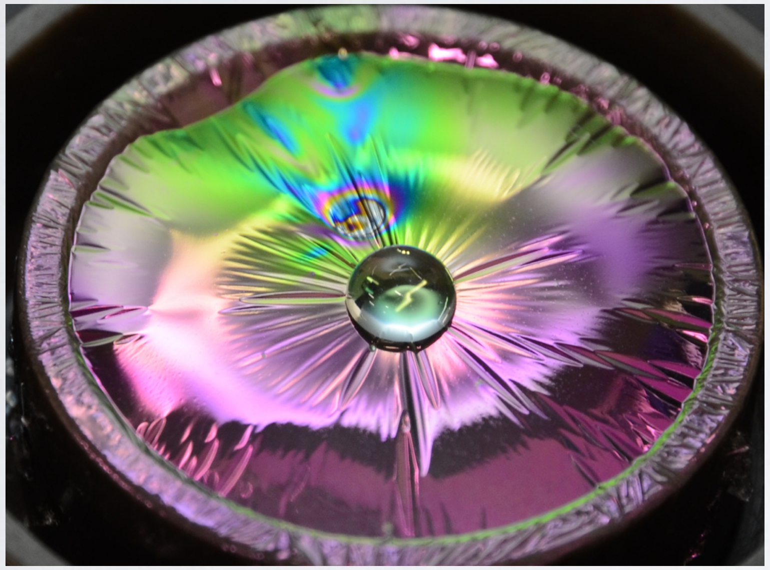
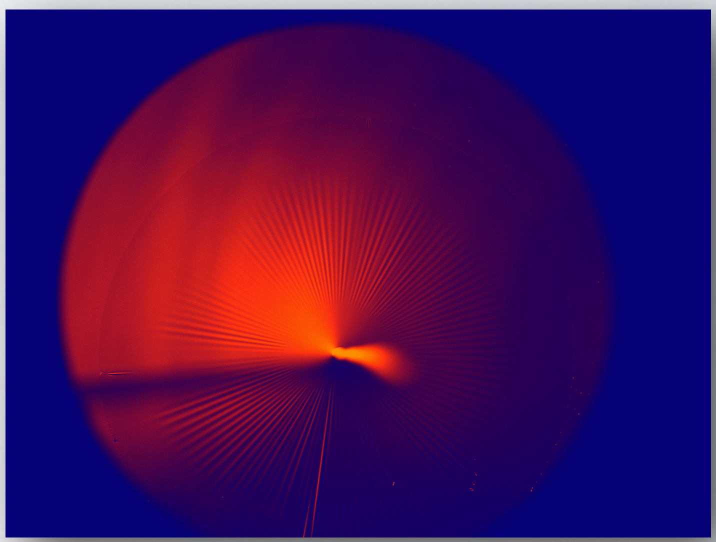
WRINKLING
NOT JUST A PRETTY PHASE?





UNIVERSITY OF OXFORD

Dominic Vella
Mathematical Institute
University of Oxford

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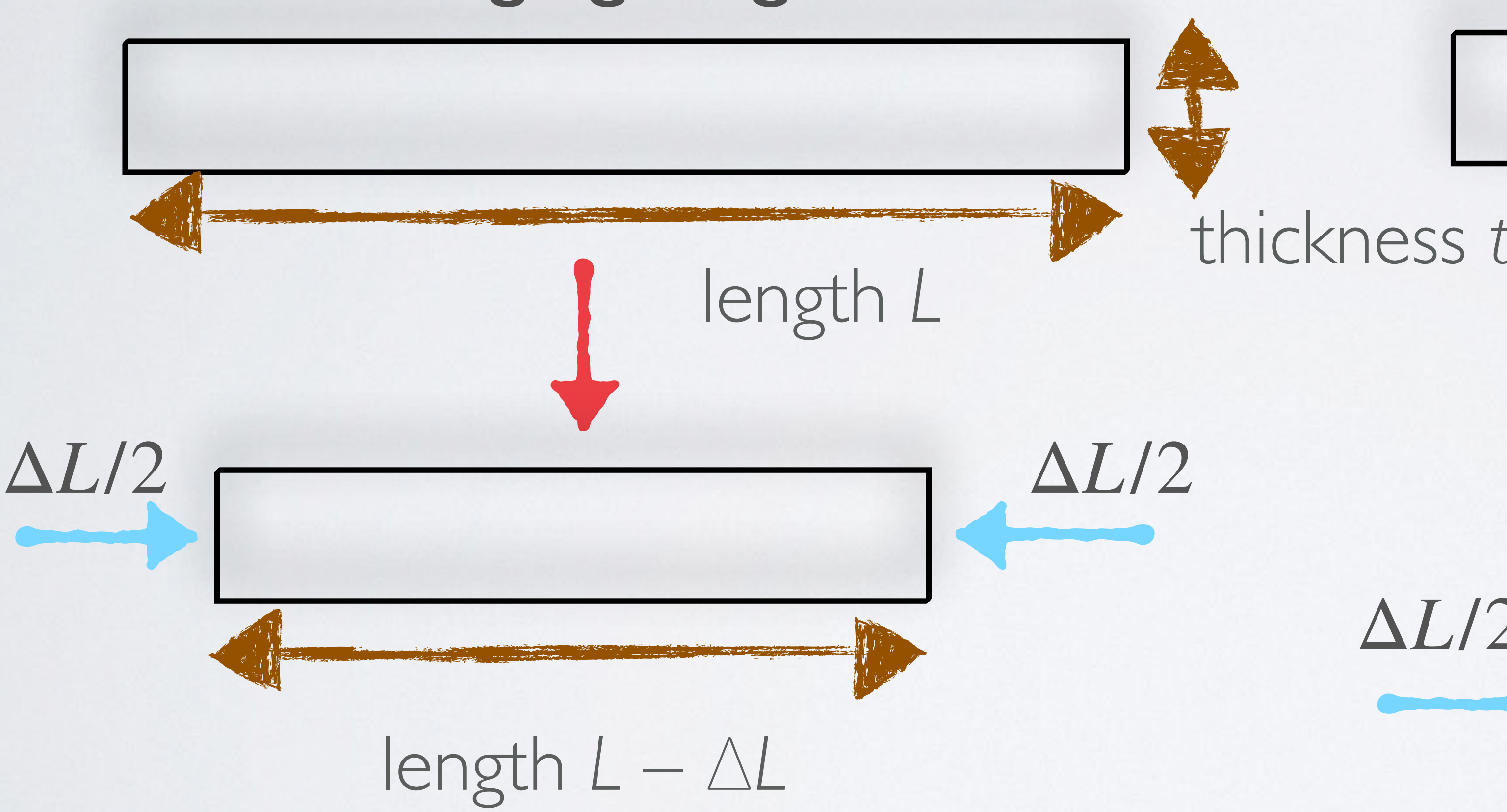
BASIC MECHANISM

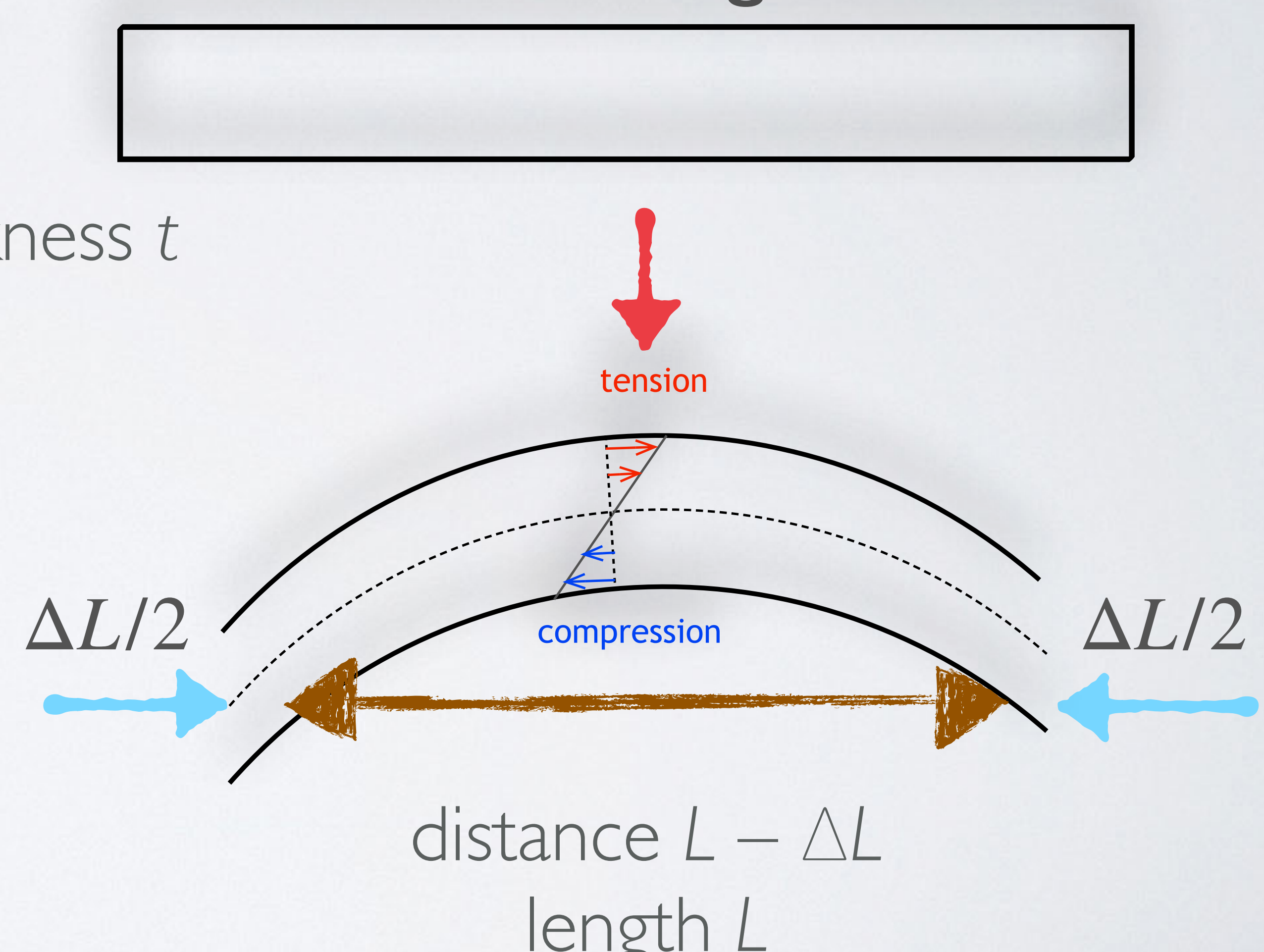
Wrinkling happens because thin layers are compressed (either by external force or by volume change)

Two possible responses to compression:

Changing length



Bending



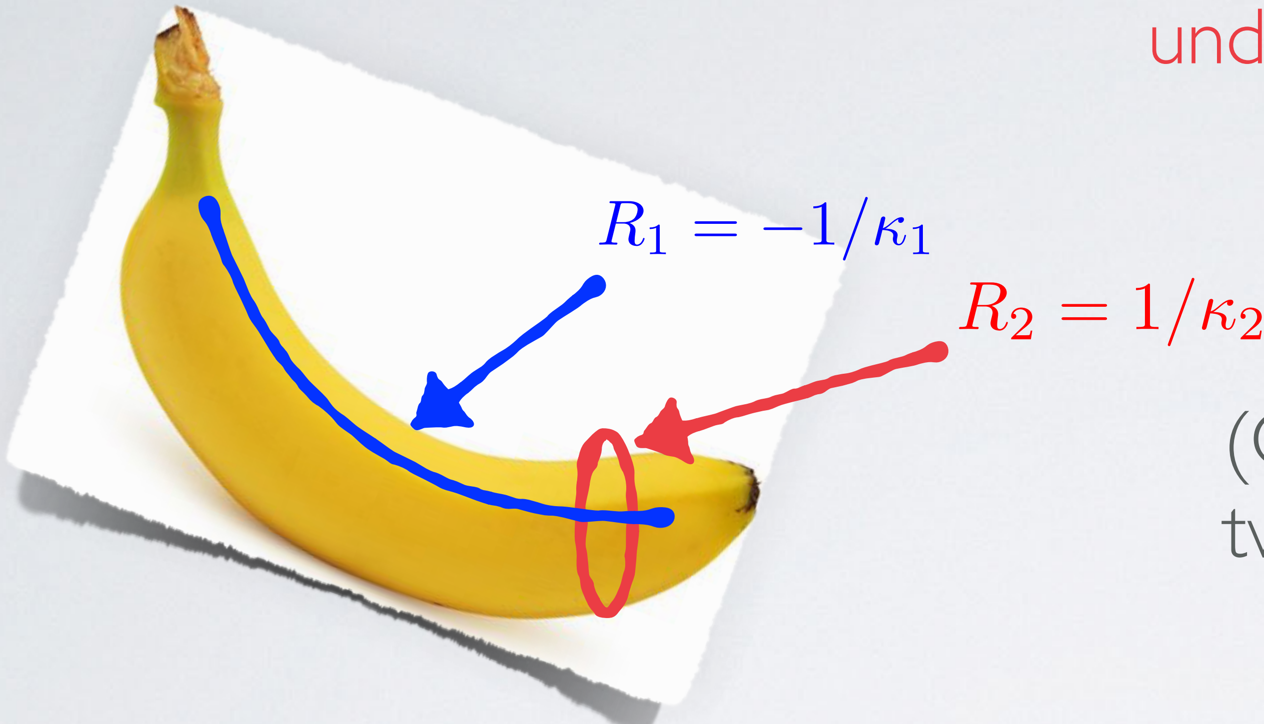
Ratio of energies is $\frac{U_{\mathrm{CL}}}{U_B} \sim \frac{L\Delta L}{t^2}$

As $t \to 0$ expect objects to deform without changing length: **study isometries**

THEOREMA EGREGIUM

Gauss' "Remarkable" Theorem

The Gaussian curvature of a surface is invariant under local isometry



(Gaussian curvature is the product of the two principal curvatures $K_G = \kappa_1 \kappa_2$)

Everyday corollary: A thin planar sheet **cannot** be deformed to a sphere

Modified by Jecowa

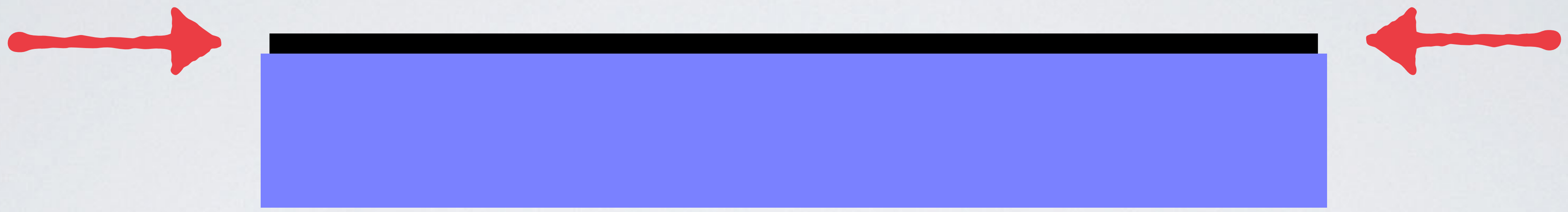
$K_G = 0$

$K_G = 1/R^2$

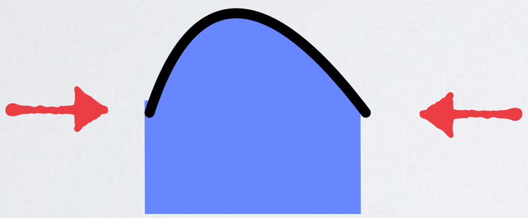
Cylindrical deformations keep $K_G = 0$ – focus on this for now

SIMPLEST WRINKLING PROBLEM

Model problem is incompressible elastic sheet floating on liquid



One big bump

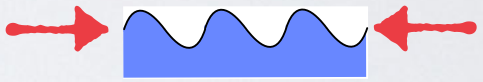


Expensive in gpe

Optimal length

$$\lambda = 2\pi\left(\frac{B}{\rho g}\right)^{1/4}$$

Many small bumps



Expensive in bending

Detailed solution (Diamant & Witten 2011) gives compressive force:

$$P = (B\rho g)^{1/2}\left[2 - \frac{\pi^2}{4}\left(\frac{\Delta L}{\lambda}\right)^2\right]$$

Force decreases with increasing compression…

(cf *elastica*, where *P* increases post buckling)

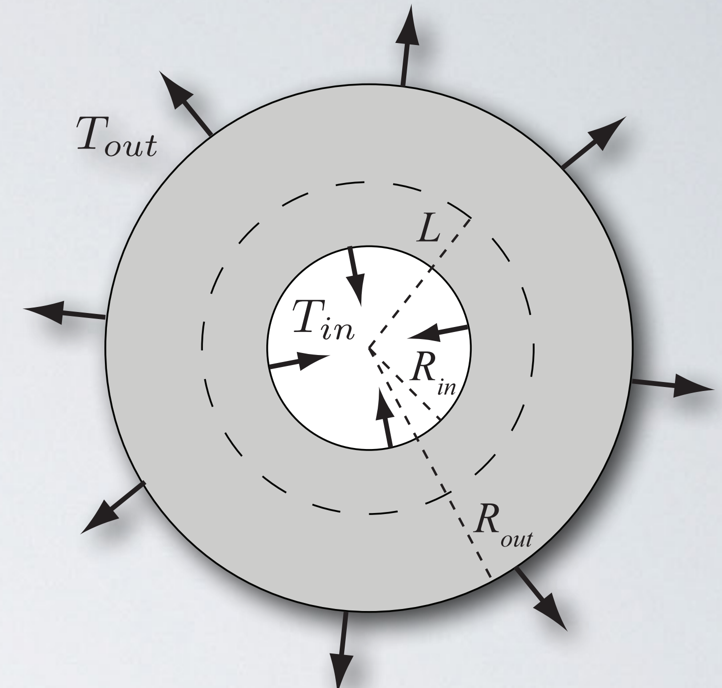
… but perturbative effect

BEYOND 1D

Simplest axisymmetric problem

An annulus with inner/outer tensions T_{in} / T_{out}

Inner hole radius R_{in} (and assume $R_{\text{out}}/R_{\text{in}} \to \infty$)

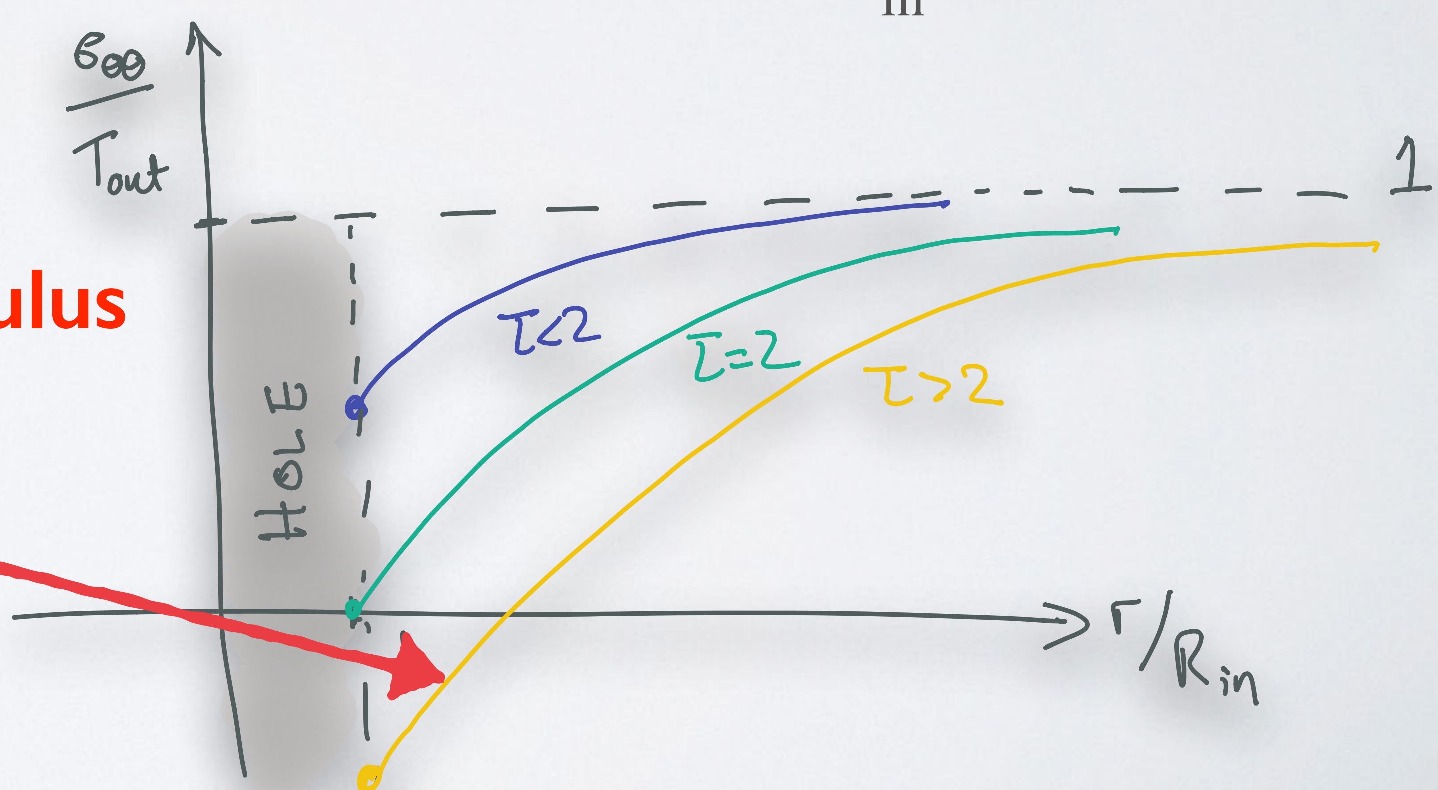


$$\nabla \cdot \sigma = 0 \implies \begin{cases} \sigma_{rr} = T_{\text{out}} + (T_{\text{in}} - T_{\text{out}})\dfrac{R_{\text{in}}^2}{r^2} \\ \sigma_{\theta\theta} = T_{\text{out}} - (T_{\text{in}} - T_{\text{out}})\dfrac{R_{\text{in}}^2}{r^2} \end{cases}$$

For $\tau = \dfrac{T_{\text{in}}}{T_{\text{out}}} > 2$ hoop stress is compressive, $\sigma_{\theta\theta} < 0$, in: $\quad 1 \leq \dfrac{r}{R_{\text{in}}} < (\tau - 1)^{1/2}$

Compressive stress in inner annulus

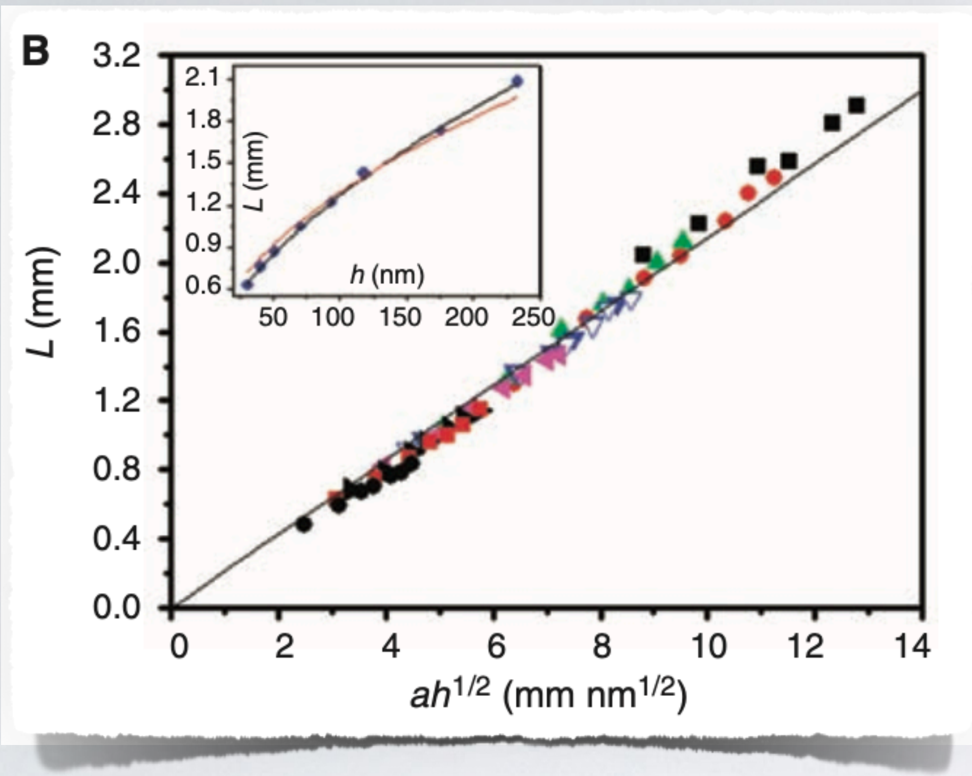
$$L_{\text{comp}} = R_{\text{in}} \left(\frac{T_{\text{in}}}{T_{\text{out}}} - 1 \right)^{1/2}$$

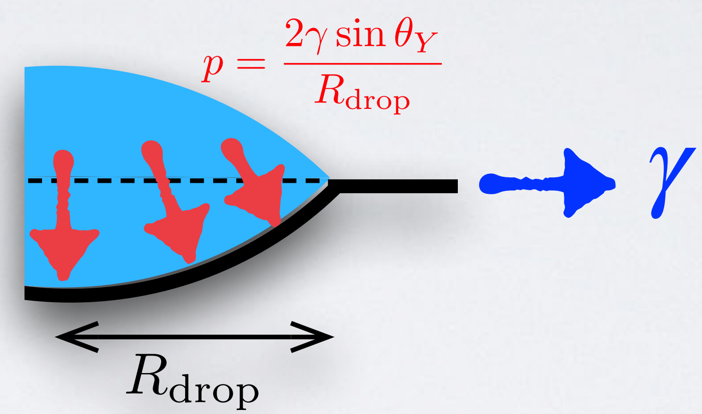


With stress determined, perform analogue of Euler buckling analysis

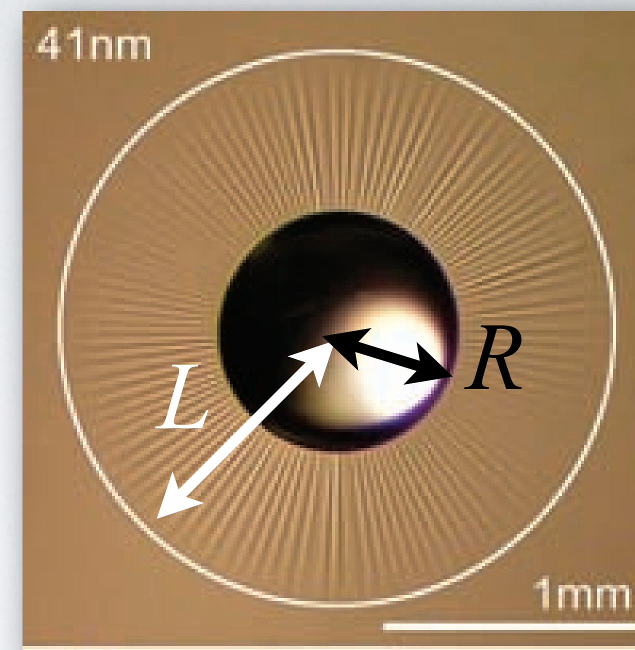
A PROBLEM

Experiments on floating sheets show wrinkles with well-defined length





Huang *et al.* 2007



Experiments

$$\frac{L}{R_{\text{in}}} \propto \left(\frac{Et}{\gamma}\right)^{1/2}$$

Lamé

$$\frac{L}{R_{\text{in}}} \propto \left(\frac{T_{\text{in}}}{T_{\text{out}}}\right)^{1/2}$$

A problem:

Scaling for wrinkle length would need T_{in} indpt of γ ???

(clearly $T_{\text{out}} = \gamma$)

WHAT TO DO?

Key idea 1: Wasted length $\implies \frac{1}{2}\int_0^{2\pi} \frac{1}{r^2}\left(\frac{\partial \zeta}{\partial \theta}\right)^2 r\, \mathrm{d}\theta = -2\pi u_r$

$\frac{\partial\zeta}{\partial\theta}$: $O(m)$ $\qquad$ $2\pi u_r$: $O(1)$

Key idea 2: $m \gg 1$

As $\dfrac{B}{T_{\text{out}} R_{\text{in}}^2} = \epsilon \to 0, m \to \infty$, but $\zeta(\theta) \to 0$ such that $\partial\zeta/\partial\theta = O(1)$

'Far-from-threshold' expansion

Expand in powers of $1/m$:

$$\zeta(r,\theta) = \bar{\zeta}(r) + \frac{1}{m}\zeta^{(1)}(r)\cos \ldots$$

$$\sigma_{ij}(r,\theta) = \sigma_{ij}^{(0)}(r) + \frac{1}{m}\sigma_{ij}^{(1)}(r,\theta)$$

Not possible in 1D – requires 2D

Find that:

$$\sigma_{\theta\theta}^0 = \sigma_{\theta\theta}^1 = 0 \implies \sigma_{\theta\theta} \ll \sigma_{rr}$$

$$\nabla \cdot \sigma = 0 \implies \sigma_{rr} \approx C/r$$

Energy minimization

$$\int \frac{r}{2} B \left(\frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2}\right)^2 + \frac{r}{2}\rho g \zeta^2 \, \mathrm{d}\theta$$

Bending term: $\sim \epsilon m^2$ $\qquad$ ζ^2 term: $\sim m^{-2}$

$\Downarrow$

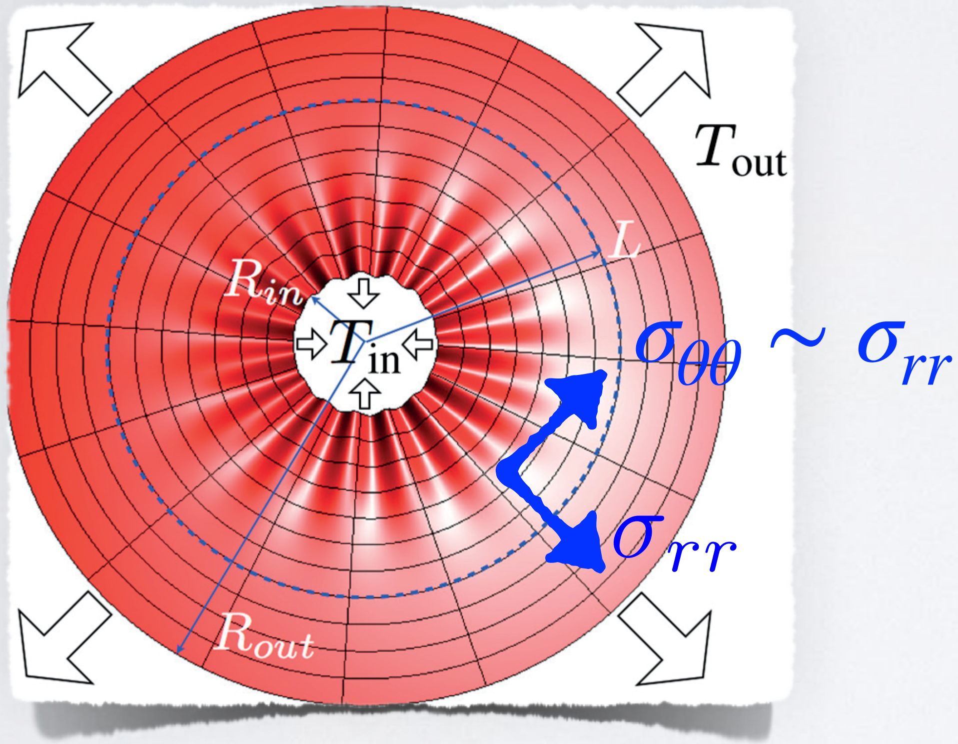
$$m \sim \epsilon^{-1/4}$$

INTERPRETATION

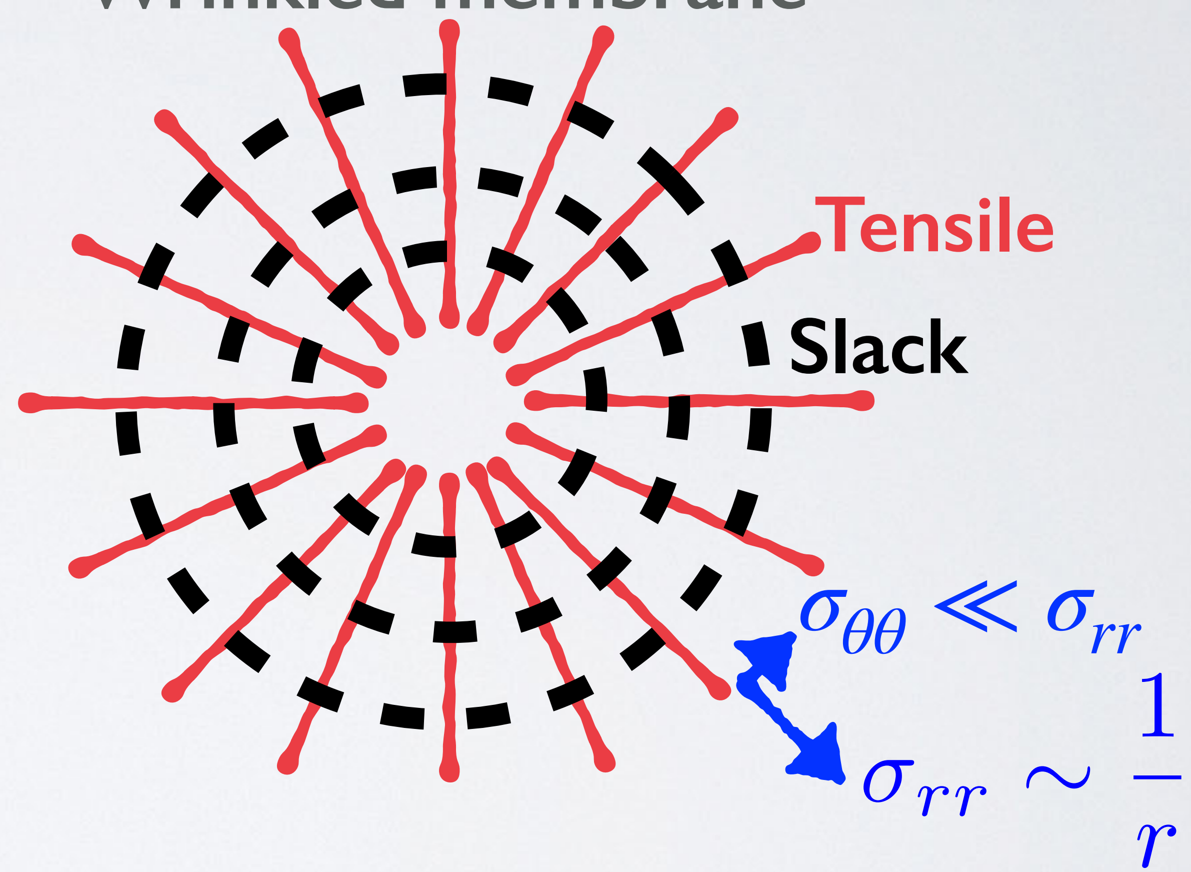
Wrinkling effectively eliminates compressive stress: $\sigma_{\theta\theta} = O(\epsilon^{1/2})$, but $\sigma_{rr} = O(1)$

– cf a spider's web

Normal membrane



Wrinkled membrane



This is leading order effect of wrinkling **not** perturbative

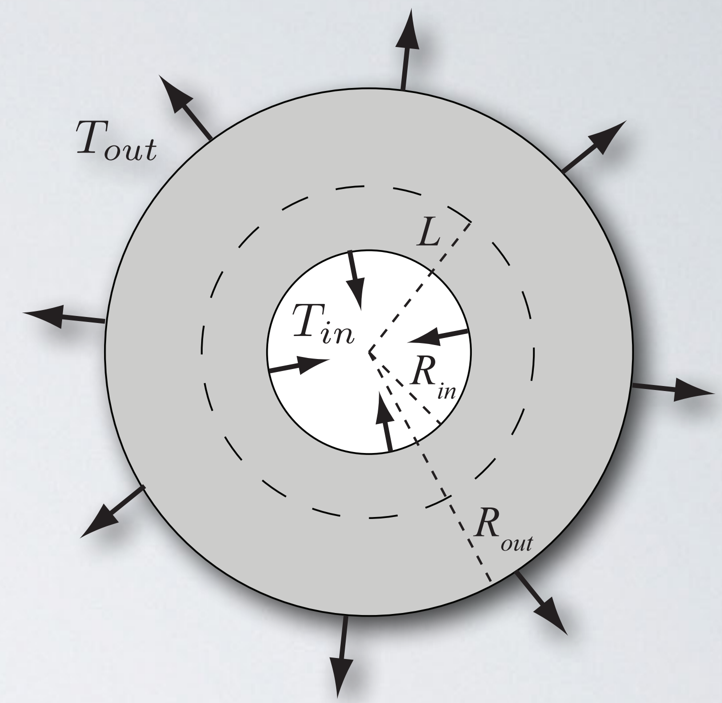
Similar to Relaxed energy functional/tension-field theory, but…

…energy of wrinkles allows determination of wrinkle number (not discussed)

LAMÉ PROBLEM: REVISITED

Wrinkling $\Longrightarrow \quad \sigma_{rr} = \dfrac{T_{\text{in}} R_{\text{in}}}{r}$ in $R_{\text{in}} < r < L$

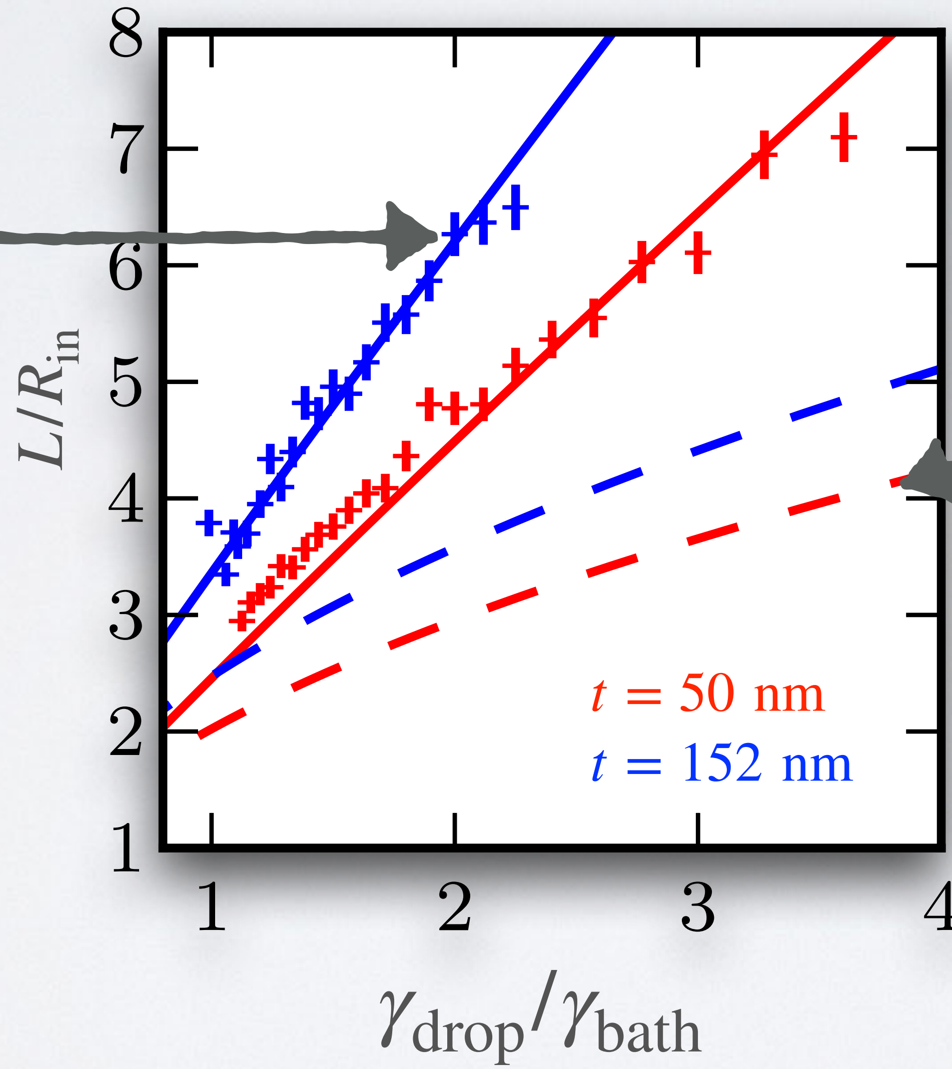
No wrinkling $\Longrightarrow \quad \sigma_{rr,\theta\theta} = T_{\text{out}} \left(1 \begin{smallmatrix} + \\ - \end{smallmatrix} \dfrac{L^2}{r^2} \right)$ in $r > L$



Continuity of σ_{rr} gives:

$$\frac{L}{R_{\text{in}}} = \frac{T_{\text{in}}}{2T_{\text{out}}}$$

(prefactor calculated)

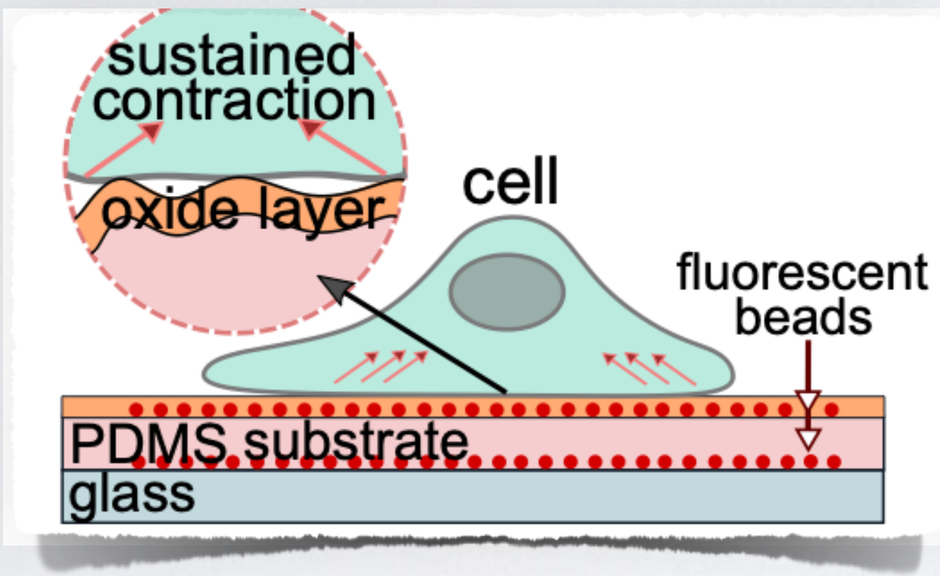


$$\frac{L}{R_{\text{in}}} = \left(\frac{T_{\text{in}}}{T_{\text{out}}} - 1 \right)^{1/2}$$

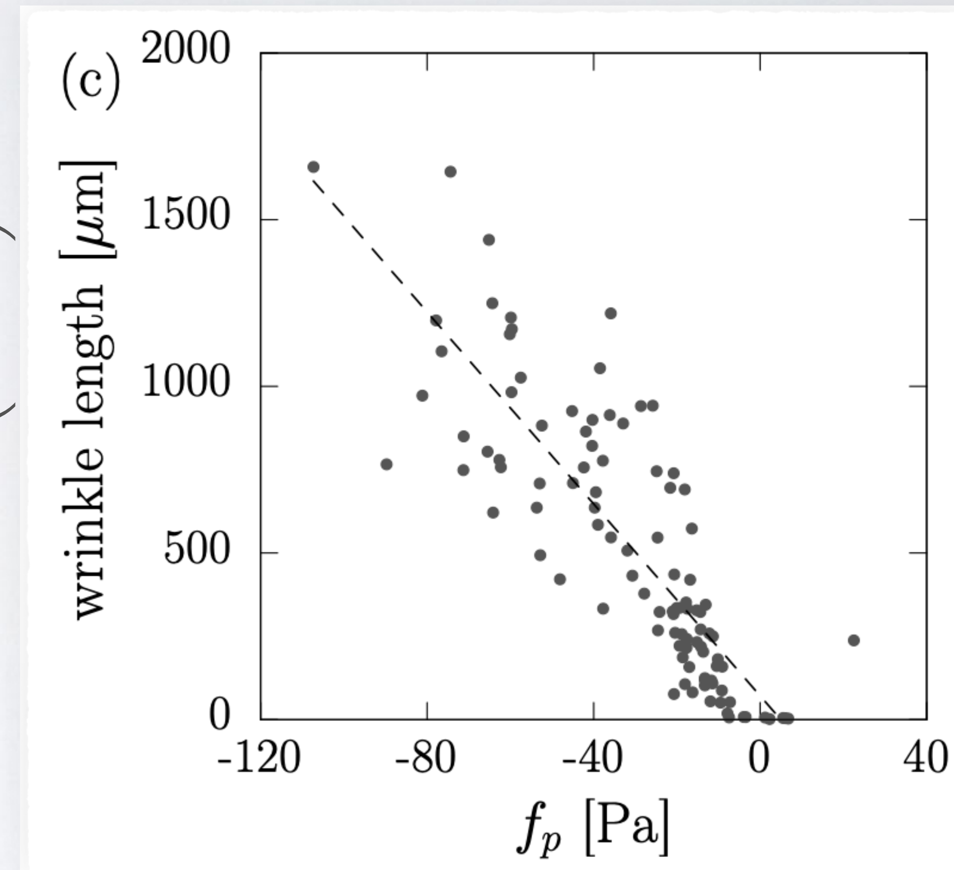
Change in stress makes wrinkles propagate significantly **further** than otherwise

IMPORTANCE FOR MEASUREMENT

Direct measurements on cells crawling on thin layers (oxide coating PDMS) also show linear trend between wrinkle length and applied force

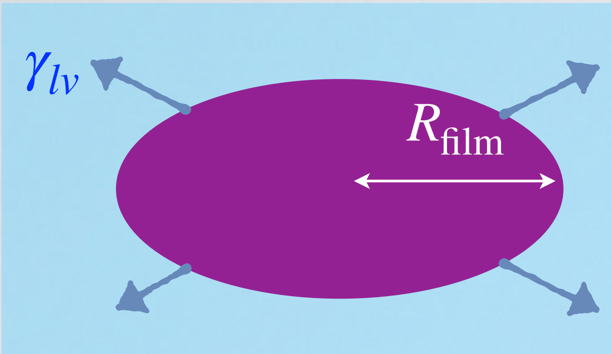


Li *et al.* (2021)

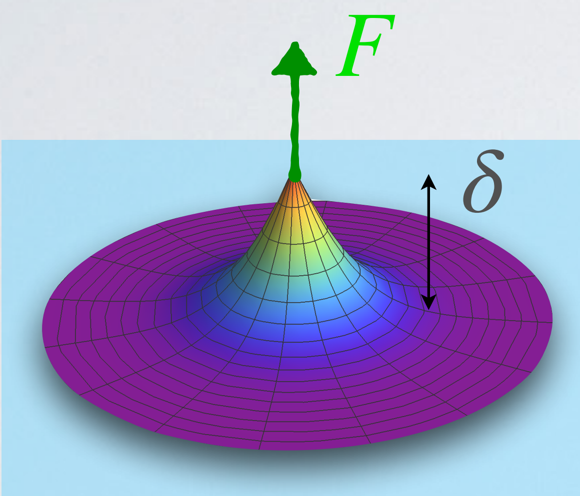


Other consequences for mechanics?

FORCING GAUSSIAN CURVATURE



Circular membrane floating and subject to a tension at its edge (surface tension)



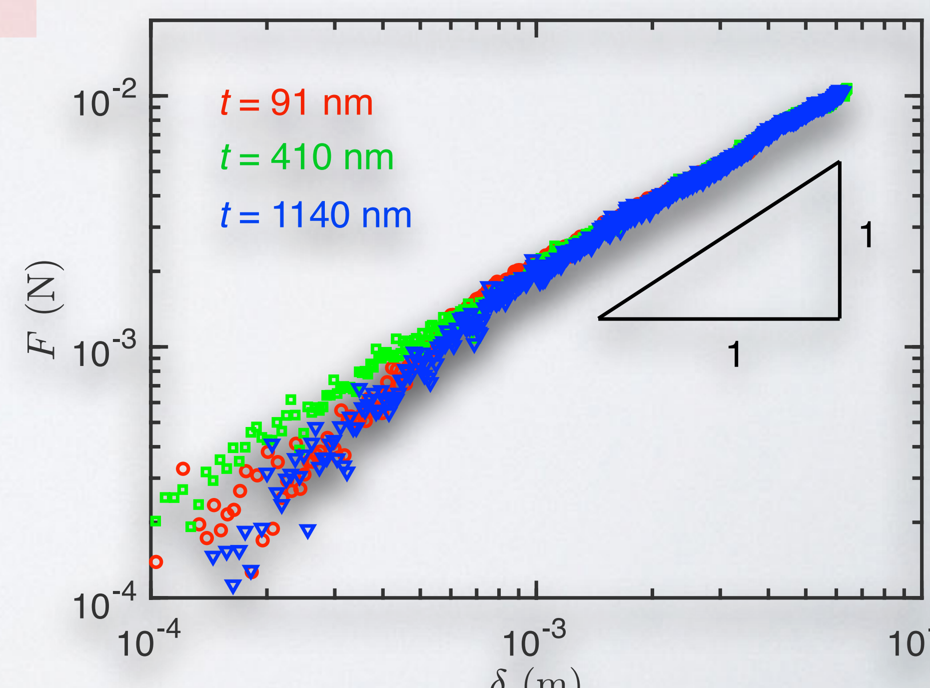
Force sheet to adopt Gaussian curvature – 'poke' height δ at a point, **expect stretching**

Stretching energy: $\mathscr{U}_s \sim E\epsilon^2 \times t\ell^2 \sim Et\delta^4/\ell^2$

Indentation force: $$F = \frac{\mathrm{d}\mathscr{U}_s}{\mathrm{d}\delta} \sim Et\delta^3/\ell^2$$

depends on sheet thickness t

First experiments (Holmes & Crosby, 2010) show indentation force linear & independent of thickness t



How is this possible?

WRINKLING MATTERS

Wrinkles at edge change stress within sheet: $\sigma_{rr} = \dfrac{\gamma_{lv} R_{\text{film}}}{r}$

$\sigma_{\theta\theta} \approx 0$

and the vertical force balance for mean membrane deflections:

$$\sigma_{rr}\frac{\mathrm{d}^2\bar{\zeta}}{\mathrm{d}r^2} + \cancelto{0}{\sigma_{\theta\theta}\frac{1}{r}\frac{\mathrm{d}\bar{\zeta}}{\mathrm{d}r}} = \rho_l g \bar{\zeta}$$

<u>Wrinkled sheet</u> $\dfrac{\gamma_{lv} R_{\text{film}}}{r}\dfrac{\mathrm{d}^2\bar{\zeta}}{\mathrm{d}r^2} = \rho_l g \bar{\zeta} \implies \bar{\zeta}(r) \propto \mathrm{Ai}(r/\ell_*)$

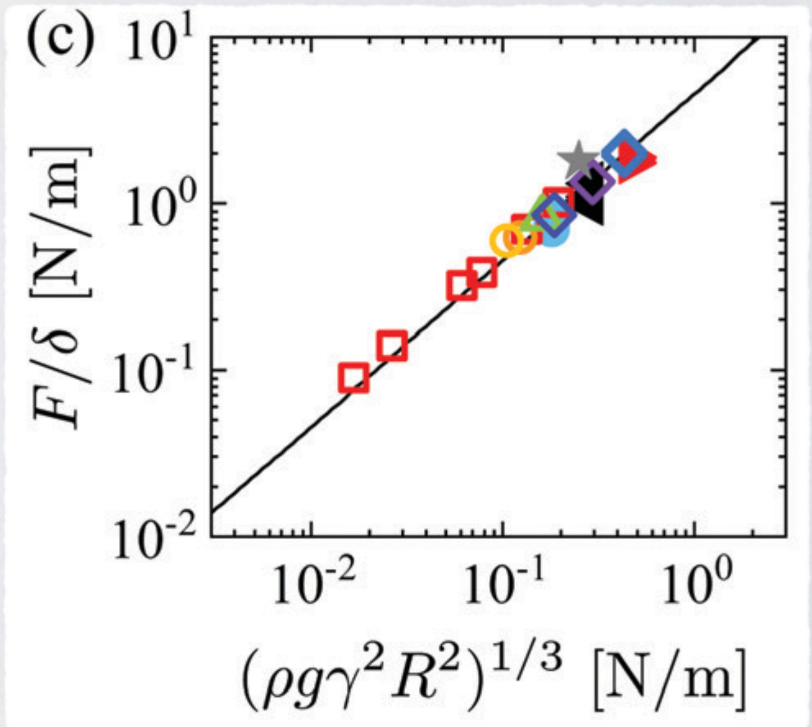
(with $\ell_* = R_{\text{film}}^{1/3}\ell_c^{2/3}$ a new length)

Find constant indentation stiffness

$$\frac{F}{\delta} \approx 4.581\,\gamma^{2/3}(\rho g)^{1/3}R_{\text{film}}^{2/3}$$

…independent of *t* and *E*

Wrinkling allows access to new mode of deformation – shape with 'apparent' Gaussian curvature but no stretching: a **'wrinkly isometry'**



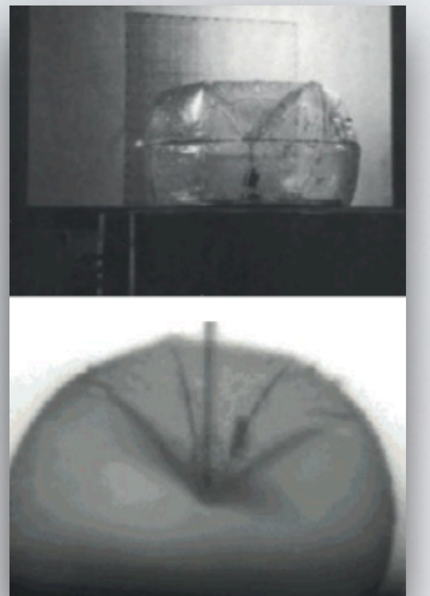
Ripp et al. (2020)

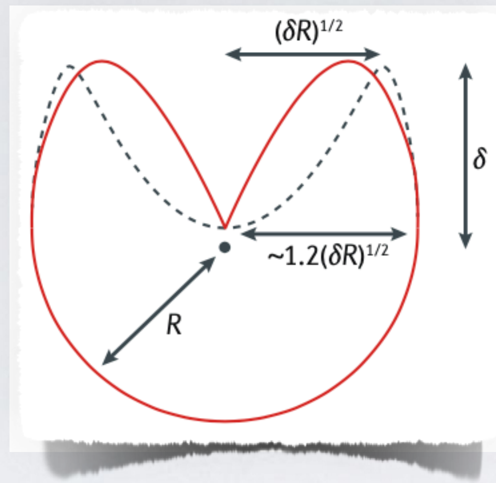
OTHER WRINKLY ISOMETRIES

Similar behaviour observed in other systems:

Pressurized shell

Szyszkowski & Glockner (1987)





- Indentation stiffness independent of elastic properties
- New isometric shape, different to mirror buckling (and with different force law)

Twisted ribbon

Chopin & Kudrolli (2013)

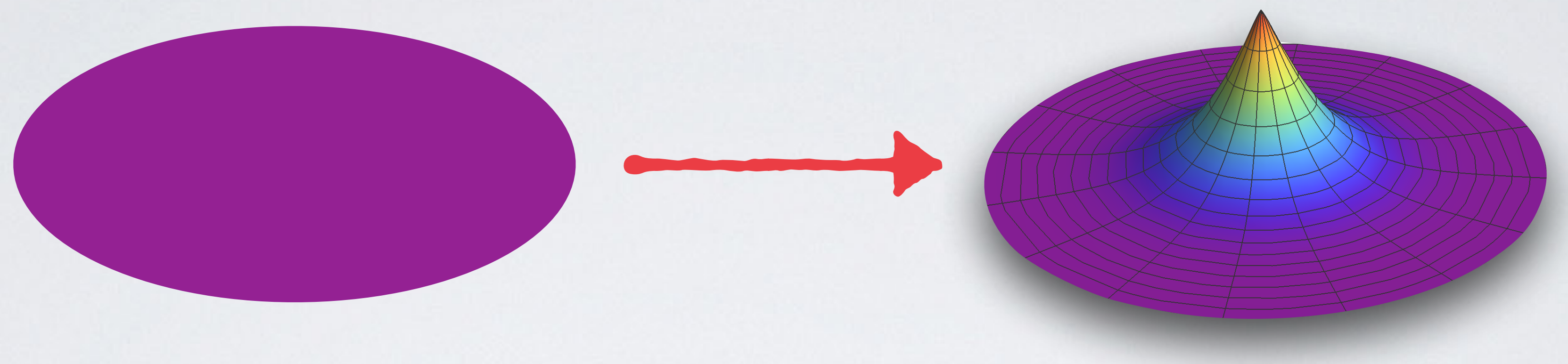
General principle:

Need a weak, but not too weak, external tension: strong enough to make buckling easy, but weak enough to not stretch the material

$$\frac{t^2}{R^2} \ll \frac{T}{Et} \ll 1$$

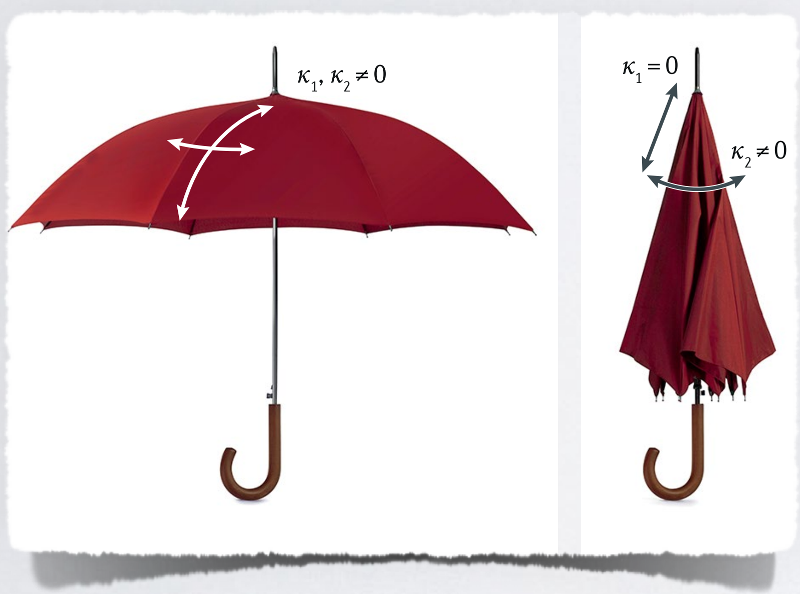
WHITHER GAUSS?

Have seen two examples in which ‘gross’ shape changes Gaussian curvature but **without** significant elastic strain



What is wrong with Gauss’ Theorem?

Focussed on gross shape (mean shape beneath fine wrinkles): to change Gauss curvature of gross shape can just ‘waste’ excess length by wrinkling



Wrinkly isometry is like closing an umbrella: you can get rid of extra length very easily

Length is ‘buffered by buckling’

Two interesting features of wrinkly isometries:

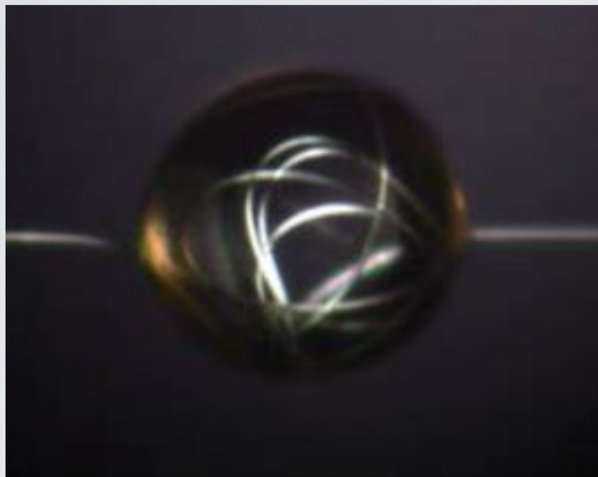
(i) The buffering structure emerges spontaneously (cf umbrella)

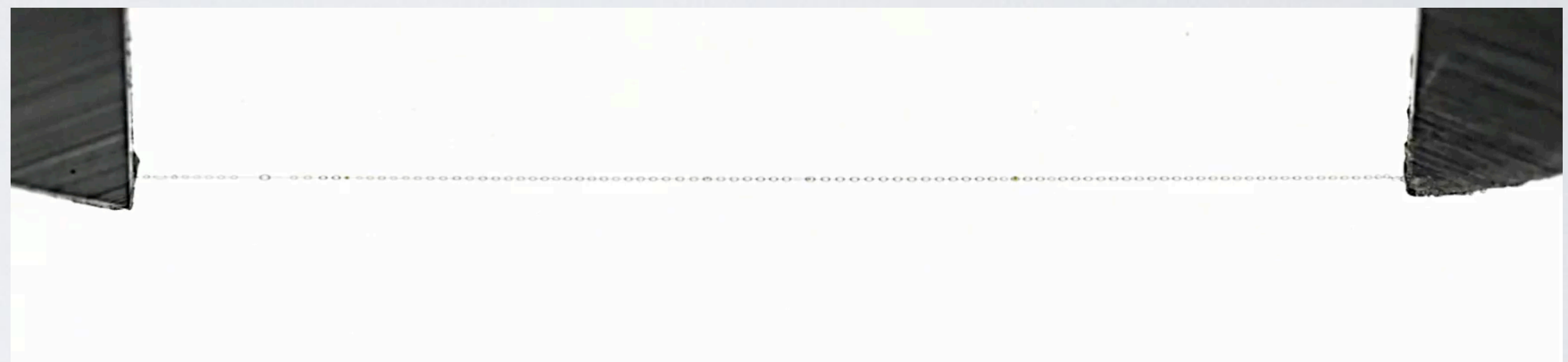
(ii) Wrinkling enables curvature ↔ curvature controls wrinkling

BUFFERING BY BUCKLING

Other examples of similar phenomenology:

Spider webs

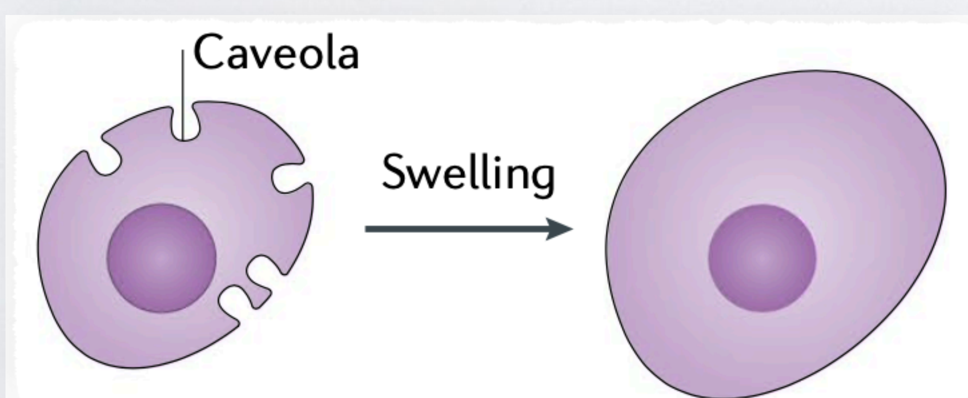


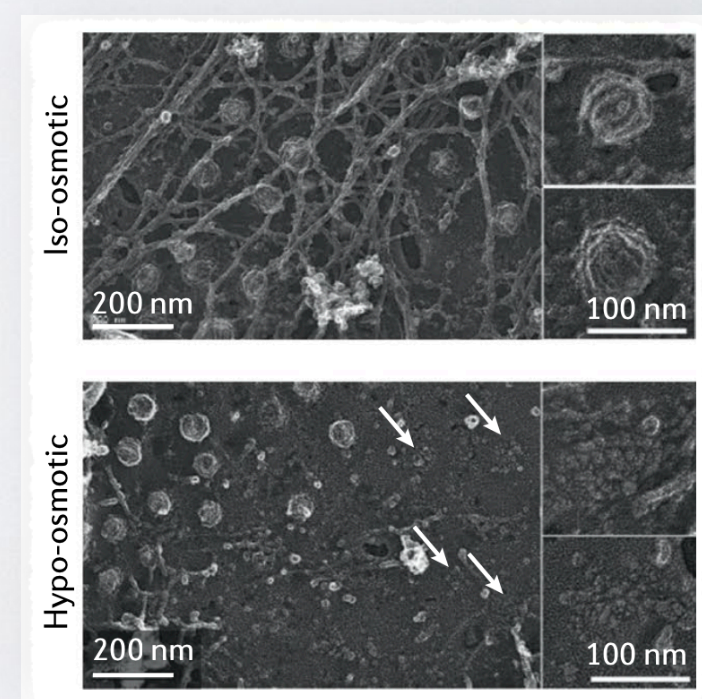


Elettro *et al.* (2016)

Surface tension of liquid on thread is sufficient to buckle the thread within droplet, but does not stretch thread: $\frac{t^2}{R^2} \ll \frac{\gamma}{Et} \ll 1$

Caveolae



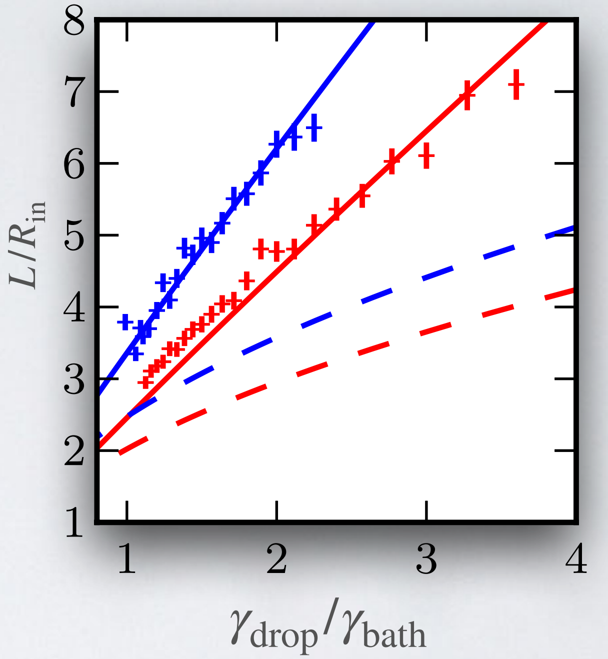


Sinha *et al.* (2011)

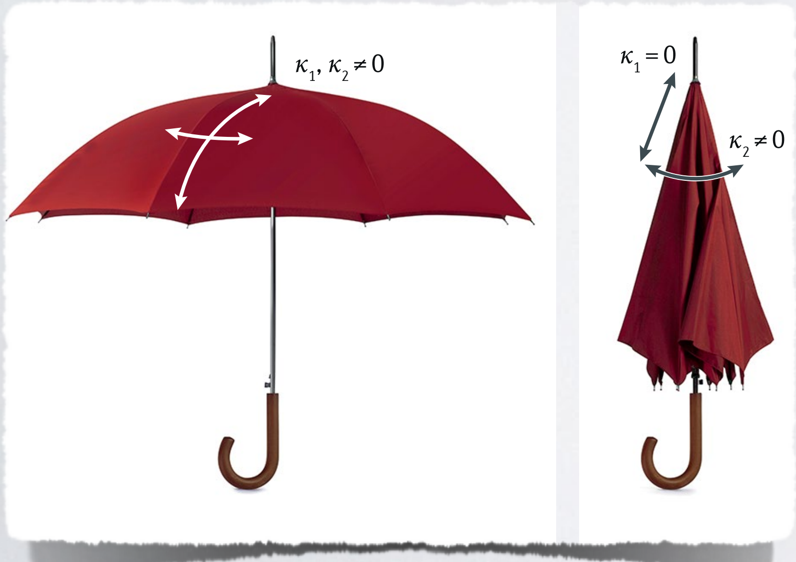
Caveolae buffer area changes in plasma membranes, maintaining constant membrane tension

TAKE HOME MESSAGES

- Wrinkling of highly bendable sheets quickly evolves away from predictions of linear stability analysis:
  - Wrinkles change stress qualitatively
  - Propagate further than might be expected



- Wrinkling buffers apparent changes of length very cheaply:
  - Can change 'apparent' K_G cheaply via **wrinkly isometries**

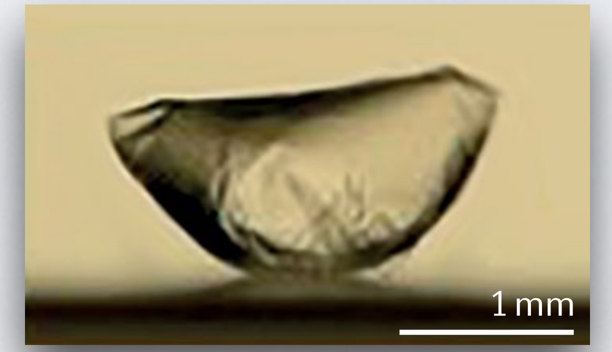


Read more details/pointers to literature @ DV, *Nat. Rev. Phys.* (2019)

OPEN QUESTIONS

- Can general statements be made about allowed shapes (replacing crude statement based on Gauss' Theorem)?

- Novel variational principles?



Paulsen *et al.* 2015

empanada/pasty

- Is `buffering by buckling' important in biological problems e.g. *caveolae*?

    - What are the active and passive mechanisms in *caveolae* formation?

    - What about zero shear rigidity? If stress is only anisotropic transiently are there dynamic analogues?