Bootstrap in matrix models and lattice Yang-Mills theory

Discussion session by

V. Kazakov and M. Kruczenski

Flux tubes, Quark Confinement and Exotic Hadrons, *Jan 2022*

Part I:
Loop equation and bootstrap methods in Lattice gauge theories

Based on 1612.08140 [NPB] P.D.Anderson and M.K.

Motivation

Can one define gauge theories purely in terms of gauge invariant quantities?
AdS/CFT gives one possibility in terms of a dual string theory.
More directly:
Wilson loops $\Longrightarrow$ Loop equation (Migdal-Makeenko)

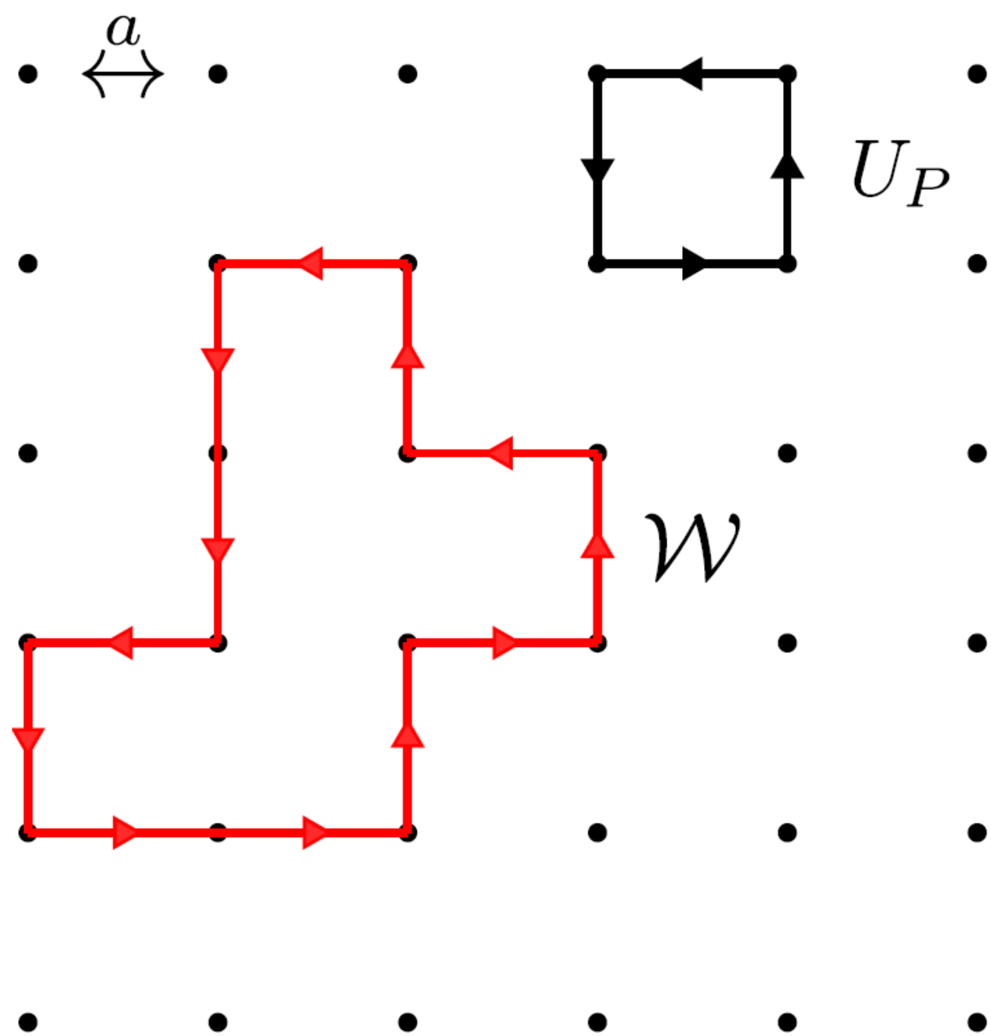
Lattice gauge theory, pure YM, large-N, cubic lattice

Action

$$S = -\frac{N}{2\lambda}\sum_P \mathrm{Tr} U_P$$

$$Z = \int \prod_{\vec{x},\mu} dU_\mu(\vec{x})\ e^{-S}$$

$$\mathcal{W}_\mathcal{C} = \frac{1}{Z}\int \prod_{\vec{x}\mu} dU_\mu(\vec{x})\ \frac{1}{N}\mathrm{Tr}(U_{\mu_1}\ldots U_{\mu_L})e^{-S}$$

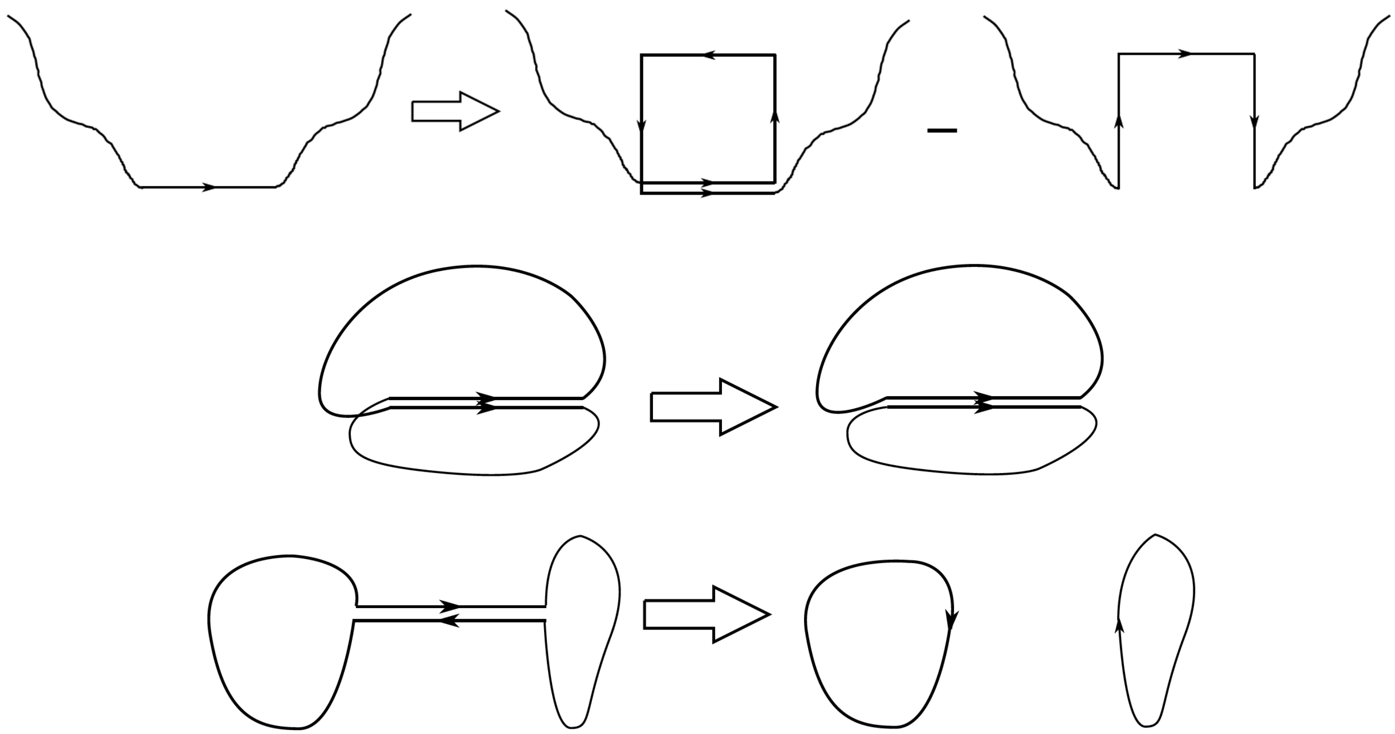


λ : ‘t Hooft coupling
is like temperature

Small λ is small temperature → lower energy
Large λ is large temperature → large entropy

Loop equation (Migdal-Makeenko, Eguchi, Foerster,…)

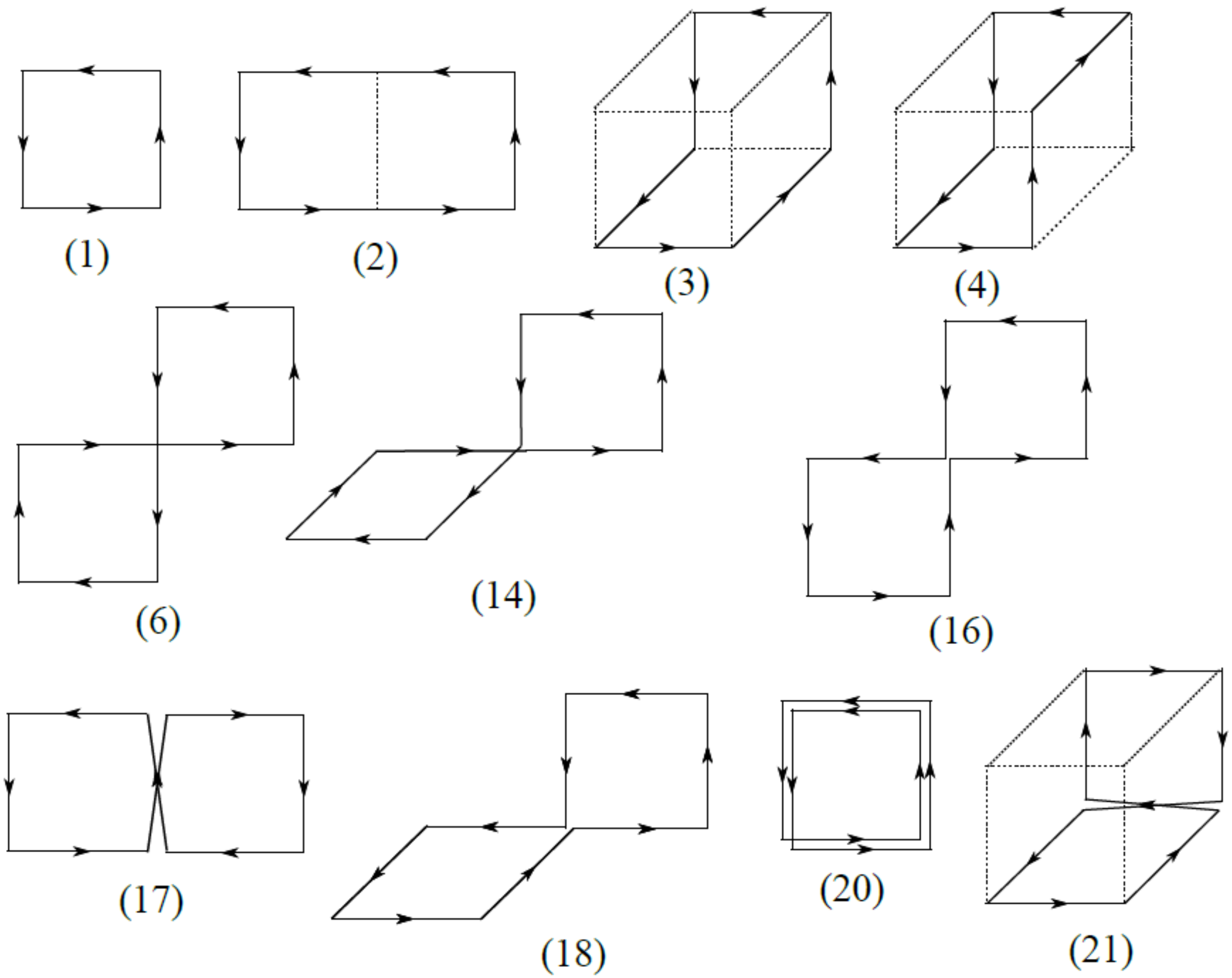
Graphic form of the equation:



Algebraic form of the equations (sum over links) :

$$\mathbb{K}_{i\to j}\mathcal{W}_j + 2\lambda\mathcal{W}_i + 2\lambda\mathbb{C}_{i\to jk}\mathcal{W}_j\mathcal{W}_k = \delta_{i1}$$

$$-\frac{1}{NL}S * \mathcal{W} + \mathcal{W} + \frac{1}{L}\sum_i \sigma_i \mathcal{W}_{1i}\mathcal{W}_{2i} = 0$$



L	# WL(4d)
10	268
12	5,324
14	142,105
16	4,483,136
18	152,322,746

$$-\mathcal{W}_0 - \mathcal{W}_2 - 4\mathcal{W}_3 + \mathcal{W}_{17} + \mathcal{W}_{20} + 4\mathcal{W}_{21} + 2\lambda\mathcal{W}_1 \;=\; 0$$

$$\mathcal{W}_2 + \mathcal{W}_6 + 4\mathcal{W}_{14} - \mathcal{W}_{16} - \mathcal{W}_{17} - 4\mathcal{W}_{18} = 0$$

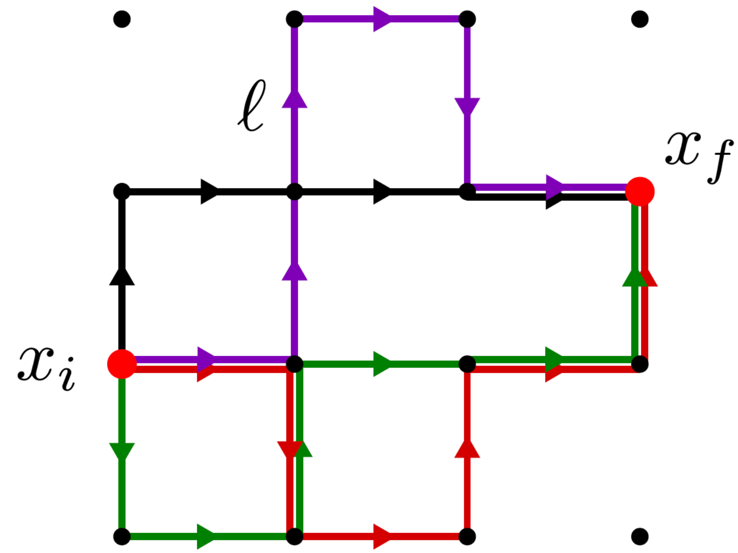
In trying to solve the equations one faces a problem:

- To define the equations properly we have to cut the set of loops, e.g. length ≤ L, and then consider $L \to \infty$.
- The equations for length L have loops of length L+4. The number of loops increases exponentially with L.
- The limit $L \to \infty$ does not seem well defined, except at strong coupling where we set the unknown loops to 0.

Better than set unknown loops to 0, leave them free and put bounds on the solution using a set of positivity constraints. This gives a well defined limit $L \to \infty$ at any coupling specially at small coupling.

Positivity constraints

$$A = \sum_{\ell=1}^{L} c_\ell U^{(\ell)}$$



$$\mathrm{Tr}\, A^\dagger A \geq 0 \quad \Rightarrow \quad \sum_{\ell\ell'} c_\ell^* c_{\ell'} \mathrm{Tr}\left[\left(U^{(\ell)}\right)^\dagger U^{(\ell')}\right] \geq 0 \quad \forall c_\ell$$

$$\boxed{\hat{\rho}^{(L)}_{\ell\ell'} = \frac{1}{L}\left\langle \mathrm{Tr}\left[\left(U^{(\ell)}\right)^\dagger U^{(\ell')}\right]\right\rangle \succeq 0}$$

$$\hat{\rho}^{(L)}_{\ell\ell} = \frac{1}{L} \Rightarrow Tr\left(\hat{\rho}^{(L)}\right) = 1$$

Closed loop:
goes along ℓ'
comes back along ℓ

ρ can be thought as a reduced density matrix obtained by tracing over color indices

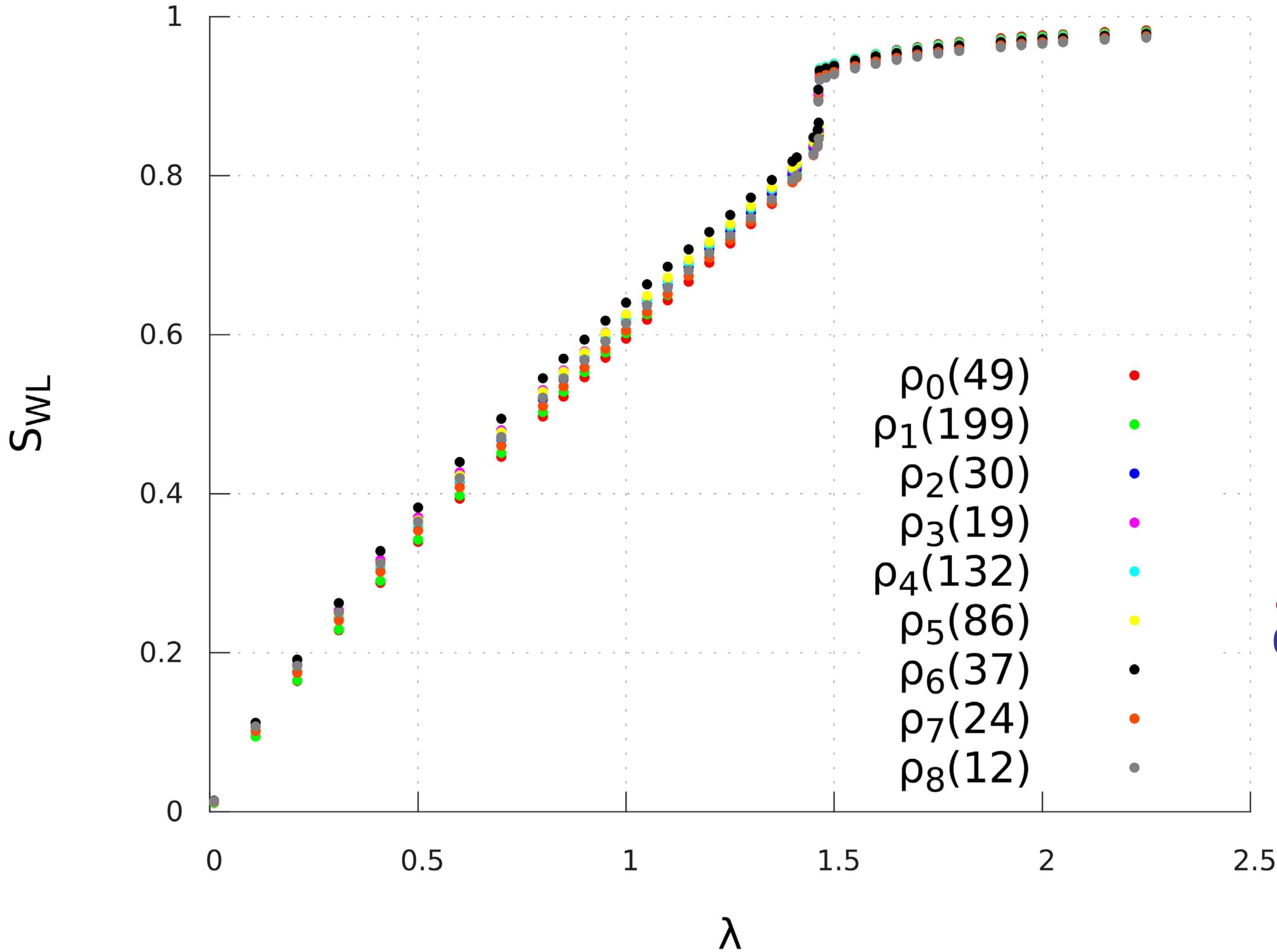
$$\hat{\rho}^{(L)}_{\ell\ell'} = \frac{1}{NL}\langle \mathrm{Tr}\left[\left(U^{(\ell)}_{ab}\right)^{*} U^{(\ell')}_{ab}\right]\rangle$$

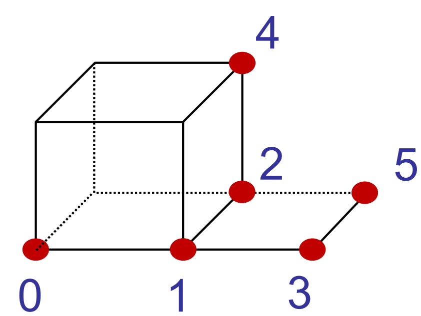
Its entropy computes the information loss due to tracing:

$$S_{WL} = -\mathrm{Tr}\,\hat{\rho}^{(L)}\log_L\hat{\rho}^{(L)}$$

When $\lambda=0$ all loops are 1, S=0, when $\lambda\to\infty$, all loops are zero, $\rho=\mathbf{I}$, S is maximal. Behaves as system entropy.

Numerically S_{WL} is approx. independent of the choice of ρ





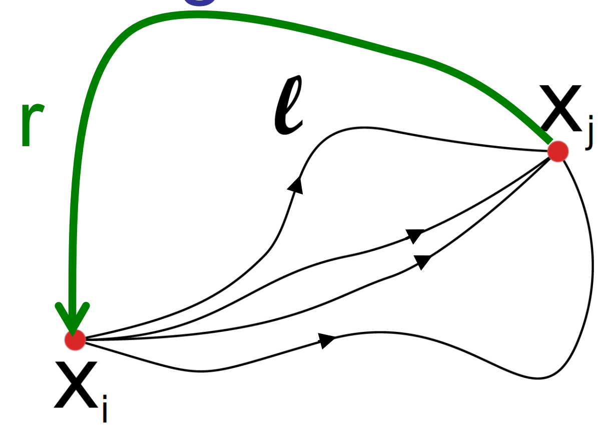
If ρ has a zero eigenvector c_0 (boundary of the domain):

$$c^*_{0\ell}\rho_{\ell\ell'}c_{0\ell'}=0 \quad \Rightarrow \quad \langle \mathrm{Tr} A_0 A_0^\dagger\rangle = 0, \quad A_0=\sum_\ell c_{0\ell}U^{(\ell)}$$

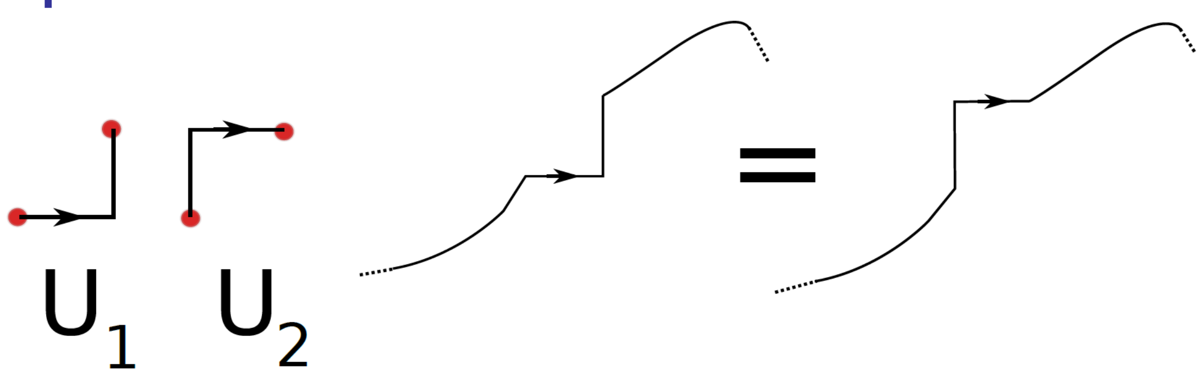
Thus $A_0 = 0$

Closing with an arbitrary path *r* we get linear equations

$$\sum_\ell c_{0\ell}\langle \mathrm{Tr}(U_r^\dagger U^{(\ell)})\rangle = 0$$



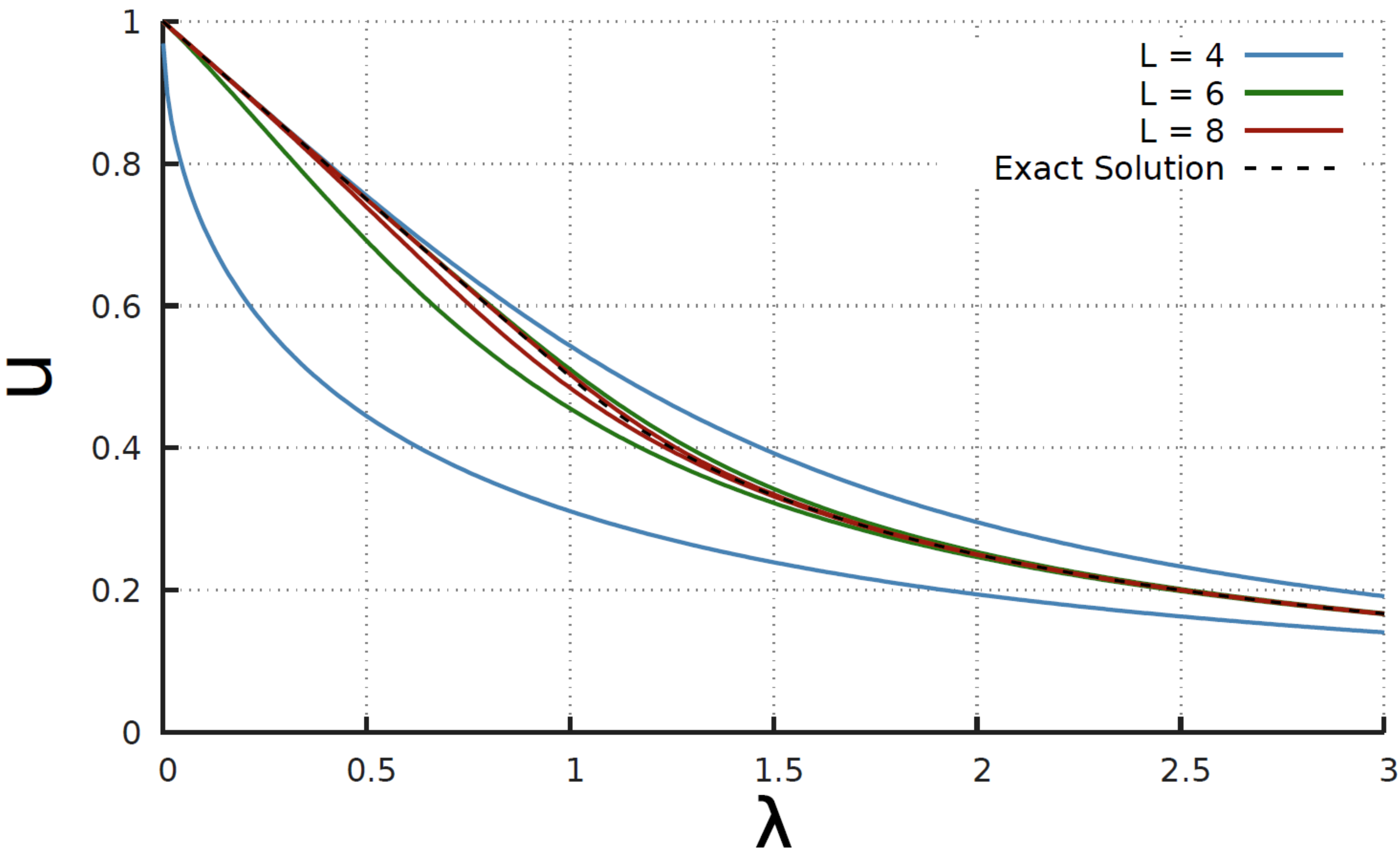
valid for arbitrary long loops. In particular, if u=1 then all loops are 1.



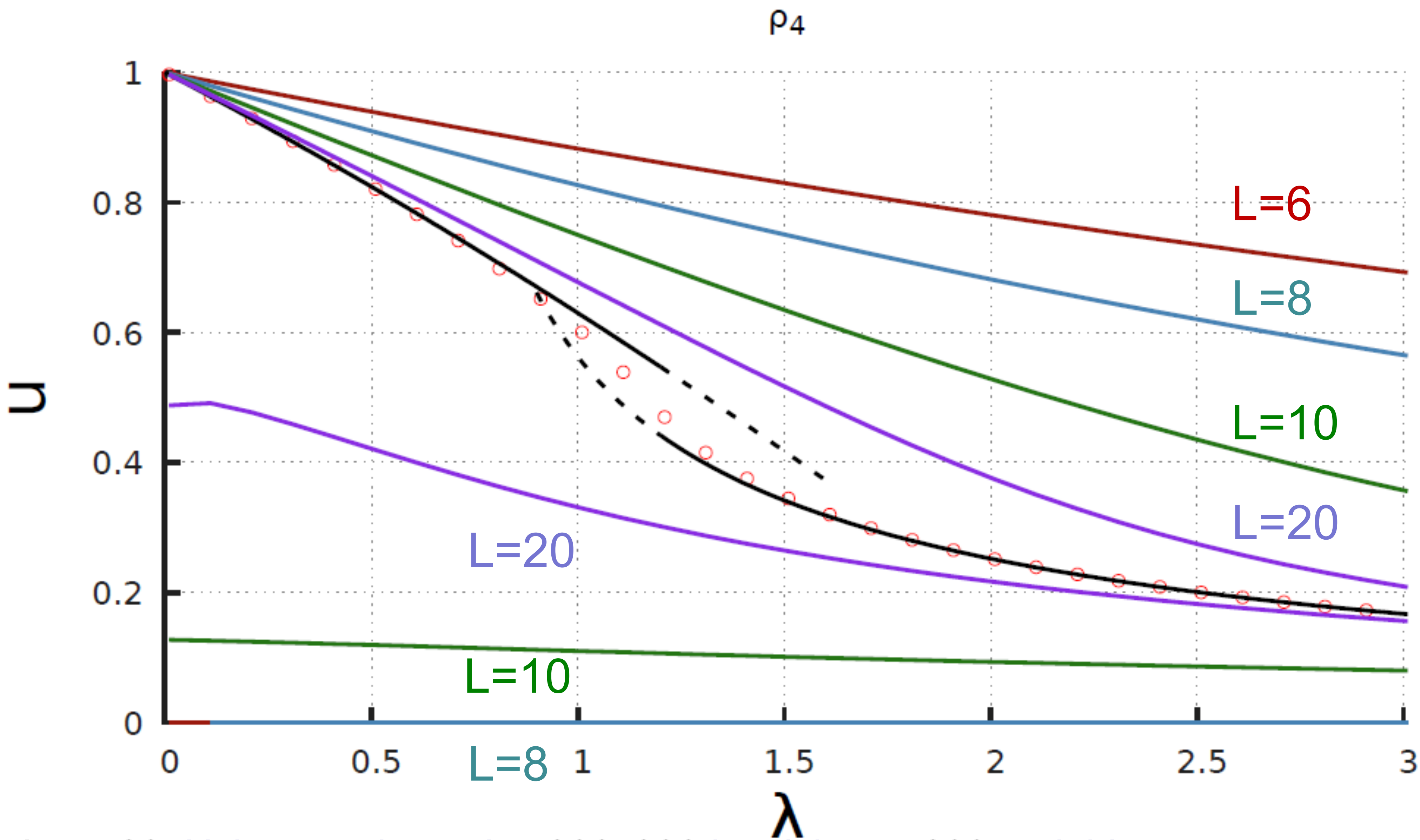
$$\rho = \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix}$$

2d case $\lambda_c = 1$, third order(Gross-Witten '80, Wadia)
(matrix model)

$$\frac{E}{V} = -\frac{d(d-1)}{2}\frac{N^2}{\lambda}u\,, \qquad u = \frac{1}{N}\langle TrU_P \rangle$$



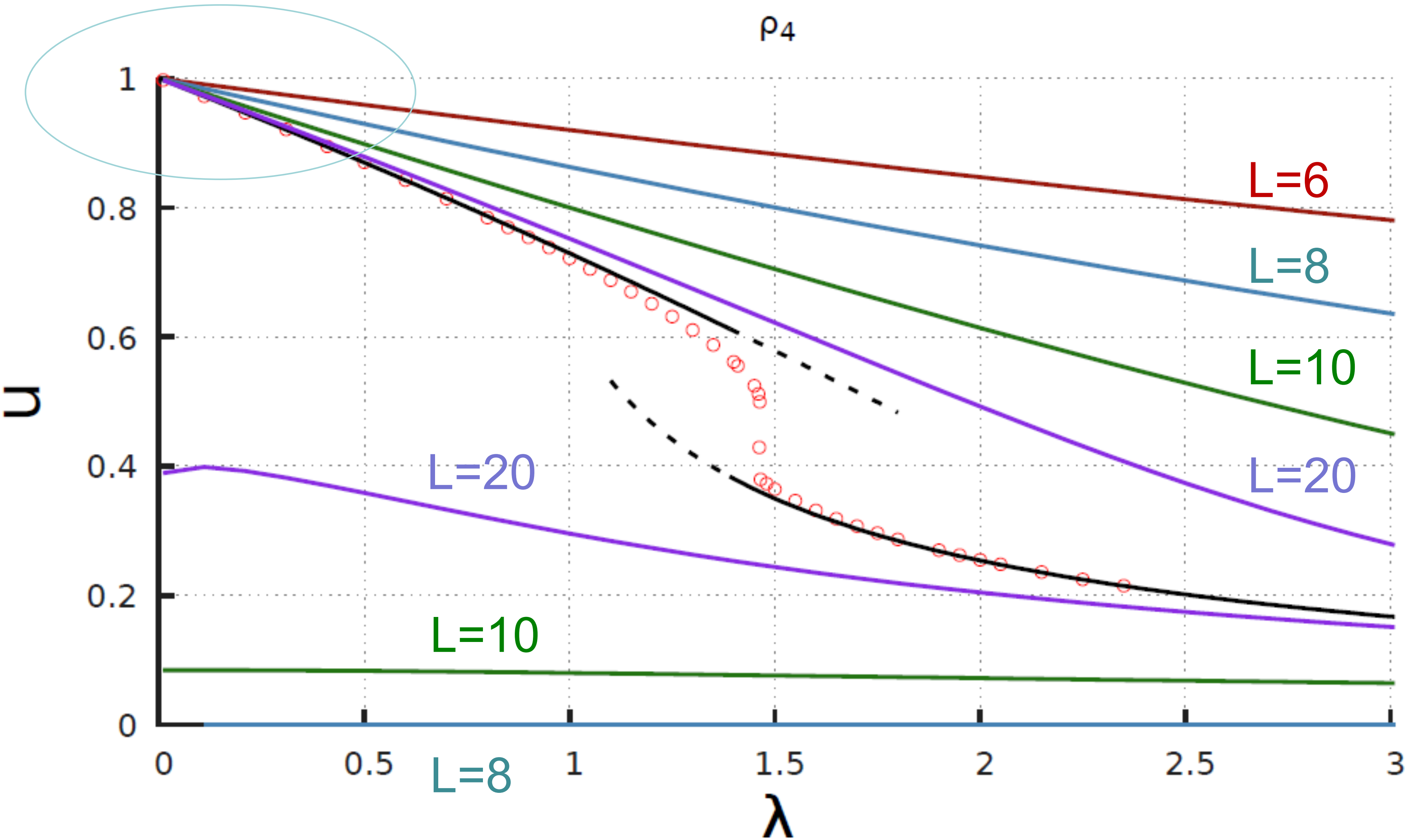
3D case $\lambda_c \sim 1.2$, third order (Teper '06)



L_{max}=20. Using matrix ρ_4 size 330x330 involving 5,299 variables.

Can be greatly improved (see Kazakov's talk next)

<u>4D case</u> $\lambda_c = 1.3904$, first order (Campostrini '99)



L_{max}=20. Using matrix ρ_4 size 786x786 involving 11302 variables.

Gradient flow and the loop equation (unpublished)

Gradient flow (Luscher) introduces smeared operators that are easier to compute in the lattice (large loops)
Given a lattice configuration we flow the links using

$$\partial_t U_{ac}(\vec{x},\mu) = -\frac{\lambda}{N}\,\partial_{\vec{x},\mu} S_W(U)_{ab} U_{bc}(\vec{x},\mu)$$

$$\partial_{\vec{x},\mu} S_W(U)_{ab} = -\frac{1}{2}\frac{\delta S_W}{\delta U_{bc}(\vec{x},\mu)} U_{ac}(\vec{x},\mu) + \frac{1}{2N}\delta_{ab}\,\frac{\delta S_W}{\delta U_{cd}(\vec{x},\mu)} U_{cd}(\vec{x},\mu)$$

For Wilson loops in the large N limit

$$\partial_t \mathcal{W}_i = -\frac{1}{2}\mathbb{K}_{i\to j}\mathcal{W}_j \quad \Rightarrow \quad \mathcal{W}(t) = e^{-\frac{1}{2}t\mathbb{K}}\,\mathcal{W}(t=0)$$

The flowed Wilson loops obey a flowed loop equation

$$\mathbb{K}_{i\to j}\mathcal{W}_j(t) + 2\lambda\mathcal{W}_i(t) + 2\lambda C(t)_{i\to jk}\mathcal{W}_j(t)\mathcal{W}_k(t) = b_i(t)$$

where

$$\begin{aligned} b_i(t) &= \left(e^{-\frac{1}{2}t\mathbb{K}}\right)_{i1} \\ C(t)_{i\to jk} &= \left(e^{-\frac{1}{2}t\mathbb{K}}\right)_{ii'} C_{i'\to j'k'} \left(e^{\frac{1}{2}t\mathbb{K}}\right)_{j'j} \left(e^{\frac{1}{2}t\mathbb{K}}\right)_{k'k} \end{aligned}$$

We get an infinite number of new positivity constraints for the flowed Wilson loops:

$$\rho_{ij}(t) = \rho_{ij,k}\mathcal{W}_k(t), \qquad \rho(t) \succeq 0, \quad \forall t$$

Are they new constraints?

Summary

-) We constructed a matrix $\rho \succeq 0$ with WLs as entries and use it to correctly formulate the problem of solving the loop equations (especially at small coupling).

-) In the weak coupling phase ρ saturates the bounds, it has zero eigenvalues whose number increases as $\lambda \to 0$ (relevant for the continuum limit?).

-) We defined an off-shell Wilson loop entropy as the entropy associated with ρ (~ indep. of particular ρ).

Conference "Fluxtubes, Quark Confinement and Exotic Hadrons"

KITP January 19, 2022

Discussion on Matrix and Yang-Mills Bootstrap

Vladimir Kazakov & Martin L. Kruczenski

Monte-Carlo was till recently the only systematic method to compute functional integrals for (non-integrable) QFTs and Matrix Models

It is (intellectually and practically) un unsatisfactory situation.

Bootstrap gives a hope for a new, more analytic approach

It gives exact inequalities on physical quantities

It combination of Ward identities (loop eqs) and positivity of correlation matrix

Large space for improvement: relaxation, reflection positivity, symmetry reductions

2-Matrix Model: "Words", Moments and Loop Equations

We studied an "unsolvable" 2MM in the planar limit

V.K. & Zechuan Zheng '21

$$Z = \lim_{N\to\infty} \int d^{N^2}A\, d^{N^2}B\ \mathrm{e}^{-N\mathrm{tr}\left(-h[A,B]^2/2+A^2/2+gA^4/4+B^2/2+gB^4/4\right)}$$

Matrix "words": $\mathrm{Word} = ABBAAABAAB\ldots$

Ward identities $\int d^{N^2}A\, d^{N^2}B\ \mathrm{tr}(\partial_M \mathrm{Word})\, \mathrm{e}^{-N\mathrm{tr}\, V(A,B)} = 0, \qquad M = \{A, B\}$

give loop equations relating various moments : $\langle \frac{\mathrm{tr}}{N}(ABBAAABAA|\ldots)\rangle$

$$\langle \mathrm{Tr}\,(\mathrm{Word}_l \times \partial_M V(A,B))\rangle = \sum_{l_1=1\ (l_1\to M)}^{l} \langle \mathrm{Tr}\,\mathrm{Word}_{l_1-1}\rangle \cdot \langle \mathrm{Tr}\,\mathrm{Word}_{l-l_1}\rangle$$

Large N factorization

For numerics, take (length of words) ≤ L. Many more words than equations!
How to complete the missing information? Positivity conditions on moments -- Matrix Bootstrap

H.Lin 2020

Non-linear bootstrap for a 2MM

Loop equations relate moments $\mathcal{W} = \{I,\, \langle \mathrm{tr} A\rangle,\, \langle \mathrm{tr} B\rangle,\, \left\langle \mathrm{tr} A^2\right\rangle,\, \left\langle \mathrm{tr} B^2\right\rangle,\, \langle \mathrm{tr} AB\rangle,\, \left\langle \mathrm{tr} AB^2\right\rangle,\, \left\langle \mathrm{tr} BA^2\right\rangle,\, \left\langle \mathrm{tr} A^2B^2\right\rangle, \ldots\}$

Positivity of the correlation : $\langle \mathrm{Tr} \mathcal{O}^\dagger \mathcal{O}\rangle \equiv \langle \mathrm{Tr} (\sum_{i=0}^{L} \alpha_i \mathcal{O}_i)^\dagger (\sum_{i=0}^{L} \alpha_i \mathcal{O}_i)\rangle = \alpha^\top \mathbb{W}_L \alpha \geq 0 \quad \Longrightarrow \quad \mathbb{W}_L \succeq 0$

where $\mathcal{O} = \alpha_1 I + \alpha_2 A + \alpha_3 B + \alpha_4 A^2 + \alpha_5 B^2 AB + \alpha_6 BA + \alpha_7 ABA + \alpha_8 ABB + \ldots$

The bootstrap process reduces to the following optimization problem:

minimize	$\sum_k c_k \mathcal{W}^{(k)}$ w.r.t. $\{\mathcal{W}^{(1)}, \mathcal{W}^{(2)}, \ldots, \mathcal{W}^{(L)}\} \in \mathbb{R}$	- physical quantity (linear function of moments)
subject to	$\mathcal{W}^\top \mathcal{A}_i \mathcal{W} + b_i^\top \cdot \mathcal{W} + a_i = 0$	- *i*-th loop equation
and	$\mathbb{W} \succeq 0, \quad (\mathbb{W}_{ij}$ – linear functions of $\{\mathcal{W}^{(l)}\})$	- positivity of correlation matrix

Loop eqs. are nonlinear -- the problem becomes highly non-convex.
Non-linear SDP (semi-definite programing) algorithms are inefficient.

Relaxation bootstrap for matrix loop equations

Replace the non-linear loop equations by linear relations plus convex inequalities

Consider $X_{kj} = \mathcal{W}^{(k)}\mathcal{W}^{(j)}$ as independent variables in loop equations, then they become linear

Relax $X_{kj} = \mathcal{W}^{(k)}\mathcal{W}^{(j)}$ to inequality $X - \mathcal{W}\mathcal{W}^{\mathsf{T}} \succeq 0$ or, equivalently, $\begin{pmatrix} 1 & \mathcal{W}^{\mathsf{T}} \\ \mathcal{W} & X \end{pmatrix} \succeq 0$

We transformed the non-linear, non-convex problem to a much simpler SDP
for which the numerical solvers are much more efficient

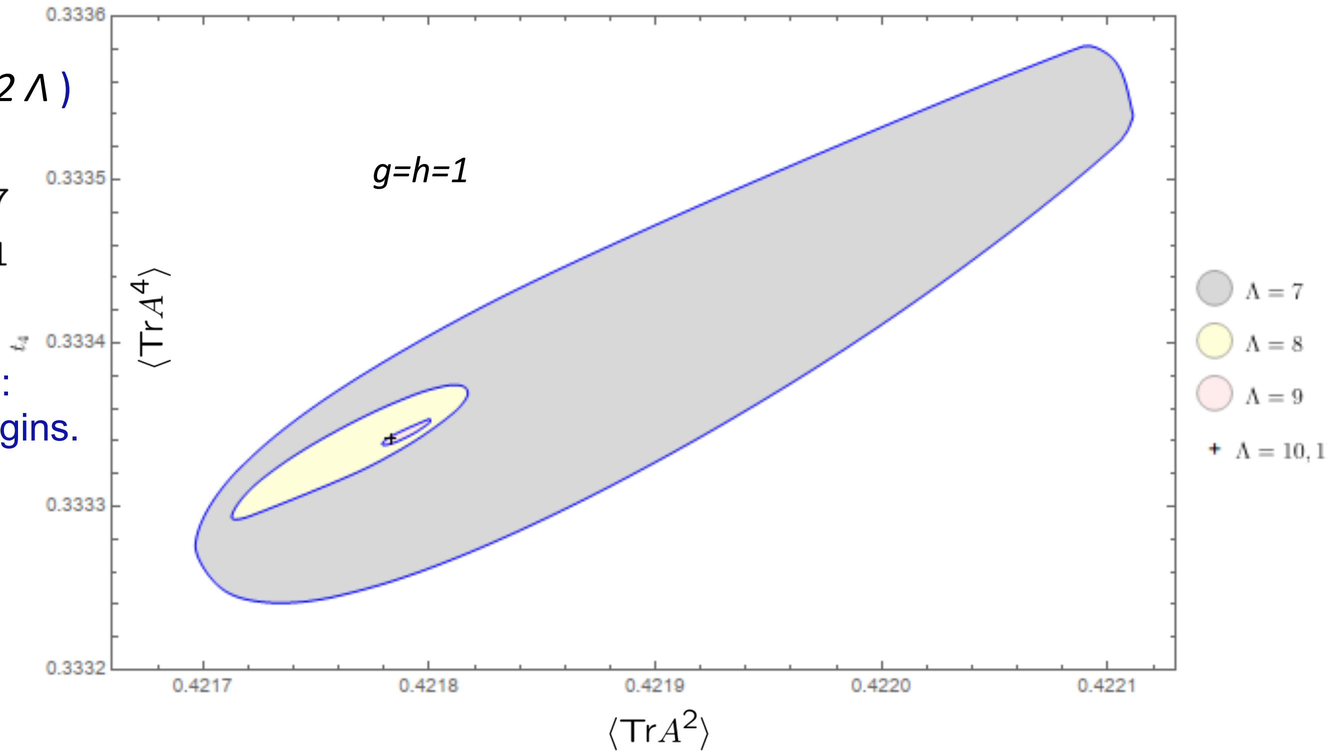
Let us apply it to our 2-matrix model

Numerical bootstrap for **2MM** (symmetric solutions)

Relaxed bootstrap gives 6-digit precision for cutoff *Λ = 11* (Length=2 *Λ*)

$$\begin{cases} 0.421783612 \leq \left\langle \mathrm{Tr} A^2 \right\rangle \leq 0.421784687 \\ 0.333341358 \leq \left\langle \mathrm{Tr} A^4 \right\rangle \leq 0.333342131 \end{cases}$$

Exact inequalities! (unlike Monte Carlo):
Increasing Λ will only improve the margins.



Input to SDPA solver: 95 variables. Correlation matrix size (blocks due to symmetry): even-even 683, odd-odd 682, even-odd 1365, relaxation matrix size 8. Still within the capability of a single laptop: ~40 hours CPU time for a single maximization cycle. Can be improved…

Compare to Monte Carlo (N=800): t2 = 0.42179(3) and t4 = 0.33336(5) (4 didgits, 80-85 hours) Roghav Govind Jha '21

Bootstrap for lattice Yang-Mills ($N_c \longrightarrow \infty$)

Anderson, Kruczenski '17
H.Lin 2020

V.K. & Zechuan Zheng
(soon to be published)

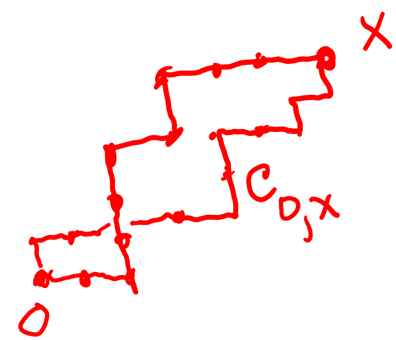
Lattice Yang-Mills $$S = -\frac{N}{2\lambda}\sum_P \mathrm{tr}\, U_P$$

Loop average $$W[C] = \langle \frac{\mathrm{tr}}{N}\prod_{l\in C} U_l\rangle$$

Makeenko-Migdal loop equation $$\sum_{\nu\perp\mu}\left(W[C_{l_\mu}\cdot\overrightarrow{\delta C^{\nu}_{l_\mu}}] - W[C_{l_\mu}\cdot\overleftarrow{\delta C^{\nu}_{l_\mu}}]\right) = \sum_{\substack{l'\in C\\ l'\sim l}} \epsilon_{ll'}W[C_{ll'}]\,W[C_{l'l}]$$

Positivity of correlation matrix $$\langle \mathrm{tr}\,(\mathcal{O}^\dagger\mathcal{O})\rangle \geqslant 0,\ \forall\alpha$$

$$\mathcal{O} = \sum_{C_{0,x}} \alpha[C_{0,x}]\,\Psi[C_{0,x}]\,, \qquad \Psi[C_{0,x}] = \prod_{l\in C_{0,x}} U_l$$



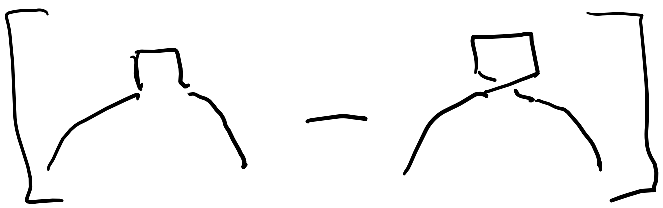
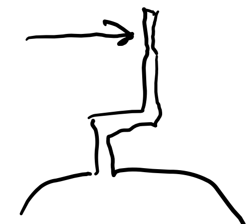
In our forthcoming work with Zechuan Zheng, we significantly modified and improved the bootstrap scheme w.r.t. the pioneering paper of Aderson & Kruczenski.

Our results look encouraging

New features in our bootstrap

V.K. & Zechuan Zheng
(soon to appear)

- We write all loop equations up to maximal length of loops L (including backtracked)

$\sum_{\text{orientations}}$  = non-linear terms 

- Relaxation trick: replace non-linear loop equations by linear inequalities (exactly as in the two-matrix model)

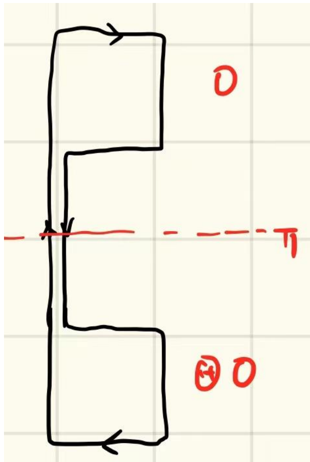
- Lattice symmetries help to block-diagonalize the correlation matrix

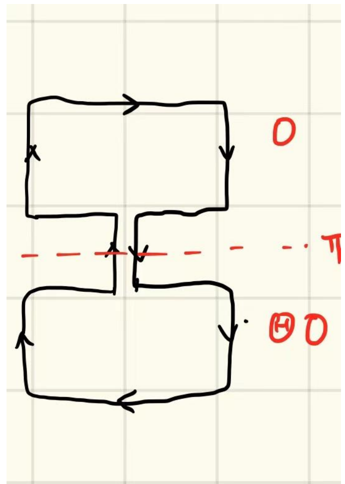
$$\langle \mathrm{tr}\,(g\circ\mathcal{O}_1)^\dagger (g\circ\mathcal{O}_2)\rangle = \langle \mathrm{tr}\,(\mathcal{O}_1)^\dagger (\mathcal{O}_2)\rangle, \quad \forall g\in G' \subset \mathbb{Z}_2\times G_d\times \mathbb{Z}^d$$

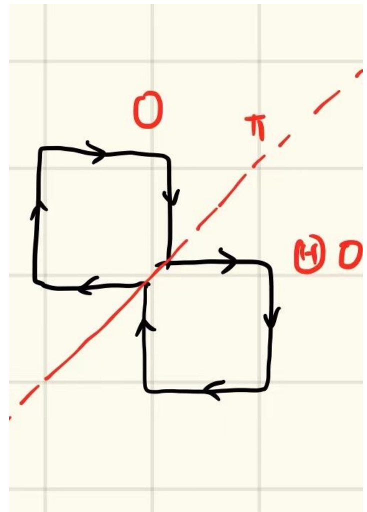
Reflection positivity of correlations

V.K. & Zechuan Zheng
(soon to be published)

Reflection positivity (no dagger): $\langle \mathrm{tr}\,[(\mathcal{O}_+ \big| \cdot (\Theta \circ \mathcal{O}_+)]\rangle \geqslant 0$

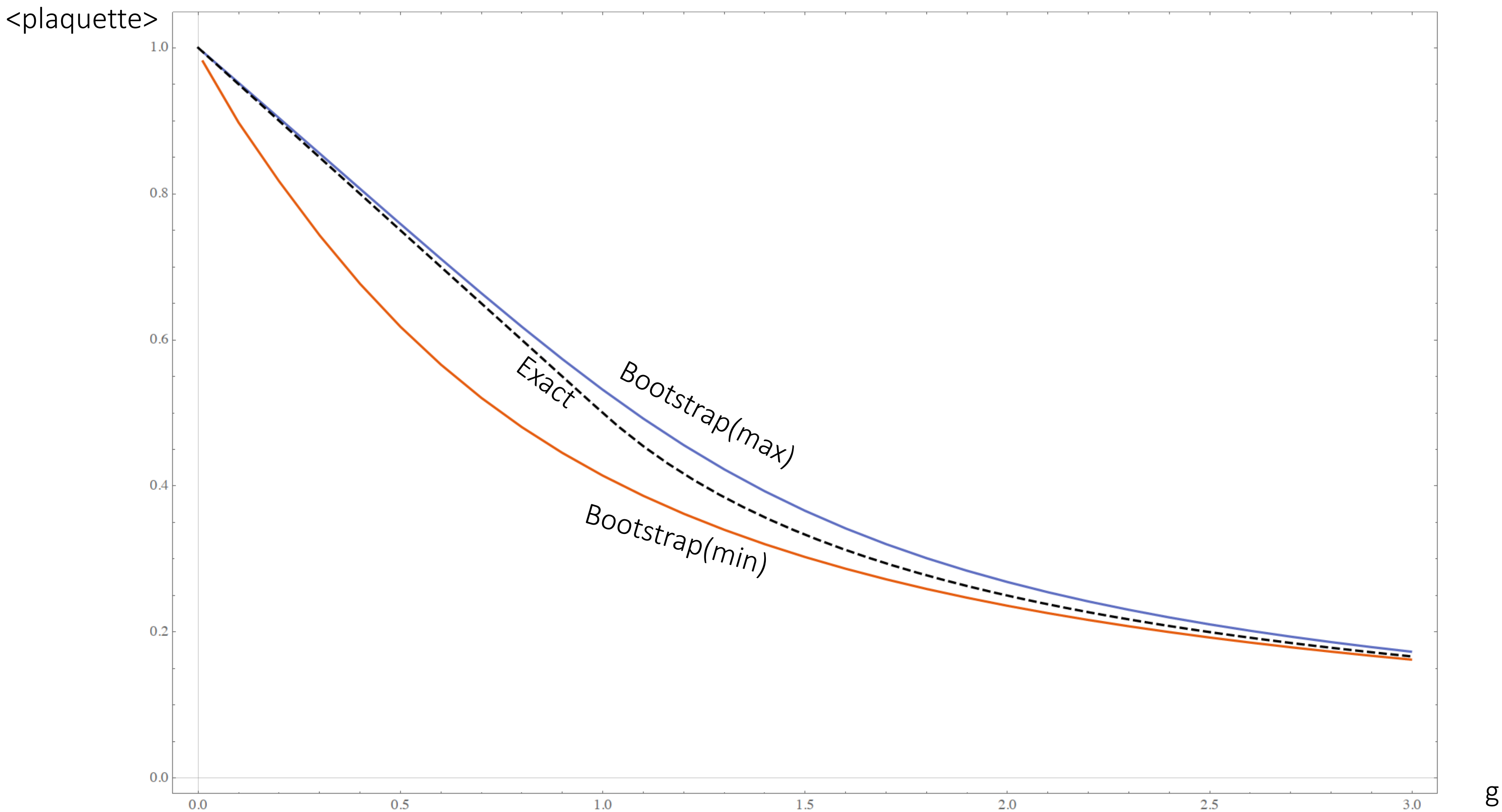




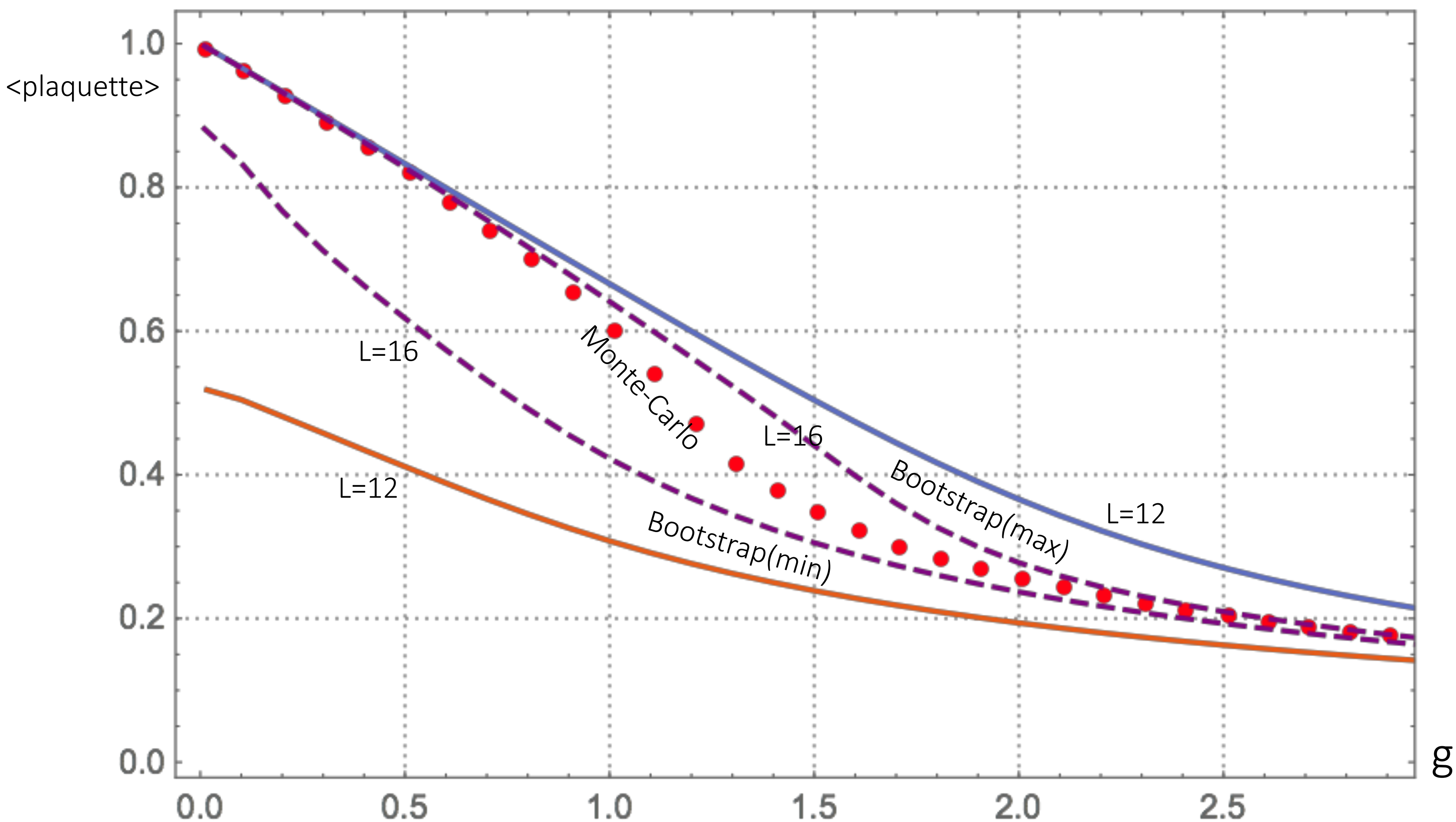


Symmetry reduction of reflection positivity: matrix splits to smaller blocks

Bootstrap for Yang-Mills: <plaquette>(g),   d=2, L=16



Bootstrap for Yang-Mills, <plaquette>(g), d=3; L=12

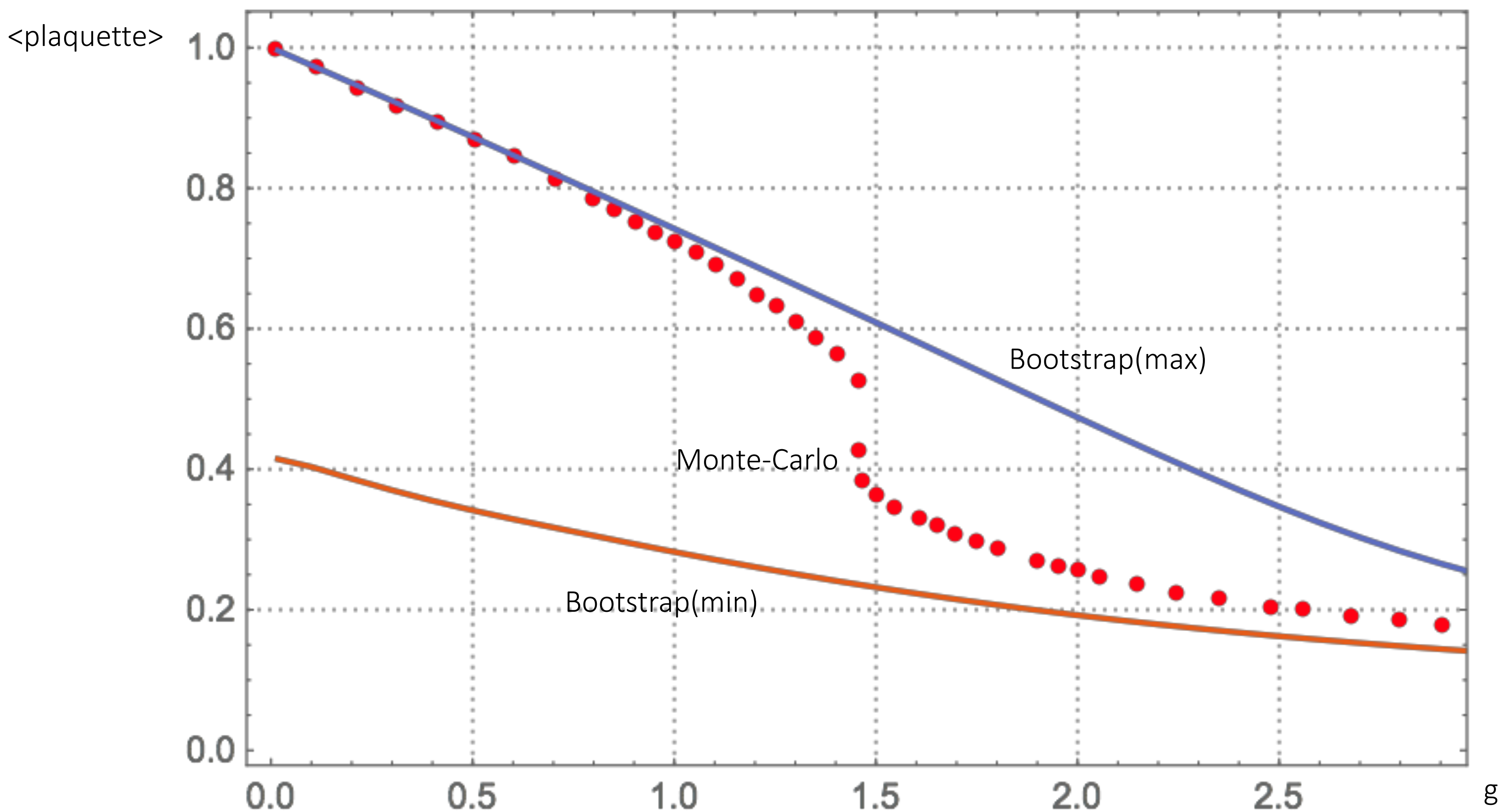


Bootstrap for d=3, L=16

V.K. & Zechuan Zheng
(soon to be published)

Full correlation matrix for the paths C_{00} has the size 6505×6505.
The invariant group G_3×Z_2 has 20 irreducible representations, so the positivity of the original correlation matrix can be reduced to the positivity condition of 20 smaller matrices with sizes:
95, 55, 71, 60, 165, 115, 201, 212, 178, 217, 64, 58, 87, 53, 151, 111, 183, 217, 206, 212

Bootstrap for Yang-Mills, d=4; L=12



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Basic open questions and future problems

- *Can we get a better precision than Monte-Carlo for d=3,4?*
- *Continuum limit? (e.g. Jevicki et al defined a continuum Hamiltonian)*
- *Prospects of computing 1/N corrections? Linear problem!*
- *Finite N bootstrap?*
- *Quarks? We compute all Wilson loops! Sum them with spinorial factor*
- *How to define gauge theory in terms of Wilson loops? Any ideas from the past (history, previous results), present and ideas for the future?*
- *Comments on the idea of the large-N limit as a "classical limit"? (Yaffe) The loop equation at large N is a classical equation that has to be supplemented by the positivity constraints*
- *What is the set of independent loop equations ?*
- *Applications to other physical problems?*

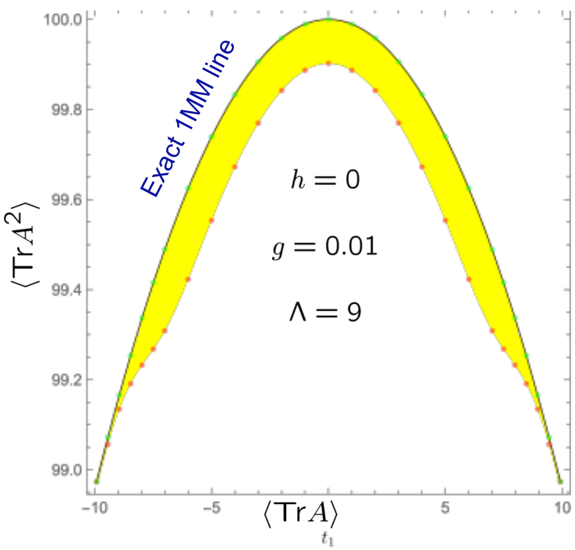
Thank You

Solutions with broken $\mathbb{Z}_2^{\otimes 3}$ symmetry $\langle \mathrm{Tr}A\rangle = \langle \mathrm{Tr}B\rangle \neq 0$

They exist only with negative signs of quadratic terms

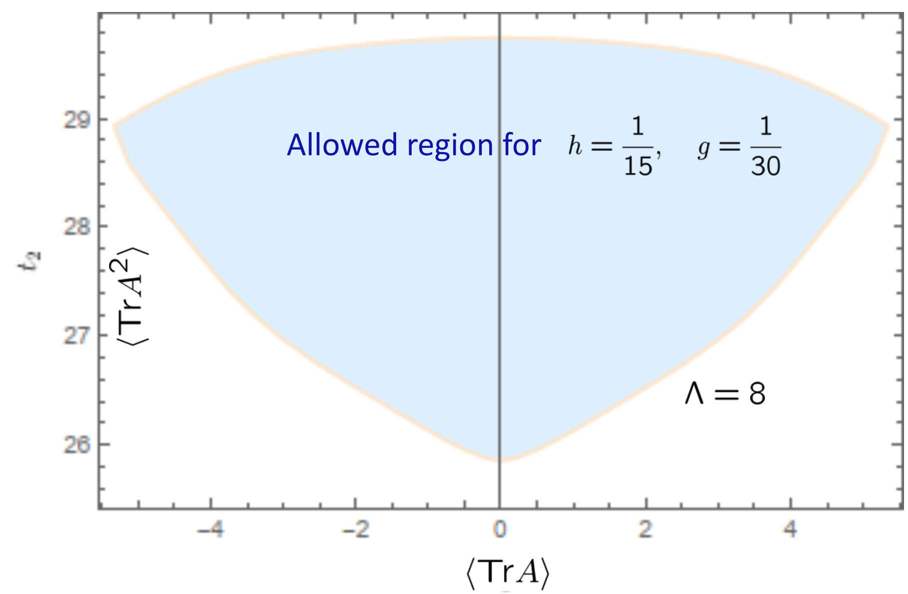
$$V(A,B) = \frac{N}{\hbar}\mathrm{tr}\left(-h[A,B]^2/2 - A^2/2 + gA^4/4 - B^2/2 + gB^4/4\right)$$

h=0: 2 decoupled 1MM's



Maximisation is much closer to the exact 1MM value

Numerical bootstrap for generic parameters



Upper limit seems to be the best estimate
to exact line of symmetry breaking solutions