Wilson loops and RG flow on line defect

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M. Beccaria, S. Giombi, AT   arXiv:1712.06874, 1903.04365, 2110.04212   and to appear

Wilson loop: important observable in $SU(N)$ gauge theory

$$W(C;g,N) = \langle\, \mathrm{P}\exp\left[i\oint_C d\tau\, \dot{x}^\mu A_\mu(x(\tau))\right]\rangle$$

$$\langle ... \rangle = \int [dA]\, e^{iS(g,A)} ...$$

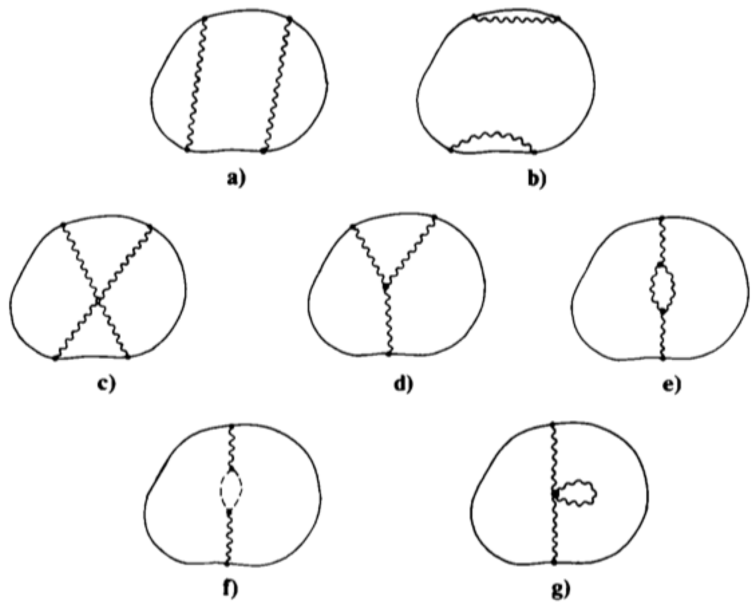
• renormalizable for smooth contours: ∞'s absorbed into g (power div factorize; absent in dim reg)

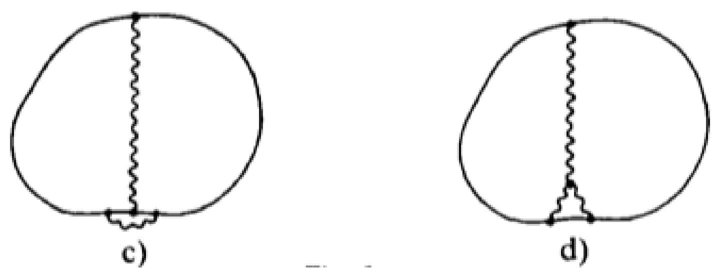
• non-local object: intricate structure of perturbation theory P-ordering + gauge th. averaging $\to$ effective non-local 1d QFT

• reduction of complexity in planar limit $N \to \infty$, $\lambda = g^2 N$=fixed still hard to sum planar diagrams

• much progress in $\mathcal{N} = 4$ SYM from susy, integrability, AdS/CFT can we use this for better understanding WL in YM theory?

• first step: consider standard non-susy WL in SYM will consider simplest regular contours – circle or line





$\mathcal{N} = 4$ SYM:

g not running, WL is conformal operator

• special analog of WL with simpler structure due to susy:

Wilson-Maldacena loop (WML)

$$iA_\mu \dot{x}^\mu \to iA_\mu \dot{x}^\mu + \phi_I |\dot{x}| \theta^I, \quad \theta^2 = 1, \quad I = 1, ..., 6$$

e.g. $\phi_I \theta^I = \phi_6 \equiv \phi$

$$\mathcal{W} = \text{P} \exp \int_C d\tau \left[i\dot{x}^\mu A_\mu(x) + |\dot{x}| \, \phi(x) \right]$$

“locally-supersymmetric", cancellation of many diagrams

• straight line ($\frac{1}{2}$ global susy): $\quad \frac{1}{N} \langle \mathcal{W} \rangle = 1$

• circular loop: exact result due to susy – only ladder diagrams contribute (graphs with int. vertices cancel) [Ericson, Semenoff, Zarembo '00]

$$
\begin{aligned}
\frac{1}{N}\langle \mathcal{W} \rangle_{N\to\infty} = & \quad \frac{2}{\sqrt{\lambda}}\, I_1(\sqrt{\lambda}) \\
\overset{\lambda \ll 1}{=} & \quad 1 + \frac{1}{8}\lambda + \frac{1}{192}\lambda^2 + \cdots \\
\overset{\lambda \gg 1}{=} & \quad \sqrt{\frac{2}{\pi}}\, \frac{e^{\sqrt{\lambda}}}{(\sqrt{\lambda})^{3/2}}\, \left(1 - \frac{3}{8\sqrt{\lambda}} + \cdots\right)
\end{aligned}
$$

also for any N: [Drukker, Gross '00; Pestun '07]

$$
\langle \mathcal{W} \rangle = N e^{\frac{\lambda}{8N}}\, L^1_{N-1}\left(-\frac{\lambda}{4N}\right)
$$

• what about standard WL? also conformal – finite for smooth contours

• circular WML had two remarkable simplifications:
(i) only ladder graphs contribute
(ii) effective propagator on line $\langle (iA+\phi)(iA+\phi)\rangle$=const
$\rightarrow$ loop equation for ladders can be solved explicitly

• no longer true for standard WL
even summing planar ladders non-trivial

• 1-parameter family of interpolating Wilson loops:
WL ($\zeta = 0$) and WML ($\zeta = 1$) [Polchinski, Sully '11]

$$\mathcal{W}^{(\zeta)}(C; g, N) = \mathrm{Tr}\,\mathrm{P}\,\exp \oint_C d\tau \left[i\, A_\mu(x)\, \dot{x}^\mu + \zeta\, \phi(x)\, |\dot{x}| \right]$$

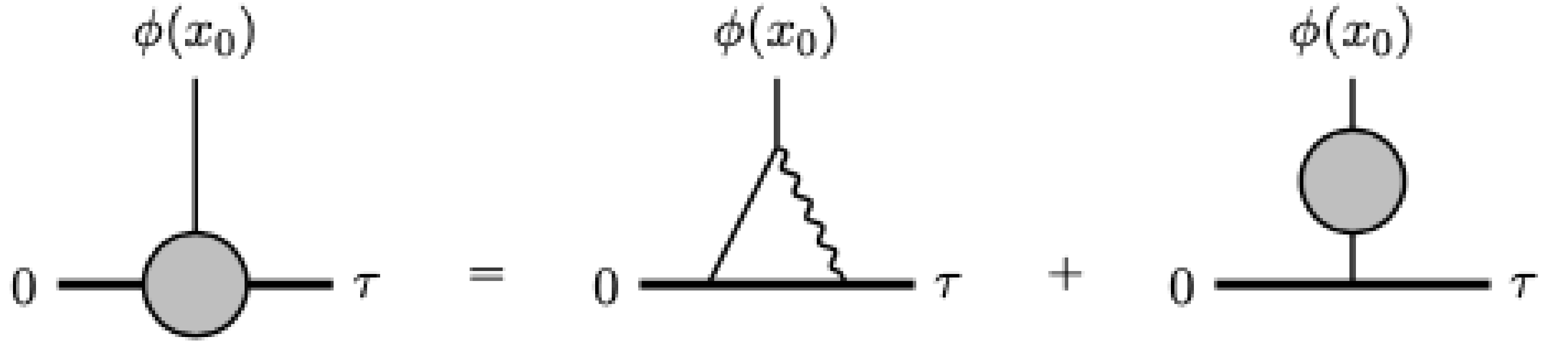
not UV finite for $\zeta \neq 1, 0$: need to renormalize ζ

$$\langle \mathcal{W}^{(\zeta)} \rangle \equiv W(\lambda; \zeta(\mu), \mu)\,, \qquad \left(\mu \frac{\partial}{\partial \mu} + \beta_\zeta \frac{\partial}{\partial \zeta} \right) W = 0$$

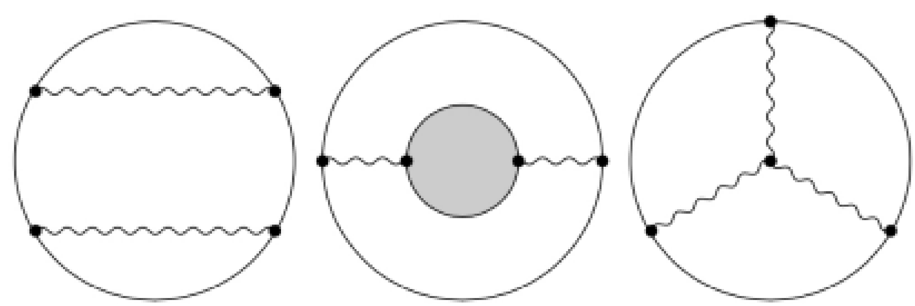
$$\beta_\zeta = \mu \frac{d\zeta}{d\mu} = -\frac{\lambda}{8\pi^2} \zeta (1 - \zeta^2) + \mathcal{O}(\lambda^2)$$

$\zeta = 0$ (WL) and $\zeta = 1$ (WML):
UV and IR conformal points
[cf. 1d QFT: conf. pert. th. by $\mathcal{O} = \zeta \phi$ near UV point $\zeta = 0$]

• one-loop β_ζ function:



• circular Wilson loop $W(\zeta)$ (all possible propagators):



• $\langle \mathcal{W}^{(\zeta)} \rangle$ on circle non-trivial function of ζ [Beccaria, Giombi, AT '17]

$$\mathrm{W}^{(\zeta)} \equiv \tfrac{1}{N}\langle \mathcal{W}^{(\zeta)} \rangle = 1 + \frac{1}{8}\lambda + \Big[\frac{1}{192} + \frac{1}{128\,\pi^2}(1-\zeta^2)^2\Big]\lambda^2 + \mathcal{O}(\lambda^3)$$

interpolates between WML at $\zeta = 1$ and WL at $\zeta = 0$:

$$\mathrm{W}^{(0)} = 1 + \frac{1}{8}\lambda + \Big(\frac{1}{192} + \frac{1}{128\,\pi^2}\Big)\lambda^2 + \mathcal{O}(\lambda^3)$$

• finiteness of 2-loop λ^2 term related to no ζ in 1-loop λ term
UV logs first appear at λ^3 order

• conf points $\zeta = 1$ and $\zeta = 0$ are extrema of $\mathrm{W}^{(\zeta)}$:

$$\frac{\partial}{\partial\zeta}\log \mathrm{W}^{(\zeta)} = \mathcal{C}\,\beta_\zeta$$

$$\beta_\zeta = -\tfrac{\lambda}{8\pi^2}\zeta(1-\zeta^2) + \dots\,, \qquad \mathcal{C} = \tfrac{1}{4}\lambda + \dots > 0$$

• $\mathrm{W}^{(\zeta)}$ as a 1d QFT partition function Z_{S^1} on S^1:
$d = 1$ case of relation $\frac{\partial F}{\partial g_i} = \mathcal{C}^{ij}\beta_j, \quad F = -\log Z_{S^d}$
• ζ marginally relevant: flow from $\zeta = 0$ in UV to $\zeta = 1$ in IR
• $\mathrm{W}^{(0)} \ > \ \mathrm{W}^{(1)}$ consistent with
F-theorem in $d = 1$ [Klebanov, Safdi, Pufu '11] and
g-theorem for line defects [Cuomo, Komargodski, Raviv-Moshe '21]
• flow driven by $\mathcal{O} = \phi(\tau)$: $\frac{\partial}{\partial\zeta}\mathrm{W}^{(\zeta)}\big|_{\zeta=0,1} = 0 \rightarrow$
$\langle\mathcal{O}\rangle\big|_{\zeta=0,1} = 0, \qquad \langle\mathcal{O}(\tau)\,\mathcal{O}(0)\rangle\big|_{\zeta=0,1} = \frac{1}{|\tau|^{2\Delta^{(0,1)}}}$
• anom dim of ϕ at $\zeta = 1$ and $\zeta = 0$ conf. points:
$\frac{\partial\beta_\zeta}{\partial\zeta}\big|_{\zeta=0,1} \ \rightarrow \ \Delta^{(1)} = 1 + \frac{\lambda}{4\pi^2} + \dots\,, \qquad \Delta^{(0)} = 1 - \frac{\lambda}{8\pi^2} + \dots$

- line defect entropy [Cuomo, Komargodski, Raviv-Moshe '21]

$$\mathrm{s} \equiv \left(1 - a\frac{\partial}{\partial a}\right) \log \mathrm{W} = \left(1 + \mu\frac{\partial}{\partial \mu}\right) \log \mathrm{W}$$

$$\left(\mu\tfrac{\partial}{\partial\mu} + \beta_\zeta\tfrac{\partial}{\partial\zeta}\right)\, \log \mathrm{W} = 0, \qquad \tfrac{\partial}{\partial\zeta}\, \log \mathrm{W} = \mathcal{C}\,\beta_\varkappa, \qquad \mathcal{C} > 0 \quad \rightarrow$$

$$\mathrm{s} = \log \mathrm{W} + \mathcal{C}\,\beta_\varkappa\beta_\varkappa \;>\; \log \mathrm{W}$$

$$\log \mathrm{W}^{(\mathrm{UV})} > \log \mathrm{W}^{(\mathrm{IR})} \quad \rightarrow \quad \mathrm{s}^{(\mathrm{UV})} > \mathrm{s}^{(\mathrm{IR})}$$

- it is s that is expected to be monotonic along the RG flow

Higher loop corrections to β_ζ

- structure of β_ζ-function in planar pert. th.: $\lambda = g^2 N \ll 1$

$$\beta_\zeta = -\tfrac{1}{8\pi^2}\,\lambda\,\zeta(1-\zeta^2) + \lambda^2\,\zeta\,(1-\zeta^2)\,(b_2 + b_3\,\zeta^2)$$
$$+\lambda^3\,\zeta\,(1-\zeta^2)\,(b_4 + b_5\,\zeta^2 + b_6\,\zeta^4) + \mathcal{O}(\lambda^4)$$

b_2, b_3 are scheme-independent

- computed from WL vertex renorm. [Dotsenko, Vergelis '80]
- highest ζ^{2n+1} powers at each λ^n order $(b_1, b_3, b_6, \dots)$

come from diagrams with max. number of scalar propagators attached to line (no internal vertices) – from scalar ladders

- described by effective ladder model with free bulk scalar ϕ

$$\mathrm{W}(g, N; \zeta) = \langle \mathrm{Tr\,P\,exp}\Big[\zeta \int d\tau\, \phi(\tau)\Big]\rangle, \qquad \phi(\tau) = \phi(x(\tau))$$

$$\langle ... \rangle = \int [d\phi]\ \exp\Big[-\frac{1}{g^2}\int d^4x\,(\partial_\mu\phi)^2\Big] ... \,, \quad \langle \phi(x)\phi(0)\rangle \sim \frac{1}{x^2}$$

• ladder terms: from planar loop equation [BGT '21]

$$\beta_\zeta^{\rm ladd} = q_1 \frac{\lambda}{4\pi^2} \zeta^3 + q_2 \left(\frac{\lambda}{4\pi^2}\right)^2 \zeta^5 + q_3 \left(\frac{\lambda}{4\pi^2}\right)^3 \zeta^7 + q_4 \left(\frac{\lambda}{4\pi^2}\right)^4 \zeta^9 + q_5 \left(\frac{\lambda}{4\pi^2}\right)^5 \zeta^{11} + \cdots ,$$

$$q_1 = \frac{1}{2}, \qquad q_2 = -\frac{1}{4}, \qquad q_3 = \frac{1}{4} - \frac{\zeta_2}{8}, \quad q_4 = -\frac{17}{48} + \frac{\zeta_2}{3} - \frac{\zeta_3}{12}$$

$$q_5 = \frac{29}{48} - \frac{37\,\zeta_2}{48} + \frac{29\,\zeta_3}{96} + \frac{25\,\zeta_4}{128}, \qquad b_{\frac{n(n+1)}{2}} = -\frac{1}{(4\pi^2)^n} q_n$$

• fixes $b_3 = \frac{1}{4}\frac{1}{(4\pi^2)^2}$; can fix combinations of b_n coefficients from relation between β_ζ and anom. dim. of scalar ϕ

• $\Delta^{(1)}$ from diagrams for cusp line [Bruser,Henn 2018] (2-loop) or from quantum spectral curve [Grabner, Gromov, Julius 2020] (several higher loop orders and interpolation to large λ)

$$\Delta^{(1)} = 1 + \frac{\lambda}{4\pi^2} - \left(\frac{\lambda}{4\pi^2}\right)^2 + d_3 \left(\frac{\lambda}{4\pi^2}\right)^3 + d_4 \left(\frac{\lambda}{4\pi^2}\right)^4 + \ldots$$

$$d_3 = 2 - \frac{7\,\zeta_4}{4}, \qquad d_4 = -5 + \zeta_2 + \frac{\zeta_3}{2} - \frac{\zeta_3\zeta_2}{2} - \frac{5\,\zeta_5}{8} + \frac{119\,\zeta_6}{16}, \ldots$$

• implies $b_2 = \frac{1}{4}\frac{1}{(4\pi^2)^2}$ $\to$ explicit form of 2-loop β_ζ [BGT '21]

$$\beta_\zeta = -\frac{\lambda}{8\pi^2}\,\zeta\,(1-\zeta^2) + \frac{\lambda^2}{64\pi^4}\,\zeta(1-\zeta^4) + \mathcal{O}(\lambda^3)$$

• leads in turn to new result: dim of ϕ inserted into WL

$$\Delta^{(0)} = 1 - \frac{\lambda}{8\pi^2} + \frac{\lambda^2}{64\pi^4} + \mathcal{O}(\lambda^3)$$

• sign-alternating λ-series: consistent with planar expansion having finite radius of convergence $|\frac{\lambda}{4\pi^2}| = 1$

• strong-coupling $\lambda \gg 1$:

[Alday, Maldacena '07; Giombi, Roiban, AT '17; Beccaria, Giombi, AT '17]

$$\Delta^{(1)} = 2 - \frac{5}{\sqrt{\lambda}} + \mathcal{O}(\frac{1}{\lambda}), \qquad \Delta^{(0)} = \frac{5}{\sqrt{\lambda}} + \mathcal{O}(\frac{1}{\lambda})$$

[Grabner, Gromov, Julius '20] found several higher-order terms in strong-coupling expansion in $\Delta^{(1)} = 2 - \frac{5}{\sqrt{\lambda}} + ...$ also from bootstrap approach [Ferrero, Meneghelli ' 20]]

Higher loop corrections to $\mathrm{W}^{(\zeta)}$

- planar expansion of $\mathrm{W}^{(\zeta)} = \langle \mathcal{W}^{(\zeta)} \rangle$ on a circle: [BGT '17; BT '18]

$$\mathrm{W}^{(\zeta)} = \mathrm{W}^{(1)} \Big[1 + \tfrac{1}{128\pi^2}\, \lambda^2\, (1-\zeta^2)^2 + \lambda^3\, (1-\zeta^2)^2 (w_2 + w_3\, \zeta^2) + \cdots \Big]$$

$$\mathrm{W}^{(1)} = \tfrac{2}{\sqrt{\lambda}}\, I_1(\sqrt{\lambda}) = 1 + \tfrac{\lambda}{8} + \tfrac{\lambda^2}{192} + \mathcal{O}(\lambda^3)$$

- $\frac{\partial}{\partial\zeta} \log \mathrm{W}^{(\zeta)} = \mathcal{C}\, \beta_\zeta, \qquad \mathcal{C} = \tfrac{1}{4}\lambda + \lambda^2 (c_2 + c_3 \zeta^2) + \mathcal{O}(\lambda^3)$
- w_3 from scalar ladders (radius of circle=1) [BT '18]

$$w_3 = -\tfrac{1}{256\pi^4} \big(\log \mu + \tfrac{5}{6} \big)$$

$\log \mu$ related to 1-loop beta-function; $\frac{5}{6}$ is scheme dep. const

- w_2 scheme-independent (currently unknown)

Details: scalar ladder contributions

• highest ζ terms at each order of λ expansion:
ladder graphs with only scalar propagators attached to line

• single effective coupling $\xi \equiv \lambda\,\zeta^2$

$$W(\xi) = \langle \mathcal{W}^{(\zeta)} \rangle^{\text{ladder}} = \Big\langle \text{Tr}\,\text{P}\,\exp \int d\tau\,\phi(\tau) \Big\rangle$$

$$\langle ... \rangle = \int d\phi\, e^{-S} ... \,, \quad S = \frac{N}{\xi} \int d^4x\, \text{Tr}(\partial_\alpha \phi \partial^\alpha \phi)\,, \quad \xi \equiv \lambda\,\zeta^2$$

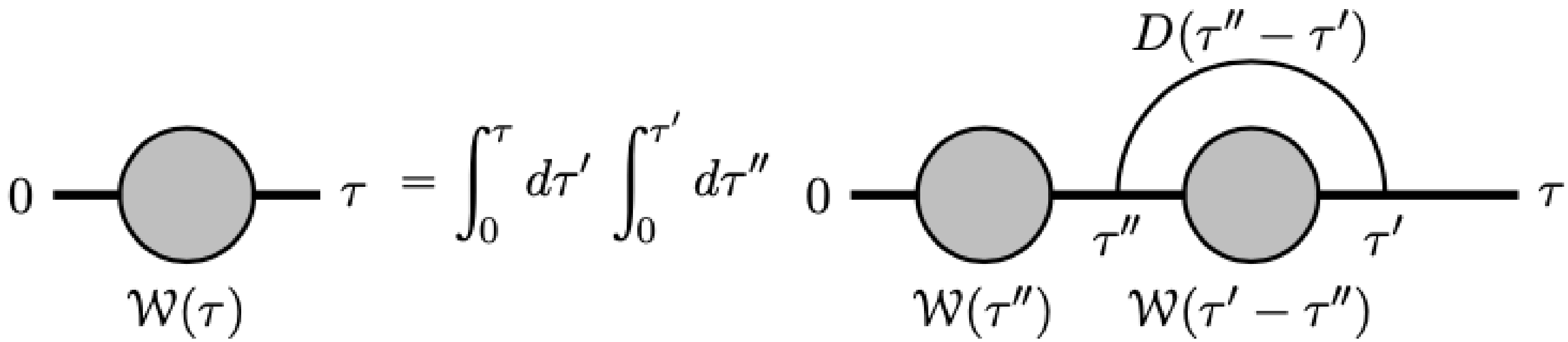
• effective propagator $\langle \phi(\tau)\,\phi(\tau') \rangle = \xi\, D(\tau - \tau')$

circle: $D(\tau) = \frac{1}{8\pi^2} \frac{1}{4\,\sin^2 \frac{\tau}{2}}$, line: $D(\tau) = \frac{1}{8\pi^2} \frac{1}{\tau^2}$

• renormalizable expansion of W: log div absorbed in ξ

• in planar limit $W(\tau) = \frac{1}{N}\langle \mathrm{Tr\,P\,exp} \int_0^\tau d\tau' \, \phi(\tau')\rangle$
obeys integral "loop equation" [Zarembo]

$$\frac{\partial \mathcal{W}(\tau)}{\partial \tau} = \xi \int_0^\tau d\tau' \, \mathcal{W}(\tau') \, \mathcal{W}(\tau - \tau') \, D(\tau - \tau')$$



• WML: $\langle [iA^a(\tau) + \phi^a(\tau)] \, [iA^b(\tau') + \phi^b(\tau')] \rangle = \delta^{ab} \, \frac{\lambda}{8\pi^2 \, N} =$ const

• for non-constant $D(\tau)$ exact solution is hard
but can be used to compute W in expansion in ξ

• to compute renormalization of ζ

may also use vertex renormalization method [Dotsenko, Vergelis '80]

in the planar limit

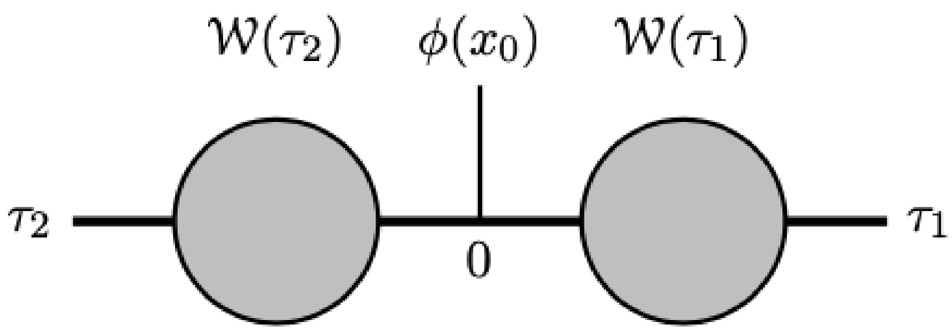
$$\langle \mathrm{Tr} \left[\phi(\tau_0)\, \mathrm{P} \exp \int_{\tau_1}^{\tau_2} d\tau \phi(\tau) \right] \rangle$$

$$= \int_{\tau_1}^{\tau_2} d\tau \, W(\tau - \tau_1) \, D(\tau_0 - \tau) \, W(\tau_2 - \tau)$$

and study renormalization of "vertex function"

$$V = \xi \, W(\tau_1) \, W(\tau_2) \, , \qquad \xi = \lambda \, \zeta^2$$

• relevant collection of diagrams ($\tau_0 = 0, \ \tau = x_0$)



• find divergent part of V and renormalize ξ or equivalently ζ

Alternative:

• cast problem of renormalization of ζ in standard QFT form representing P-ordering by functional integral over 1d fermions $\psi_i(\tau)$ in fundamental rep: $(\partial_\tau)^{-1} = \theta(\tau)$ [Gervais, Neveu '80]

• integrating over free adjoint bulk scalar ϕ
get effective non-local 1d action with coupling $\xi = \lambda\zeta^2$

$$S = \int d\tau\, \bar{\psi}_i \partial_\tau \psi_i + \xi \int d\tau\, d\tau'\, \bar{\psi}_j(\tau)\psi_i(\tau)\, \frac{1}{|\tau - \tau'|^2}\, \bar{\psi}_i(\tau')\psi_j(\tau')$$

• introducing cutoff (e.g. $\frac{1}{(|\tau-\tau'|+\varepsilon)^2}$) compute β_ξ and thus β_ζ

Generalization to higher reps and finite N

- so far Tr was in fund. rep of $SU(N)$ and planar limit

general rep. R of $SU(N)$ and finite N?

- for circular WML ($\zeta = 1$) in rep R in $\mathcal{N} = 4$ SYM

(normalizing on dim R) [Fiol, Martinez-Montoya, Fukelman '18]

$$\mathrm{W}^{(1)} = 1 + C_{\mathrm{R}} \frac{g^2}{4} + \Big(C_{\mathrm{R}}^2 - \frac{1}{6} C_{\mathrm{R}} C_{\mathrm{A}}\Big) \frac{g^4}{32} + \Big(C_{\mathrm{R}}^3 - \frac{1}{2} C_{\mathrm{R}}^2 C_{\mathrm{A}} + \frac{1}{12} C_{\mathrm{R}} C_{\mathrm{A}}^2\Big) \frac{g^6}{384} + \dots$$

$T^a T^a = C_{\mathrm{R}}\, \dim \mathrm{R}$; $SU(N)$: adj. $C_{\mathrm{A}} = N$, fund. $C_{\mathrm{F}} = \frac{N^2-1}{2N}$

- for any ζ [BGT '22]

$$\mathrm{W}^{(\zeta)} = \mathrm{W}^{(1)} \Big[1 + C_{\mathrm{R}} C_{\mathrm{A}} (1-\zeta^2)^2 \frac{g^4}{64\pi^2} + (1-\zeta^2)^2 (\mathrm{w}_2 + \mathrm{w}_3 \zeta^2)\, g^6 \Big] + \dots$$

$$\mathrm{w}_3 = -\frac{1}{128\pi^4} C_{\mathrm{R}} C_{\mathrm{A}}^2 \big(\log \mu + \frac{5}{6}\big)$$

- structure of ladder part of β_ζ:

$$\beta_\zeta^{\rm ladd} = q_1'\, C_A \zeta^3 \frac{g^2}{4\pi^2} + (q_2'\, C_A^2 + q_2''\, C_A C_R)\, \zeta^5 \left(\frac{g^2}{4\pi^2}\right)^2 + \left(q_3'\, C_A^3 + q_3''\, C_A^2 C_R + q_3'''\, C_A C_R^2 + q_3''''\, Q_R\right) \zeta^7 \left(\frac{g^2}{4\pi^2}\right)^3 + \ldots$$

$$Q_R \equiv \frac{d_A^{abcd} d_R^{abcd}}{C_R\, \dim R}\,, \qquad d_R^{abcd} = {\rm Str}(T^a T^b T^c T^d)$$

- dependence on R starts at three loops only

$$\beta_\zeta^{\rm ladd} = \frac{1}{2}\, C_A \zeta^3 \frac{g^2}{4\pi^2} - \frac{1}{4}\, C_A^2\, \zeta^5 \left(\frac{g^2}{4\pi^2}\right)^2 + \left[\frac{1}{4}\, C_A^3 - \frac{\pi^2}{2}\, Q_R\right] \zeta^7 \left(\frac{g^2}{4\pi^2}\right)^3 + \ldots$$

- consistent with $\left(\mu \frac{\partial}{\partial\mu} + \beta_\zeta \frac{\partial}{\partial\zeta}\right) W^{(\zeta)} = 0$
- to fix coeffs. consider special case:

R = *k*-symmetric rep S_k of $SU(N)$

$$C_{S_k} = \frac{k(N-1)(N+k)}{2N}, \qquad \dim S_k = \frac{(N+1-k)!}{k!\,(N-1)!}$$

1d path integral for scalar ladder WL in k-symm rep

[interesting 1d model: relation to RG flow on line defects]

- Tr in k-symm. rep: use N periodic 1d bosons $\chi_i(\tau)$

(e.g. [Gomis, Passerini '06])

$$W = \langle W_k \rangle, \qquad W_k = \int [d\chi\, d\bar\chi]\, \delta(\bar\chi\chi - R^2)\, e^{-S}$$

$$S = \int_0^{2\pi} d\tau \Big[\bar\chi\, \partial_\tau \chi + \zeta\, \phi^a(\tau)\, \bar\chi\, T^a\, \chi \Big], \qquad R^2 \equiv k + \tfrac{1}{2} N$$

$$\langle ... \rangle = \int [d\phi] \exp \Big[-\frac{1}{g^2} \int d^4x\, \mathrm{Tr}(\partial_\alpha \phi)^2 \Big] ... , \qquad \phi(\tau) = \phi(x(\tau))$$

- integrating out ϕ $\rightarrow$ interacting non-local 1d theory

$$S = \int d\tau\, \bar\chi\, \partial_\tau \chi - \zeta^2 g^2 \int d\tau\, d\tau'\, D(\tau - \tau')\, \bar\chi(\tau) T^a \chi(\tau)\, \bar\chi(\tau') T^a \chi(\tau')$$

$$D(\tau - \tau') = \langle \phi(\tau)\, \phi(\tau') \rangle_{\text{circle}} \sim \frac{1}{[2 \sin \frac{\tau - \tau'}{2}]^2} \text{ or } \sim \frac{1}{(\tau - \tau')^2} \text{ on line}$$

• rescaling χ by R: constraint $\bar{\chi}_i \chi_i = 1$

$$S = R^2 \Big[\int d\tau\, \bar{\chi}\, \partial_\tau \chi - \varkappa \int \frac{d\tau\, d\tau'}{(\tau-\tau')^2}\, \bar{\chi}(\tau) T^a \chi(\tau)\, \bar{\chi}(\tau') T^a \chi(\tau') \Big]$$

$$\varkappa \equiv \frac{\zeta^2 g^2 R^2}{8\pi^2} = \frac{\zeta^2 g^2 (k + \frac{1}{2}N)}{8\pi^2}$$

• develop large R^2 or large k pert. theory at fixed $\varkappa$ and N

• in particular UV cutoff scheme [BGT '22]

$$\beta_\varkappa = \mu \frac{d\varkappa}{d\mu} = \frac{2N}{R^2} \frac{\varkappa^2}{1+\pi^2\varkappa^2} - \frac{2N^2}{R^4} \frac{\varkappa^3 (1-\pi^2\varkappa^2)}{(1+\pi^2\varkappa^2)^3} + \mathcal{O}\big(\frac{1}{R^6}\big)$$

• as g and R^2 not running, $\beta_\varkappa$ is directly related to $\beta_\zeta^{\rm ladd}$:
$\varkappa$ expansion $\to$ agreement with 1-loop ζ^3 and 2-loop ζ^5
and allows to fix coeff of 3-loop term in $\beta_\zeta^{\rm ladd}$

• renormalized value of ladder WL on circle of radius a (normalized to $\dim S_k = \frac{(N+k-1)!}{(N-1)!\,k!}$)

$$\mathrm{W} = (1+\pi^2\varkappa^2)^{\frac{N-1}{2}} \Big[1 + \frac{\mathrm{v}_1}{R^2}\frac{N(N-1)\,\varkappa^3}{(1+\pi^2\varkappa^2)^2} + \mathcal{O}(\frac{1}{R^4})\Big]$$

$$\mathrm{v}_1 = -2\pi^2\big[\log(\mu\,a) + \tfrac{5}{6}\big]$$

$(1+\pi^2\varkappa^2)^{1/2}$ in $SU(2)$ case found earlier [Komargodski et al '21]

• $\log\mu$ term directly related to one-loop $\beta_\varkappa$

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta_\varkappa\frac{\partial}{\partial\varkappa}\Big)\,\mathrm{W} = 0\,,$$

$$\frac{\partial}{\partial\varkappa}\log\mathrm{W} = \mathcal{C}\,\beta_\varkappa\,, \qquad\qquad \mathcal{C} = \frac{(N-1)\pi^2R^2}{2N\varkappa} + ... > 0$$

- RG flow: large k, fixed $\varkappa$ expansion

$$\frac{d\varkappa}{dt} = \frac{2N}{k}\frac{\varkappa^2}{1+\pi^2\varkappa^2} + ..., \qquad t = \log\mu$$

$$\varkappa(t) = \gamma t + \tfrac{1}{\pi}\sqrt{1+\pi^2\gamma^2 t^2}, \qquad \gamma \equiv \tfrac{N}{\pi^2 k}$$

$$\text{IR}: \quad \varkappa(t\to-\infty) \;\to\; \tfrac{1}{2\pi^2\gamma|t|} \to 0$$

$$\text{UV}: \quad \varkappa(t\to+\infty) \;\to\; 2\gamma t \to \infty$$

- asymptotics stable under $1/k$ corrections: for exact $\beta_\varkappa$

$$\beta_\varkappa\big|_{\varkappa\to 0} \to 0\,, \qquad \beta_\varkappa\big|_{\varkappa\to\infty} \to \tfrac{2N}{k+\frac{1}{2}N} = \text{const}$$

- IR: $\mathrm{W}\big|_{\varkappa\to 0} \to 1$

  UV: $\mathrm{W}\big|_{\varkappa\to\infty} \to \varkappa^{N-1} \to \infty$

$$\log \mathrm{W}^{(\mathrm{UV})} > \log \mathrm{W}^{(\mathrm{IR})}$$

- consistent with 1d F-theorem for W as part. f. on S^1

- g-theorem for line defects [Cuomo, Komargodski, Raviv-Moshe '21]

$$\mathrm{s} \equiv (1 - a\frac{\partial}{\partial a}) \log \mathrm{W} = (1 - \mu\frac{\partial}{\partial \mu}) \log \mathrm{W}$$

$$(\mu\tfrac{\partial}{\partial\mu} + \beta_\varkappa\tfrac{\partial}{\partial\varkappa})\ \log \mathrm{W} = 0, \quad \tfrac{\partial}{\partial\varkappa} \log \mathrm{W} = \mathcal{C}\,\beta_\varkappa, \quad \mathcal{C} > 0 \quad \to$$

$$\mathrm{s} = \log \mathrm{W} + \mathcal{C}\,\beta_\varkappa\beta_\varkappa \ > \ \log \mathrm{W}\,, \qquad\qquad \mathrm{s}\big|_{\beta_\varkappa=0} = \log W\big|_{\beta_\varkappa=0}$$

$$\log \mathrm{W}^{(\mathrm{UV})} > \log \mathrm{W}^{(\mathrm{IR})} \ \to \ \mathrm{s}^{(\mathrm{UV})} > \mathrm{s}^{(\mathrm{IR})}$$

$$\mathrm{s}(\varkappa) = \tfrac{1}{2}(N-1)\log(1+\pi^2\varkappa^2) + \tfrac{2N(N-1)}{k}\tfrac{\pi^2\varkappa^3}{(1+\pi^2\varkappa^2)^2} + \mathcal{O}(\tfrac{1}{k^2})$$

$\mathrm{s}(\varkappa)$ is monotonic function on RG trajectory:
decreases towards IR

Wilson-Fisher fixed points:

1d defect line in bulk scalar theory in $d = 4 - \epsilon$ dim:
coupling g^2 and thus $\varkappa \sim g^2 \zeta^2 k$ get dim $\epsilon \to 0$

- in addition to trivial $\varkappa = 0$ two new fixed points

$$\beta_\varkappa = -\epsilon\,\varkappa + \frac{2N}{k}\frac{\varkappa^2}{1+\pi^2\varkappa^2} + \mathcal{O}\big(\frac{1}{k^2}\big)$$

$$\beta_\varkappa = 0 : \qquad \varkappa_\pm = \frac{N}{\pi^2 k\,\epsilon}\Big(1 \pm \sqrt{1 - \frac{2\pi^2 k^2\,\epsilon^2}{N^2}}\Big) + \mathcal{O}\big(\frac{1}{k^2}\big)$$

$$\text{UV} : \quad \varkappa_+ = \frac{2N}{\pi^2 k}\frac{1}{\epsilon} \to \infty$$

$$\text{IR} : \quad \varkappa_- = \frac{k}{2N}\epsilon \to 0$$

- fixed points are again stable under higher corrections to $\beta_\varkappa$
- values of W at fixed points? need to include ϵ-corrections

Details about k-symmetric representation

• periodic bosons χ^i in fundamental representation of $SU(N)$

operator quantization: $[\chi^j, \bar{\chi}_i] = \delta_i^j$,

$Z = \mathrm{tr}_\chi [\mathrm{T} \exp i \int d\tau \, \mathcal{H}(\tau)]$, $\mathcal{H}(\tau) \equiv \hat{\phi} = -i \, \bar{\chi} \phi \chi$, $\mathrm{T} = \mathrm{P}$

$$Z = \mathrm{tr}_\chi \left[\mathrm{P} \exp \left(i \int_0^{2\pi} d\tau \, \hat{\phi}(\tau) \right) \right]$$

state space: from $\chi^i |0\rangle = 0$ acting with $\bar{\chi}_i$.

Z = sum of partition functions in sectors with $\nu = \bar{\chi}_i \chi^i$ = fixed

$\nu = k$: $\bar{\chi} T^a \chi \to T^a$ in k-symm. rep. [Gomis, Passerini '06]

$$Z = \sum_{k=0}^{\infty} W_k \, , \qquad W_k = \mathrm{Tr}_k \, \mathrm{P} \exp \left(\int_0^{2\pi} d\tau \, \phi(\tau) \right)$$

$$Z = \int D\chi D\bar{\chi} \, e^{i \int_0^{2\pi} d\tau \mathscr{L}} , \qquad \mathscr{L} = i \, \bar{\chi} \partial_\tau \chi + i \, \phi^a(\tau) \bar{\chi} T^a \chi$$

• select W_k: constraint on $\bar{\chi}\chi$ (shift $\frac{1}{2}N$ due to Weyl ordering)

$$\mathcal{L} = i\,\bar{\chi}\partial_\tau\chi + i\bar{\chi}\phi(\tau)\chi + A(\bar{\chi}\chi - R^2)\,, \qquad R^2 \equiv k + \tfrac{1}{2}N$$

Free theory: normalization of the measure

$$W_{k,0} = \int D\chi D\bar{\chi}\, e^{-\int d\tau\, \bar{\chi}\partial_\tau\chi}\, \delta(\bar{\chi}\chi - R^2) = \dim \mathrm{S}_k = \tfrac{(N+k-1)!}{k!\,(N-1)!}$$

• use “Lagrange multiplier” A (1d $U(1)$ gauge field)

$$W_{k,0} = \int DA\, D\chi\, D\bar{\chi}\ \exp\Big(\int_0^{2\pi} d\tau\, \big[-\bar{\chi}\partial_\tau\chi + iA(\bar{\chi}\chi - R^2)\big]\Big)$$

invariant (due to constraint) under

$$\chi^i \to e^{i\alpha}\chi^i\,, \qquad \bar{\chi}_i \to e^{-i\alpha}\bar{\chi}_i\,, \qquad A \to A + \partial_\tau\alpha\,, \qquad \alpha = \alpha(\tau)$$

$$\chi(\tau) = \chi(\tau + 2\pi) \;\rightarrow\; \alpha(\tau) = \alpha_0(\tau) + n\tau\,, \qquad \alpha_0(2\pi) = \alpha_0(0)$$

"small" gauge transformation α_0: gauge fix $A = \mu$ =const

- "large" gauge transformation $\alpha(\tau) = n\tau$:

$A \rightarrow A + n$: symmetry for $R^2 = k + \frac{N}{2}$

- redundancy fixed by restricting μ to $[0, 1]$

$$W_{k,0} = \int_0^1 d\mu \int D\chi D\bar{\chi}\; \exp\left(i \int_0^{2\pi} d\tau\, \left[i\bar{\chi}\partial_\tau\chi + \mu(\bar{\chi}\chi - R^2)\right]\right)$$

- integral over χ and $\bar{\chi}$: $\left[\,\det(i\partial_\tau + \mu)\right]^{-N}$ in ζ-function reg.

$$\det(i\partial_\tau + \mu) = \prod_{n=-\infty}^{\infty}(n + \mu) \;\rightarrow\; = \mu \prod_{n=1}^{\infty}(\mu^2 - n^2) = -2i\sin(\pi\mu)$$

$$W_{k,0} = \int_0^1 d\mu\; \frac{e^{-2\pi i\mu R^2}}{\left[-2i\sin(\pi\mu)\right]^N} = \int_0^1 d\mu\; \frac{e^{-2\pi ik\mu}}{(1 - e^{2\pi i\mu})^N} = \frac{(N+k-1)!}{k!\,(N-1)!}$$

Including interactions: $\frac{1}{R^2} \sim \frac{1}{k}$ expansion

- ladder model on S^1 ($a = 1$): integrate out free bulk scalar

$$S = \int d\tau \Big[i\,\bar{\chi}\partial_\tau\chi + \mu(\bar{\chi}\chi - R^2) \Big]$$
$$- \frac{i\zeta^2 g^2}{8\pi^2} \int \frac{d\tau\, d\tau'}{4\sin^2\frac{\tau-\tau'}{2}}\, \bar{\chi}(\tau) T^a \chi(\tau)\, \bar{\chi}(\tau') T^a \chi(\tau')$$

rescale χ by R $\rightarrow$ can use perturbation theory in $1/R^2$

$$S = iR^2 \Big[\int d\tau\, \bar{\chi}\partial_\tau\chi - \varkappa \int \frac{d\tau\, d\tau'}{4\sin^2\frac{\tau-\tau'}{2}}\, \bar{\chi}(\tau) T^a \chi(\tau)\, \bar{\chi}(\tau') T^a \chi(\tau') \Big]$$

$$\bar{\chi}_i\chi_i = 1\,, \qquad \varkappa \equiv \frac{\zeta^2 g^2 R^2}{8\pi^2} = \text{fixed}, \qquad R^2 = k + \tfrac{1}{2}N$$

• separate constant part:

$$\chi = n + \tfrac{1}{R}\chi', \;\; \bar\chi = \bar n + \tfrac{1}{R}\bar\chi', \;\; \int d\tau\, \chi' = 0, \;\; \bar n n = 1$$

$$\int [d\chi\, d\bar\chi] \;\to\; \int [d\chi'\, d\bar\chi'] \int dn\, d\bar n\; \delta\Big(\bar n n - 1 + \tfrac{1}{2\pi R^2} \int d\tau\, \bar\chi'\chi'\Big)$$

One-loop order:

• $\bar n_i n_i = 1$, $(\bar\chi_1\, T^a \chi_2)\,(\bar\chi_3\, T^a \chi_4) = \tfrac{1}{2}(\bar\chi_1\chi_4)(\bar\chi_3\chi_2) - \tfrac{1}{2N}(\bar\chi_1\chi_2)(\bar\chi_3\chi_4)$

$$S^{(2)} = i \int d\tau\, \bar\chi'\partial_\tau\chi' - \frac{i}{2}\varkappa \int [d^2\tau]\Big[\big(1 - \tfrac{1}{N}\big)\big(\chi_i'(\tau')\bar n_i \bar n_j \chi_j'(\tau) + \bar\chi_i'(\tau') n_i n_j \bar\chi_j'(\tau)\big) + 2\bar\chi_i'(\tau')\big(\delta_{ij} - \tfrac{1}{N} n_i \bar n_j\big)\chi_j'(\tau)\Big]$$

• mode expansion $\chi'(\tau) = \sum_{\ell\neq 0} a(\ell)\, e^{i\ell\tau}$

$i\int d\tau\, \bar\chi'\partial_\tau\chi' = 2\pi \sum_{\ell\neq 0} \ell\, \bar a(\ell)\, a(-\ell)\,, \quad \frac{1}{4\sin^2\frac{\tau}{2}} = \sum_{\ell=1}^{\infty}(-\ell)\cos(\ell\tau)$

integrate over $a(\ell)$ and compute det

$\prod_{\ell\neq 0} c = \prod_{\ell=1}^{\infty} c^2 = \exp\left(\zeta(0)\log c^2\right) = c^{-1}$

$$W_k = W_{k,0}\left(1+\pi^2\varkappa^2\right)^{\frac{N-1}{2}}\left[1+\tfrac{1}{R^2}\Gamma_2+\tfrac{1}{R^4}\Gamma_4+...\right]$$

Two-loop order: Γ_2

fix $n=\bar{n}=(0,\ldots,0,1), \quad \chi_i' = (\eta_1,\ldots,\eta_{N-1},\varphi)$

$$\mathcal{D}_{\varphi\varphi}(\tau) = -\frac{N-1}{2N}\varkappa\sum_{\ell\neq 0}\frac{1}{|\ell|}\,e^{i\ell(\tau-\tau')}, \qquad \mathcal{D}_{\bar{\varphi}\varphi} = \sum_{\ell\neq 0}\left(\frac{i}{2\pi\ell}+\frac{N-1}{2N}\varkappa\frac{1}{|\ell|}\right)e^{i\ell(\tau-\tau')}$$

$$\mathcal{D}_{\eta\eta} = \frac{1}{2\pi}\sum_{\ell\neq 0}\frac{i}{\ell+i\pi\varkappa|\ell|}\,e^{i\ell(\tau-\tau')}$$

involved computation of Γ_2 with mode cutoff $\sum_\ell e^{-\varepsilon\,\ell}$... gives

$$\Gamma_2 = 2\pi^2 N(N-1)\,\frac{\varkappa^3}{(1+\pi^2\varkappa^2)^2}\,\log\varepsilon + ...$$

- divergence absorbed into renormalization of $\varkappa$

$$\varkappa \equiv \varkappa_{\text{bare}} \to \varkappa(\mu) - \frac{2N}{R^2} \frac{\varkappa^2(\mu)}{1+\pi^2\varkappa^2(\mu)} \log(\mu\,\varepsilon) + \mathcal{O}\big(R^{-4}\big)$$

- W_k satisfies $\left(\mu\frac{\partial}{\partial\mu} + \beta_\varkappa \frac{\partial}{\partial\varkappa}\right)\mathrm{W}_k = 0$ with

$$\beta_\varkappa = \mu\frac{d\varkappa}{d\mu} = \frac{2N}{R^2}\frac{\varkappa^2}{1+\pi^2\varkappa^2} + \mathcal{O}(R^{-4})$$

- $\varkappa^3 \log\mu \sim \zeta^6 g^6 \log\mu$ in $\log \mathrm{W}_k$ in agreement with

$$\frac{1}{128\pi^4} \mathrm{C_R C_A^2}\, \zeta^6 g^6 = \frac{2\pi^2}{k} N(N-1)\,\varkappa^3 + ...$$

- also get

$$\frac{\partial}{\partial\varkappa}\log \mathrm{W}_k = \bar{\mathcal{C}}\,\beta_\varkappa\,, \qquad\qquad \bar{\mathcal{C}} = \frac{(N-1)\pi^2 R^2}{2N\varkappa} + ...$$

Comments

• study case of k-antisymm. rep.: χ_i here fermions

• $W^{(\zeta)}$ family of Wilson loops:
AdS/CFT perspective deserves further study [BGT '17]

• associated boundary CFT_1:
correlators of operators along WL at $\zeta = 0$ and $\zeta = 1$
as correlators in effective defect 1d CFT
relation to conformal bootstrap in $d = 1$
[Drukker, Kawamoto; Cooke, Dekel ,Drukker; Giombi, Roiban, AT;
Giombi, Komatsu; Bianchi et al; Barrat et al; ...]

• role of integrability? in SYM 1-loop spin chain integrability
only for $\zeta = 1$ and $\zeta = 0$ – WML and WL [Correa, Leoni, Luque '18]

• RG flow in other similar non-local 1d QFT's ?
e.g. SYK-like models [Gross, Rosenhaus '17]