Classical loop models & quantum phase transitions:

Néel to VBS and VBL transitions in SU(n) magnets

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Overview

Close-packed loop models & 3D classical stat mech

Transitions between short-loop phases and extended phases

Continuum description as CP^{n-1} sigma model

Loop models & SU(n) spin systems in (2+1) dimensions

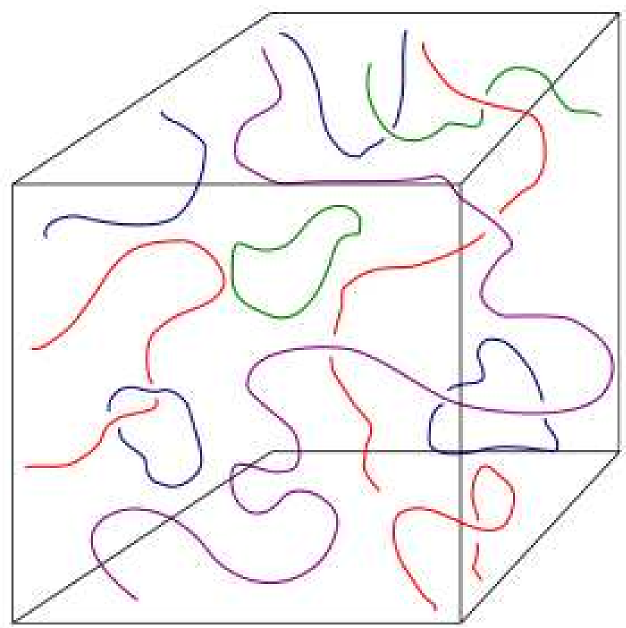
Valence bond liquid to Néel transitions

Continuous transition at $n = 3$

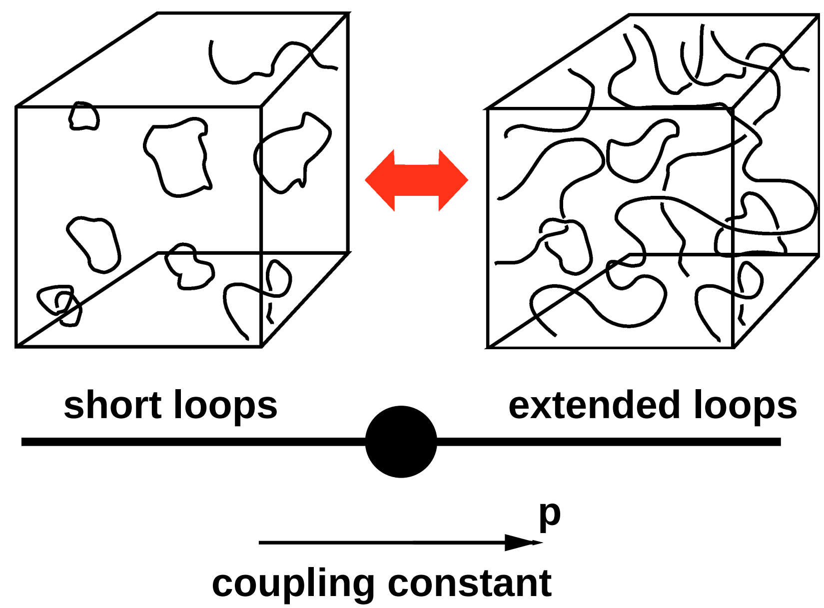
Valence bond solid to Néel transition and deconfined critical point

Loop models

Continuum problem



Phase transition

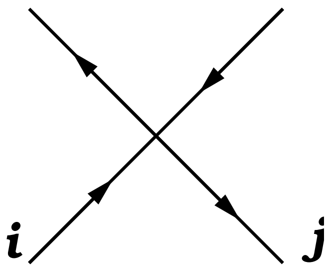


Lattice formulation

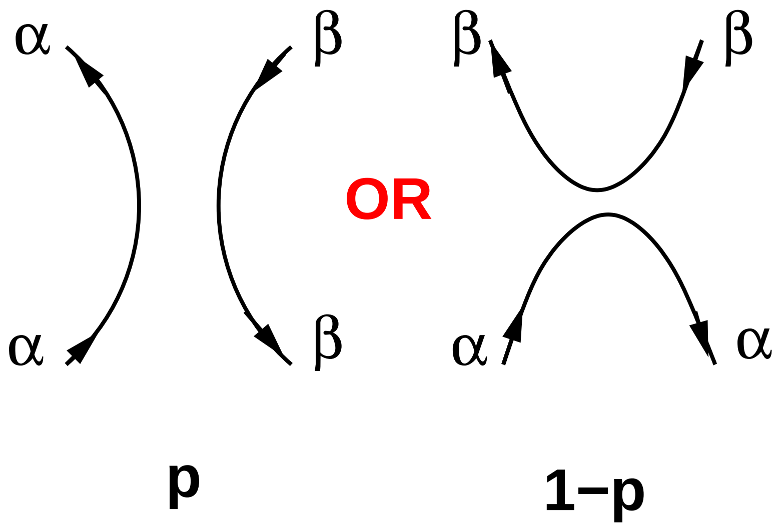
Close-packed loops with n colours on lattice of (directed) links

Loops on lattices

Lattice node



Two decompositions



Colours $\alpha, \beta = 1 \ldots n$

Loops on lattices

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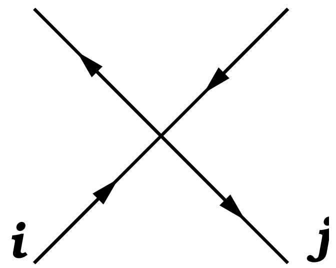

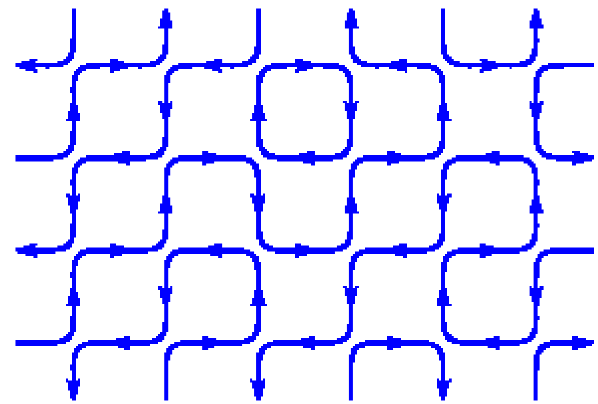

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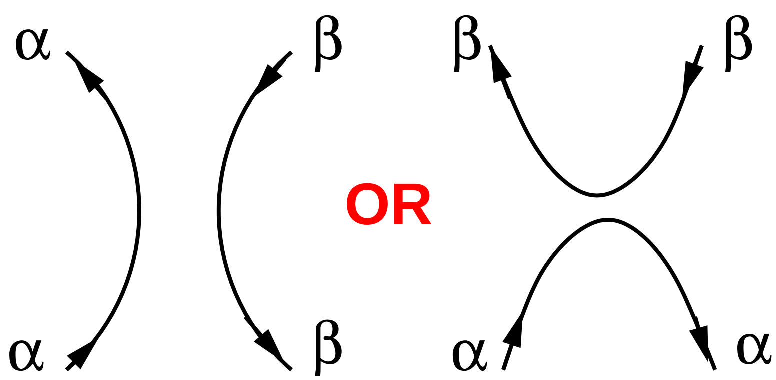
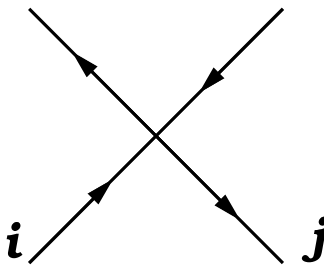


p **1−p**

Colours $\alpha, \beta = 1 \ldots n$

Loops on lattices & singlets

Lattice node



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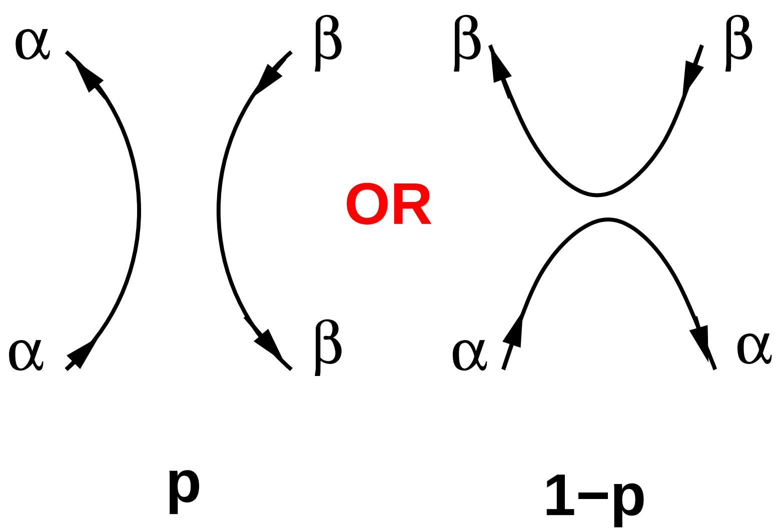

Hilbert space basis at site i

$$|\alpha\rangle_i \qquad \alpha = 1 \ldots n$$

SU(n) rotations

$$|\alpha\rangle_i \to \mathrm{U}_{\alpha\beta}|\alpha\rangle_i$$

$$|\alpha\rangle_j \to \mathrm{U}^*_{\alpha\beta}|\alpha\rangle_i$$

SU(n) singlet

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} |\alpha\rangle_i \otimes |\alpha\rangle_j$$

Loops on lattices & singlets

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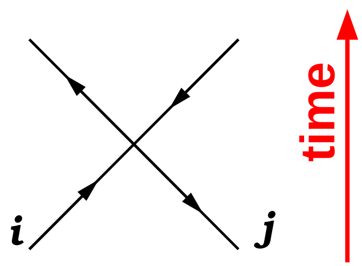

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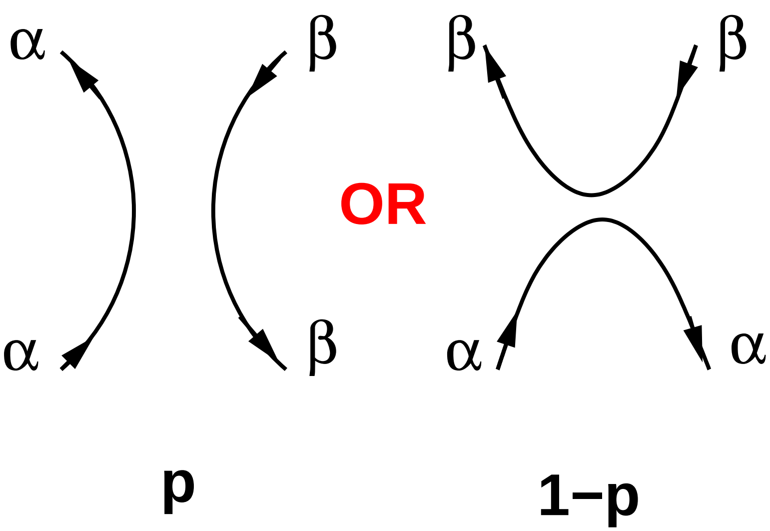

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Transfer matrix

$$\mathrm{T} = p \cdot 1 + (1-p)\mathrm{P}_{\text{singlet}} \equiv e^{-\beta\mathcal{H}}$$

$$\mathcal{H} = J \sum_{\langle ij \rangle} \mathrm{P}_{\text{singlet}}$$

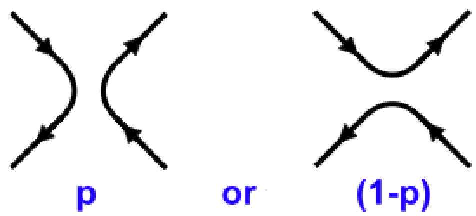
Equivalent states

3D loop model		(2+1)D SU(n) spin model
Short loop phase	$\Leftrightarrow$	Dimerised state
Extended phase	$\Leftrightarrow$	Néel ordered state

Loop models

- all-positive weights (c.f J-Q model)
- dynamical exponent $z = 1$ built-in

Phase transitions in loop models

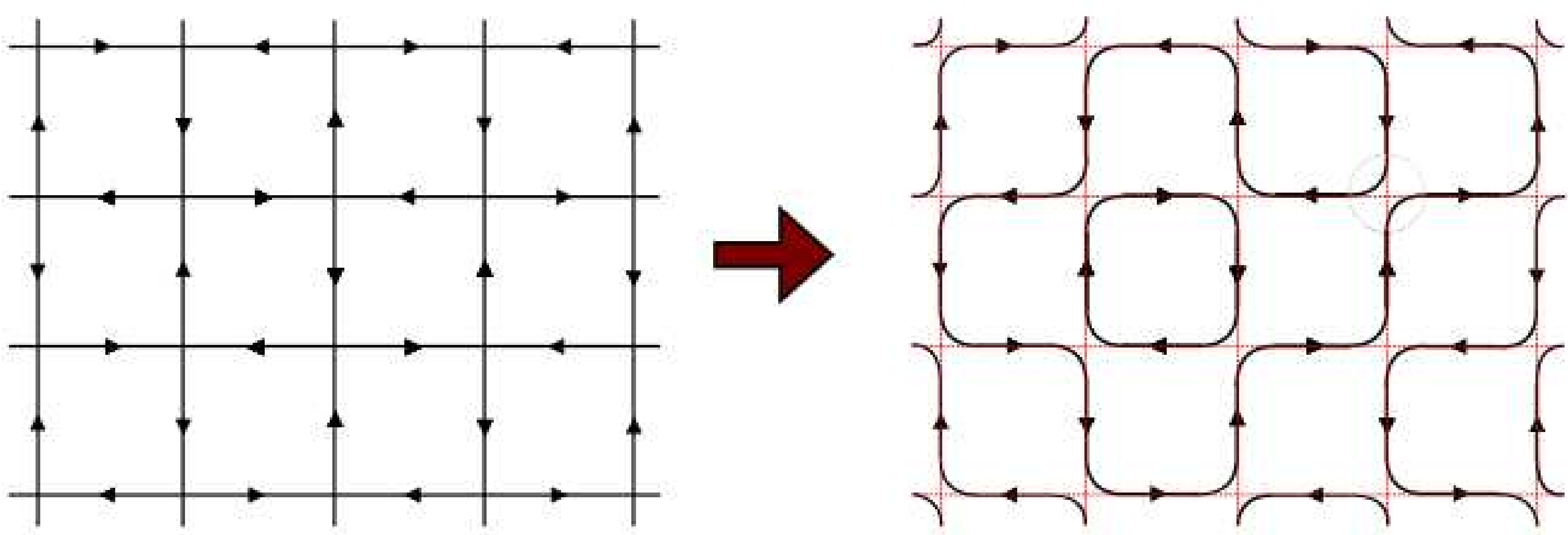


$$Z = \sum\nolimits_{\text{configs}} p^{n_{\text{p}}} (1-p)^{n_{1-\text{p}}} n^{n_{\text{loops}}}$$

To define model: specify lattice, link directions and nodes

2D model

Sample configuration



3D loop models

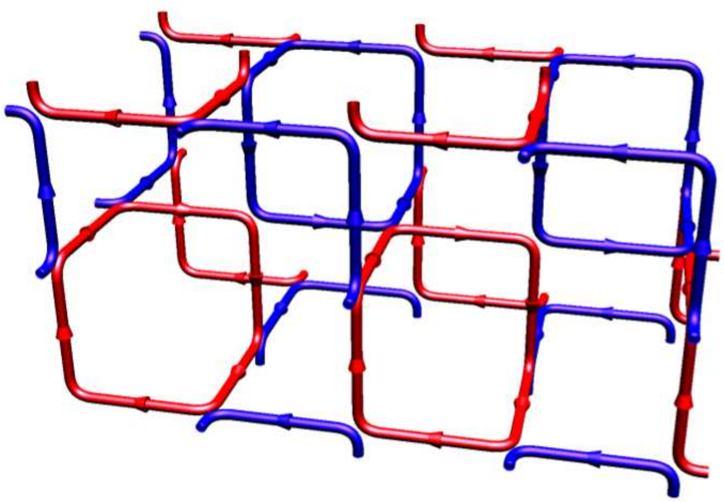
Study two lattices: '3D L-lattice' & '3D Manhattan lattice'

Both designed so that:

$p = 0$ all loops short

$p > 0$ long loops possible

Configuration of 3D model



3D L-lattice

Symmetry under $p \leftrightarrow [1-p]$

3D Manhattan lattice

All loops extended at $p = 1$

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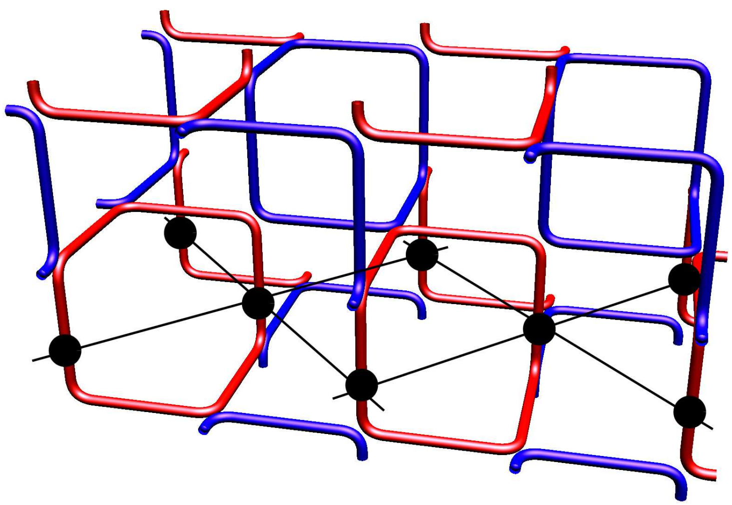

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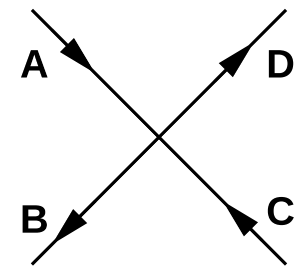
Local Description and Continuum Theory

$$Z = \sum_{\text{configs}} p^{n_{\text{p}}} (1-p)^{n_{1-\text{p}}} n^{n_{\text{loops}}}$$

Introduce n **component complex**

unit vector $\vec{z}_l$ **on each link** l



Calculate $\mathcal{Z} = \mathcal{N} \prod_l \int \mathrm{d}\vec{z}_l \; \mathrm{e}^{-\mathcal{S}}$

with $\mathrm{e}^{-\mathcal{S}} = \prod_{\text{nodes}} \left[p(\vec{z}_A^\dagger \cdot \vec{z}_B)(\vec{z}_C^\dagger \cdot \vec{z}_D) + (1-p)(\vec{z}_A^\dagger \cdot \vec{z}_D)(\vec{z}_C^\dagger \cdot \vec{z}_B) \right]$

Expand $\prod_{\text{nodes}} [\ldots]$ **Loops contribute factors**

$$\sum_{\alpha,\beta,\ldots\gamma} \int \mathrm{d}\vec{z}_1 \ldots \int \mathrm{d}\vec{z}_L \; z_1^{*\alpha} z_2^{\alpha} z_2^{*\beta} \ldots z_L^{*\gamma} z_1^{\gamma}$$

Hence: **(i) factor of** n **per loop** **(ii) invariance under** $\vec{z}_l \to \mathrm{e}^{\mathrm{i}\varphi_l} \vec{z}_l$

Local Description and Continuum Theory

$$Z = \sum_{\text{configs}} p^{n_{\text{p}}} (1-p)^{n_{1-\text{p}}} n^{n_{\text{loops}}}$$

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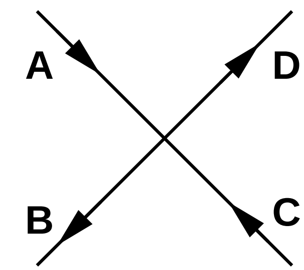

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Continuum limit **CP(n-1) model**

$S = \frac{1}{g} \int \mathrm{d}^d \mathbf{r} \, |(\nabla - \mathrm{i}A)\vec{z}|^2$ with $A = \frac{\mathrm{i}}{2}(z^{*\alpha} \nabla z^{\alpha} - z^{\alpha} \nabla z^{*\alpha})$

with $|\vec{z}|^2 = 1$ and invariance under $\vec{z} \to \mathrm{e}^{\mathrm{i}\varphi(\mathrm{r})} \vec{z}$

see also: Candu, Jacobsen, Read and Saleur (2009)

Phase transitions in CP^{n-1} model

Gauge-invariant degrees of freedom: 'spins' $Q \equiv zz^\dagger - 1/n$

(Mapping to Heisenberg model for $n = 2$ via $S^\alpha = z^\dagger \sigma^\alpha z$)

Order parameter – prob. link lies on extended trajectory

Correlations $\langle \mathrm{tr}\, Q(\mathbf{0}) Q(\mathbf{r}) \rangle \propto G(r)$ – prob. $\mathbf{0}$ & $\mathbf{r}$ on same loop

Paramagnetic phase
— only finite loops

$$G(r) \sim \frac{1}{r} \mathrm{e}^{-r/\xi}$$

Critical point
— fractal loops

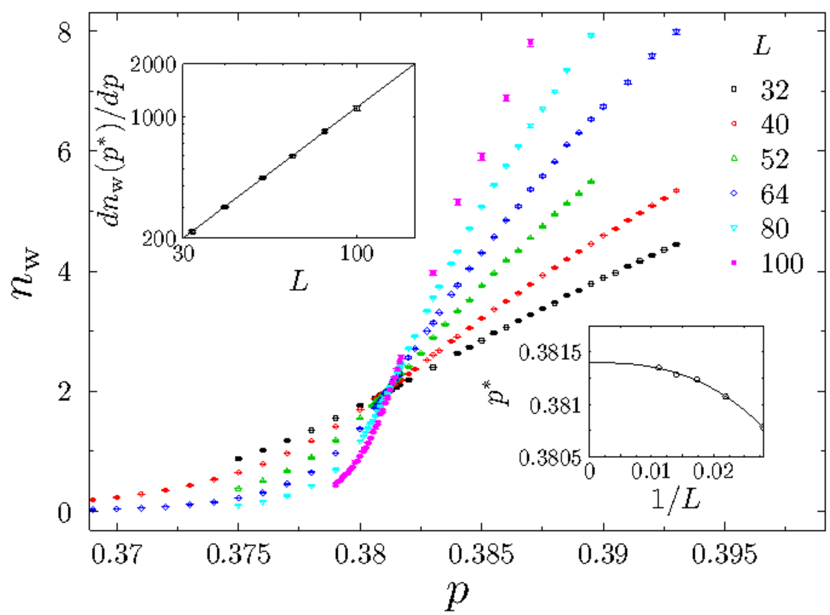
$$G(r) \sim r^{-(1+\eta)} \qquad d_f = \frac{5-\eta}{2}$$

Ordered phase
— Brownian loops
escape to infinity

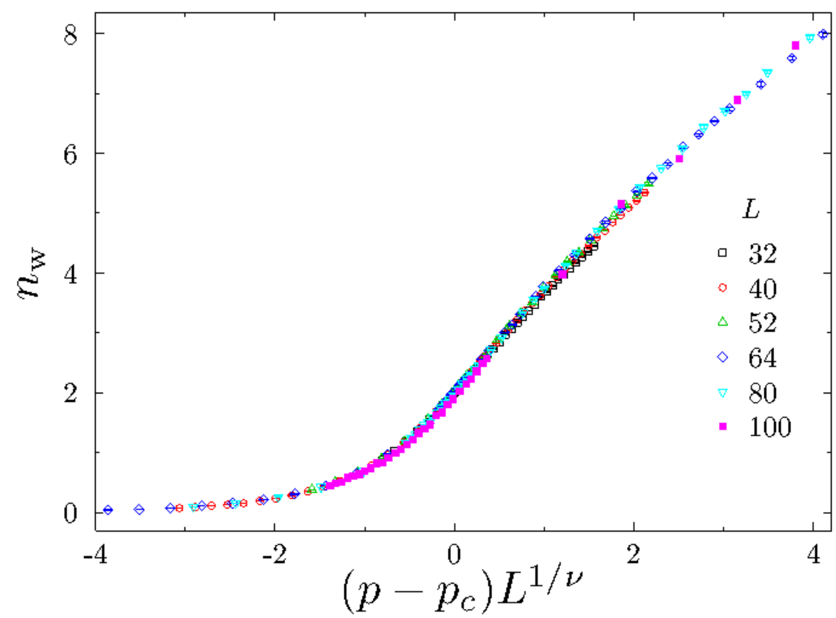
$$G(r) \sim r^{-2}$$

Testing CP^{n-1} description: $n = 2$

Winding number vs coupling const



Scaling collapse with Heisenberg exponents



Fitted exponents $\nu = 0.708(6) \quad \gamma = 1.39(1)$

Consistent with best estimates for

classical Heisenberg model $\nu = 0.7112(5) \quad \gamma = 1.3960(9)$

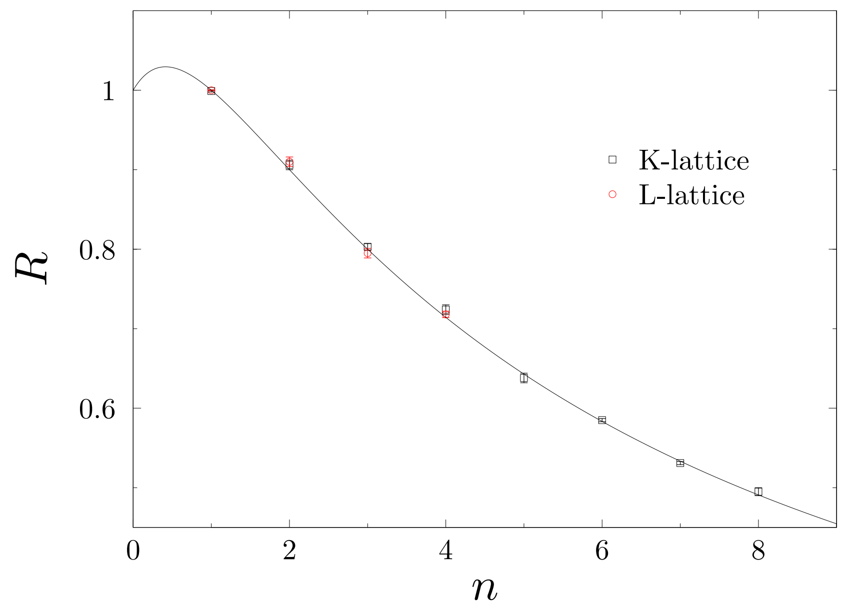
Testing CP^{n-1} description: extended phase

Moments of loop length distribution

$$R = \frac{\langle \sum_{\text{loops}} (\text{length})^4 \rangle}{\langle \sum_{\text{loops}} (\text{length})^2 \rangle^2} = \frac{6(n+1)}{n^2 + 5n + 6}$$

Derived from average over orientations of CP^{n-1} order parameter

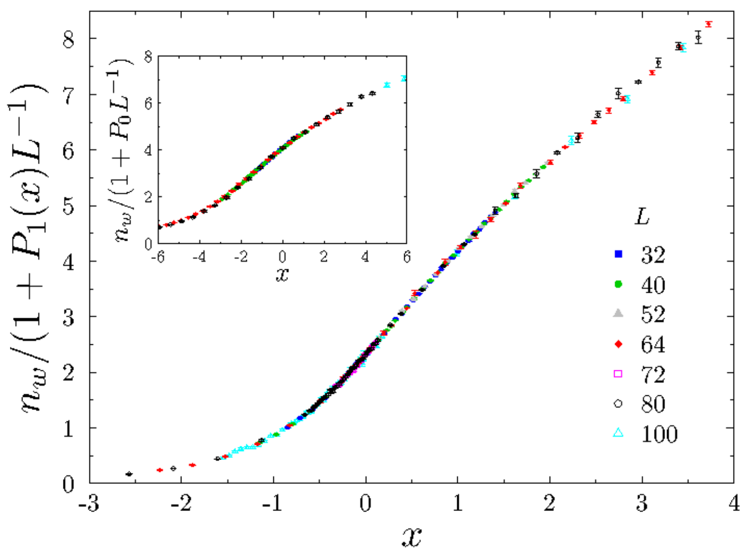
Results from simulations



New critical point at $n = 3$

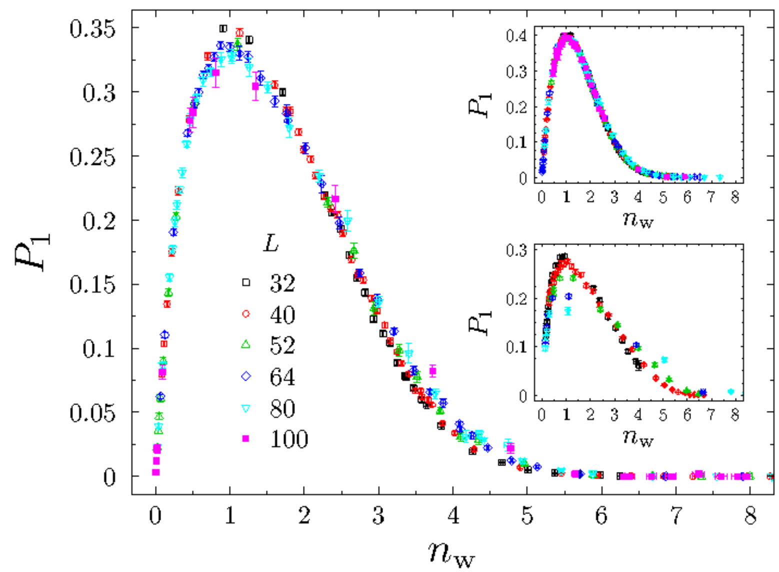
Cubic invariant in Landau theory for $n > 2 \Rightarrow$ 1st order transition?

Evidence for critical point



Scaling collapse of winding number

main: $n = 2$, inset: $n = 3$



Parameter free scaling collapse

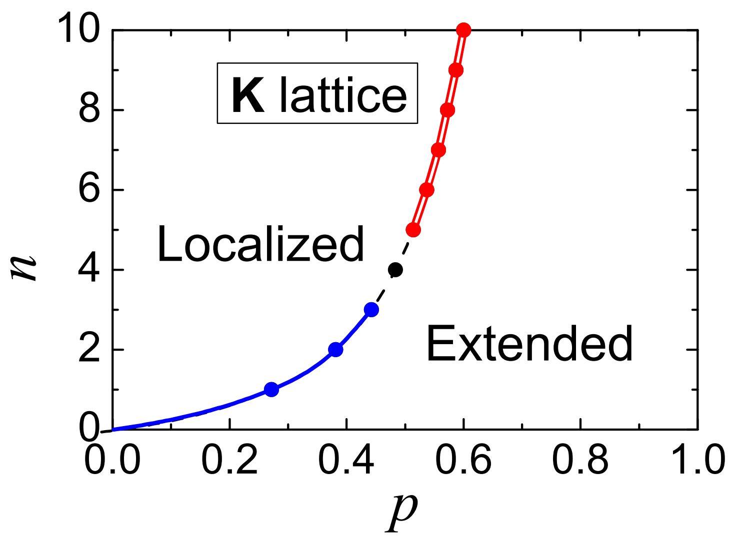
P(1 spanning curve) vs. $\langle$ no. spanning $\rangle$

main: $n = 3$, insets: $n = 2 \,\&\, 4$

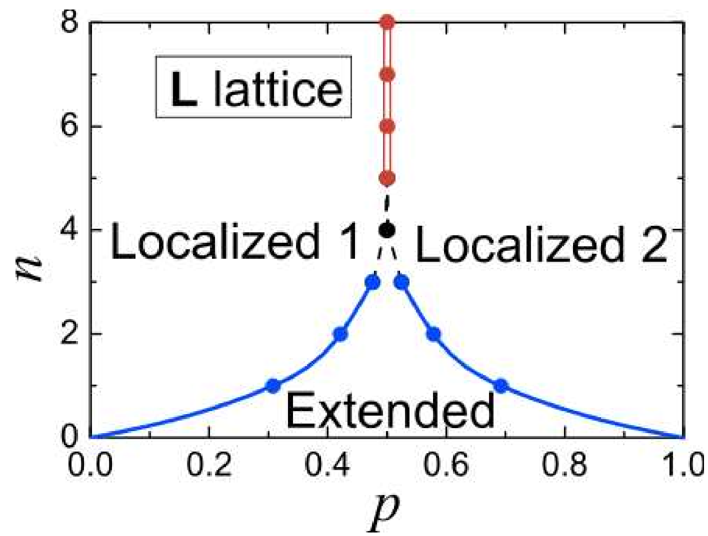
See also Kaul (2012)

Phase diagrams

3D Manhattan lattice



3D L-lattice



n	ν	γ
1	0.9985(15)	2.065(18)
2	0.708(6)	1.39(1)
3	0.50(2)	1.01(2)

Key

'localised' $\equiv$ dimerised

'extended' $\equiv$ Néel ordered

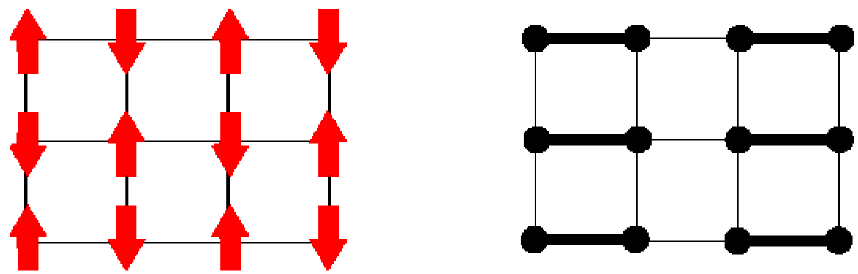
Dimerisation pattern enforced by staggered exchange

— except on L-lattice at $p = 1/2$

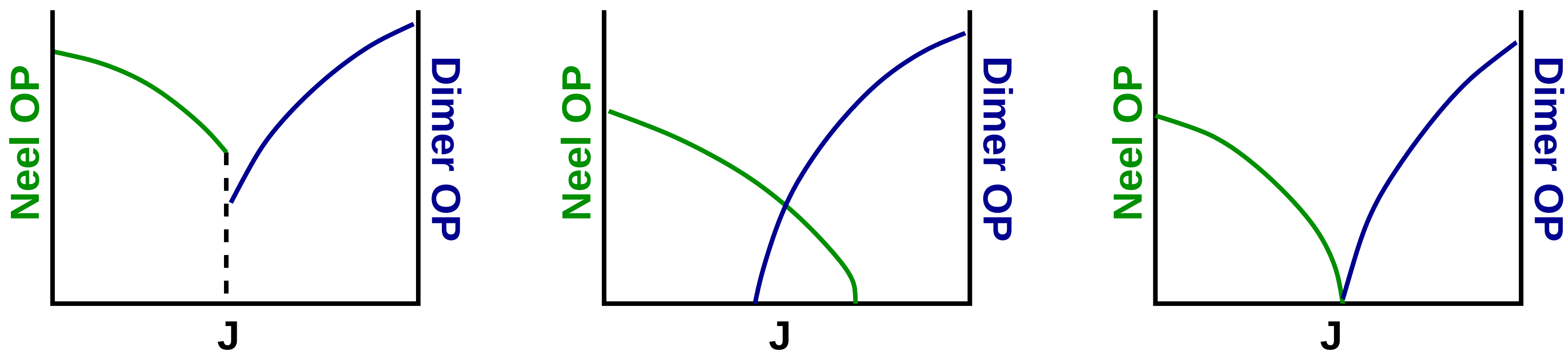
Loop description of deconfined critical point

Model **L-lattice at $p = 1/2$ + extra coupling J to favour dimerisation**

Néel to VBS transition – four dimerised states break lattice symmetries

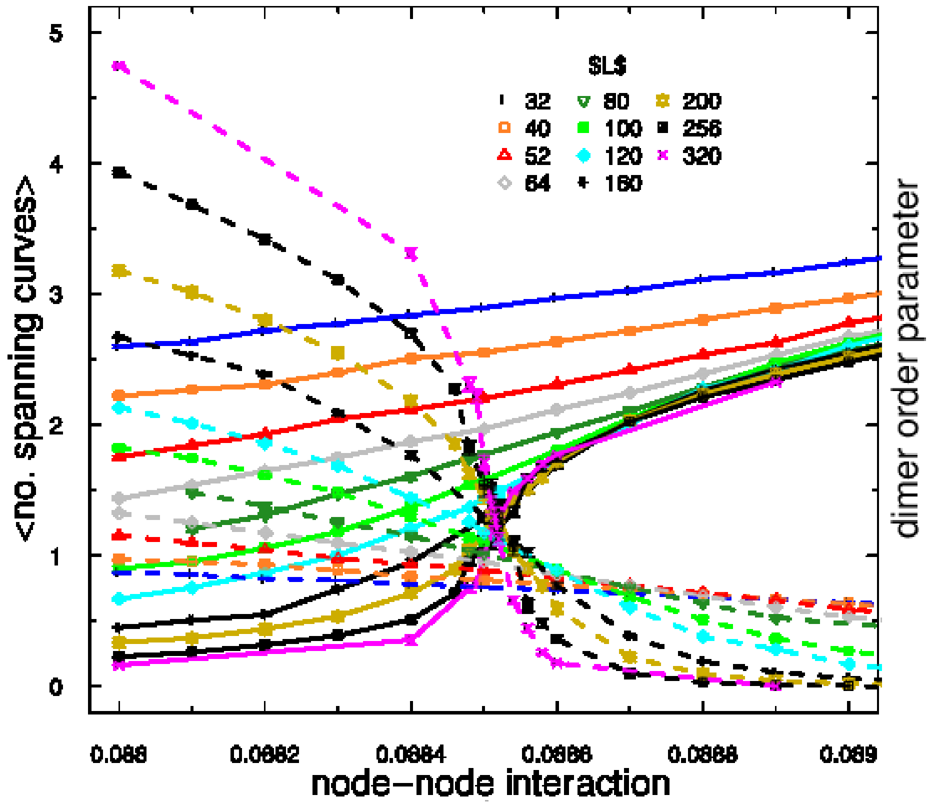


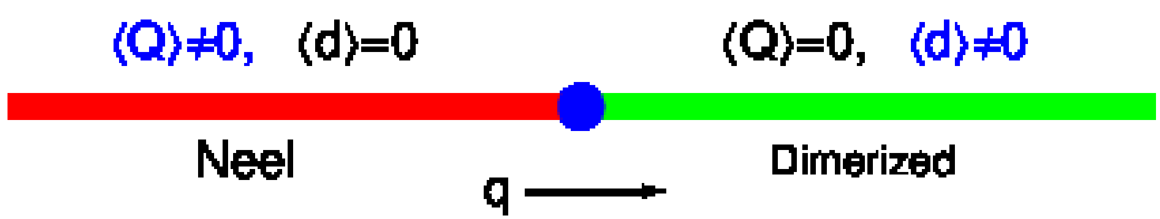
Possible phase diagrams



Senthil et al (2004)

Signatures of deconfined critical point – I

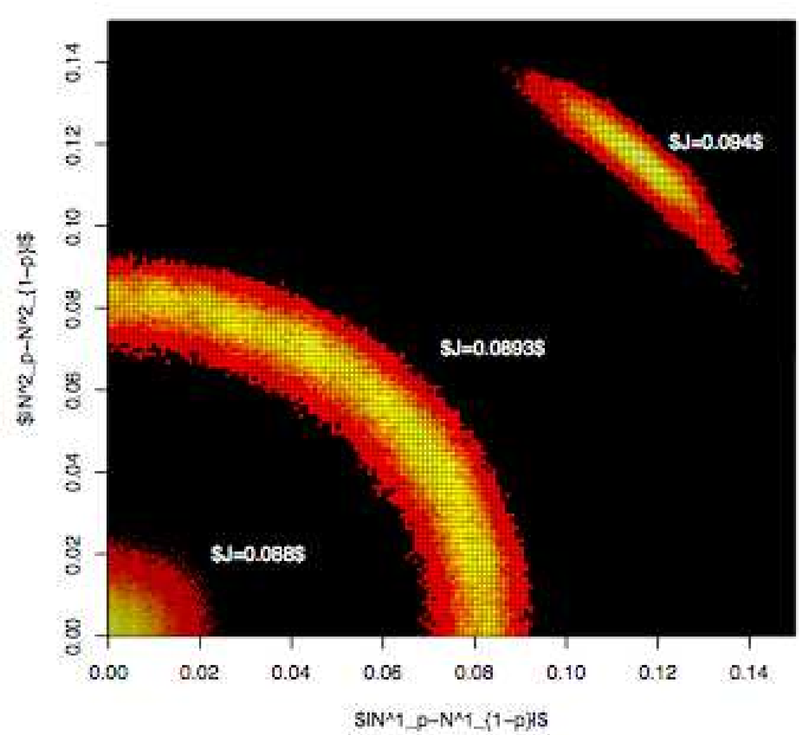




Signatures of deconfined critical point – II

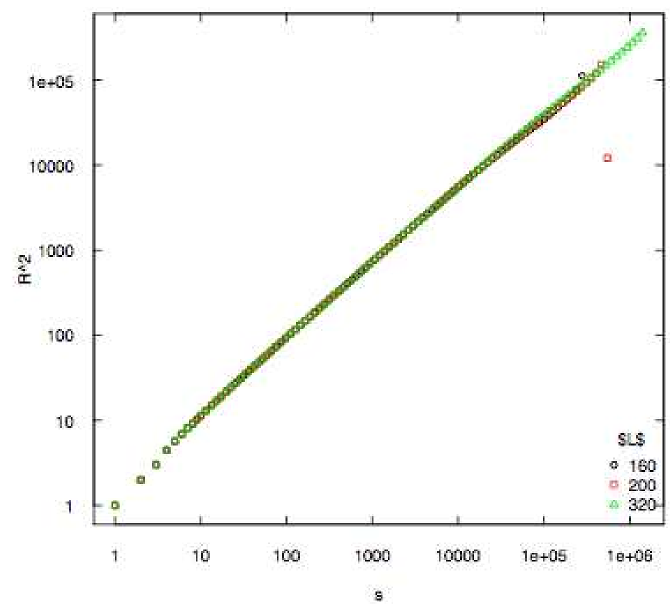
Emergent U(1) symmetry

for dimer order parameter



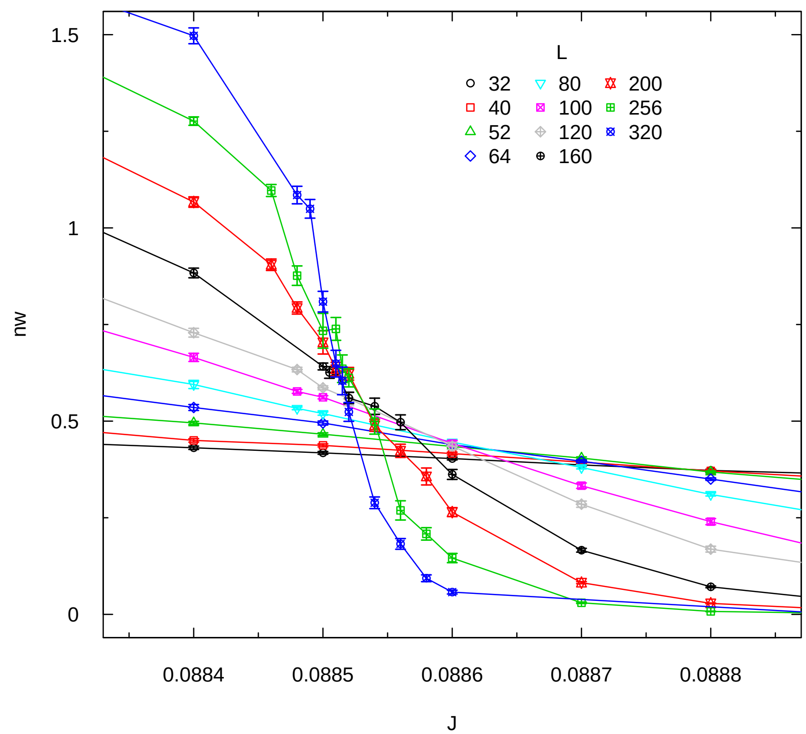
Large η via

fractal dimension of loops



$$\eta_{\text{Neel}} = 0.38(6)$$

. . . but large corrections to scaling



Drift in crossing points with system size

— similar behaviour in $J-Q$ model (Sandvik, Kaul)

— argued to be evidence of first order transition (Kuklov et al, 2008)

but $\xi_{\rm max}$ very large (system sizes up to $L=320$ here)

Summary

Loop models capture universal aspects of SU(n) magnets

- **In Néel phases**
- **At Néel to VBL & VBS transitions**

Advantages

- **No sign problem**
- **Dynamical exponent $z = 1$ guaranteed**

Critical behaviour in place of 1st order transition for

- **SU(3) system**
- **Néel to VBS at $n = 2$**

Related work: geometrical phase transition in loop model at $n = 1$