Symmetry classification of gapped Z_2 spin liquids

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Classification of spin liquids

- Why classify?
- Long-term goal: develop general framework to understand states of quantum matter beyond Landau symmetry-breaking theory
- Shorter-term: sharp distinctions among various spin liquids

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Types of spin liquids

- Gapless states (d=3 Coulomb phases, gapless Z_2 spin liquids, algebraic spin liquids, spinon Fermi surface)
- Gapped, topologically ordered states (e.g. gapped Z_2 spin liquids)
- (Symmetry-protected topological phases.)

Program of this talk

Ingredients:

Local bosonic model
(i.e. spin model)

Fixed topological order
(focus on d=2 Z_2 spin liquids)

Symmetries (space group,
time reversal, spin rotation)

Classify distinct phases
with these ingredients

- Will give a *symmetry classification*. (Not a full classification.)
- This means: classify states by how symmetry acts on the topological degrees of freedom.
- Two phases in different symmetry classes are different phases.
- Two phases in the same class may or may not be different phases.

Symmetry classes ⟺ Distinct types of quantum number fractionalization

Prior work: projective symmetry group (PSG) classification

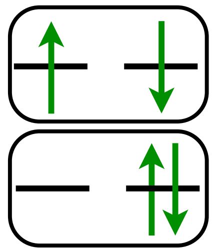
(X. G. Wen)

- Consider *e.g.* S=1/2 spin model, represent with S=1/2 fermionic partons

Hilbert space

$$\vec{S} = \frac{1}{2} f^\dagger_\alpha \vec{\sigma}_{\alpha\beta} f_\beta$$

$$f^\dagger_\alpha f_\alpha = 1$$



S=1/2 doublet (physical states) S=1/2, G=0

Unphysical doublet S=0, G=1/2

- Mean-field Hamiltonian: $H_{MFT} = \sum_{(r,r')} \left[t f^\dagger_{r\alpha} f_{r'\alpha} + \Delta (f^\dagger_{r\uparrow} f^\dagger_{r'\downarrow} + f^\dagger_{r'\uparrow} f^\dagger_{r\downarrow}) + \cdots \right]$

- Symmetry: *e.g.* square lattice space group + time reversal + spin rotation.

Action of symmetry:

1. Non-trivial gauge transformations: $T_x : f_{r\alpha} \rightarrow e^{i\lambda_r} f_{r+\boldsymbol{x},\alpha}$
2. Acts projectively: $T_x T_y = e^{i\phi} T_y T_x$

- Classify distinct ways symmetry can act, up to unitary (gauge) equivalence.

- Each such class is called a "PSG." Really, PSGs comprise a particular class of projective *representations* of the symmetry group

PSG pros and cons

- PSG gives a classification of mean-field theories.
- Beyond mean-field:
  1. Couple partons to fluctuating gauge field
  2. Gutzwiller projected wavefunctions
- PSG pros: Organizes construction of effective theories, wavefunctions. Works for various types of spin liquids.
- PSG cons: Tied to partons. May not hold beyond mean-field.

Symmetry classification of this talk

- Pros: Not tied to any particular formalism or mean-field theory.
- Cons: Probably limited to gapped phases. Not constructive.

- Other prior work: Alexei Kitaev, Ann. Phys. 2006, Appendix F
- Results agree where they overlap

Outline

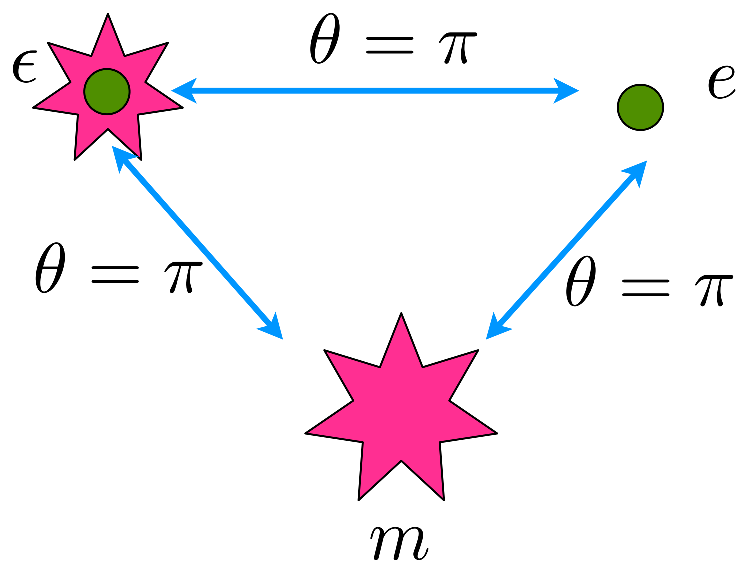
1. Review: topological properties of Z_2 spin liquids
2. Symmetry classification

Topological particle types

- Two bosons (*e* and *m*). One fermion (ε).
- In Z_2 gauge theory, *m* is the vison (Z_2 flux)
- Z_2 gauge theory with bosonic matter: *e* is electric charge, ε is charge-flux bound state
- Or, with fermionic matter: ε is electric charge, *e* is charge-flux bound state

  - Fusion rules: $\epsilon \times \epsilon = m \times m = e \times e = 1$
    $\epsilon \times m = e \ , \epsilon \times e = m \ , e \times m = \epsilon$

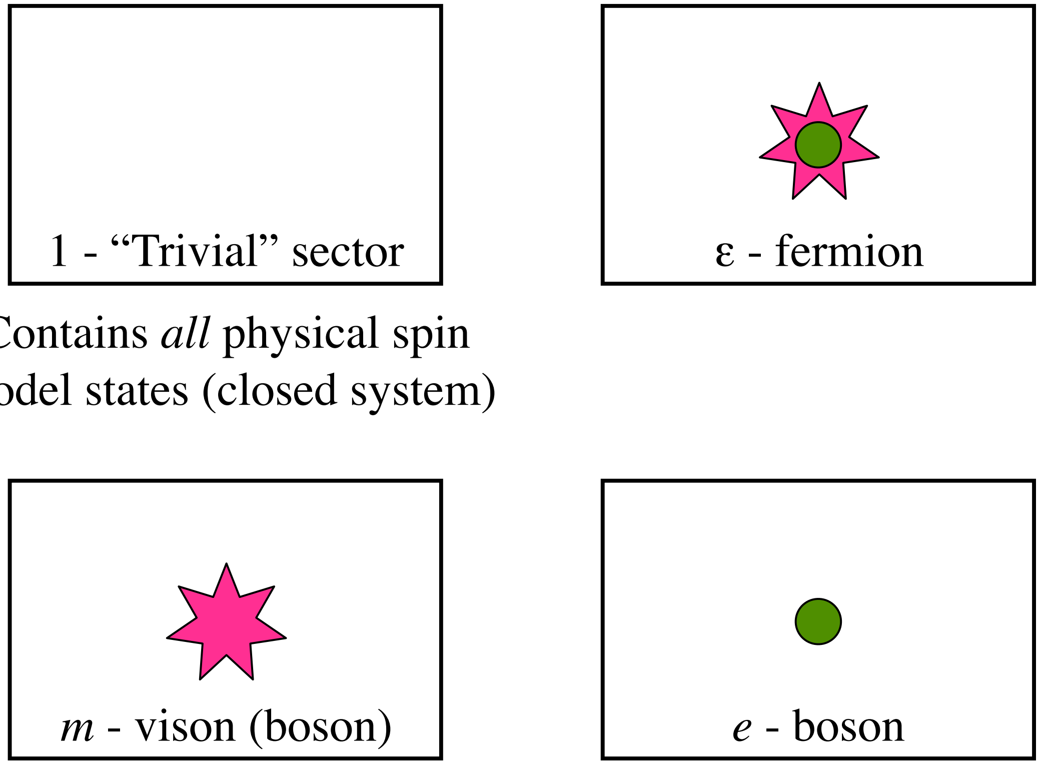
  - Mutual statistics:



- Note: no real distinction between *e* and *m*, can relabel $e \leftrightarrow m$

Superselection sectors

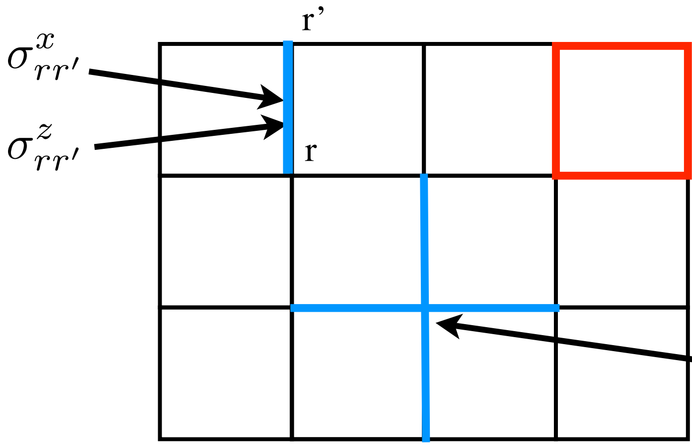
- Topological superselection sectors



- Cannot locally create single isolated *e*, *m* or ε. Create in pairs and separate.
- Sectors are closed under action of local operators (in particular, no matrix elements of *any* term in Hamiltonian *between* sectors).

Concrete model: toric code

A. Kitaev



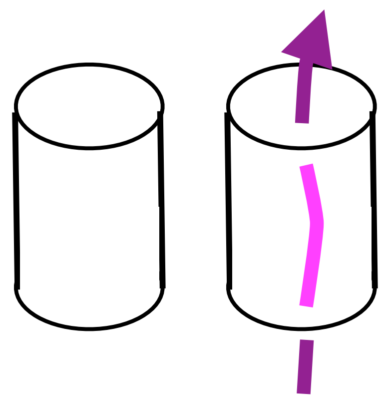
$$P_\square = \prod_{rr' \in \square} \sigma^z_{rr'}$$

$$v_r = \prod_{rr' \in +} \sigma^x_{rr'}$$

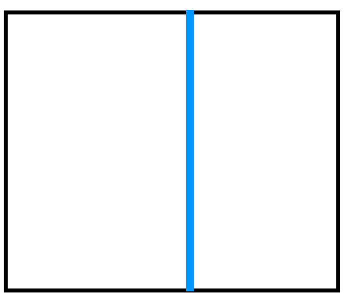
$$H = -u \sum_r v_r - K \sum_\square P_\square$$

$$s_v = \text{sign}\, u \quad s_P = \text{sign}\, K$$

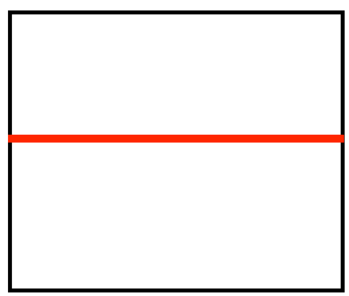
$$[v_r, P_\square] = 0$$



- Ground state has: $P_\square = s_P \ , \ v_r = s_v$
- 4-fold degenerate ground states on torus (threading of visons)
- Loop operators/algebra:



$$L^m_y = \prod_y \sigma^x$$



$$L^e_x = \prod_x \sigma^z$$

$$\{L^e_x, L^m_y\} = 0$$
$$\{L^e_y, L^m_x\} = 0$$

D=4 irrep (4-fold ground state degeneracy)

Outline

1. Review: topological properties of Z_2 spin liquids
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Basic Idea

- To start, consider just translation symmetry: $T_x T_y = T_y T_x$
  $$T_x T_y T_x^{-1} T_y^{-1} = 1$$

- Quantum mechanics: $T_x T_y T_x^{-1} T_y^{-1} = e^{i\phi}$
- Trivial superselection sector, $\phi = 0$, since *e.g.* $T_x : \vec{S}_r \to \vec{S}_{r+\boldsymbol{x}}$
- However, on other sectors, may have $\phi \neq 0$
- $T_x T_y T_x^{-1} T_y^{-1}$ must be *constant* on each sector. Otherwise, could find a local operator on which $T_x T_y T_x^{-1} T_y^{-1}$ acts nontrivially, a contradiction.
- Will see this constant takes discrete values, therefore can't change as long as sectors remain well-defined (gap remains open) → Universal property

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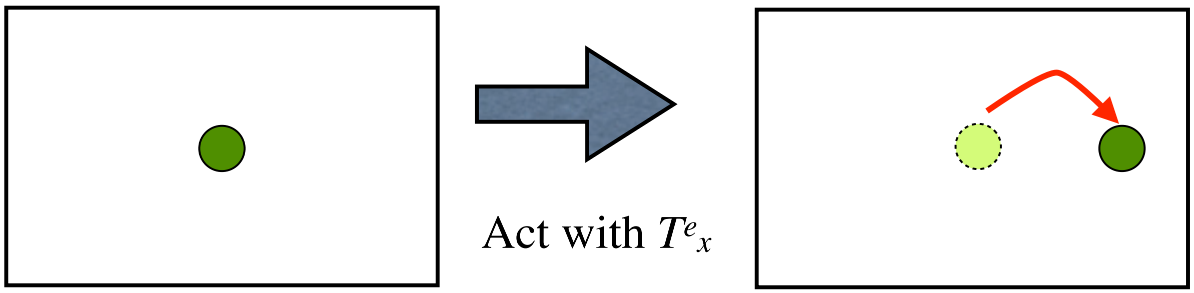
- Note: will not discuss case where some symmetries interchange *e* and *m* particles

Meaning of sector-translation operators & fusion constraints

- Physical states live in trivial sector. How to define, say, T_x in other sectors?



Move e-particle with respect to background of the ground state

- Acting on 1-sector state with 2 e-particles: $T_x \simeq T_x^e(1) T_x^e(2)$
- Implies: $T_x^e T_y^e (T_x^e)^{-1} (T_y^e)^{-1} = \sigma_e = \pm 1$
- This is a constraint imposed by $e^2 = 1$ fusion rule. Other fusion rules imply:

$$T_x^m T_y^m (T_x^m)^{-1} (T_y^m)^{-1} = \sigma_m = \pm 1$$

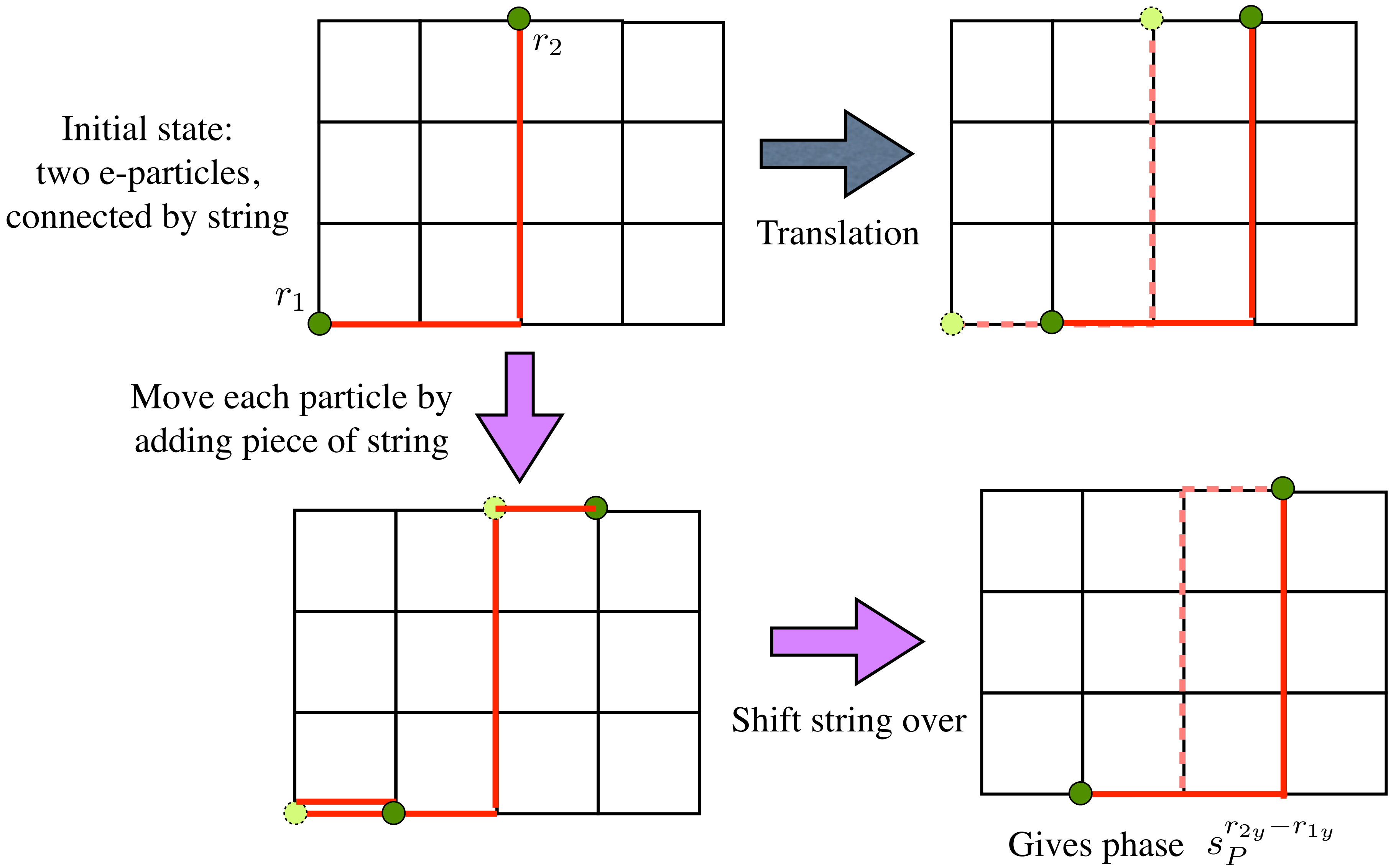
$$T_x^\epsilon T_y^\epsilon (T_x^\epsilon)^{-1} (T_y^\epsilon)^{-1} = \sigma_\epsilon = \pm 1$$

$$\sigma_\epsilon = \sigma_e \sigma_m$$

Two Z_2 parameters
$\rightarrow$ 4 possibilities

Actually only 3 classes, since two related by relabeling $e \leftrightarrow m$

Construction of sector-translations for toric code



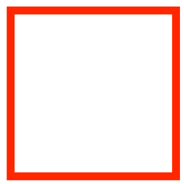
Define: $T_x^e(r) = (-1)^{r_y}\sigma^z_{r,r+\boldsymbol{x}}$ $\Rightarrow$ $T_x = T_x^e(r_1)T_x^e(r_2)$

Construction of sector-translations for toric code

- Work out group relation (illustrate for case of two e-particles):

$$T_x T_y T_x^{-1} T_y^{-1} = \left[T_x^e(1) T_y^e(1) (T_x^e)^{-1}(1) (T_y^e)^{-1}(1)\right]\left[(1 \to 2)\right] = \left[s_P\right]\left[s_P\right]$$

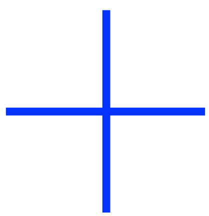
Translation around
a plaquette



$$= P_\square \to s_P$$

- Make analogous definition for $T^m{}_{x,y}$

Translation around
a vertex



$$T_x^m(1) T_y^m(1) (T_x^m)^{-1}(1) (T_y^m)^{-1}(1) = v_r \to s_v$$

- Finally, can show:

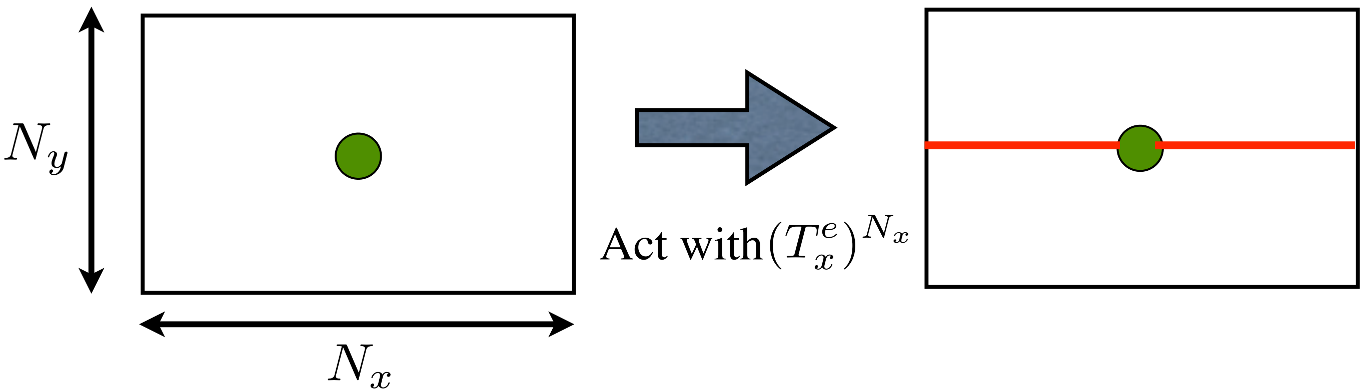
$$T_x = \sum_{n_e, n_m} \sum_{\{r_i^e\}, \{r_i^m\}} T_x^e(r_1^e) \cdots T_x^e(r_{n_e}^e) T_x^m(r_1^m) \cdots T_x^m(r_{n_m}^m) P_{n_e, n_m}(r_1^e, \ldots, r_{n_e}^e; r_1^m, \ldots, r_{n_m}^m)$$

Projects onto n_e e-particles, n_m
m-particles at given positions

- Can show: sector-relations unchanged for arbitrary number/configuration of particles

Relation to ground state quantum numbers

- Degenerate ground states can have nontrivial quantum numbers



- Suggests associations: $L^e_x \simeq (T^e_x)^{N_x}\ ,\ L^m_x \simeq (T^m_x)^{N_x}\ ,\ \ldots$

- Action of symmetry on loop operators: $T_y L^e_x T_y^{-1} \to T^e_y (T^e_x)^{N_x} (T^e_y)^{-1}$

- For toric code, calculating with this and this gives same answers.

- Note: this only gives info on *relative* momenta among 4 ground states

General symmetry group

- Some mathematics...
- Consider symmetry group G, elements $g \in G$, projective representation $\Gamma(g)$

$$\Gamma(g_1)\Gamma(g_2) = \omega(g_1, g_2)\Gamma(g_1 g_2),\ \omega(g_1, g_2) \in Z_2$$

"Factor set"

From fusion rules

- Associativity constraint: $\omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\omega(g_2, g_3)$
- Abelian group structure: $(\omega_A \omega_B)(g_1, g_2) = \omega_A(g_1, g_2)\omega_B(g_1, g_2)$
- "Gauge" transformation:

$$\Gamma(g) \to \lambda(g)\Gamma(g) \implies \omega(g_1, g_2) \to \lambda^{-1}(g_1)\lambda^{-1}(g_2)\lambda(g_1 g_2)\omega(g_1, g_2)$$

- Classify factor sets up to "gauge" equivalence.
- Factor set classes also form Abelian group: $H^2(G, Z_2)$

2nd group cohomology group, coefficients in Z_2

Symmetry class (for one sector) $\Longleftrightarrow$ Element of $H^2(G, Z_2)$

Meaning of factor set classes: fractionalization classes

- Factor set class is not tied to particular group representation
- For a given class there is a list of (projective) irreps belonging to that class.

  Aside: each class associated with distinct "central extension" of symmetry group. Projective reps in a given class are ordinary reps of the central extension group.

- Projective representations $\leftrightarrow$ quantum number fractionalization
- Factor set classes $\leftrightarrow$ distinct types of fractionalization

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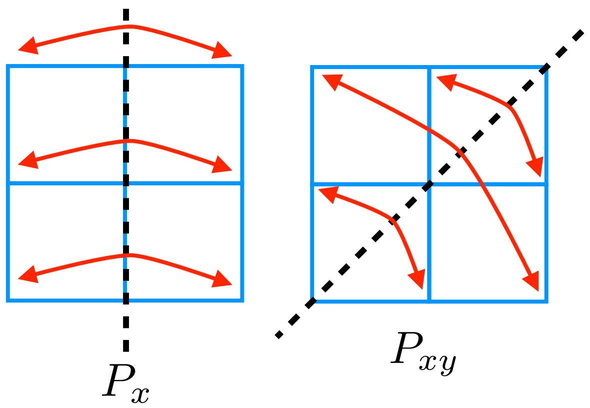
- Familiar example: spin rotation
- Two classes: $R_s(2\pi\hat{n}) = \pm 1$
- Integer vs. half-odd integer spin

Space group + time reversal + spin rotation

- Square lattice space group generators: T_x, P_x, P_{xy}
- Note that: $T_y = P_{xy} T_x P_{xy}^{-1}$
- Time reversal $\mathcal{T}$
- Spin rotation (by θ about $\hat{n}$-axis): $R(\theta \hat{n})$



- Generators + relations specify the symmetry class in one sector:

$$
\begin{aligned}
P_x^2 &= \sigma_{px} \\
P_{xy}^2 &= \sigma_{pxy} \\
(P_x P_{xy})^4 &= \sigma_{pxpxy} \\
T_x T_y T_x^{-1} T_y^{-1} &= \sigma_{txty} \\
T_x P_x T_x P_x^{-1} &= \sigma_{txpx} \\
T_y P_x T_y^{-1} P_x^{-1} &= \sigma_{typx} \\
\mathcal{T}^2 &= \sigma_T
\end{aligned}
$$

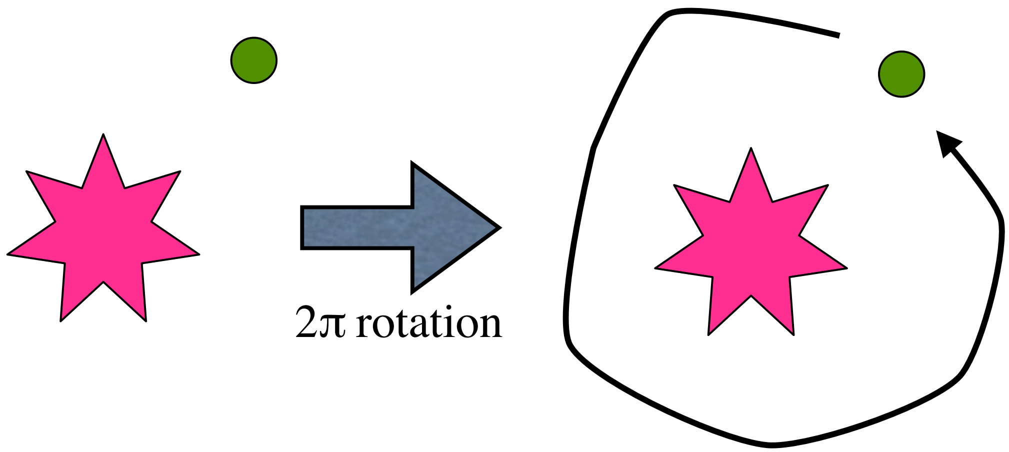
$$
\begin{aligned}
\mathcal{T} T_x \mathcal{T}^{-1} T_x^{-1} &= \sigma_{Ttx} \\
\mathcal{T} P_x \mathcal{T}^{-1} P_x &= \sigma_{Tpx} \\
\mathcal{T} P_{xy} \mathcal{T}^{-1} P_{xy} &= \sigma_{Tpxy} \\
R(2\pi \hat{n}) &= \sigma_R \\
R(\theta \hat{n}) \mathcal{T} &= \mathcal{T} R(\theta \hat{n}) \\
R(\theta \hat{n}) P_x &= P_x R(\theta \hat{n}) \\
R(\theta \hat{n}) P_{xy} &= P_{xy} R(\theta \hat{n}) \\
R(\theta \hat{n}) T_x &= T_x R(\theta \hat{n})
\end{aligned}
$$

(+ Lie algebra of spin rotations)

- Here the σ's = ±1
- Specify class this way in two sectors (view as specifying two elements of $H^2(G, Z_2)$, or one element of $H^2(G, Z_2 \times Z_2)$. 2^{22} classes.
- Third class follows, "almost" a product of other two.

Role of mutual statistics (topological spin)

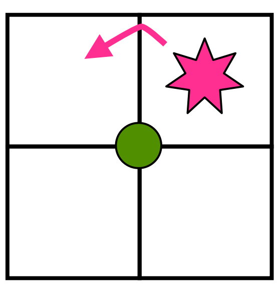
- Continuum:



- Square lattice: $(P^e_x P^e_{xy})^4 = \sigma_e$, $(P^m_x P^m_{xy})^4 = \sigma_m$ $\qquad (P^\epsilon_x P^\epsilon_{xy})^4 = -\sigma_e \sigma_m$

Sign from mutual statistics

- Simple way to see this in toric code:



Localize single *e* and *m*, preserving point group symmetry

Work out two-particle effective Hamiltonian, *m* sees *e* as π-flux

Can read off factor set from this Hamiltonian.

PSG classification revisited

- For any PSG in some parton approach, can find symmetry class of corresponding spin liquid (use effective gauge theory)
- On square lattice, Wen found 272 PSGs (for a *single* sector). Should be compared with 2^{10} classes for same sector (all have $S=1/2$, fixes one parameter).
- Some classes not realized.
- There are distinct PSGs belonging to same class. But in all cases I know, one PSG is gapless.

Open issues

- Allow for symmetries to interchange e and m (in progress)

- Chiral and/or non-Abelian topological order
- Three dimensions?
- How can symmetry class be determined given ground state wavefunction, excited states?