# Schwinger boson mean field theory:

## numerics for the energy landscape and gauge excitations in two-dimensional antiferromagnets

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# Schwinger boson representation & mean-field approximation

- Spin operators

$$\vec{S}_i = \tfrac{1}{2} b^+_{i\alpha} \vec{\sigma}_{\alpha\beta} b_{i\beta} \qquad \alpha, \beta = \uparrow, \downarrow$$

$$+ \text{ constraint} \qquad b^+_{i\uparrow} b_{i\uparrow} + b^+_{i\downarrow} b_{i\downarrow} = 2S \quad \text{at each site}$$

- Heisenberg interaction

$$\vec{S}_i \cdot \vec{S}_j = S^2 - 2 \hat{A}^+_{ij} \hat{A}_{ij} \dashrightarrow \hat{A}^+_{ij} = \frac{1}{2}\left( b_{\uparrow i}{}^+ b_{\downarrow j}{}^+ - b_{\downarrow i}{}^+ b_{\uparrow j}{}^+ \right)$$

**Singlet** bond operator

- Mean-field approximation [Arovas & Auerbach 1988, Read & Sachdev 1989]

$$A^+_{ij} \hat{A}_{ij} \xrightarrow{MF} \hat{A}^+_{ij} \underbrace{\left\langle \hat{A}_{ij} \right\rangle}_{= a_{ij} = -a_{ji}} + \left\langle \hat{A}^+_{ij} \right\rangle \hat{A}_{ij} - \left| \left\langle A_{ij} \right\rangle \right|^2$$

- Self-consistency equations:

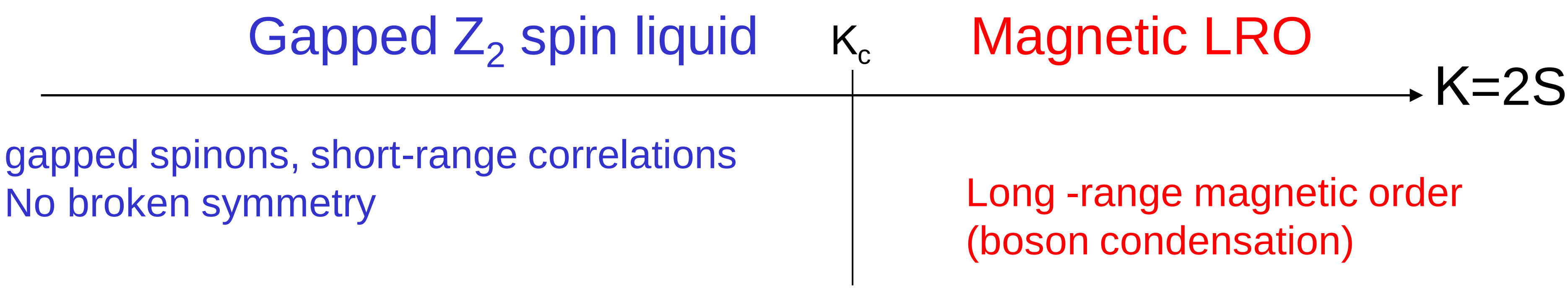
$$\begin{cases} \dfrac{1}{2} \left\langle b^+_{\uparrow i} b^+_{\downarrow j} - b^+_{\downarrow i} b^+_{\uparrow j} \right\rangle_{MF} = a_{ij} \\ \left\langle b^+_{\uparrow i} b_{\uparrow i} + b^+_{\downarrow i} b_{\downarrow i} \right\rangle_{MF} = 2S \end{cases}$$

chemical potential $\lambda_i$

# Typical mean-field phase diagram

- So far, most studies have focused on the (mean-field) ground-state properties, assuming the solution is *uniform* -- or with a small unit cell.

- Example of phase diagram on non-partite lattices, such as triangular or kagome [Sachdev 92, Wang & Vishwanath 2006]




- What is missing: gauge degrees of freedom (confinement, …), instabilities to VBC, …

# Mean field & large-N

- Introduce **N "flavors"** of bosons [Sp(N) Read & Sachdev 1991]

$$\left(b^+_{i\uparrow}, b^+_{i\downarrow}\right) \xrightarrow[SU(2)\to Sp(N)]{} \left(b^+_{i\uparrow 1}, b^+_{i\downarrow 1}\right), \left(b^+_{i\uparrow 2}, b^+_{i\downarrow 2}\right) \cdots \left(b^+_{i\uparrow N}, b^+_{i\downarrow N}\right)$$

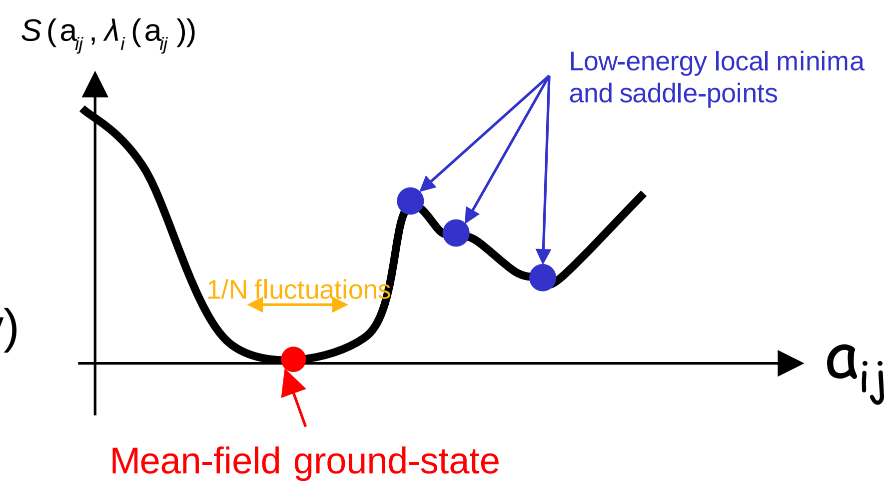
- Partition function

$$Z_N = \int D\left[a_{ij}(\tau),\ \lambda_i(\tau)\right] \exp\left( -N \underbrace{S(a_{ij}, \lambda_i)}_{\substack{\text{single-flavour} \\ \text{boson free energy} \\ \text{in presence of } a_{ij} \ \& \ \lambda}} \right)$$

- Large N: $Z_N$ dominated by **saddle-point** of S

- Finding (time-independent) saddle-points is equivalent to solving self-consistently the mean-field Hamiltonian.

- Goal: **explore numerically this "energy landscape"** and (hopefully) get some insight about the large but finite-N physics



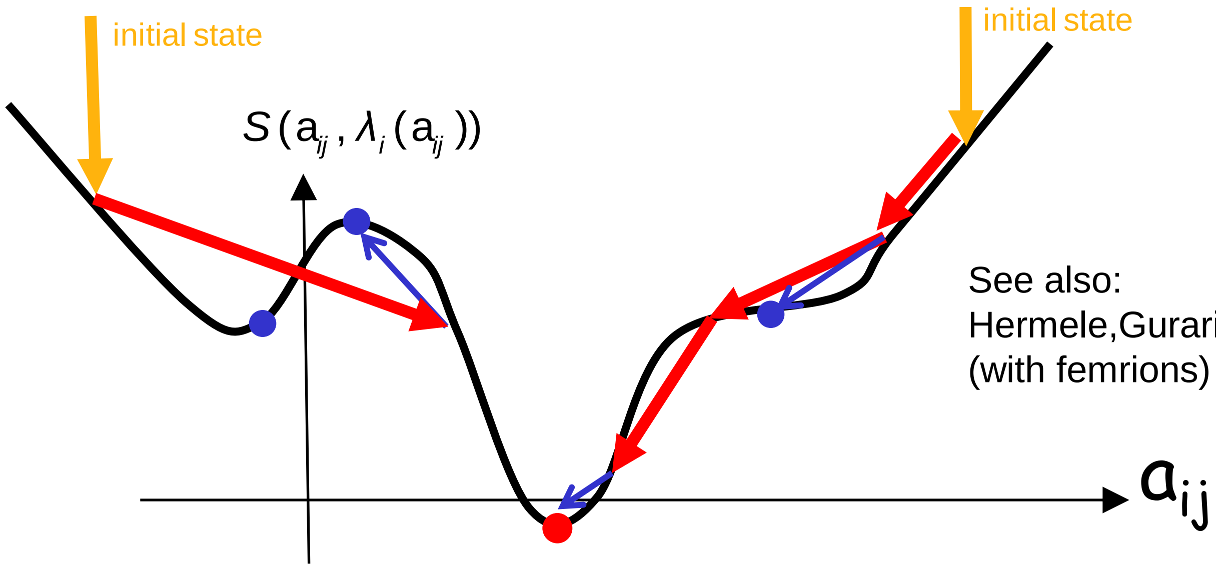
# An algorithm to search for saddle-points

- Two stages:

1. iterations of the self-consistency equations: $a_{ij} \to \left\langle \hat{A}_{ij} \right\rangle \to a_{ij} \to \left\langle \hat{A}_{ij} \right\rangle \to \cdots$
2. Minimize the energy derivatives using the Levenberg-Marquardt method.
   (= adaptative steepest-descend + Gauss-Newton)

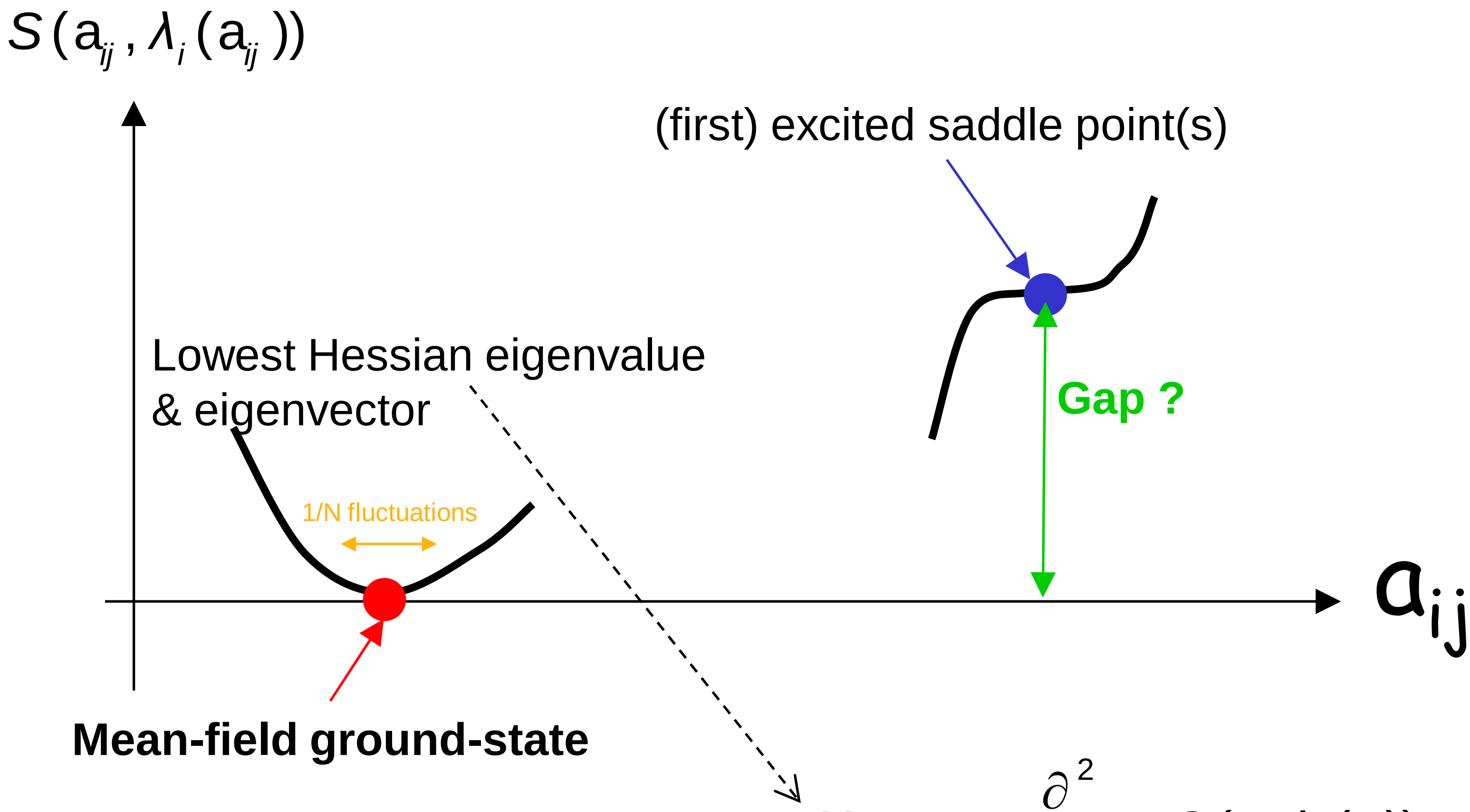
NB: No so fast, because each step requires to compute the $\lambda_i$ as a function of the $a_{ij}$

- Repeat with thousands of random initial conditions
  until no new low-energy solution is found.
  
  $\to$ list of low-energy local extrema and saddle-points




See also:
Hermele,Gurarie & Rey PRL 2009
(with femrions)

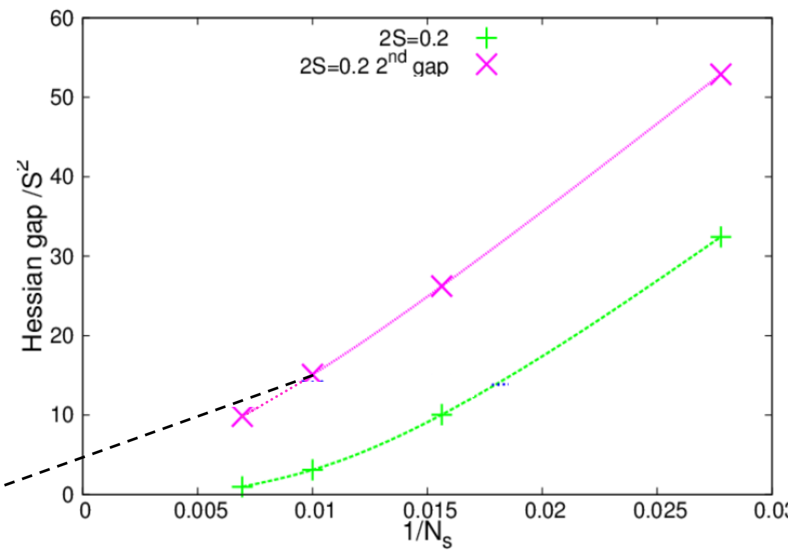
# Mean-field energy landscape




$$K_{\alpha\beta} = \frac{\partial^2}{\partial a_\alpha \partial a_\beta} S(a, \lambda_i(a))$$

# Square lattice $S<S_c$: Hessian

Mean-field ground-state: uniform, gapped spinons, unstable toward VBC at finite N [Read & Sachdev 89]




- ❑ Lowest Hessian eigenvalue(s): gapless in the thermodynamic limit

- ❑ corresponding Hessian eigenvector: gauge mode

$|a_{ij}|$ = constant but $a_l \rightarrow a_l e^{i\theta_l}$

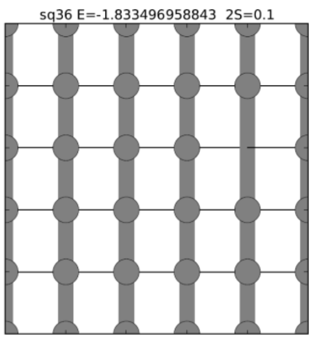
Long-wavelength ($k \sim 2\pi/L$) modulation of the flux on each plaquette $\phi_{1234} = \arg\left(a_1 \bar{a}_2 a_3 \bar{a}_4\right)$

→ U(1) gauge degrees of freedom
(Invariant Gauge Group IGG=U(1) [X.-G. Wen])

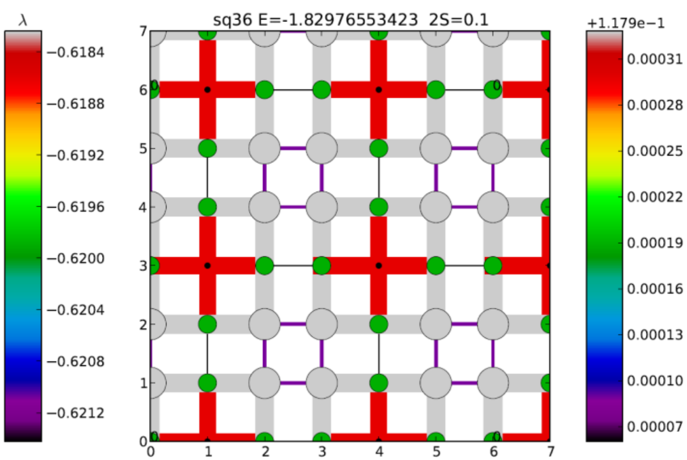
# Square lattice $S<S_c$: excited saddle-points

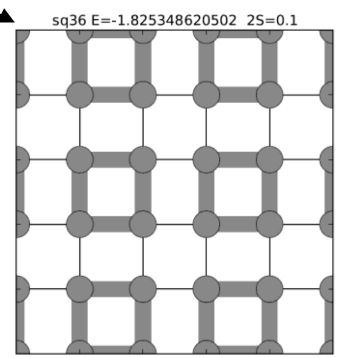
- High density of saddle points: energy spacing << spin gap Δ

(observe the tiny spatial modulations → high numerical accuracy needed)



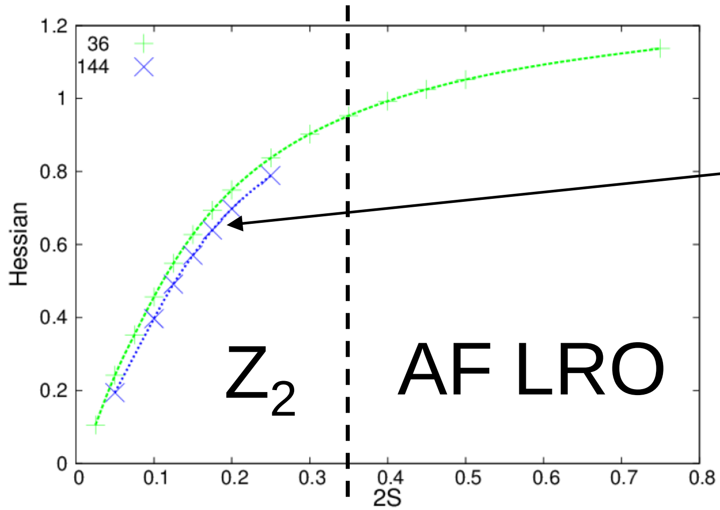

| $E$ | $\Delta$ | $H$ | d | $N_\lambda$ | $\min_\lambda$ | $\max_\lambda$ | $N_{\vert A\vert}$ | $\min_{\vert A\vert}$ | $\min_{\vert A\vert}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| -1.83492169860 | 0.40562447718 | 0.0514862091 | 1 | 1 | -0.62322660824 | -0.62322660824 | 1 | 0.11828994799 | 0.11828994799 | R |
| -1.83349695884[a] | 0.43766690295 | -0.0513757991 | 2 | 1 | -0.62174106995 | -0.62174106995 | 2 | 0.11707639293 | 0.11940834877 | R |
| -1.83265972180 | 0.42301616405 | 3.3891521814e-05 | 36 | 6 | -0.62283957972 | -0.61901747505 | 12 | 0.11749362973 | 0.11895083747 | C |
| -1.83265969367 | 0.42301757051 | -3.3879532685e-05 | 72 | 12 | -0.62289751868 | -0.61872552844 | 25 | 0.11742978176 | 0.11900687718 | C |
| -1.83265966553 | 0.42301846049 | -3.3877477372e-05 | 36 | 10 | -0.62295399258 | -0.61842513968 | 12 | 0.11745616476 | 0.11898150831 | C |
| -1.83162471056 | 0.41456789451 | -0.2513880759 | 36 | 6 | -0.62576828556 | -0.60318829800 | 12 | 0.10370572327 | 0.13006171701 | C |
| -1.83121683711 | 0.41375814481 | -0.2872095650 | 72 | 12 | -0.63567288324 | -0.58931901135 | 25 | 0.03077944153 | 0.13945180759 | C |
| -1.83088353771 | 0.43032855876 | -0.2737625727 | 72 | 9 | -0.62660943591 | -0.60282776054 | 24 | 0.10017310034 | 0.13081740450 | C |
| -1.83047599540 | 0.42976957584 | -0.3191268237 | 72 | 12 | -0.63670770779 | -0.58921985280 | 25 | 0.01544819043 | 0.13979752631 | C |
| -1.83036799495 | 0.43107796748 | -0.3414892108 | 72 | 12 | -0.63601360482 | -0.58745590945 | 25 | 0.00522910878 | 0.14104278833 | C |
| -1.83004203619 | 0.43897814318 | -0.2774384716 | 36 | 6 | -0.62530891168 | -0.60252710044 | 12 | 0.10277137746 | 0.13077306110 | C |
| -1.82976553423[c] | 0.44236450064 | 0.0004197766 | 9 | 3 | -0.62142264885 | -0.61823831215 | 4 | 0.11796026878 | 0.11822830533 | C |
| . . . | | | | | | | | | | |
| -1.82534862050[b] | 0.45598298556 | 0.0049547827 | 4 | 1 | -0.61651000360 | -0.61651000360 | 2 | 0.11510636037 | 0.12084117665 | C |



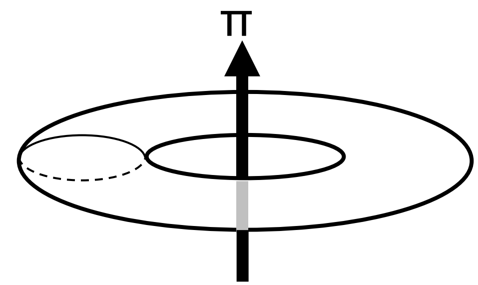




- Relation to valence-bond crystal formation at finite N ? [Read & Sachdev 89, 91]

# Triangular lattice S<S$_{c:}$ $Z_2$ liquid phase




- Gapped Hessian (discrete IGG=$Z_2$)




Topologically degenerate ground-states

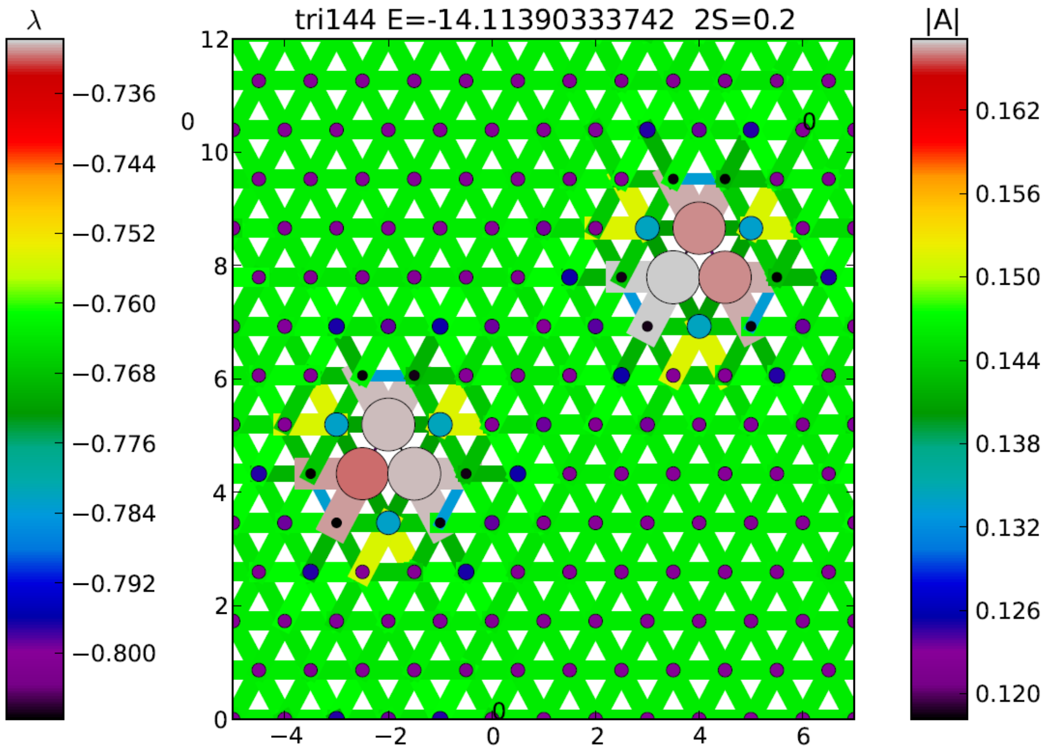
- Excited saddle points (N=36, 2S=0.1)

| $E$ | $\Delta$ | $H$ | d | $N_\lambda$ | $\min_\lambda$ | $\max_\lambda$ | $N_{\vert A\vert}$ | $\min_{\vert A\vert}$ | $\max_{\vert A\vert}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| -1.77115513853 | 0.39183413870 | 0.2418792448 | 1 | 1 | -0.63927958247 | -0.63927958247 | 1 | 0.09721004221 | 0.09721004221 | R |
| -1.76857445623 | 0.43969994315 | 0.1404951036 | 3 | 1 | -0.63620536809 | -0.63620536809 | 2 | 0.09694182615 | 0.09756074413 | R |
| -1.76558449027 | 0.40008559551 | -0.0136673394 | 108 | 13 | -0.64036695527 | -0.61163933625 | 30 | 0.08631046645 | 0.10220320383 | C |
| -1.76556438644 | 0.39881698544 | -0.0662494156 | 216 | 21 | -0.64094448900 | -0.60807754792 | 57 | 0.07142251452 | 0.11045947514 | C |
| -1.76519815707 | 0.40323866935 | -0.0476017475 | 216 | 21 | -0.64101212209 | -0.62149883058 | 57 | 0.08166138572 | 0.10941122095 | C |
| -1.76517640350 | 0.40348584568 | -0.0574617718 | 432 | 36 | -0.64242077150 | -0.62028199544 | 108 | 0.08102625728 | 0.11240116020 | C |
| -1.76512811482 | 0.40255745524 | -0.0675887556 | 216 | 21 | -0.64418307267 | -0.62192577975 | 57 | 0.07214207918 | 0.11553853523 | C |
| -1.76498033616 | 0.40426965800 | -0.0822323880 | 216 | 18 | -0.64027903292 | -0.62235496321 | 56 | 0.07818420245 | 0.11549233664 | C |
| -1.76494386569 | 0.40647240030 | -0.0077675537 | 108 | 12 | -0.64031335557 | -0.61976563929 | 31 | 0.08656330909 | 0.10201782841 | C |
| -1.76490270376 | 0.41103320490 | 0.0552851450 | 216 | 18 | -0.64030870204 | -0.62463756694 | 56 | 0.08904499619 | 0.10413653140 | C |
| -1.76486322625 | 0.40885376933 | -0.0517770911 | 432 | 36 | -0.64144886953 | -0.62152076605 | 108 | 0.08370457723 | 0.10627245220 | C |
| -1.76481070231 | 0.40714839383 | -0.0588744760 | 216 | 18 | -0.64059451058 | -0.62076171059 | 56 | 0.08285231897 | 0.10673538608 | C |

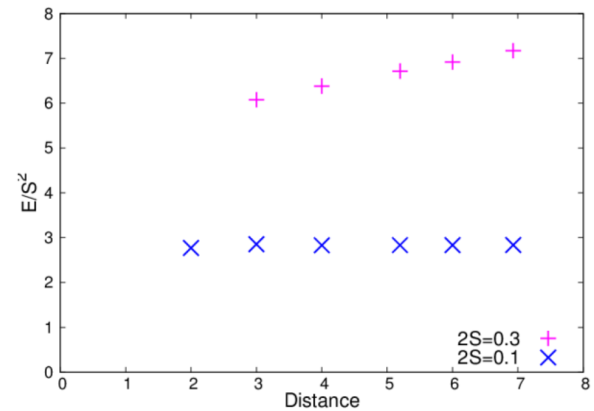
?

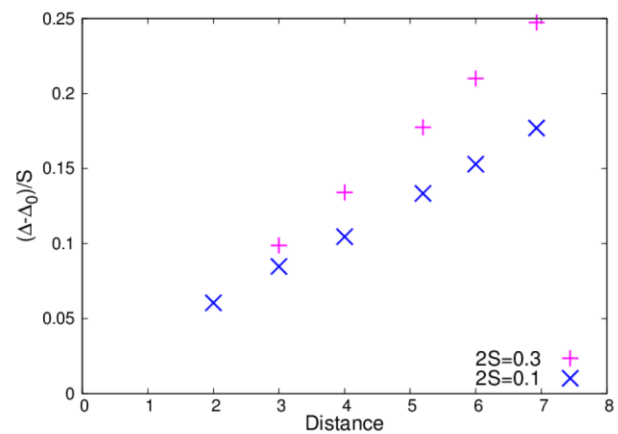
# Triangular lattice: $Z_2$ vortices in the gapped phase

- Bond amplitude (and λ) modulations in a vison-pair saddle-point




- Energy as a function distance



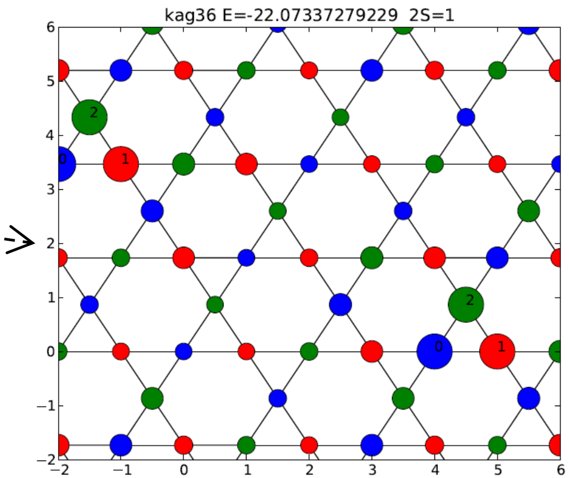




- The spinon gap Δ in presence of a vison-pair is *larger* than the gap $\Delta_0$ in absence of visons → spinon-vison repulsion

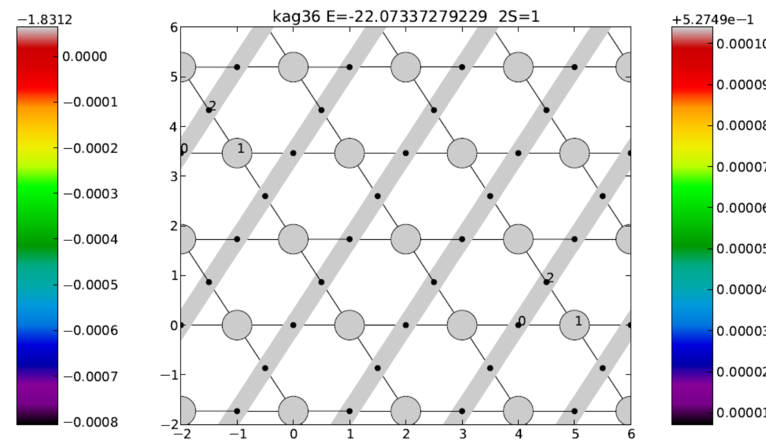
See also: Huh, Punk & Sachdev 2011

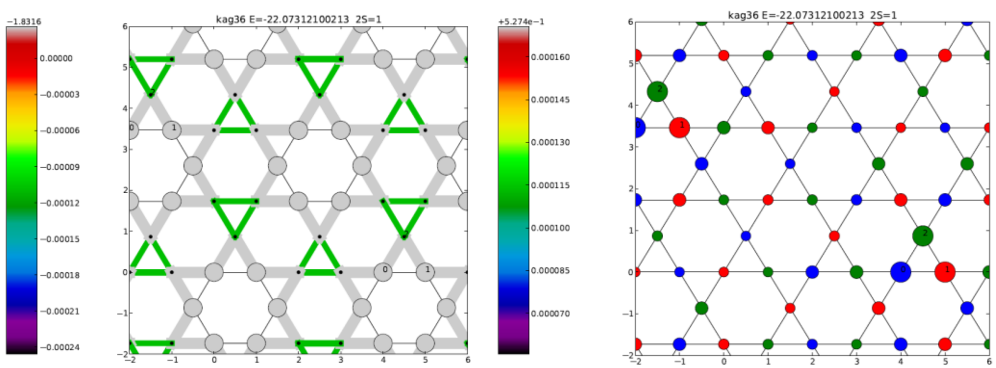
# Kagome lattice - preliminary results

- The mean-field ground-state is not the √3x√3 state ! (at least on 36, 48 and 108 sites clusters and S>0.2)




- High density of low-energy saddle-points
  - Some saddle-points are associated to classical planar configuration,
  but some other are not (complex fluxes / chiral)…
  - connection between VBC patterns & classical planar states



# Summary & future directions

So far:

- On small systems (<50 sites), we can find numerically the first low-energy saddle-points (+Hessian), without any symmetry assumtpion.
- The energy landscape gives some information concerning finite-N gauge degrees of freedom (photons, visons).

Next steps:

- More numerics on the kagome lattice
- Instanton calculation for tunneling processes between local minima ?
  (vison dynamics, …)