Spin-Majorana (and other) Dualities, Holography, and Deconfinement

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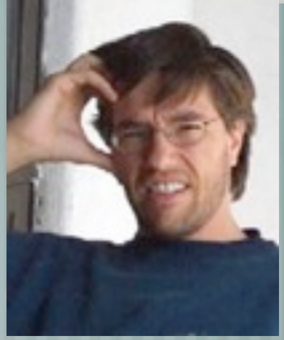

Gerardo Ortiz: Indiana University

NSF (DMR-1106293)

KITP - September 20, 2012

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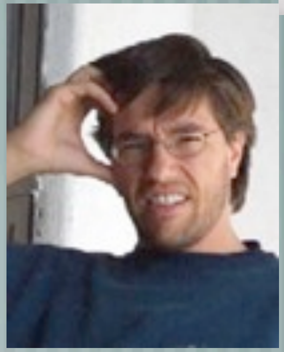

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Some routes to spin-liquid phases (and general topological orders)

General- dimensional reductions: low-dimensional systems do not exhibit long range order (symmetries rigorously lead to spin liquid type behavior)

Exact solutions: Including systems directly derived from the Hubbard model on pyrochlore lattices which exhibit exact deconfined excitations and “Kitaev like” systems

Dimensional reductions and spin liquids

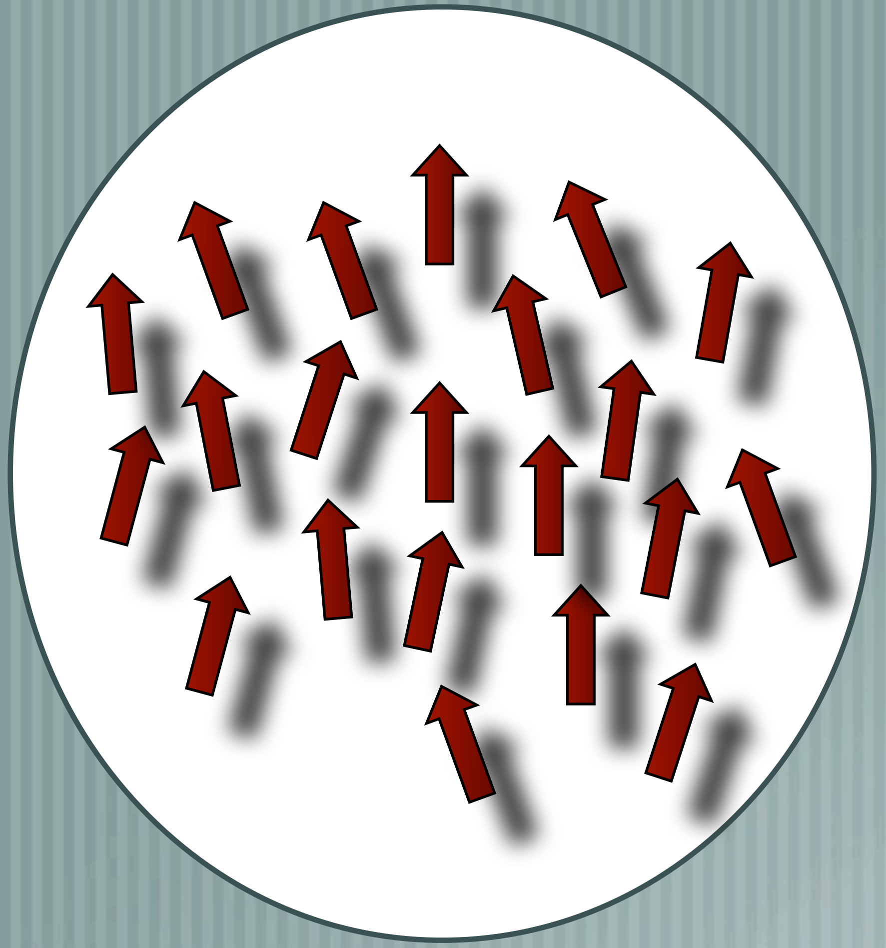
- Global Symmetry Breaking Orders (e.g. Magnets)
  Landau paradigm to matter classification
  in terms of an Order Parameter
- Topological Order (e.g., Spin Liquids, Quantum Hall, Gauge Theories) - no obvious broken symmetry

What characterizes topological orders ?

Non-local Order Parameters?????

Symmetry and Phase Transitions

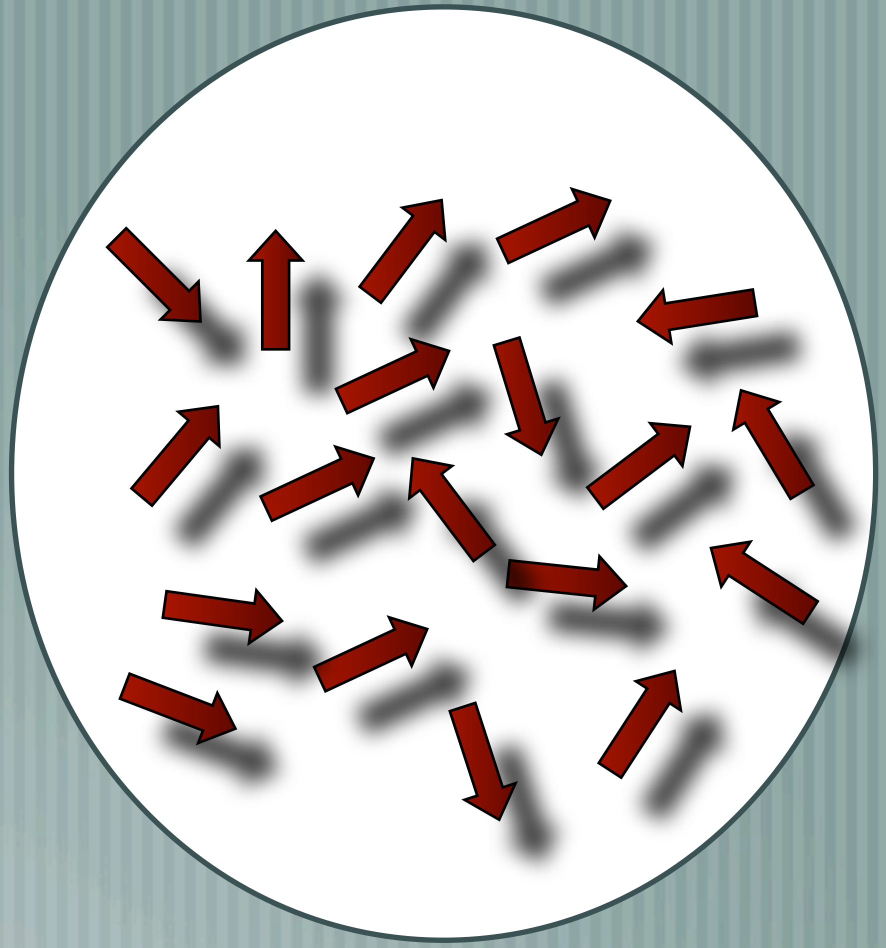
$T < T_c$



Broken Symmetry Phase

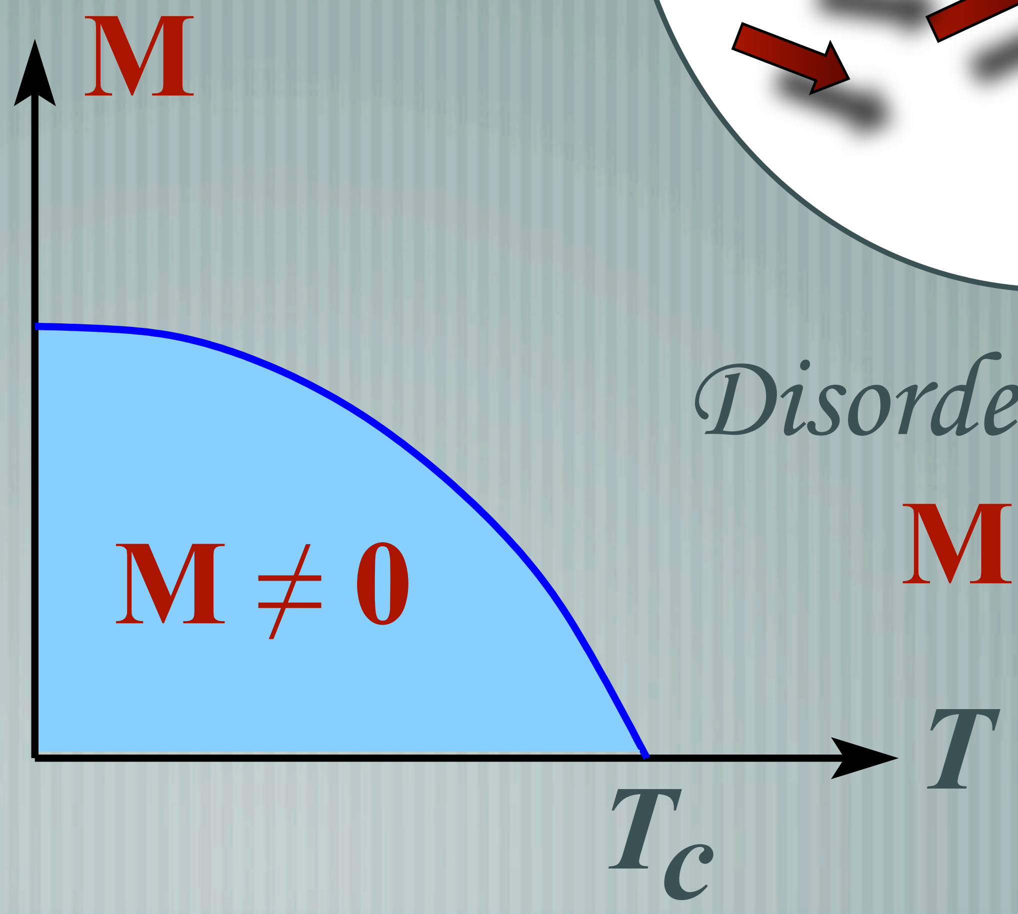
$M \neq 0$

$T > T_c$



Disordered Phase

$M = 0$



Local order parameters

In a ferromagnet, a local expectation value is different for different orthogonal ground states (GSs)

$$\langle g_\alpha|\hat{M}|g_\alpha\rangle \neq \langle g_\beta|\hat{M}|g_\beta\rangle \qquad T=0$$

Applying different boundary conditions can lead, at sufficiently low temperatures to spontaneous symmetry breaking

$$\langle \hat{M}\rangle_\alpha \neq \langle \hat{M}\rangle_\beta \qquad T\neq 0$$

Local Measurements can distinguish the GSs

What is TQO?

Colloquially, TQO is often very loosely referred to as order whose GS degeneracy depends on the surface topology of the manifold on which the physical system is embedded.

Our working definition: Robustness

Non-Distinguishability: Given a quasi-local operator $\hat{V}^m$

$$\boxed{\langle g_\alpha | \hat{V}^m | g_\beta \rangle = c\,\delta_{\alpha\beta},\ \forall\,\alpha,\beta \in \mathcal{S}_0,}$$

Perturbation Theory:

$$\langle g_\alpha | \underbrace{\hat{V}\bar{G}_0\hat{V}\ldots\bar{G}_0\hat{V}}_{m \text{ factors } \hat{V}} | g_\beta \rangle = c\,\delta_{\alpha\beta},\ \forall\,\alpha,\beta \in \mathcal{S}_0$$

$$\bar{G}_0 = (\epsilon_0 - H_0)^{-1}\hat{P}_\perp$$

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What is TQO?

Order is evident only in non-local (topological) quantities

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What is TQO?

Order is evident only in non-local (topological) quantities

Order hidden to ordinary local probes

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Non-Distinguishability: Given a quasi-local operator $\hat{V}^m$

$$\langle g_\alpha | \hat{V}^m | g_\beta \rangle = c \, \delta_{\alpha\beta}, \ \forall \, \alpha, \beta \in \mathcal{S}_0,$$

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$$\bar{G}_0 = (\epsilon_0 - H_0)^{-1} \hat{P}_\perp$$

Gauge-Like-Symmetries (lead to dim. reductions and TQO)

Given a D-dim theory:

A d-dim **GLS** is a group of transformations that leave the theory invariant such that the minimum non empty set of fields that are changed under the symmetry operation occupies a d-dim region

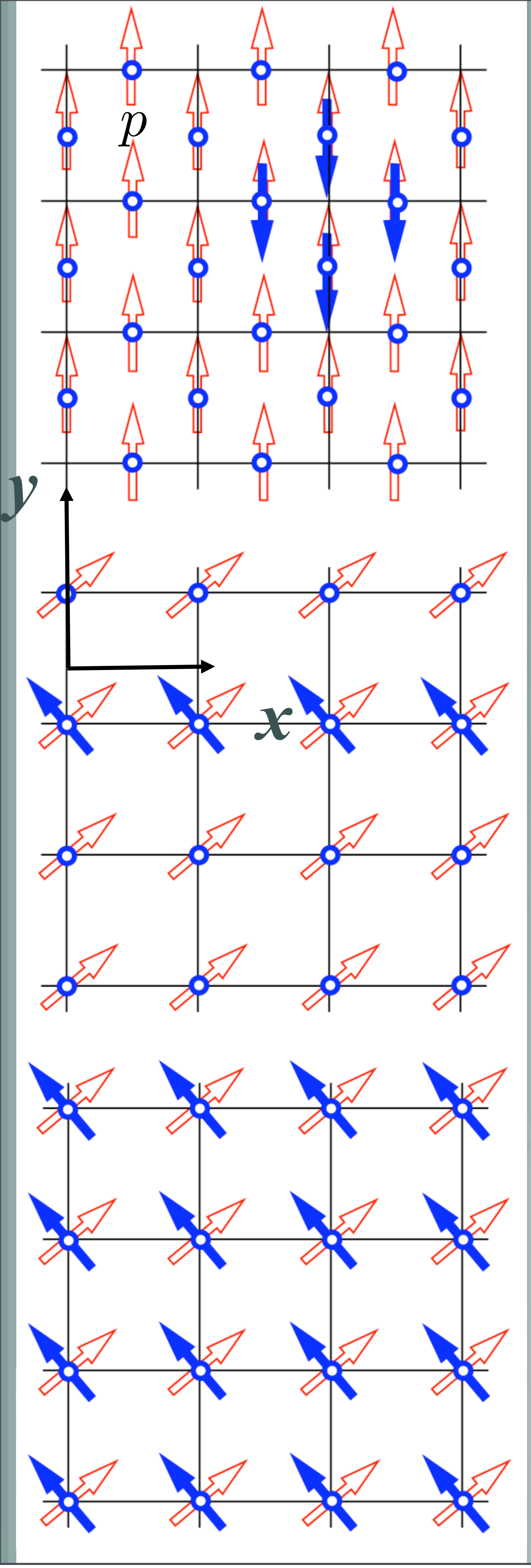
$$d \leq D$$

d=0 (Gauge)

$d < D$ (Gauge-Like)

d=D (Global)

Group: $\mathcal{G}_d$



Gauge-Like-Symmetries $D = 2$

$d = 0$ (Ising Gauge Theory)

$$H = -K \sum_{p} \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li} \qquad G_i = \prod_{s \in \mathrm{nn}} \sigma^x_{is}$$

$d = 1$ (Orbital Compass Model)

$$H = -\sum_{i} [J_x \sigma^x_i \sigma^x_{i+\hat{e}_x} + J_y \sigma^y_i \sigma^y_{i+\hat{e}_y}]$$

$$O^x = \prod_{j \in C_x} i\sigma^x_j \qquad O^y = \prod_{j \in C_y} i\sigma^y_j$$

$d = D = 2$ (XY model)

$$H = -J \sum_{\langle ij \rangle} [\sigma^x_i \sigma^x_j + \sigma^y_i \sigma^y_j]$$

$$U(\theta) = \prod_{j} \exp[-(i/2)\theta\sigma^z_j]$$

Exactly solvable systems with fractionalized deconfined excitations.

Example: the half-filled Hubbard model on the pyrochlore

$$H_{\mathrm{Hubb}} = -t \sum_{\langle ij\rangle,\sigma} d^{\dagger}_{i\sigma} d_{j\sigma} + U \sum_{i} n_{i\uparrow} n_{i\downarrow},$$

$$\tilde{H}_{\mathrm{Hubb}} = H + J_3 \sum_{\langle\langle ij\rangle\rangle} \vec{S}_i \cdot \vec{S}_j$$ - effective 4th order Hamiltonian at half-filling

$$\begin{aligned} J_1 &= \frac{4t^2}{U} - \frac{160t^4}{U^3} + \mathcal{O}\left(\frac{t^6}{U^5}\right), \quad J_3 = \frac{4t^4}{U^3} + \mathcal{O}\left(\frac{t^6}{U^5}\right) \\ J_2 &= \frac{40t^4}{U^3} + \mathcal{O}\left(\frac{t^6}{U^5}\right). \end{aligned}$$

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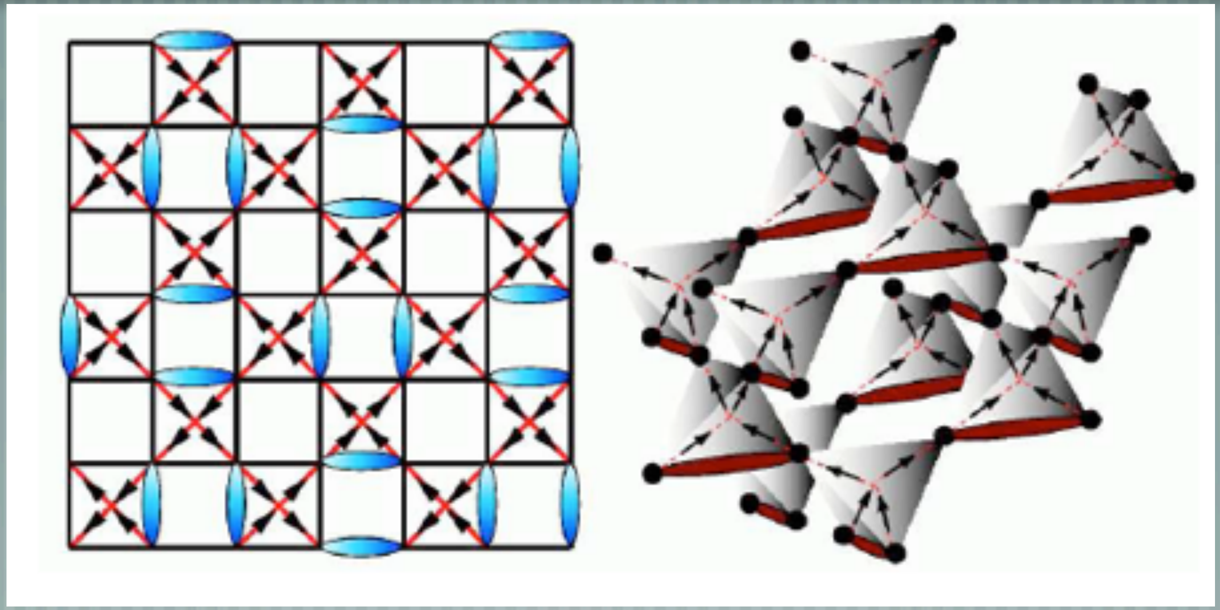
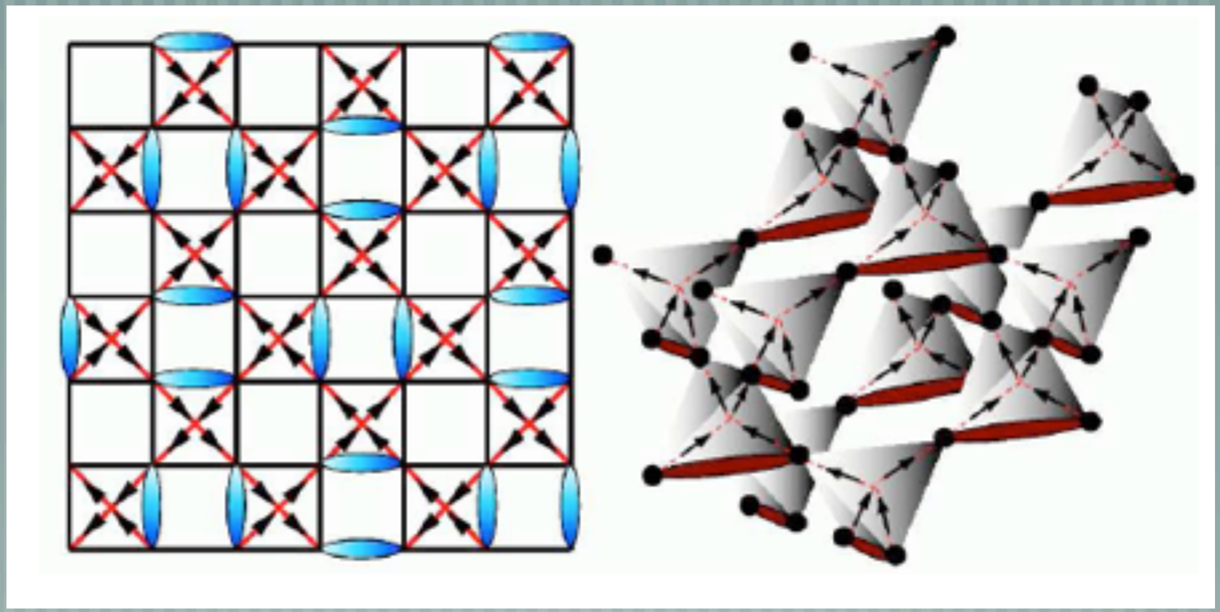
Fractionalization and deconfinement on the pyrochlore lattice



$$H_{Klein} = \frac{J_1}{2}\sum_{\boxtimes}\vec{S}_{\boxtimes}^{2} + \frac{J_2}{4}\sum_{\boxtimes}\vec{S}_{\boxtimes}^{4}$$

$\vec{S}_{\boxtimes}$ is the total spin of a tetrahedral unit



$$H_{Klein} = \frac{J_1}{2}\sum_{\boxtimes}\vec{S}^2_{\boxtimes} + \frac{J_2}{4}\sum_{\boxtimes}\vec{S}^4_{\boxtimes}$$

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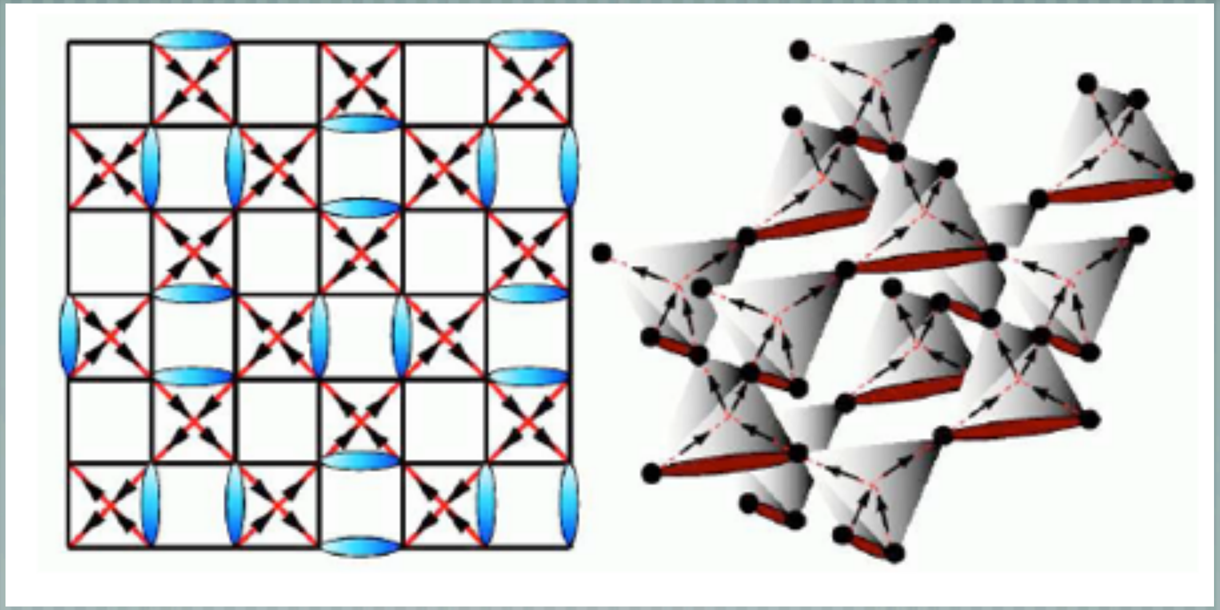
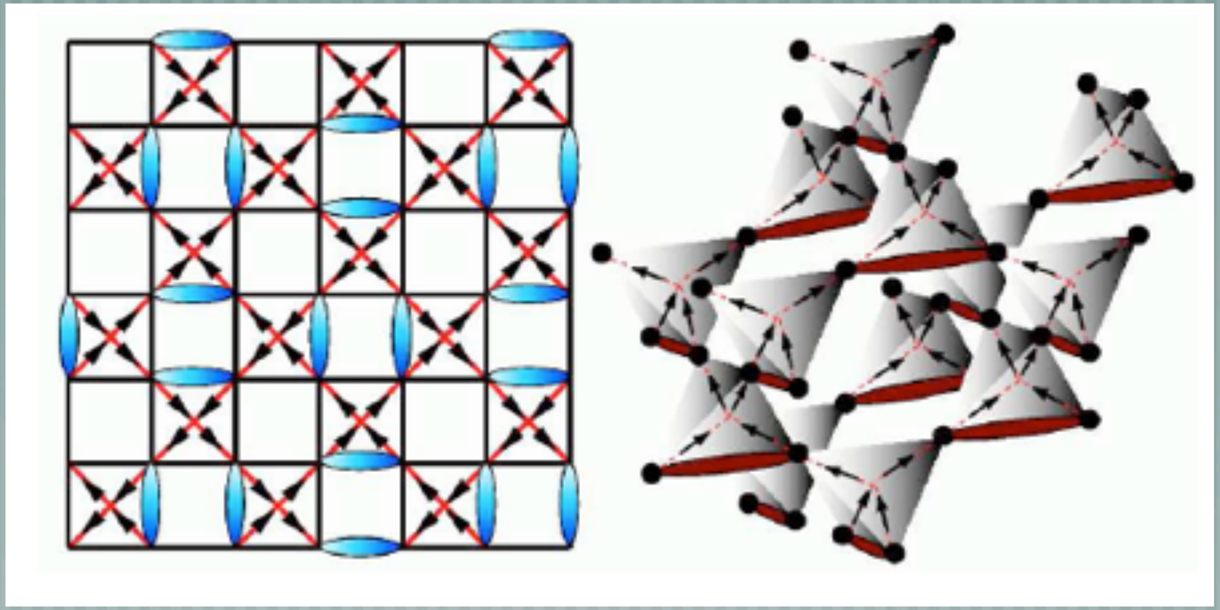


$J_2 = -J_1$ $\qquad$ $(K = K_c = 4J/5)$

Intra-unit projection operator onto maximal total spin

$$H_K = \frac{12}{5} J \sum_{\boxtimes} \mathcal{P}^{\boxtimes}$$



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Fractionalization and deconfinement on the pyrochlore lattice

All ground states are linear superpositions of dimer states.

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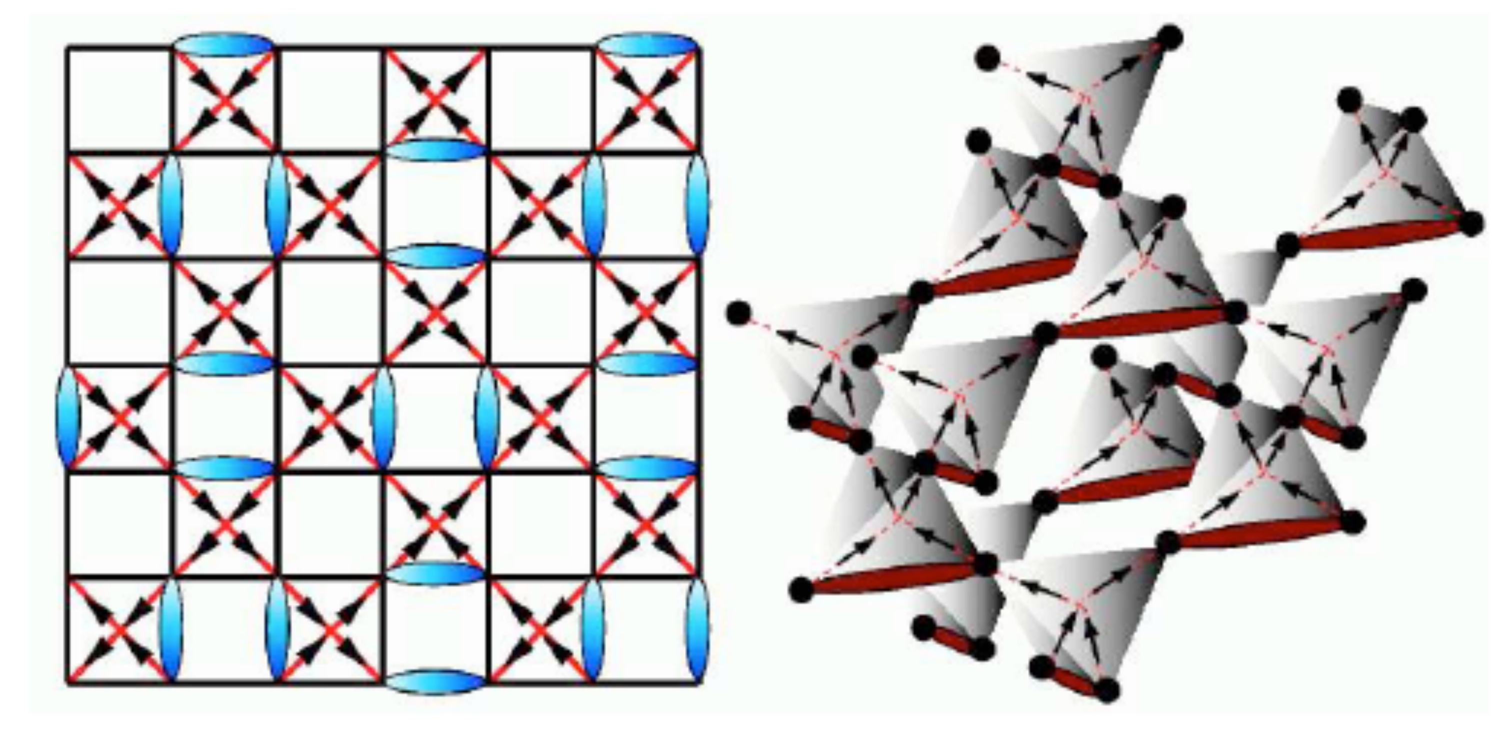

Provable consequences: deconfined excitations, spin-charge separation,..., extensive degeneracy and critical correlations in an extended finite temperature region about solvable point (the latter assuming a gap and linear independence)

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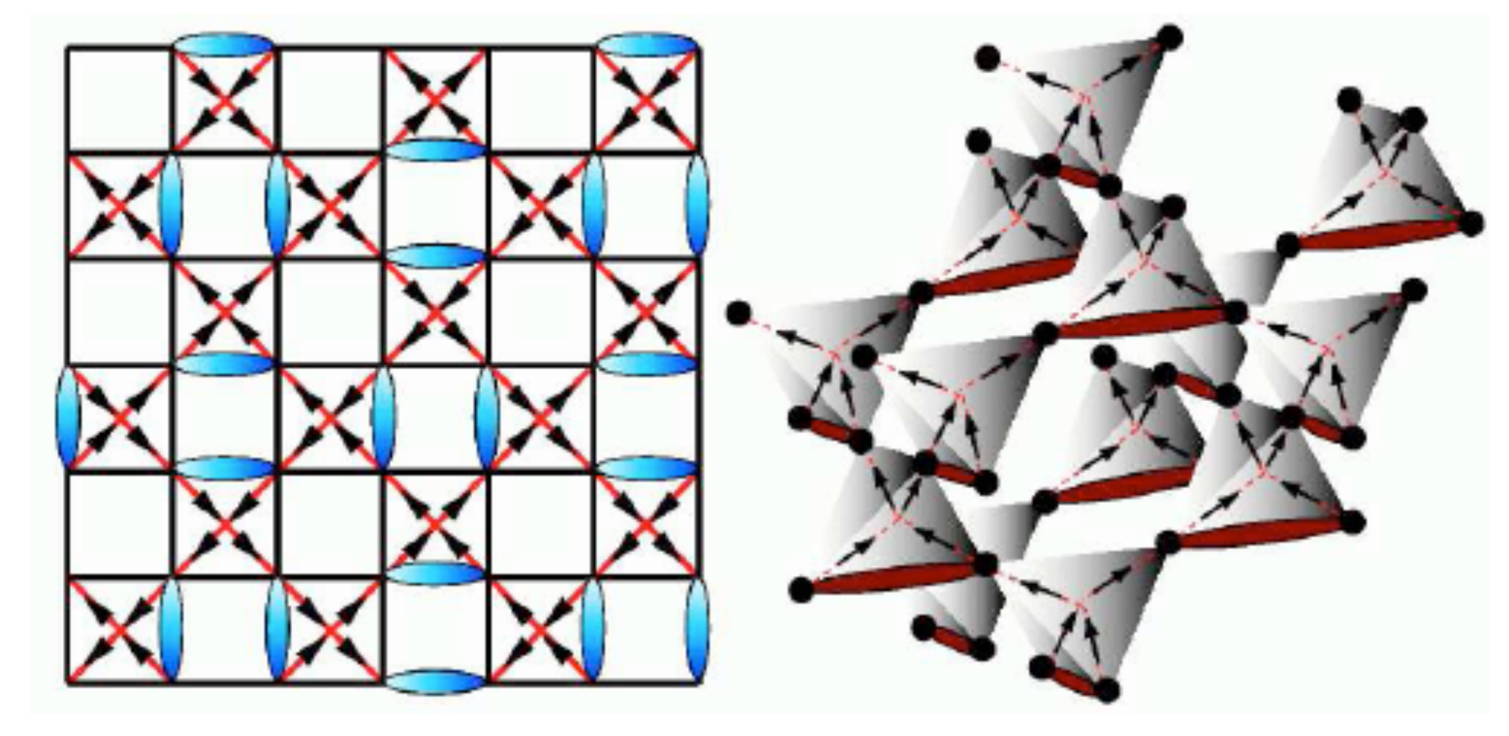

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Fractionalization and deconfinement on the pyrochlore lattice

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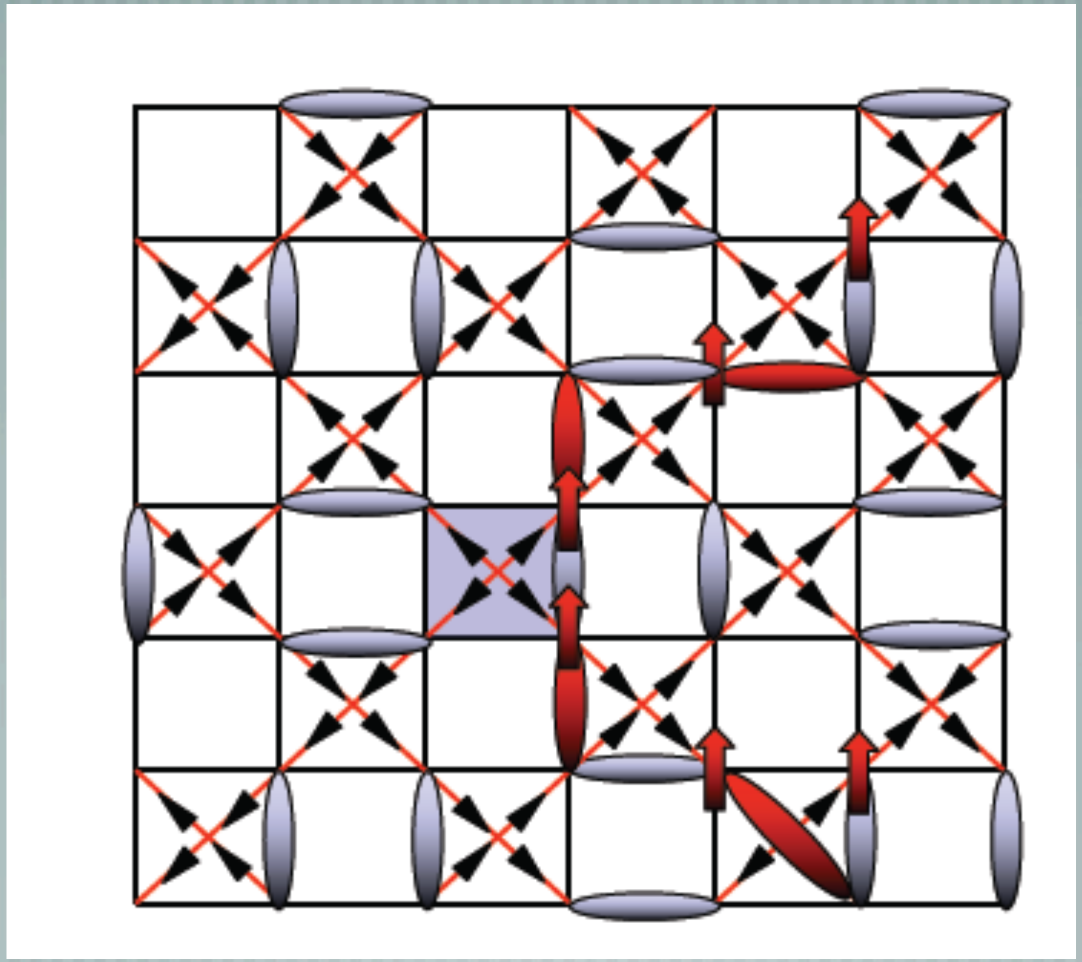
Trivially exact deconfined excitations



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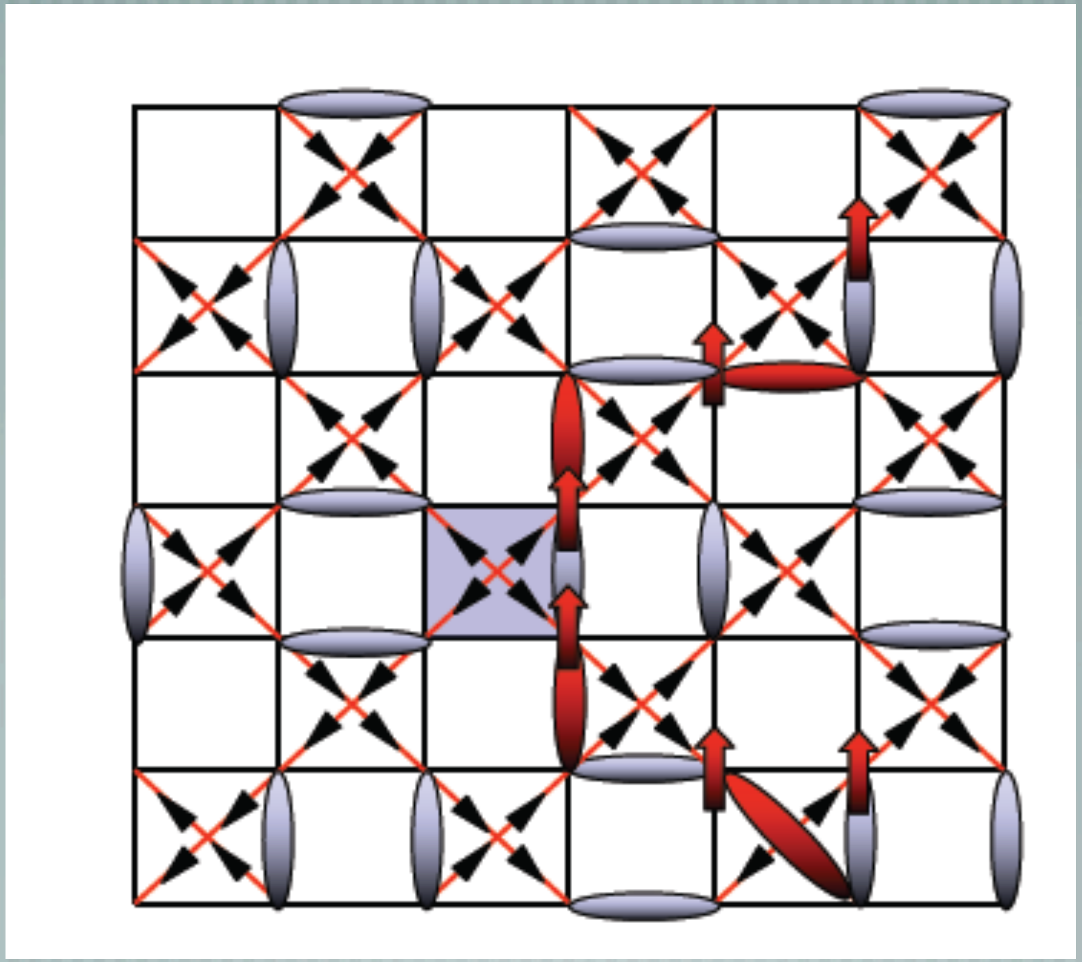

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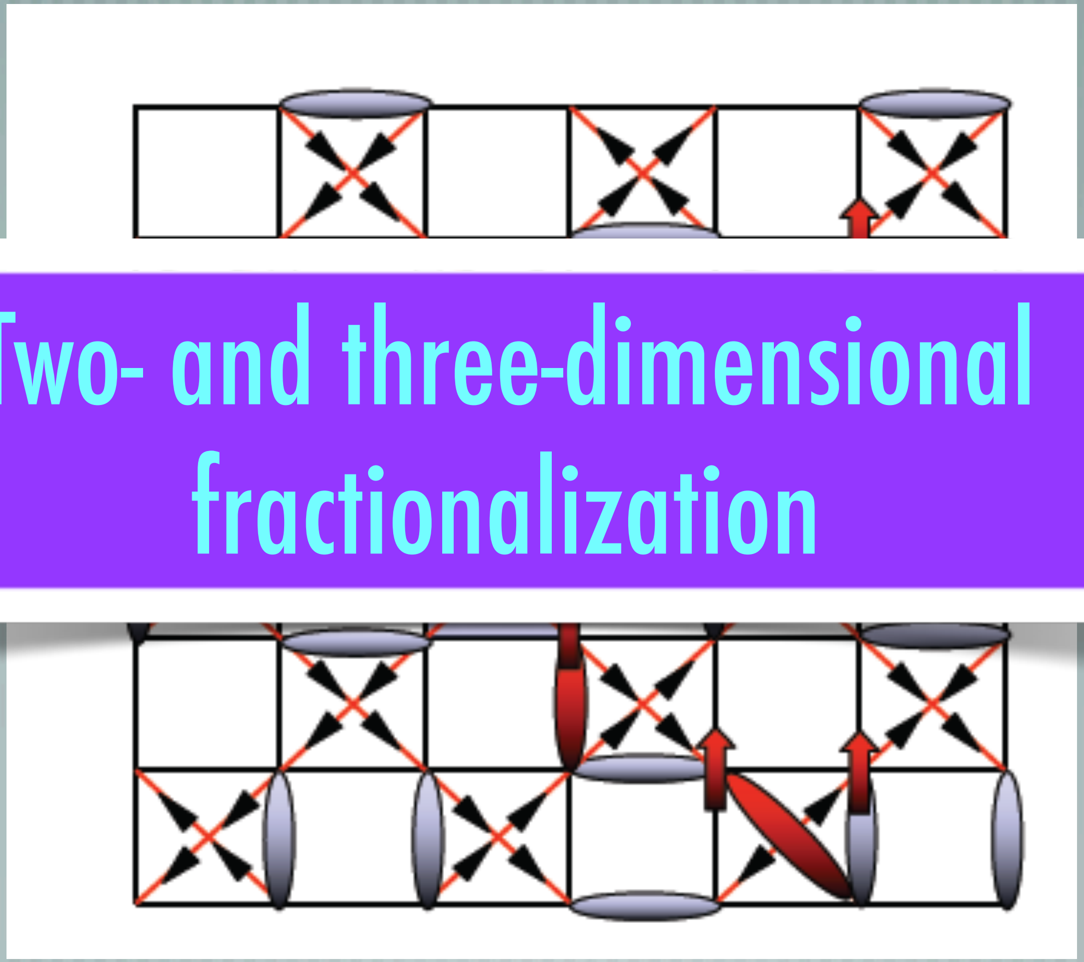


Two- and three-dimensional fractionalization

Some solvable spin-liquids (e.g., Kitaev's honeycomb model) are simple Majorana systems in disguise

The elusive Majorana fermion

Ettore Majorana: 1906-1938(?)

1937: "Real" counterpart to a Dirac fermion

$$\{c_{li}, c_{l'i'}\} = 2\delta_{l,l'}\delta_{i,i'}, \qquad c_{li}^{\dagger} = c_{li}$$

The elusive Majorana fermion

The "real and imaginary parts" of a Dirac fermion are Majorana fermions.

A representation:

$$d_l \quad = \quad \frac{1}{\sqrt{2}}(c_{l1} + ic_{l2}),\ d_l^\dagger = \frac{1}{\sqrt{2}}(c_{l1} - ic_{l2})$$

Hilbert space dimension of N_s Majorana fermions scales as

$$2^{N_s/2}$$

The elusive Majorana fermion

High energy physics: neutrino(?)

Condensed matter: p-wave superconductors(?), interface between topological insulators and s-wave superconductors(?), Quantum Hall states(?), semiconductor wires on s-wave superconductors

V. M. Mourik et al (Science 2012)

This talk: Majorana-Pauli spin dualities

Most of the work to date focuses on non-interacting Majorana fermions. We wish to map interacting Majorana systems in an arbitrary number of dimensions to Pauli spin systems for which much is known.

Intermezzo

the tool: the bond-algebraic approach to dualities (incl. fermionization)

“Bond algebras” and their symmetries

Quantum Hamiltonians are built as a sum of quasi-local operators
We call these **BONDS**:

$$H = \sum_R J_R \mathcal{O}_R$$

A bond algebra for H is the set of all linear combinations of products of bonds

$$\mathcal{A}_H = \{1, \alpha\mathcal{O}_{\mathcal{R}}, \beta\mathcal{O}_R\mathcal{O}_{R'}, \mathcal{O}_R - \mathcal{O}_R\mathcal{O}_{R'}\mathcal{O}_{R''}, \cdots\}$$

Exposing Dualities

Exposing Dualities

Bonds are more fundamental objects than the elementary degrees of freedom

The special character of various systems including statistics of their basic constituents [Bose, Fermi (Dirac or Majorana), spin, or other], etc. may be irrelevant. In the calculation of most physically measurable quantities such as various non-vanishing correlation functions, entropies, complexities, and free energies, **only composite quantities (the bonds) appear.**

Space-time, momentum, spin (or other) coordinates are (generally non-unique) labels for bonds. Bonds can automatically be gauge or Lorentz invariant. It is possible to reformulate the quantum (and classical) problem using only measurable quantities. We reformulated electrodynamics with only the gauge invariant interaction terms (the bonds).

When are two Hamiltonian dual?

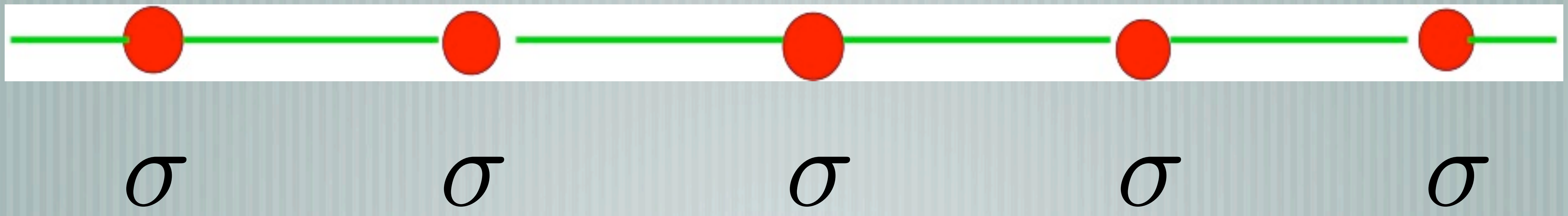
H_1 **and** H_2 **are dual if there is an homomorphism between their bond algebras**

DUALITIES **are one-to-one, onto mappings between bond algebras that preserve every algebraic relation between bonds:**

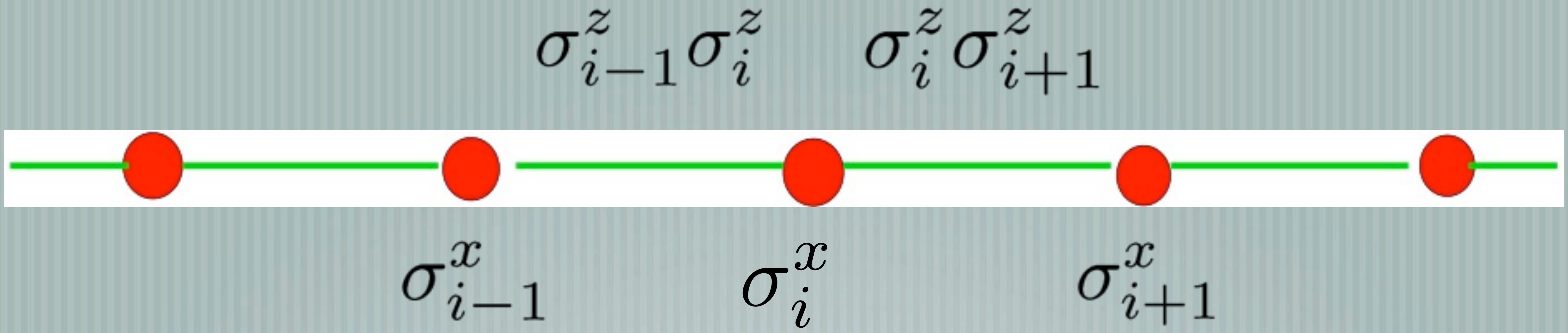
$$\mathcal{O}^1_{R_1} \leftrightarrow \mathcal{O}^2_{R_2}$$

Example of Self-Duality:
Ising chain in a transverse field

$$H[j,h] = \sum_i j\sigma_i^z \sigma_{i+1}^z + h\sigma_i^x$$



BOND ALGEBRA



$\sigma^z_{i-1}\sigma^z_i$ $\sigma^z_i\sigma^z_{i+1}$

σ^x_{i-1} σ^x_i σ^x_{i+1}

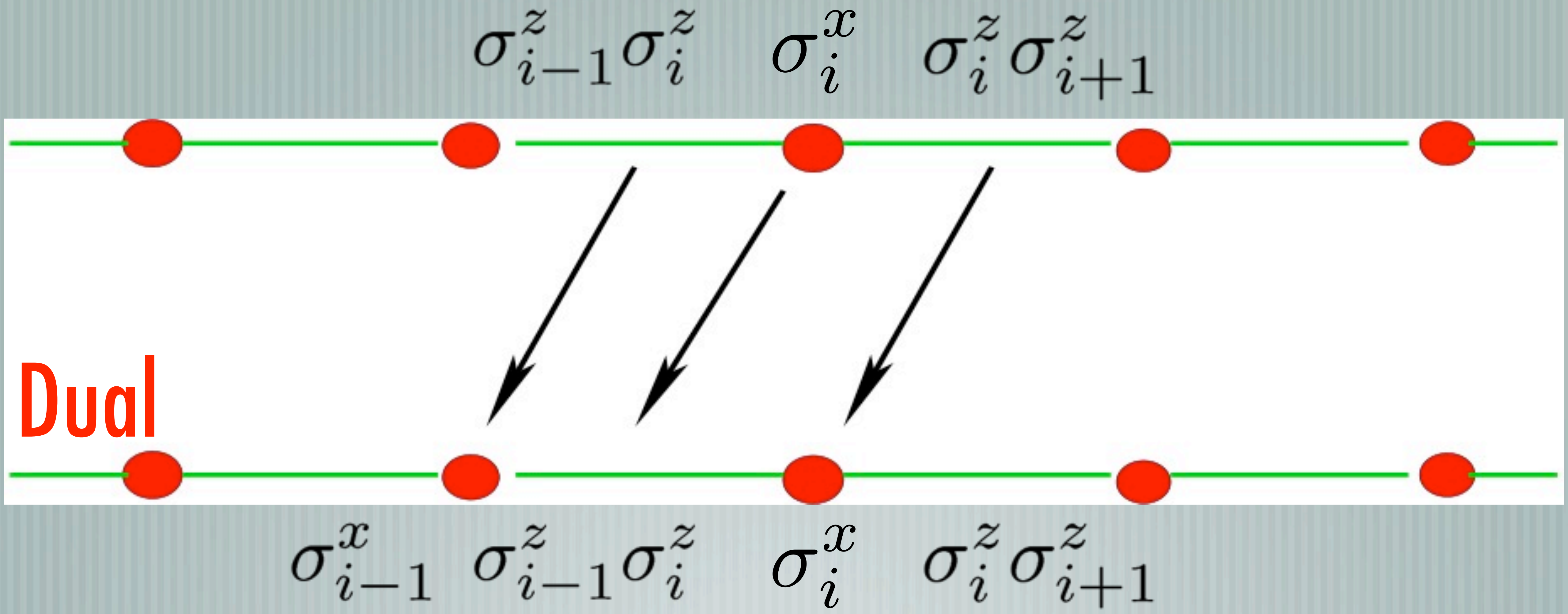
Every bond $\sigma^z\sigma^z$ anti-commutes with two bonds σ^x

Every bond σ^x anti-commutes with two bonds $\sigma^z\sigma^z$

SELF-DUALITY AUTOMORPHISM

Homomorphism Φ_D:

$$\sigma^z_i \sigma^z_{i+1} \mapsto \sigma^x_i \qquad\qquad \sigma^x_i \mapsto \sigma^z_{i-1}\sigma^z_i$$



Mapping is Unitarily implementable

$$\mathcal{U}_D \sigma^z_i \sigma^z_{i+1} \mathcal{U}^\dagger_D = \sigma^x_i \qquad \mathcal{U}_D \sigma^x_i \mathcal{U}^\dagger_D = \sigma^z_{i-1} \sigma^z_i$$

Ising chain in a transverse field is self-dual, meaning:

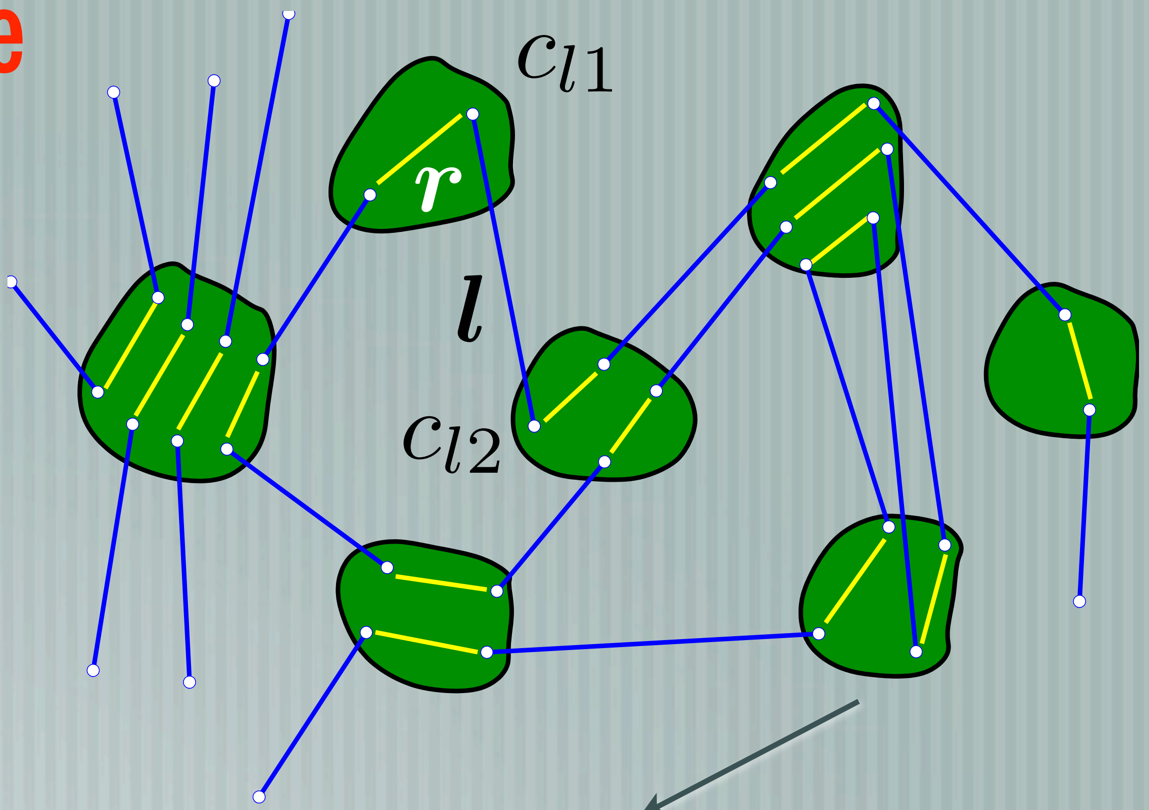
$$\mathcal{U}_D H[j, h] \mathcal{U}^\dagger_D = H[h, j]$$

$$j \leftrightarrow h$$

Interacting Majorana fermion - - Pauli spin Dualities

Interacting Majorana Wire Networks

Consider a semiconductor **Majorana wire network** in any number of dimensions:



$$H_{\mathsf{M}} = -i \sum_{l} J_l c_{l1} c_{l2} - \sum_{r} h_r \mathcal{P}_r,$$

Josephson tunneling

Charging energy

z_r wires per SC grain

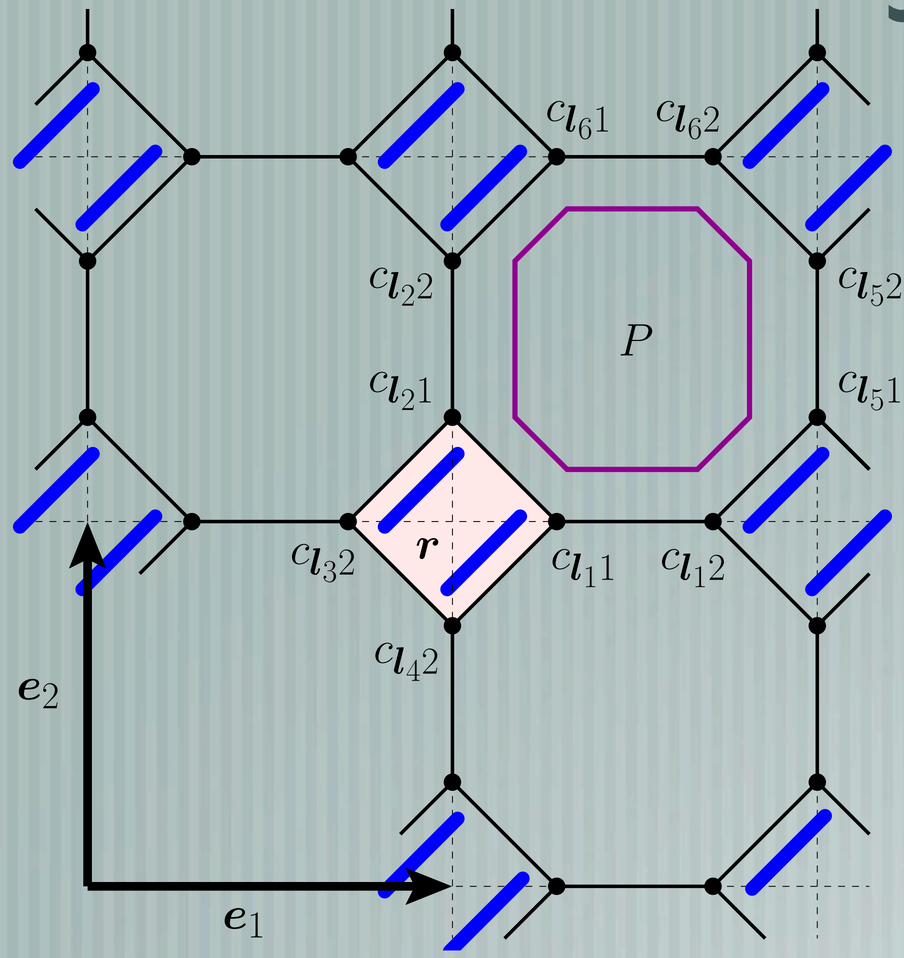
$$z_{r_2} = \begin{cases} 0 & \text{if} \quad z_r \text{ is even,} \\ 1 & \text{if} \quad z_r \text{ is odd.} \end{cases}$$

$$\mathcal{P}_r \equiv i^{z_{r_2}} \; c_{l_1 i_1} c_{l_2 i_2} \cdots c_{l_{q_r} i_{q_r}}, \; r \in l_1, \cdots, l_{q_r} \quad (q_r = 2 z_r)$$

Examples in D=2

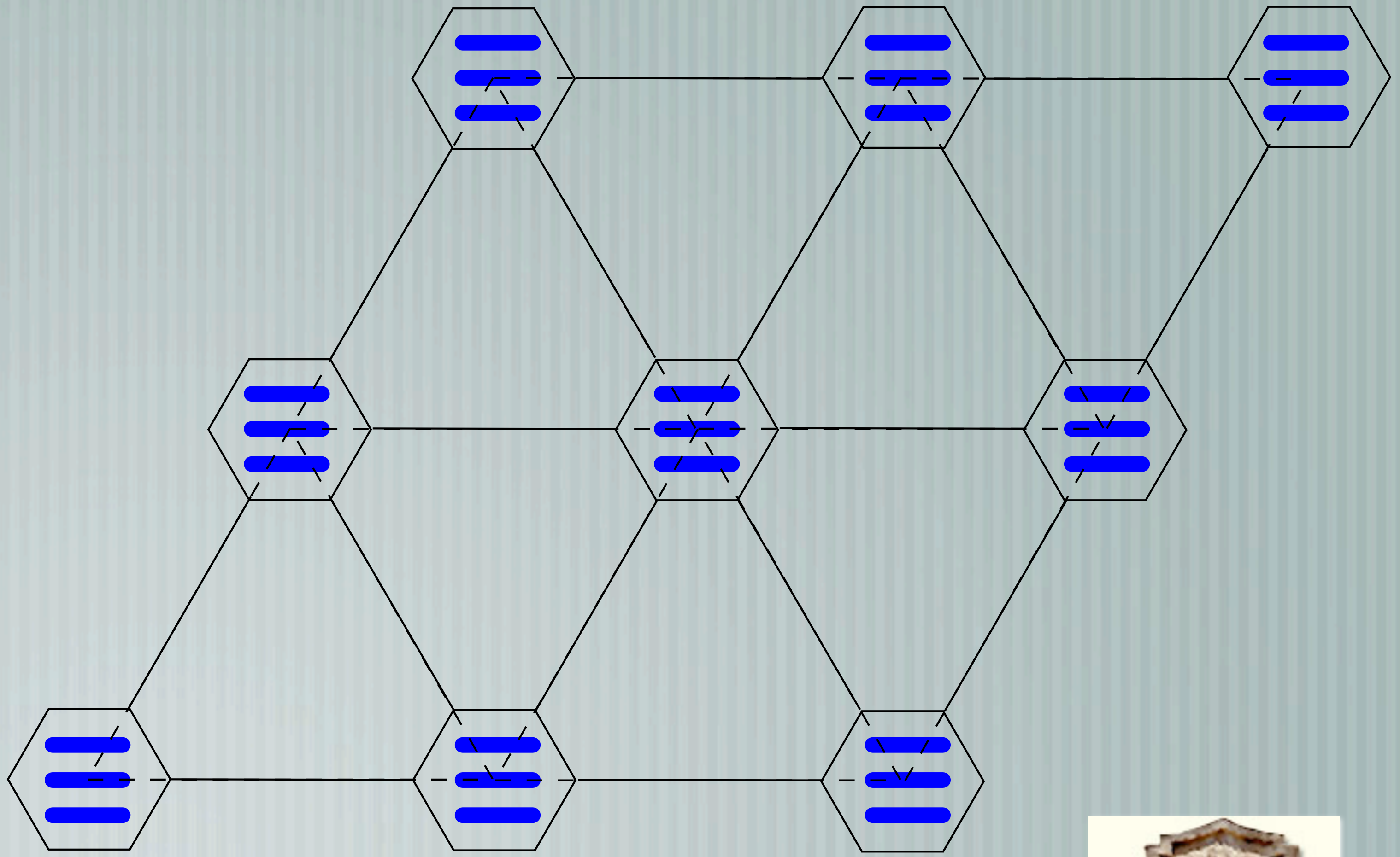
Square lattice Majorana wire architecture

(B. Terhal et al, arXiv:1201.3757)



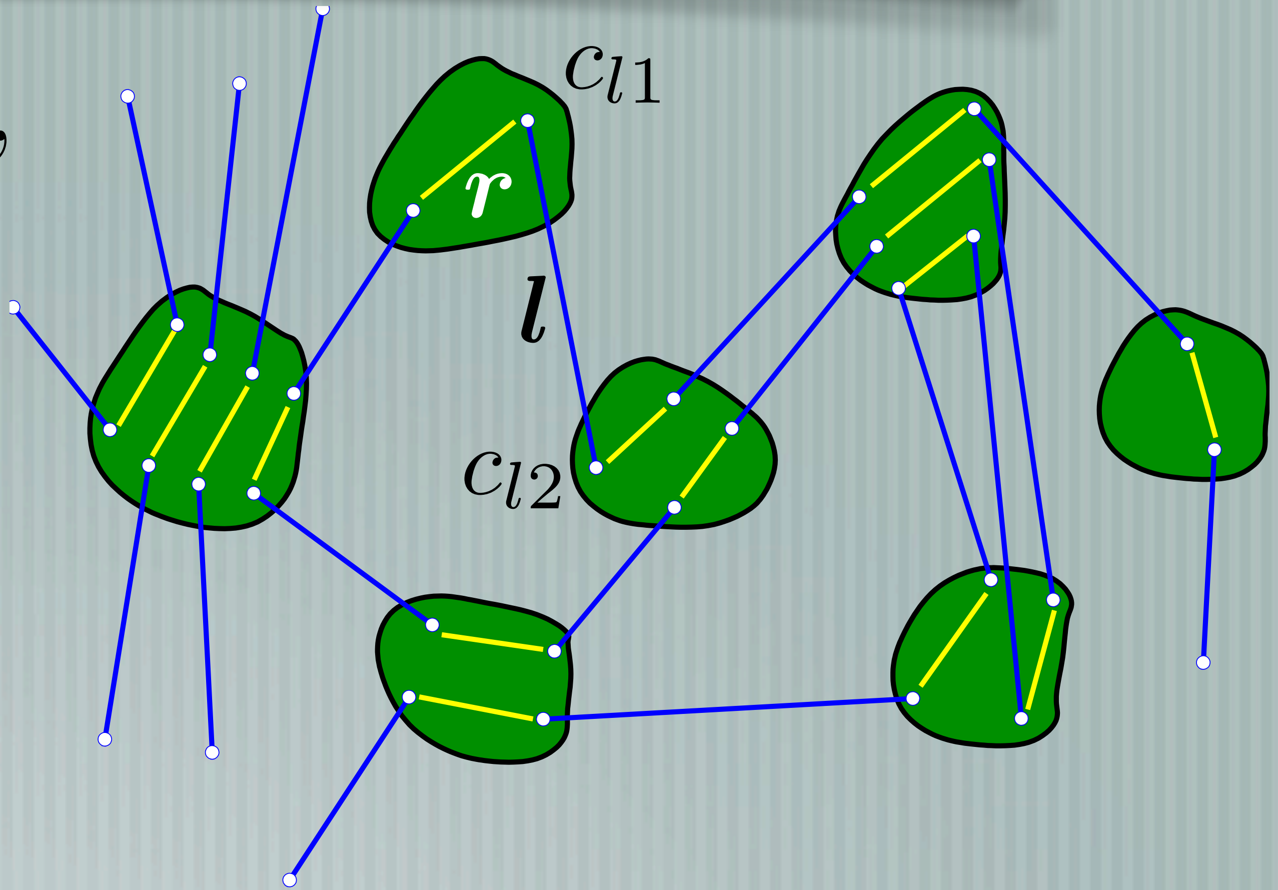
Triangular network

(Z. Nussinov et al, arXiv:1203.2983)



Interacting Majorana Bond Algebra

$$H_{\mathsf{M}} = -i \sum_{l} J_l c_{l1} c_{l2} - \sum_{r} h_r \mathcal{P}_r,$$



Bond algebra:

A. $(ic_{l1}c_{l2})^2 = 1 = (\mathcal{P}_r)^2,$

B. for $r, r' \in l$

$$\{\mathcal{P}_r, ic_{l1}c_{l2}\} = 0 = \{\mathcal{P}_{r'}, ic_{l1}c_{l2}\},$$

C. for $r \in l_i, \ i = 1, 2, \cdots, q_r$

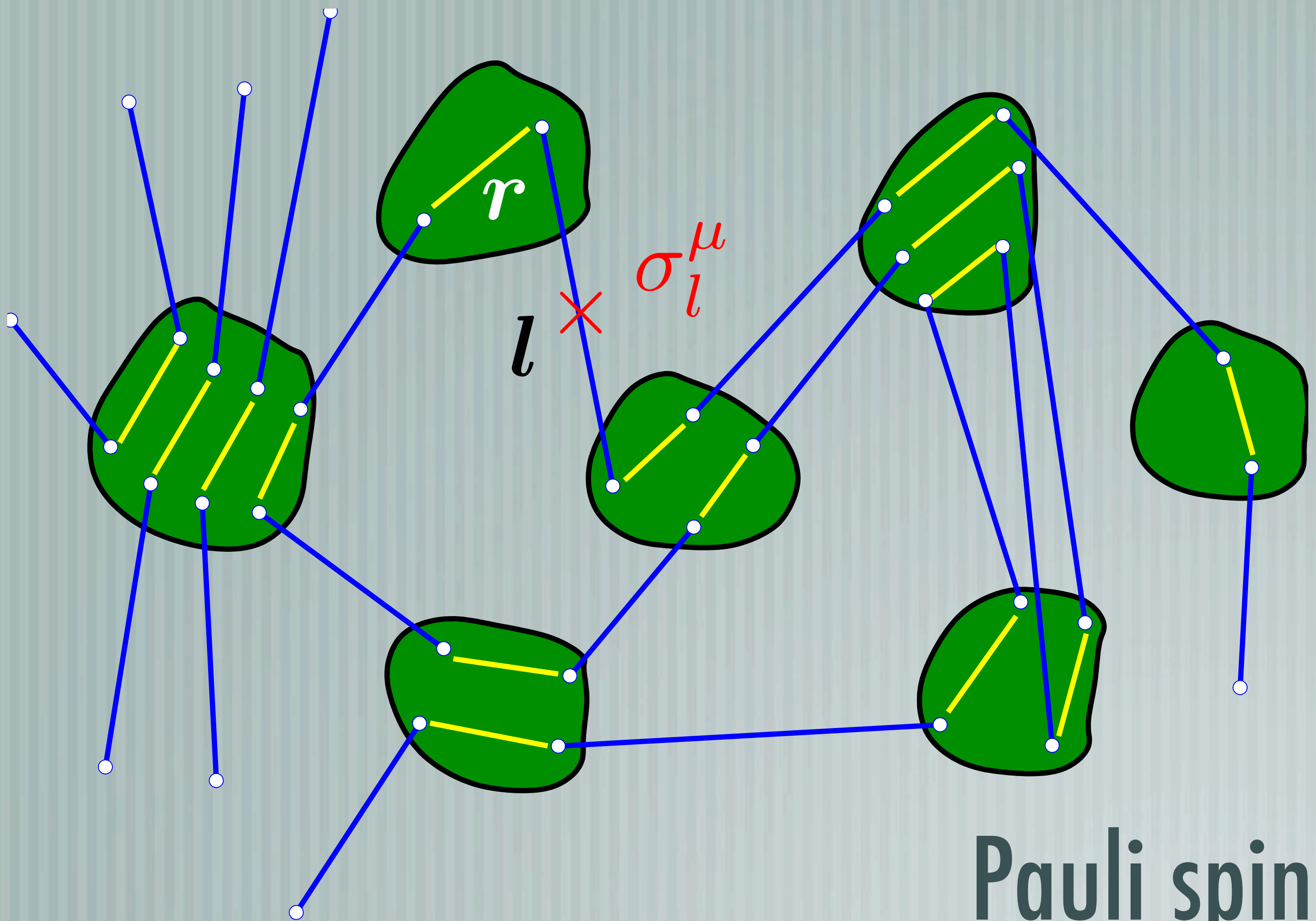
$$\{\mathcal{P}_r, ic_{l_i1}c_{l_i2}\} = 0.$$

Quantum Ising Gauge theories on planar networks

$$H_{\text{QIG}} = -\sum_{l} J_l \sigma_l^x - \sum_{r} h_r \widetilde{\mathcal{P}}_r$$

where

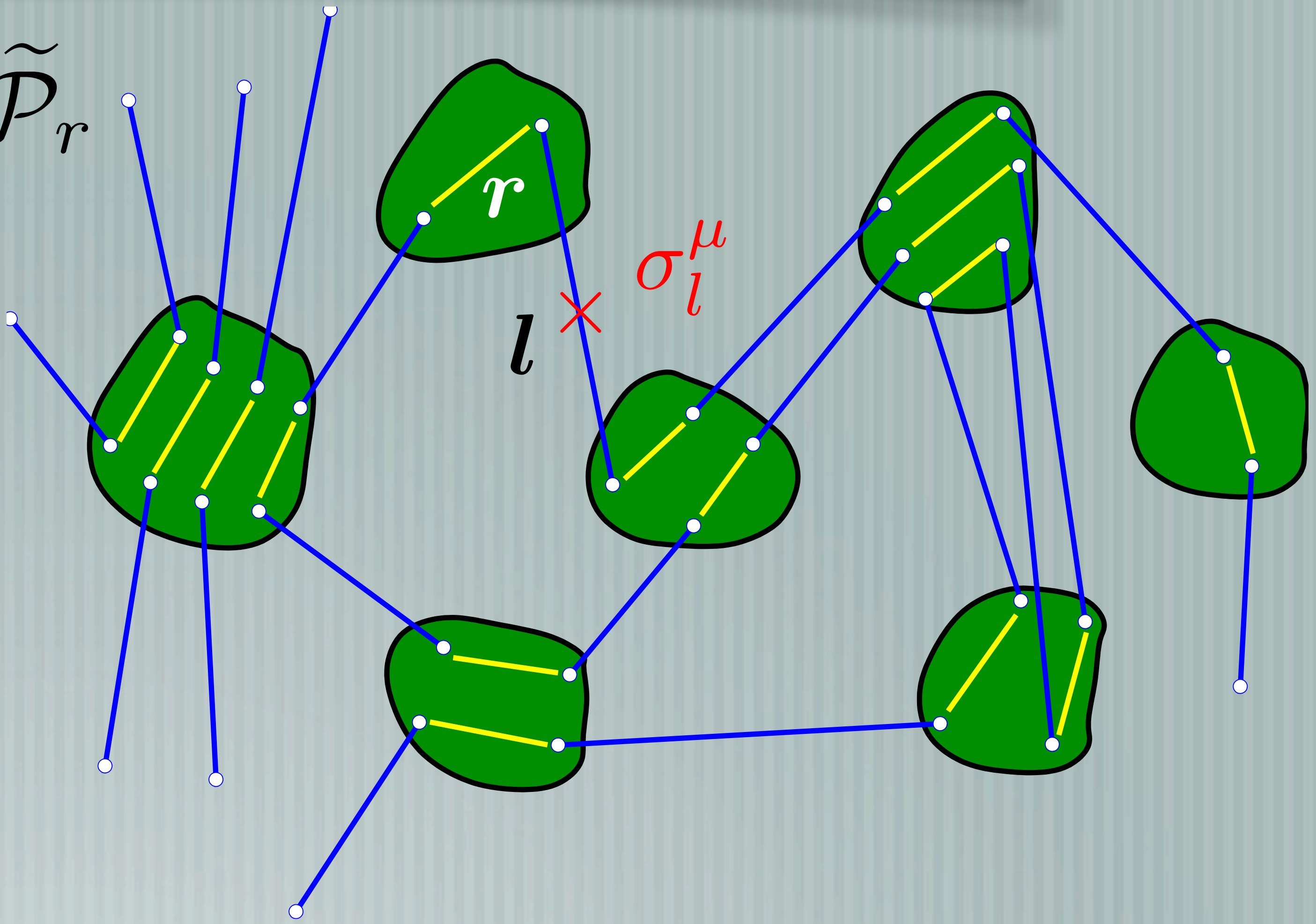
$$\widetilde{\mathcal{P}}_r = \prod_{\{l|r\in l\}} \sigma_l^z \,.$$



Pauli spin operators at the centers × of all inter-grain links

Pauli-spin Bond Algebra

$$H_{\mathsf{QIG}} = -\sum_l J_l \sigma_l^x - \sum_r h_r \widetilde{\mathcal{P}}_r$$



Bond algebra:

A. $(\sigma_l^x)^2 = 1 = (\widetilde{\mathcal{P}}_r)^2,$

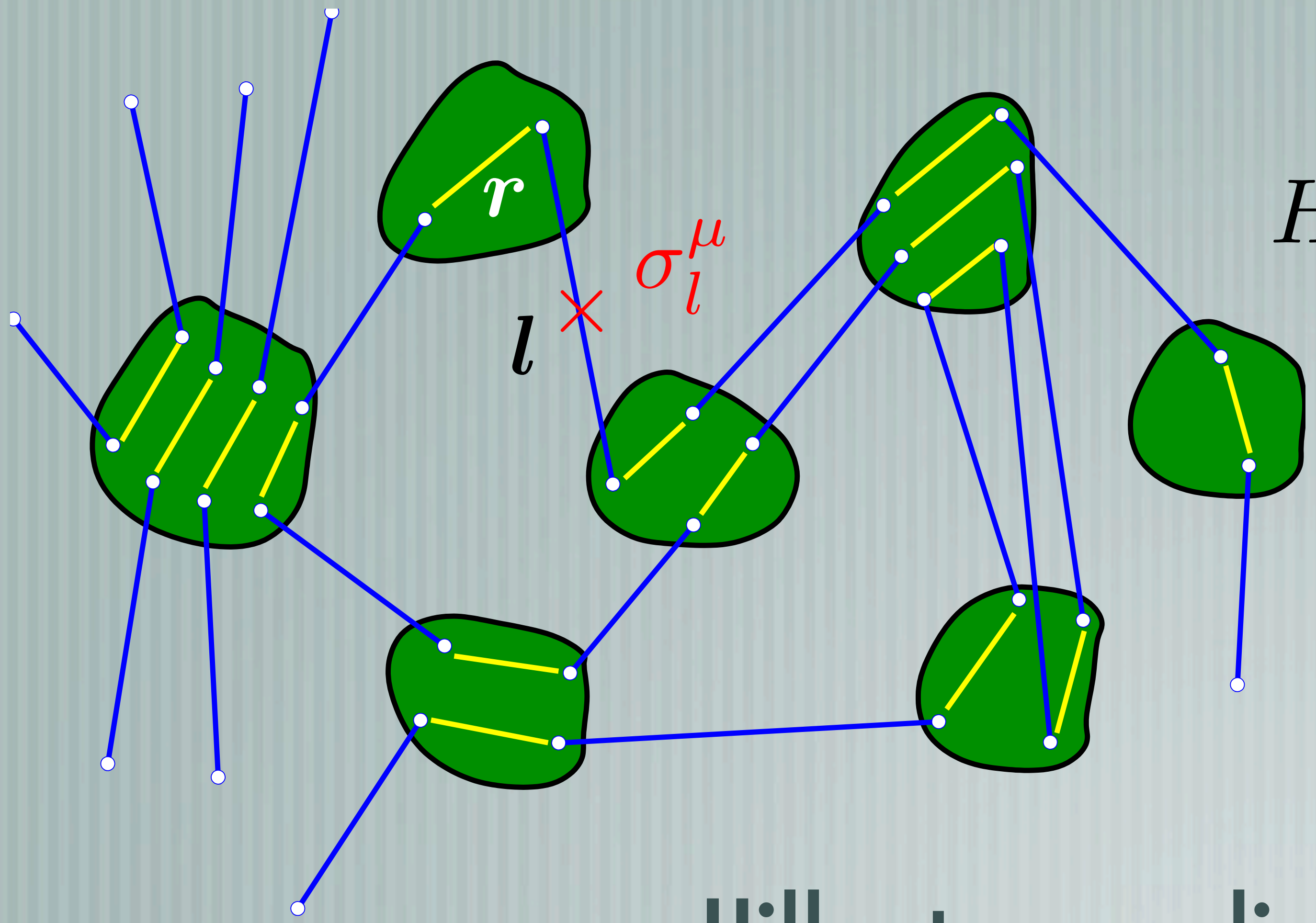
B. for $r, r' \in l$

$$\{\widetilde{\mathcal{P}}_r, \sigma_l^x\} = 0 = \{\widetilde{\mathcal{P}}_{r'}, \sigma_l^x\},$$

C. for $r \in l_i, \ i = 1, 2, \cdots, q_r$

$$\{\widetilde{\mathcal{P}}_r, \sigma_{l_i}^x\} = 0.$$

Majorana network to QIG duality on general planar graphs

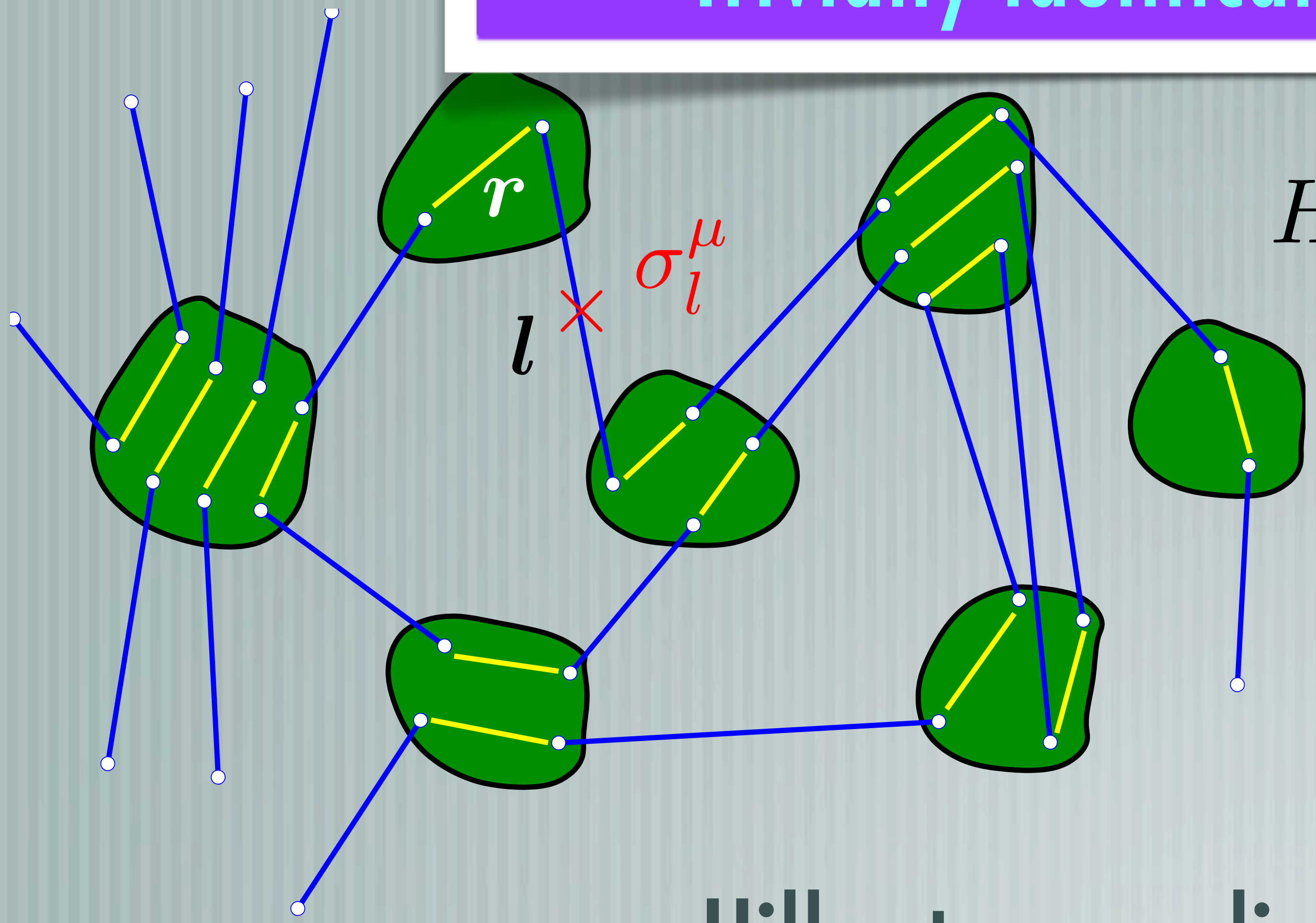


$$\begin{array}{rcl} H_{\mathsf{M}} &=& -i\sum_l J_l c_{l1} c_{l2} - \sum_r h_r \mathcal{P}_r, \\ && \qquad\qquad\quad \updownarrow \qquad\qquad\qquad \updownarrow \\ H_{\mathsf{QIG}} &=& -\sum_l J_l \sigma_l^x - \sum_r h_r \widetilde{\mathcal{P}}_r \end{array}$$

Hilbert space dimensions are the same

Majorana network to QIG duality on general planar graphs

Trivially identical bond algebras!



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$$\updownarrow \qquad\qquad \updownarrow$$

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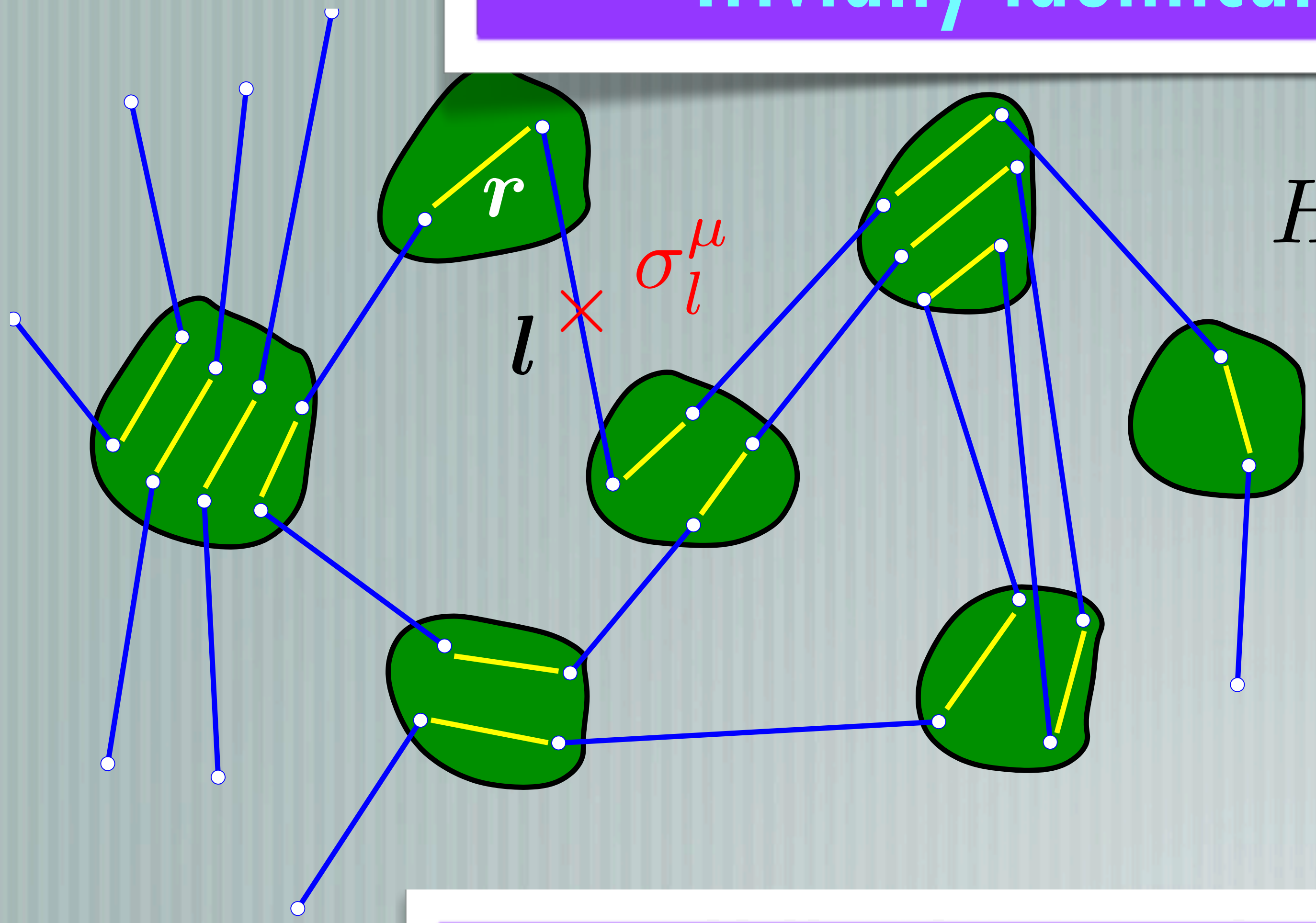

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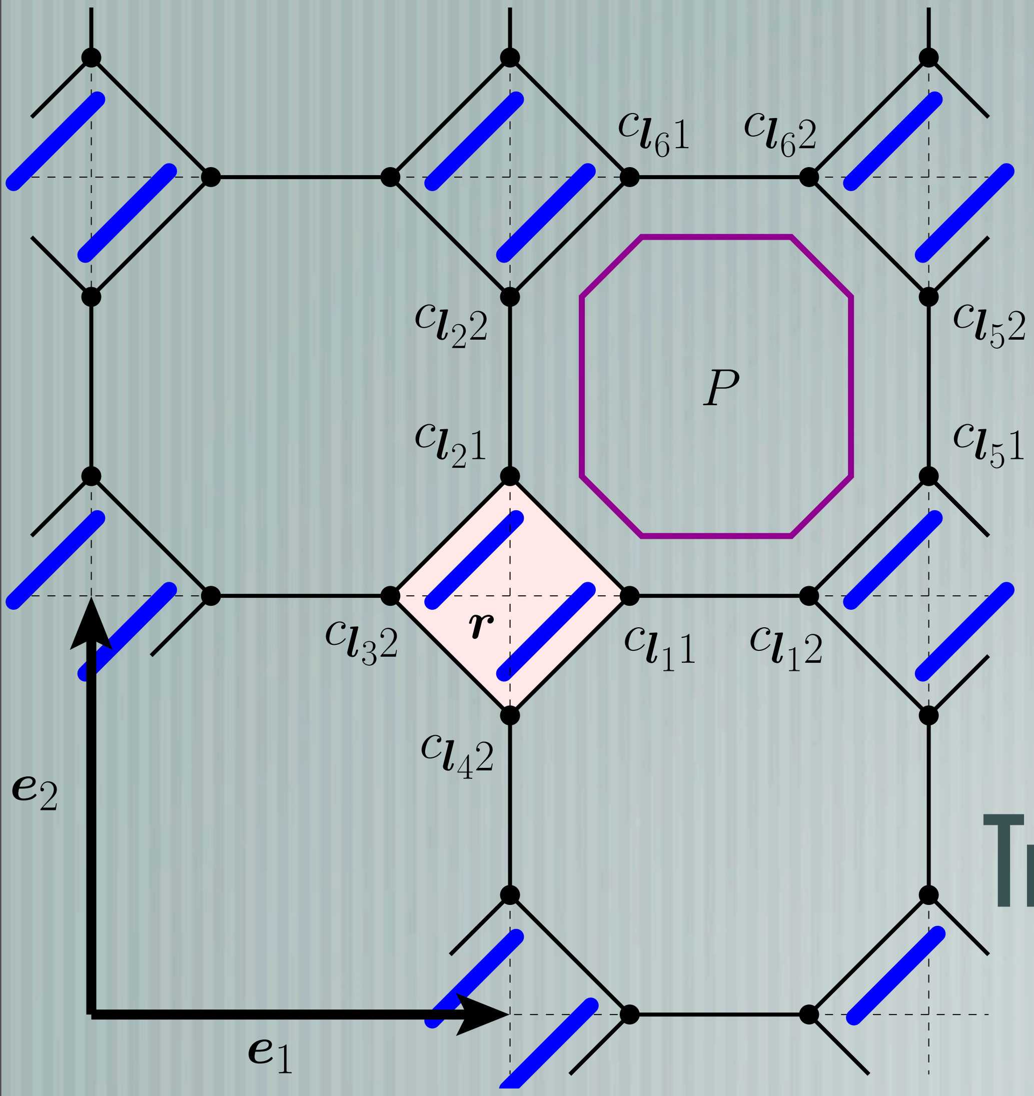
$$\updownarrow \qquad \updownarrow$$

$$H_{\mathsf{QIG}} = -\sum_l J_l \sigma_l^x - \sum_r h_r \widetilde{\mathcal{P}}_r$$

Trivially dual systems!

Majorana network to QIG duality on the square lattice



$$H_{\mathsf{M}} = -\sum_l J_l (i c_{l1} c_{l2}) - \sum_r h_r c_{l_1 1} c_{l_2 1} c_{l_3 2} c_{l_4 2}$$

dual ↕ dual ↕

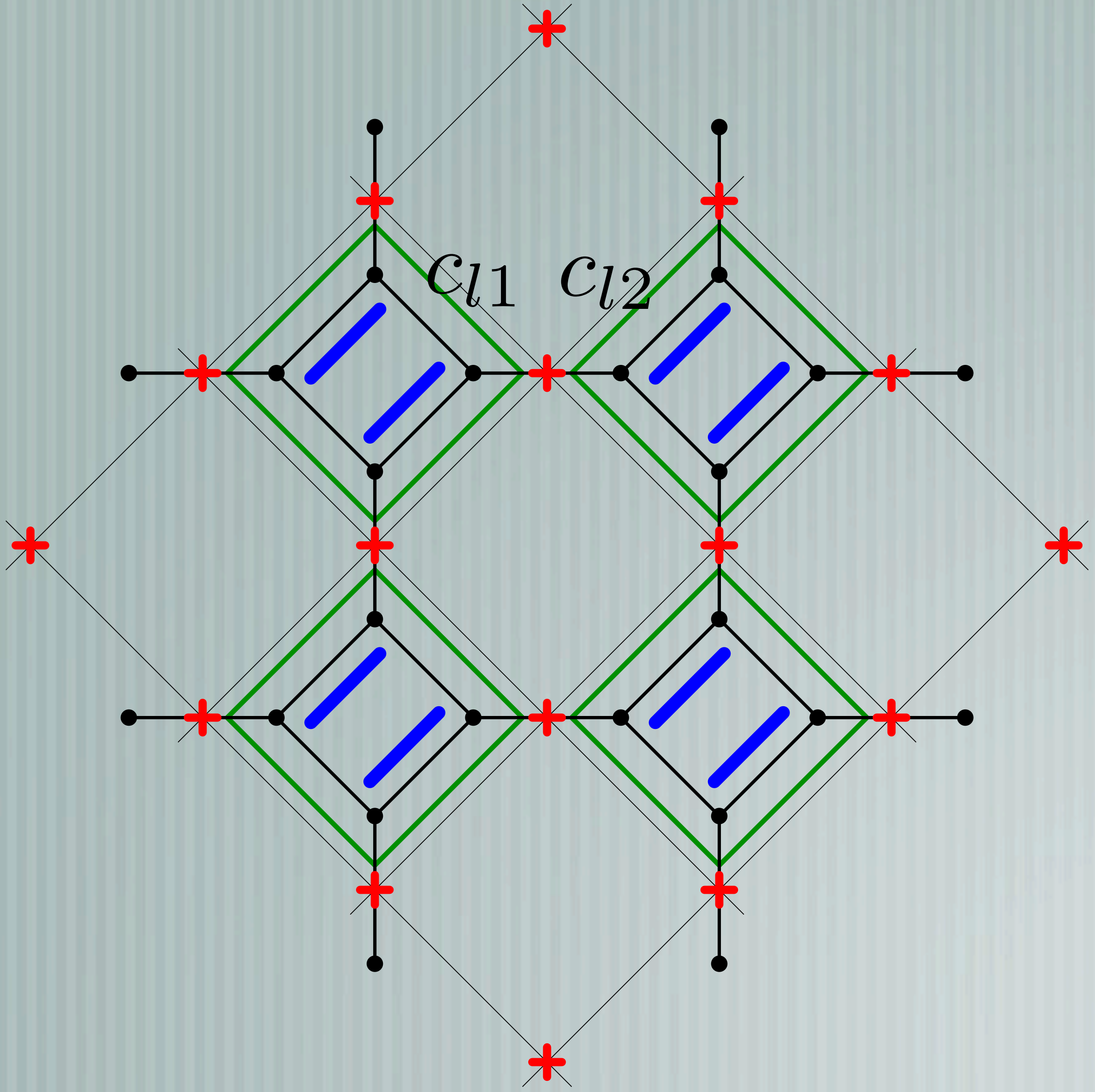
$$H_{\mathsf{QIG}} = -\sum_l J_l \sigma_l^x - \sum_r h_r \sigma_{l_1}^z \sigma_{l_2}^z \sigma_{l_3}^z \sigma_{l_4}^z$$

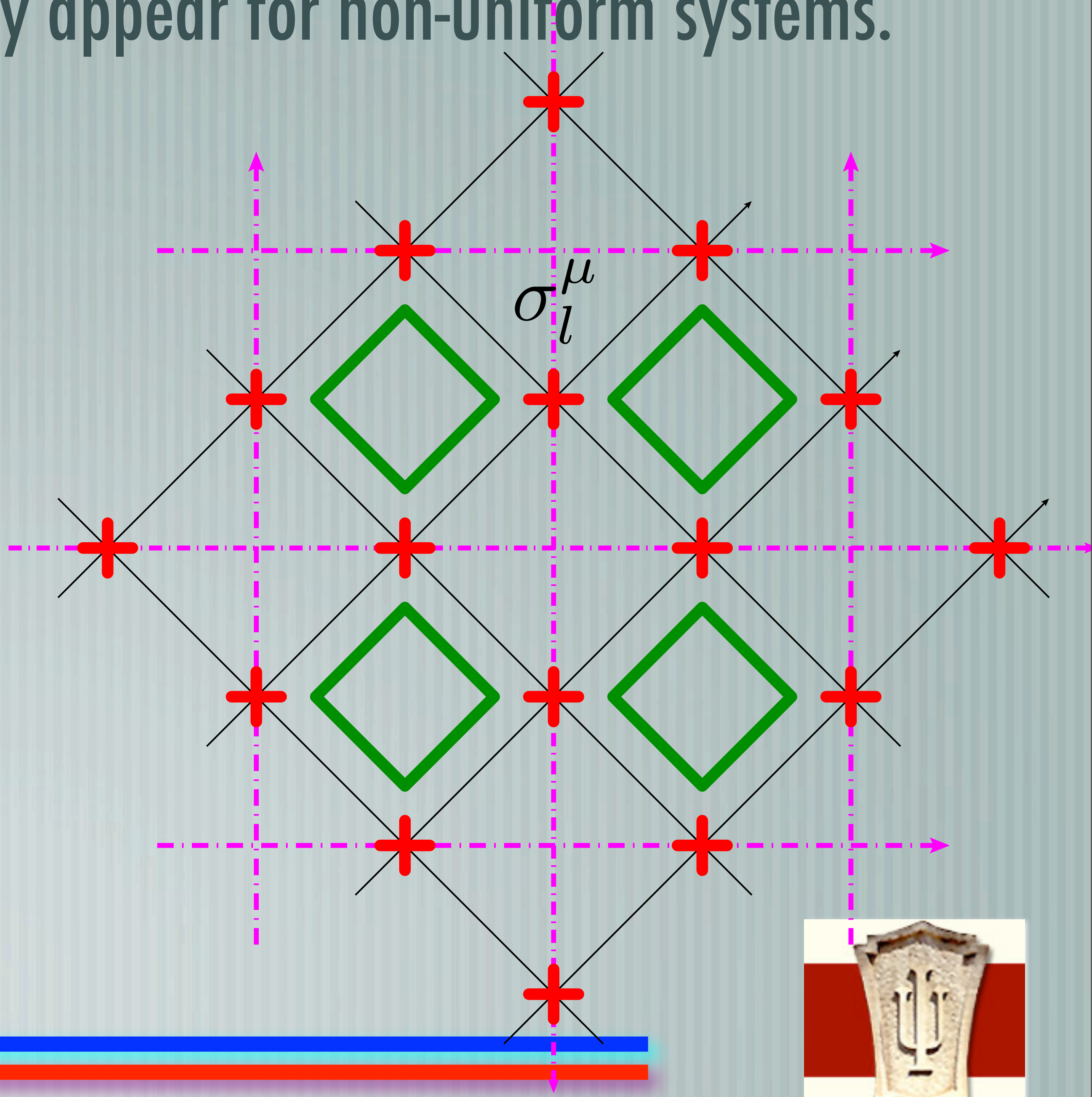
Transverse field

Plaquette term of lattice gauge theories

Critical Behavior

The uniform square lattice Ising gauge theory lies in the 3D Ising universality class (and thus so does a spatially uniform Majorana network). Spin-glass behavior may appear for non-uniform systems.





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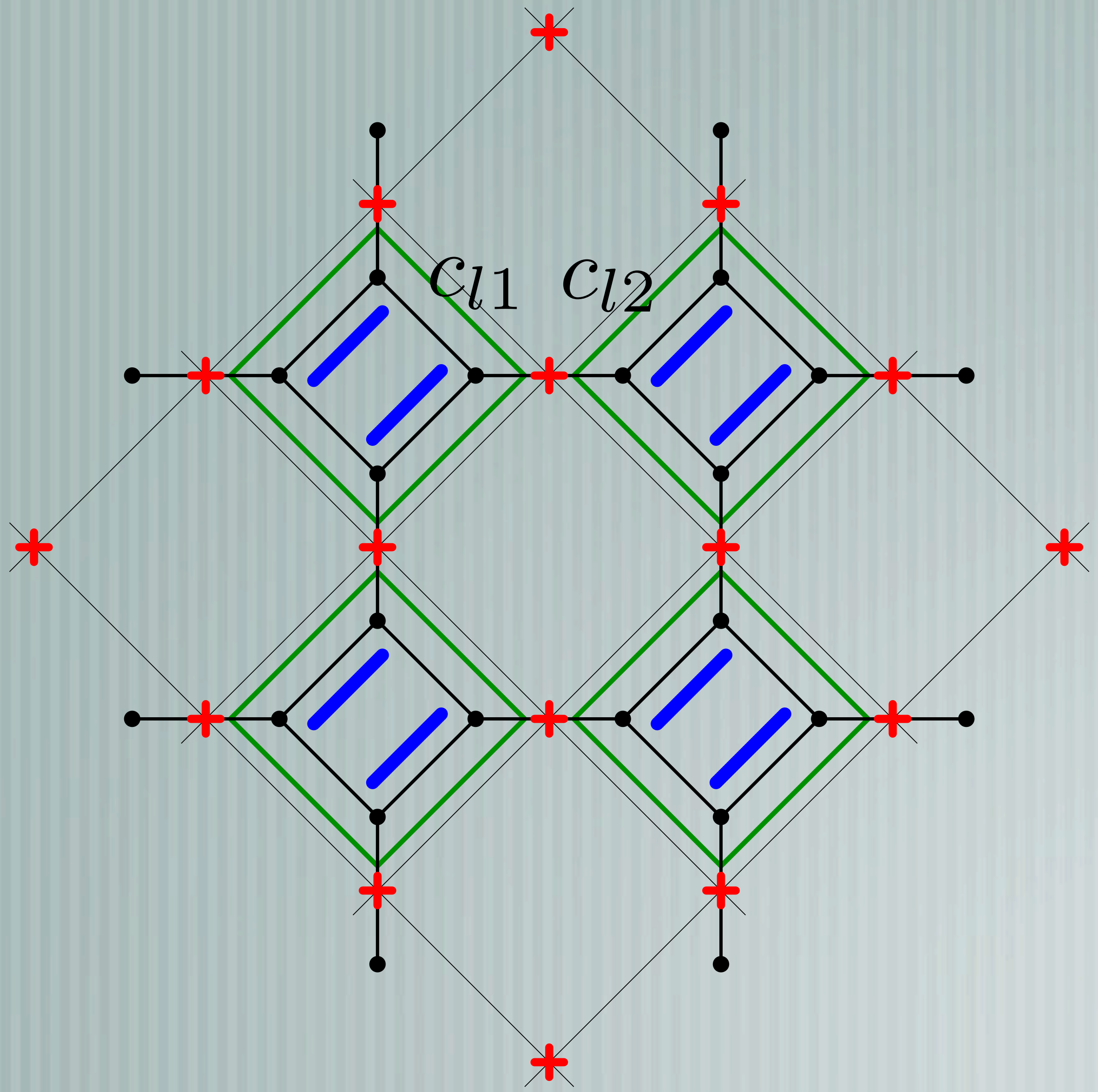

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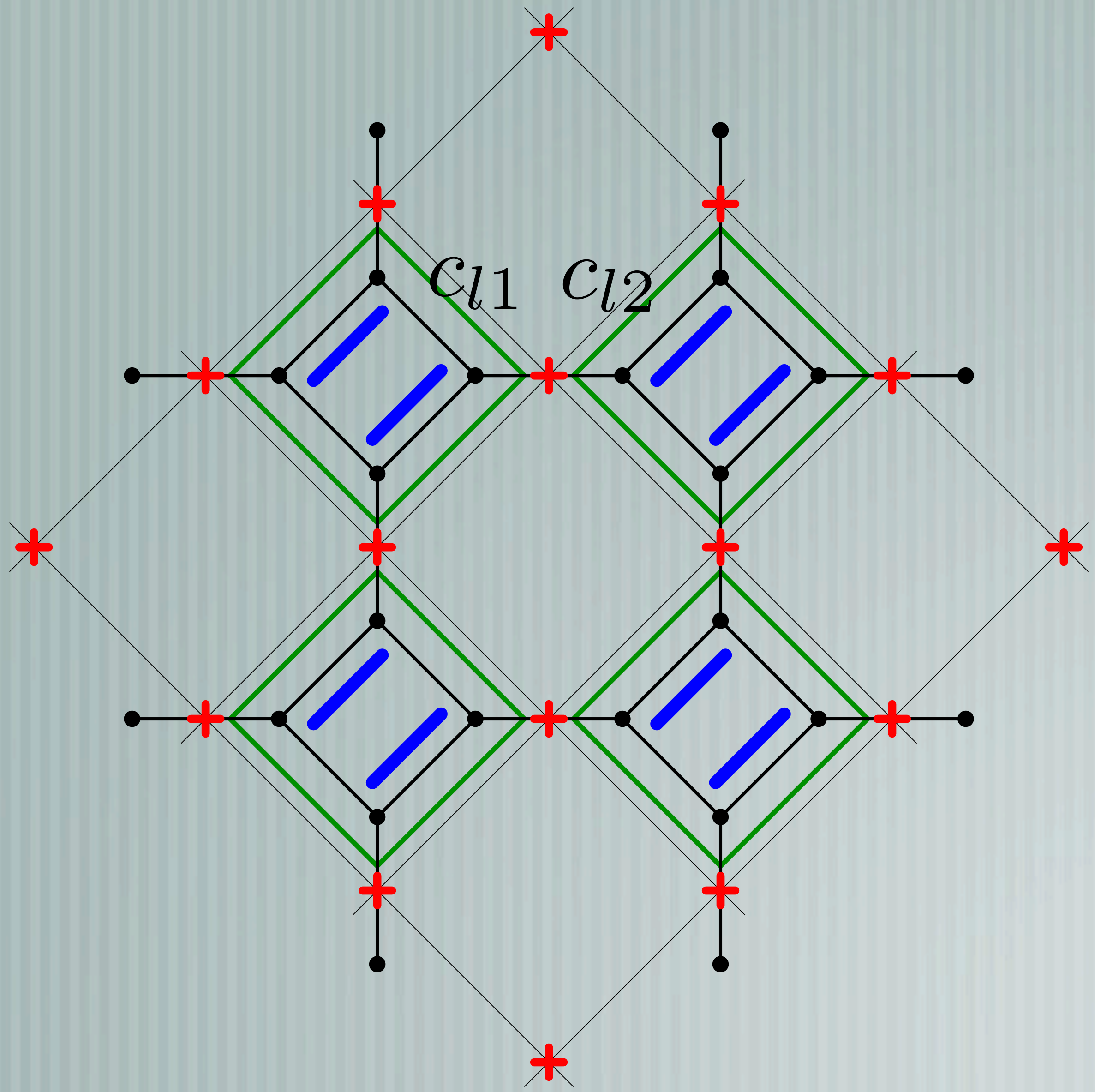
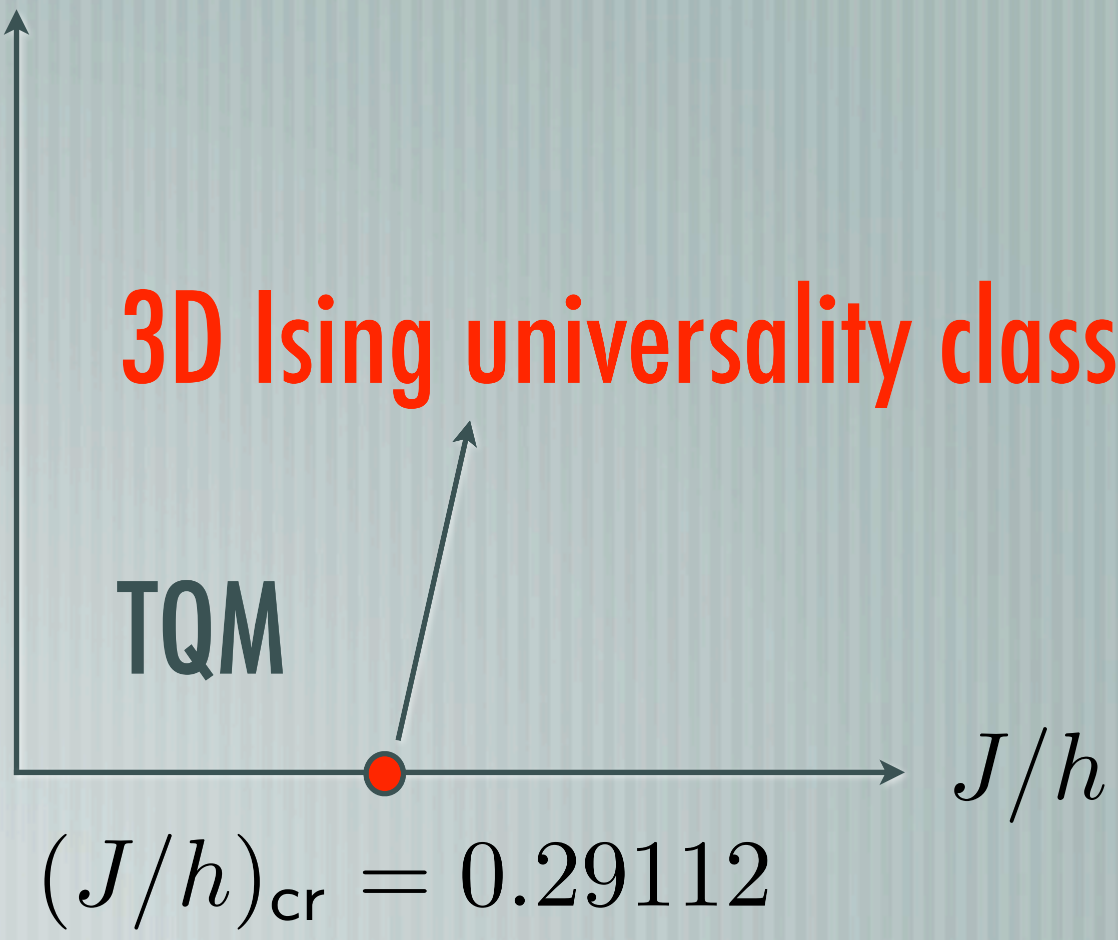
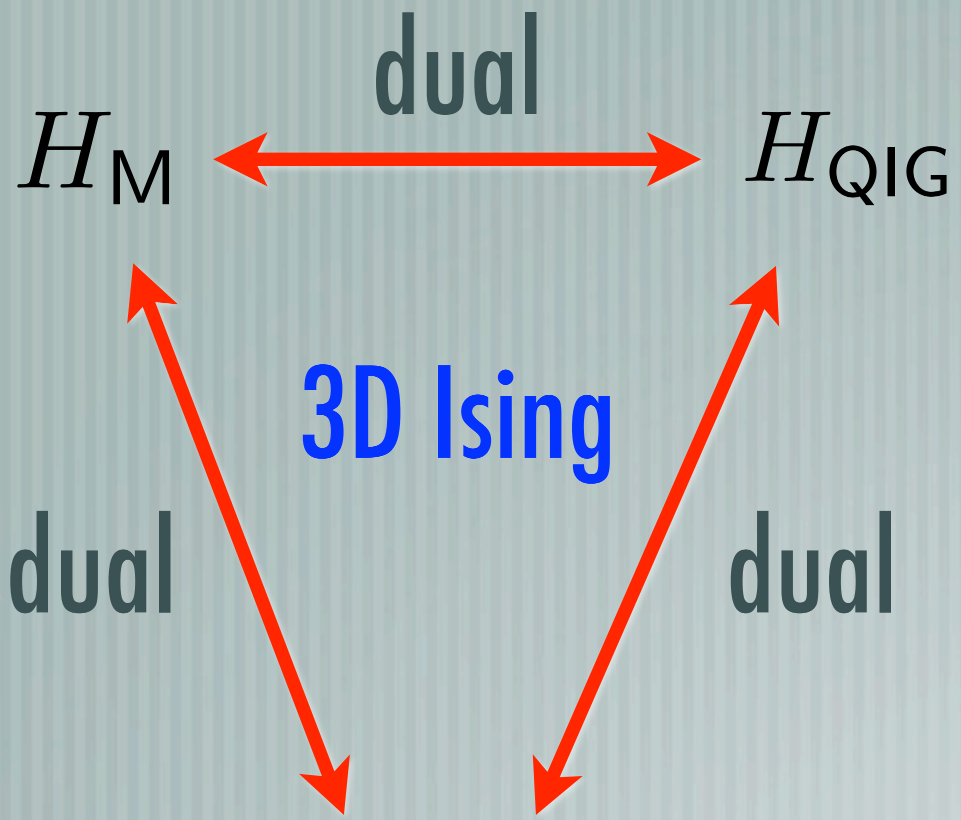




$(J/h)_{\text{cr}} = 0.29112$

Majorana network to XXZ Honeycomb compass model duality on the square lattice

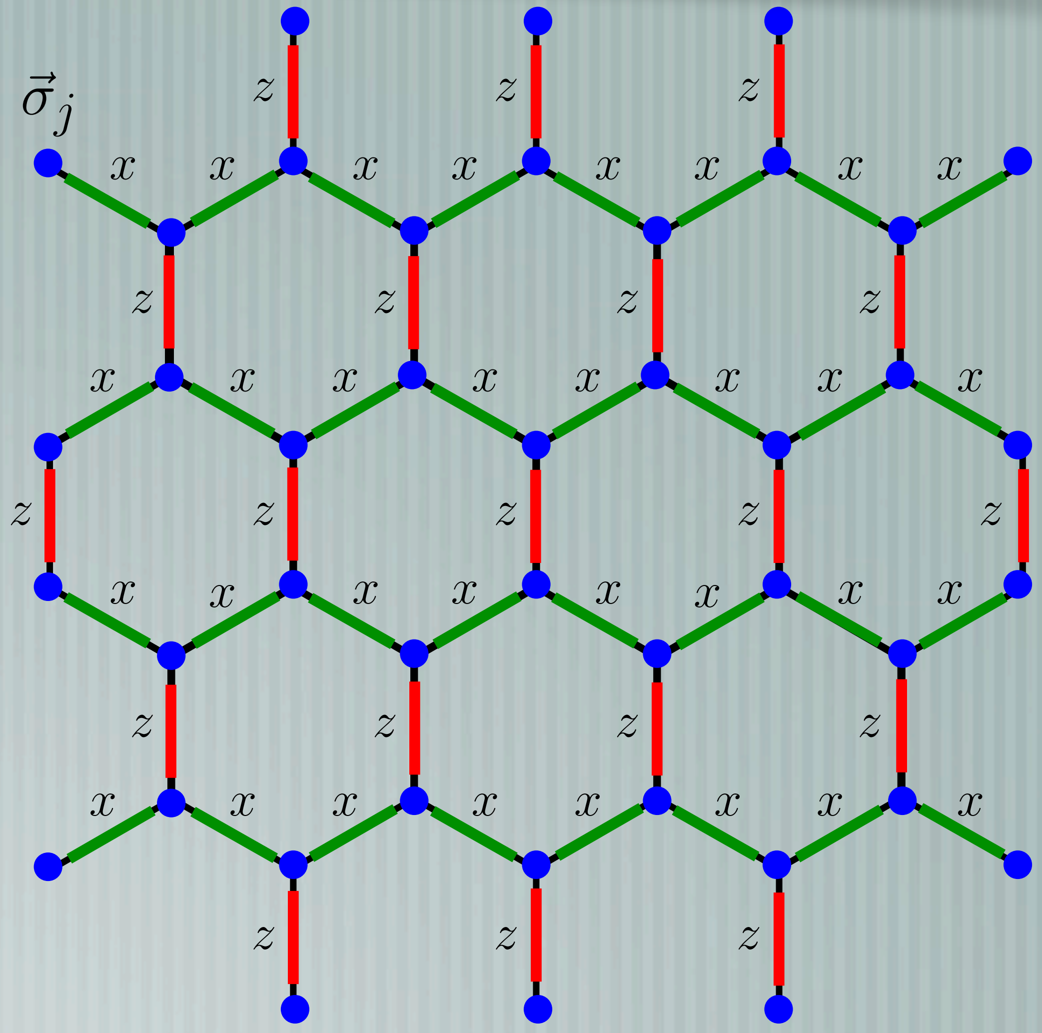


H_{M} dual H_{QIG}

3D Ising

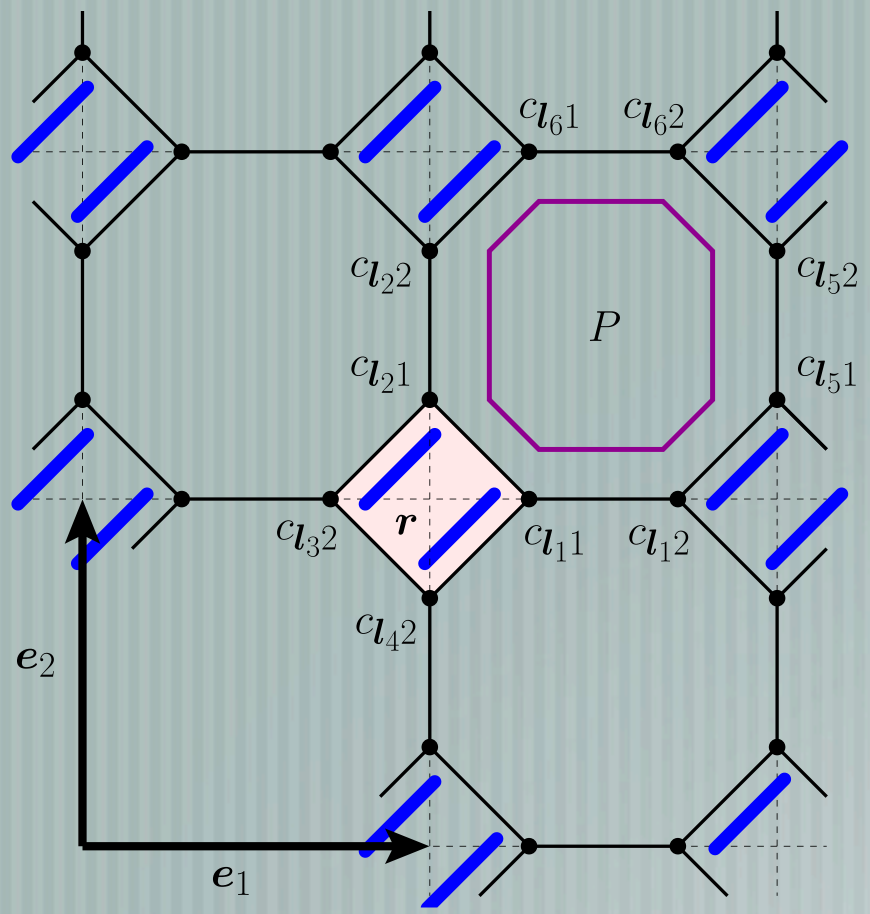
dual

dual



$$H_{\mathrm{XXZh}} = -\sum_{\text{non-vertical links}} J_l\, \sigma^x_r \sigma^x_{r+\hat{e}_l} - \sum_{\text{vertical links}} h_r\, \sigma^z_r \sigma^z_{r+\hat{e}_z}$$

Quantum Simulation of Hubbard-type models



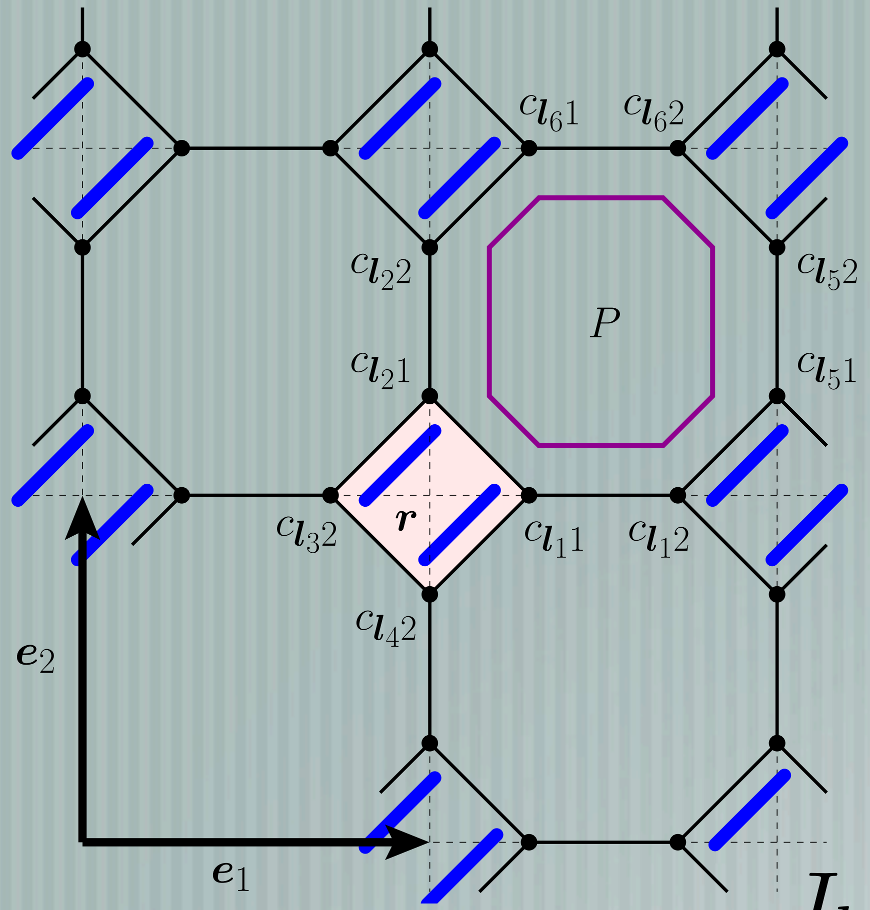
Mapping of Majorana to Dirac fermions leads to interacting Hubbard-type models

A possible mapping is (see Fig.):

$$\begin{aligned} d_{r\uparrow} &= \frac{1}{\sqrt{2}}(c_{l_1 1} + i c_{l_3 2}), d^{\dagger}_{r\uparrow} = \frac{1}{\sqrt{2}}(c_{l_1 1} - i c_{l_3 2}), \\ d_{r\downarrow} &= \frac{1}{\sqrt{2}}(c_{l_2 1} + i c_{l_4 2}),\ \ d^{\dagger}_{r\downarrow} = \frac{1}{\sqrt{2}}(c_{l_2 1} - i c_{l_4 2}) \end{aligned}$$

$$n_{r\sigma} = d^{\dagger}_{r\sigma} d_{r\sigma}$$

Hubbard-type Dictionary



$$U_r(n_{r\uparrow} - 1)(n_{r\downarrow} - 1) = U_r(\mathcal{P}_r - 1)$$

Intra-grain Majorana fermion interaction

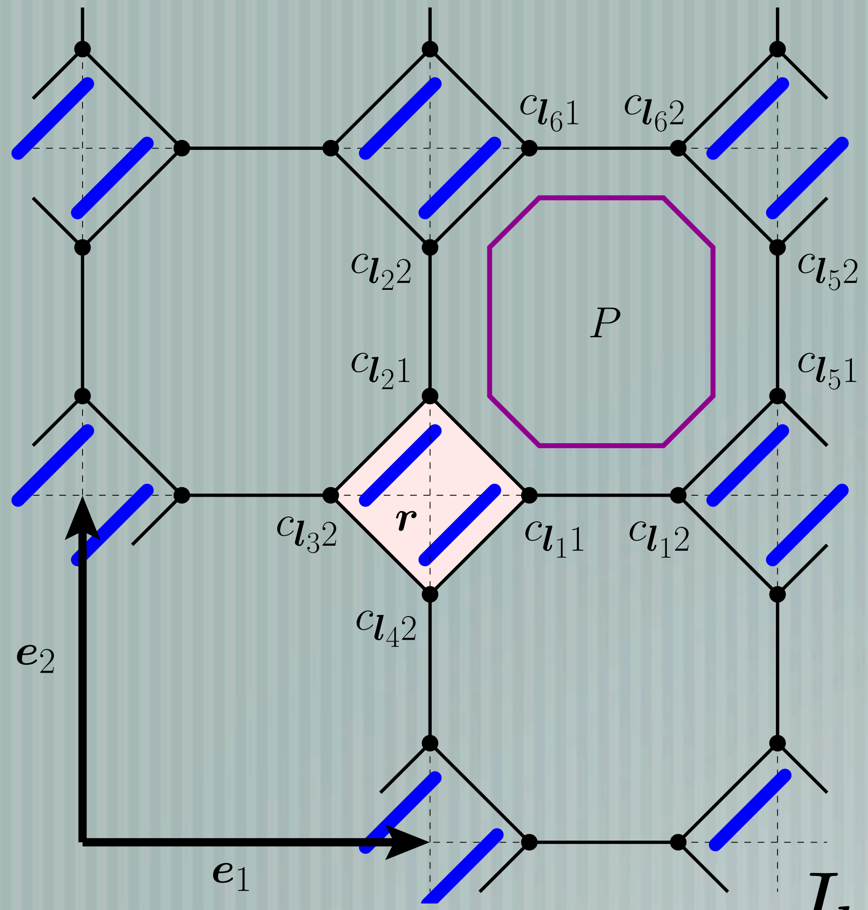
Inter-grain Majorana fermion term

$$\frac{J_l}{2}(d^\dagger_{r\downarrow} + d_{r\downarrow})(d^\dagger_{r+\hat{e}_2\downarrow} - d_{r+\hat{e}_2\downarrow}) = -iJ_l c_{l_2 1} c_{l_2 2}$$

Spin polarization dependent electronic hopping and pairing

(compass-type terms)

Hubbard-type Dictionary



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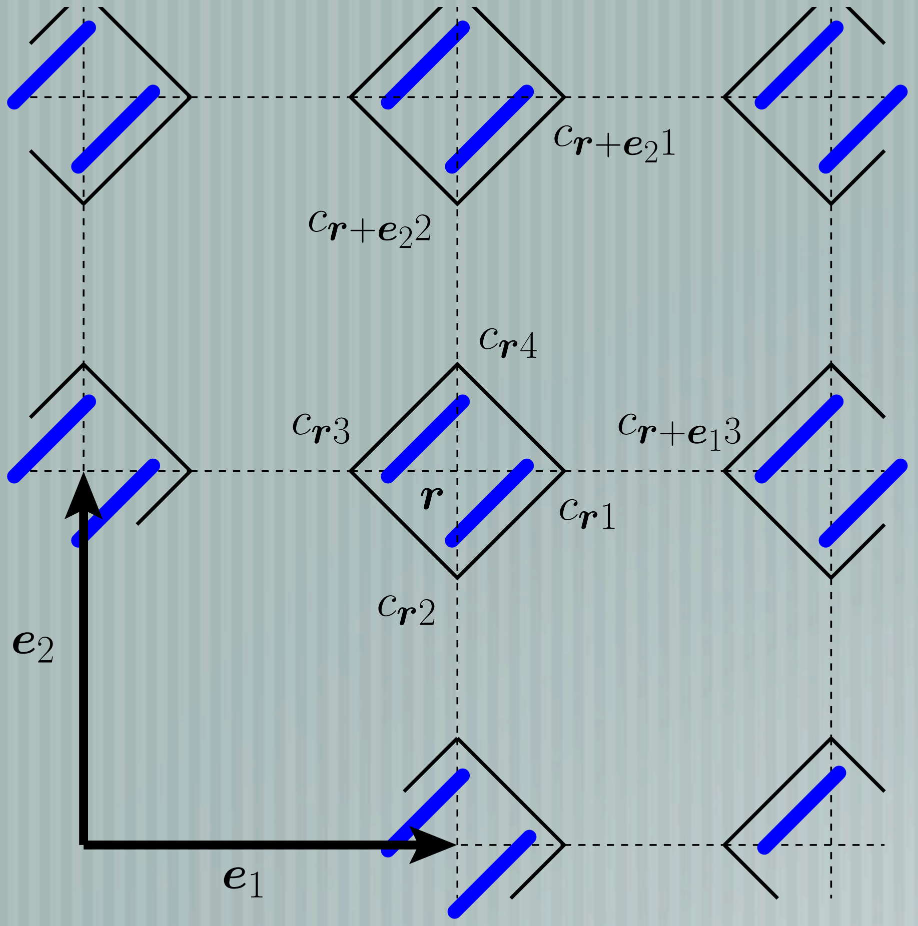
This Hubbard-type system lies in the 3D Ising universality class

(compass-type terms)

How about simulating the standard Hubbard model ?



It can be simulated in principle but requires additional Josephson couplings

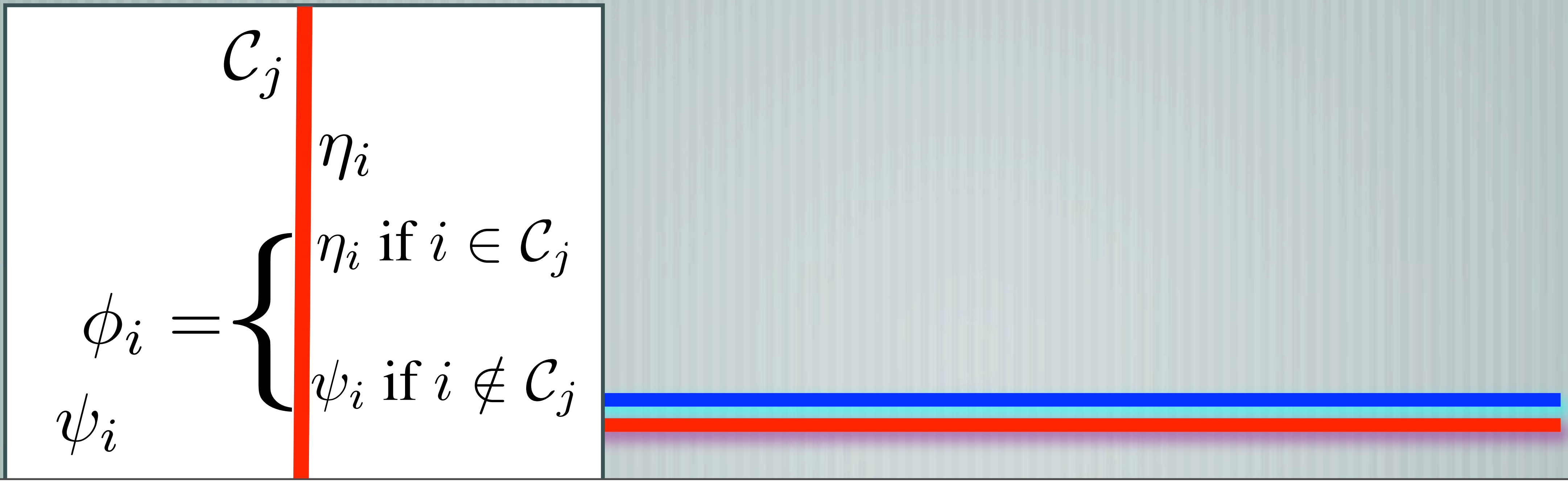
$$
\begin{aligned}
H_{\text{Hubbard}} = & - \quad t \sum_{r,\alpha,a=1,2} i(c_{ra} c_{r+\boldsymbol{e}_\alpha a+2} + c_{r+\boldsymbol{e}_\alpha a} c_{ra+2}) \\
& + \quad U \sum_{r} (\mathcal{P}_r - 1)
\end{aligned}
$$

Summary of Main Results

- Lightning review of Majorana fermions
- Quick Introduction to the Bond algebra technique
- Dualities between Majorana networks and quantum Ising gauge theories. Adduce Ising, spin-glass, and other behavior.
- The XXZ honeycomb compass model = 3D Ising model
- Square lattice Hubbard compass model = 3D Ising
- Hubbard model might be simulated by Majorana network

Dimensional reduction and holography

The average of any quasi-local quantity f is bounded from above by the absolute value of the mean of the same quantity when this quasi-local quantity is computed with a d-dimensional Hamiltonian that preserves the range of the interactions in the original D-dim system



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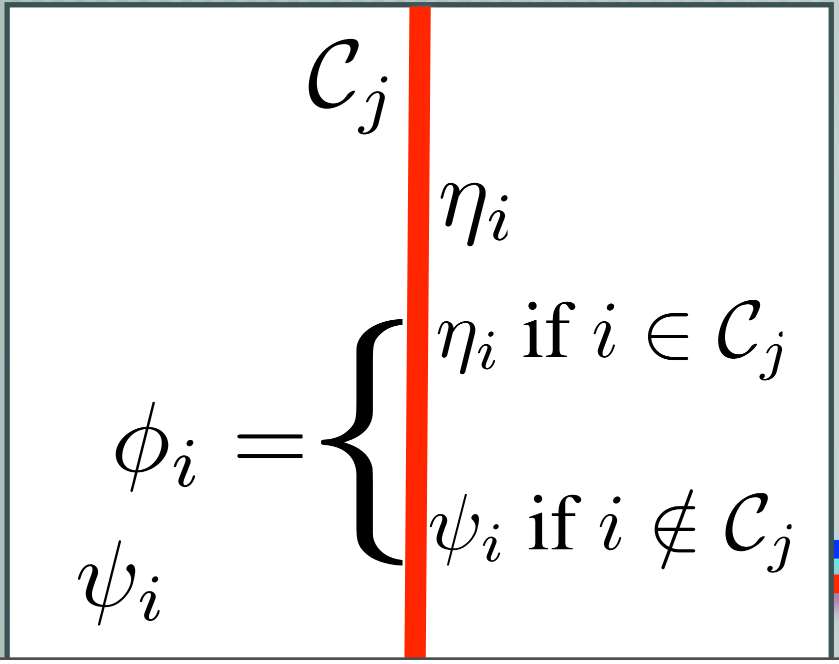

$$\phi_i = \begin{cases} \eta_i \text{ if } i \in \mathcal{C}_j \\ \psi_i \text{ if } i \notin \mathcal{C}_j \end{cases}$$

$$|\langle f(\phi_i(t))\rangle_{H_D}| \leq |\langle f(\eta_i(t))\rangle_{H_d}|$$

Dimensional reduction

Dimensional reduction and holography

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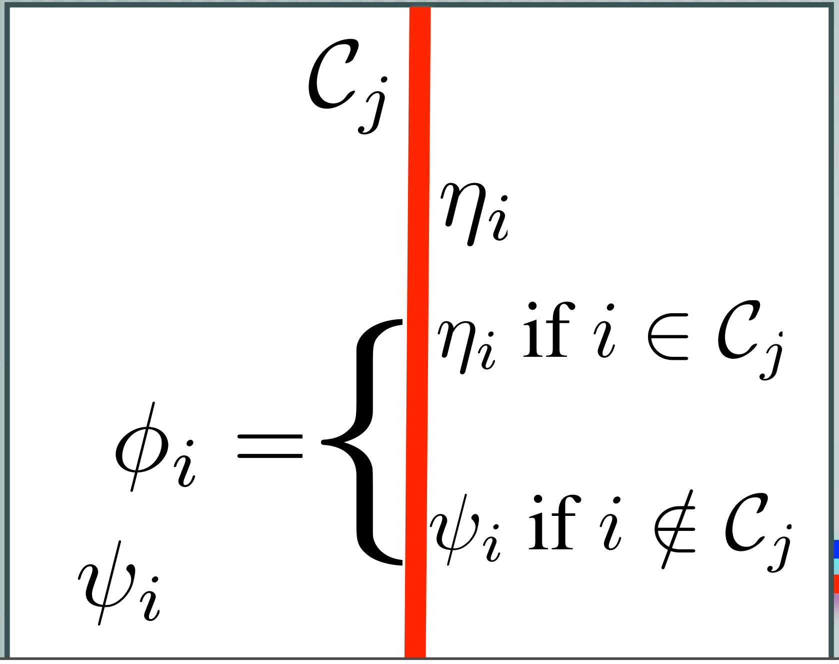

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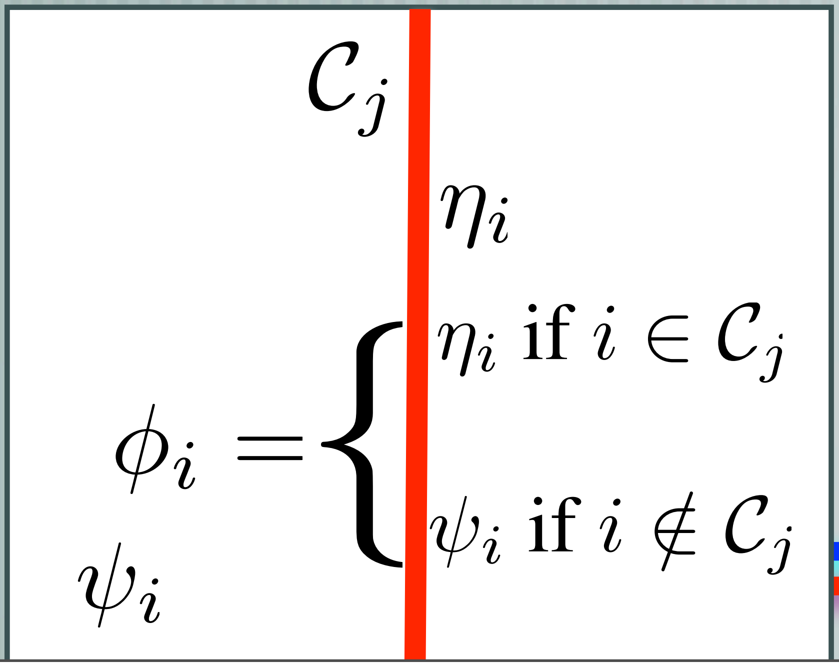
A "boundary to bulk correspondence" as a bound

Dimensional reduction

Dimensional reduction inequalities

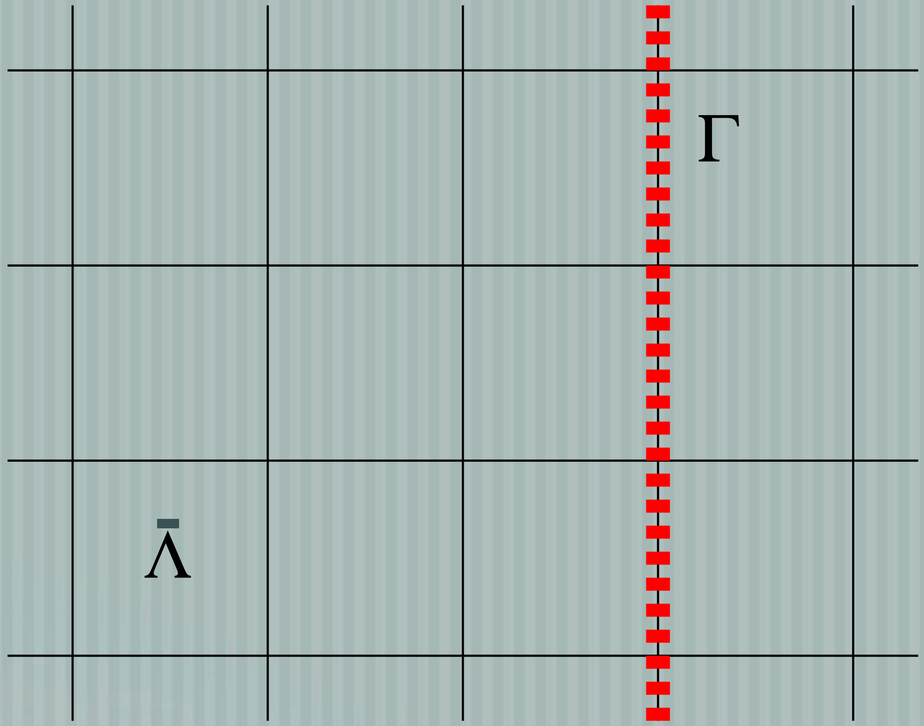
The average of any quasi-local quantity f is bounded from above by the absolute value of the mean of the same quantity when this quasi-local quantity is computed with a d-dimensional Hamiltonian that preserves the range of the interactions in the original D-dim system



A "boundary to bulk correspondence" as a bound

Dimensional reduction

Dimensional reduction in classical systems:



$$\phi(\boldsymbol{x}) = \begin{cases} \phi_0(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \Gamma \\ \psi(\boldsymbol{x}) & \text{if} \quad \boldsymbol{x} \in \bar{\Lambda} \end{cases}$$

$f[\phi] = f[\phi_0]$ localized observable

$$\langle f \rangle^D = \sum_{\{\psi\}} \sum_{\{\phi_0\}} f[\phi_0] \frac{e^{-\beta E[\phi_0,\psi]}}{\mathcal{Z}} = \sum_{\{\psi\}} \frac{z[\psi]}{\mathcal{Z}} \frac{\sum_{\{\phi_0\}} f(\phi_0) e^{-\beta E[\phi_0,\psi]}}{z[\psi]}$$

$$\langle f \rangle_l^d \equiv \mathsf{min}_\psi \, \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\mathsf{min}}], \qquad \langle f \rangle_u^d \equiv \mathsf{max}_\psi \, \langle f \rangle^d[\psi] = \langle f \rangle^d[\psi_{\mathsf{max}}]$$

$$\boxed{\langle f \rangle_l^d \leq \langle f \rangle^D \leq \langle f \rangle_u^d}$$

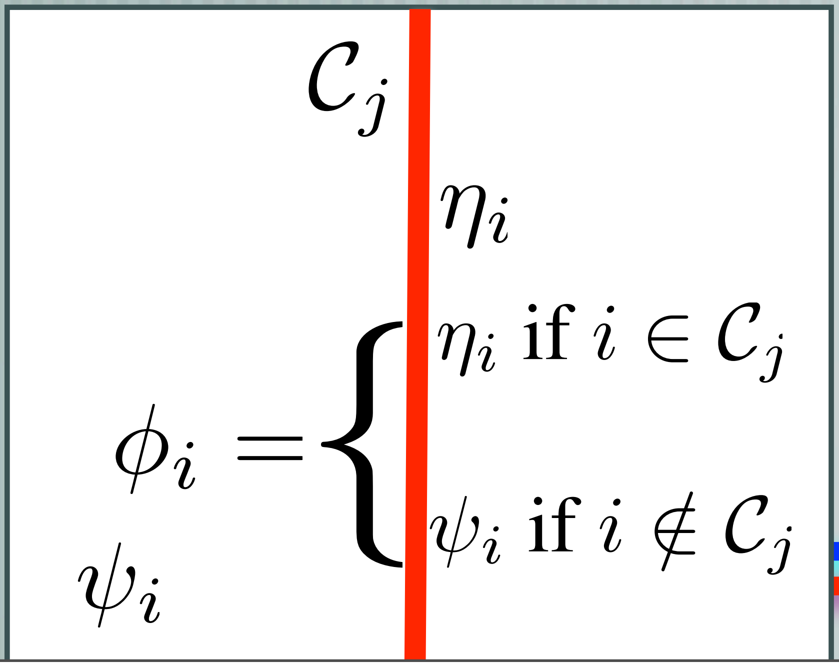
$$\langle f \rangle_l^d \; : \; E_l[\phi_0, \psi_{\mathsf{min}}] \quad \text{and} \quad \langle f \rangle_u^d \; : \; E_u[\phi_0, \psi_{\mathsf{max}}]$$

Local effective boundary theories

Dimensional reduction inequalities

In some cases, due to symmetries both upper and lower bounds scale in the same way.

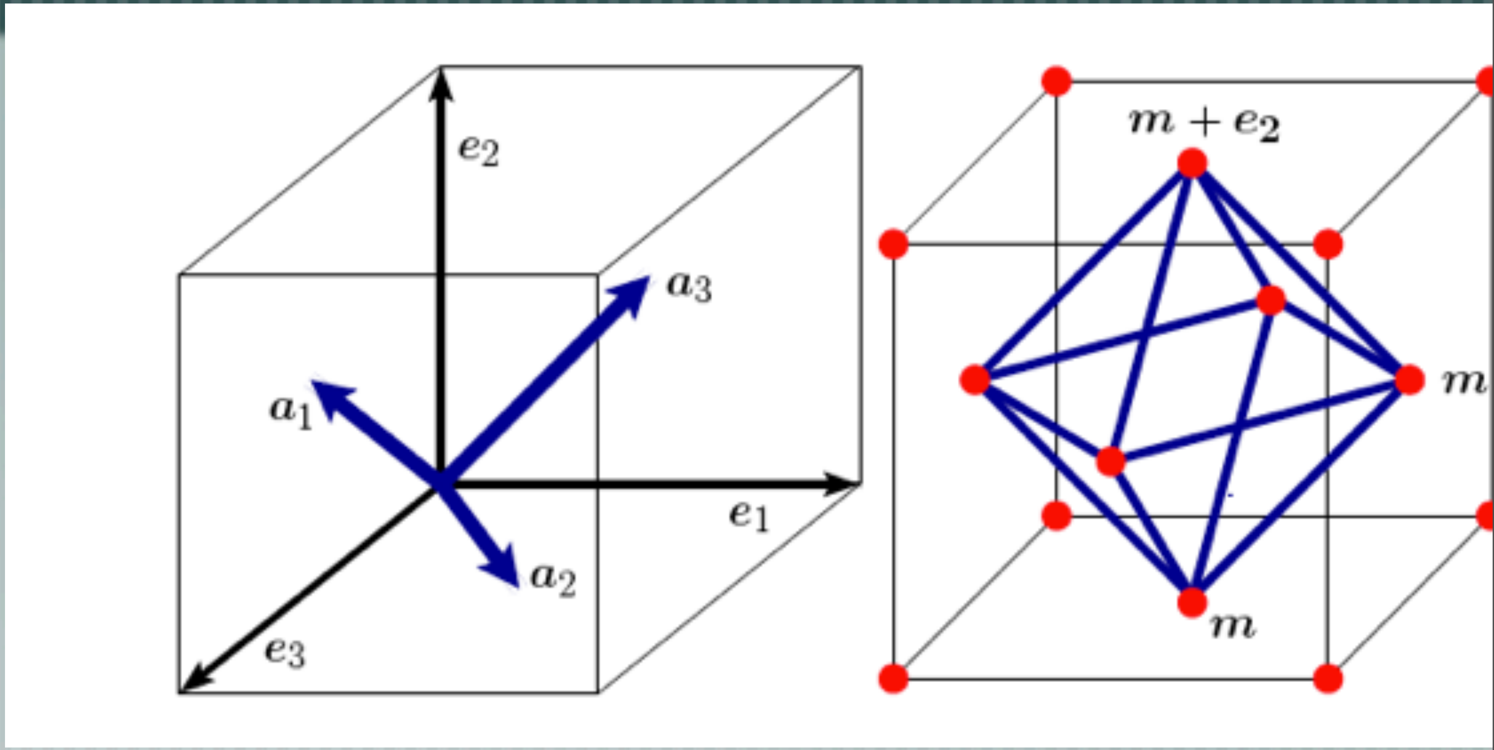
In other systems, stringent upper bounds (on, e.g., autocorrelation functions) can be derived due to “lower dimensional symmetries”. The effect of any additional symmetry breaking perturbations can be quantified with the bounds.



$$\phi_i = \begin{cases} \eta_i \text{ if } i \in \mathcal{C}_j \\ \psi_i \text{ if } i \notin \mathcal{C}_j \end{cases}$$

When combined with the d-dimensional GLSs noted earlier in this talk, this allows proofs of topological quantum order.

Exact Dimensional Reduction



XXYYZZ model (Chamon; Bravyi, Leemhuis, Terhal)

$$\mathbf{a}_1 = \frac{\hat{e}_2 + \hat{e}_3}{2}\ ,\ \mathbf{a}_2 = \frac{\hat{e}_1 + \hat{e}_3}{2}\ ,\ \mathbf{a}_3 = \frac{\hat{e}_1 + \hat{e}_2}{2}$$

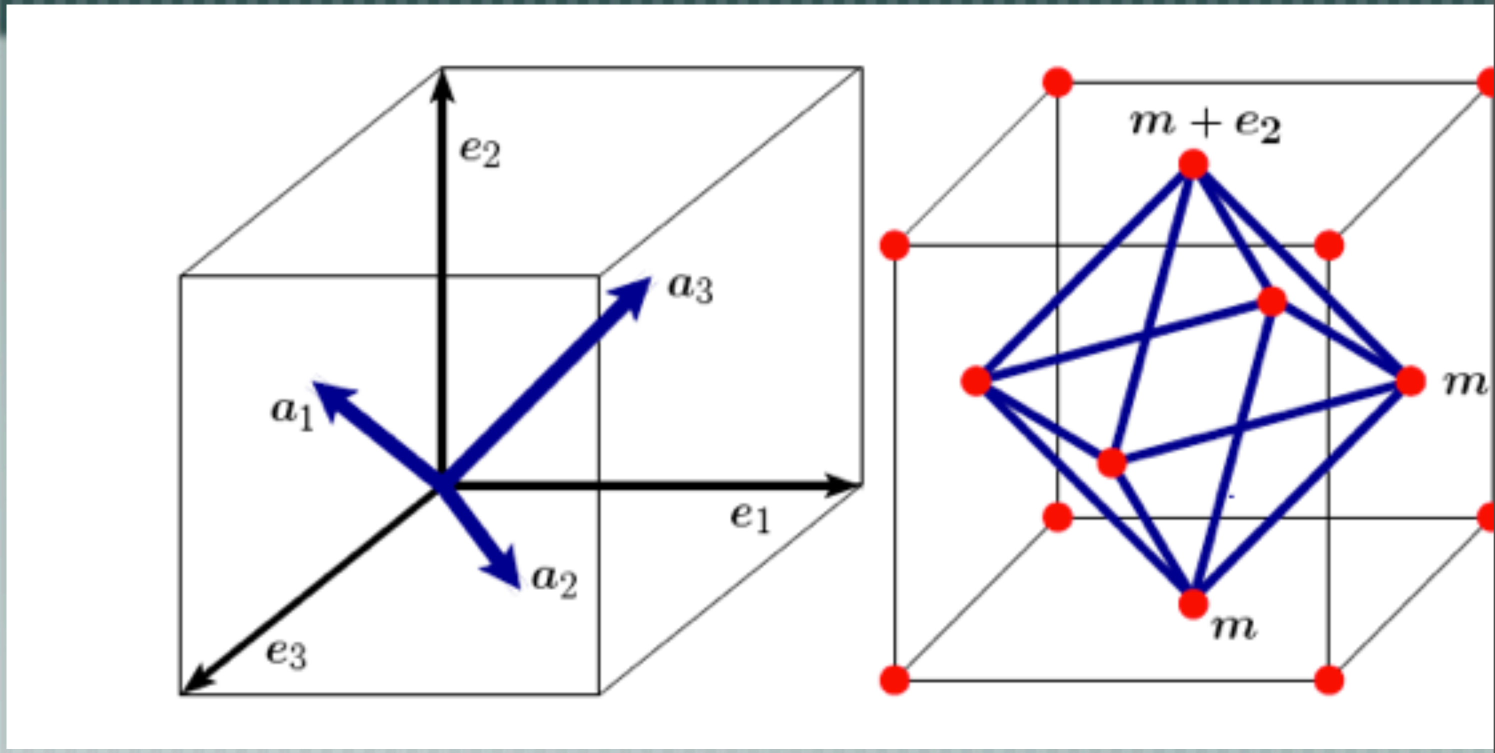
$$O_m = \sigma^x_{m+a_1-a_2}\sigma^x_{m+a_3}\sigma^y_m\sigma^y_{m+a_2}\sigma^z_{m+a_3-a_2}\sigma^z_{m+a_1}$$

$$H_{XXYYZZ} = -J \sum_{m\in\Lambda^P_{fcc}} O_m$$

$$H_{4IP} = -J\sum_{\kappa=1}^{4}\sum_{m=1}^{N_s/4}\sigma^z_{\kappa,m}\sigma^z_{\kappa,m+1}$$

Exact Dimensional Reduction

XXYYZZ model (Chamon; Bravyi, Leemhuis, Terhal)



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$$\Big\downarrow$$

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Duality connecting the two theories

Exact Dimensional Reduction

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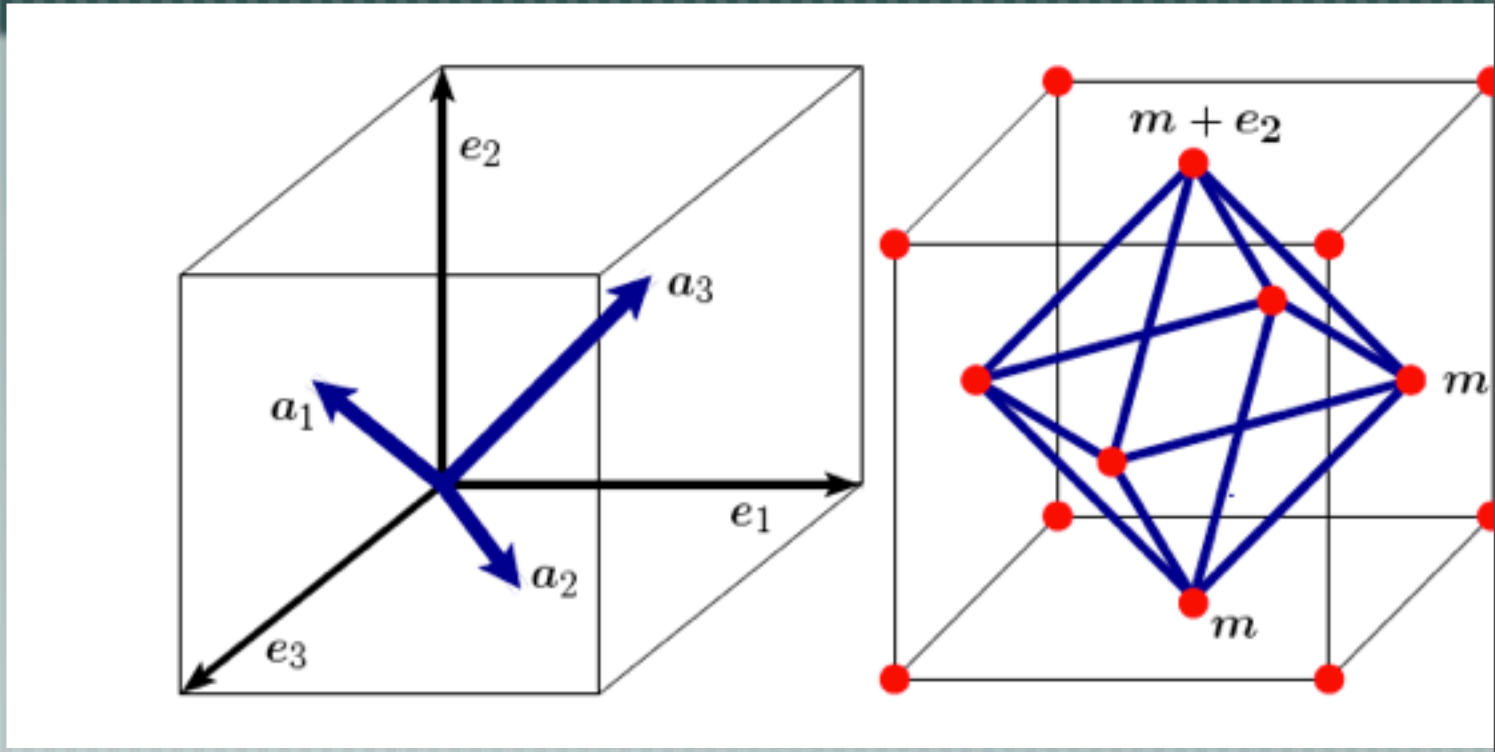

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$4 \quad N_s/4$

Duality connecting the two theories

One-dimensional system (four decoupled Ising chains)

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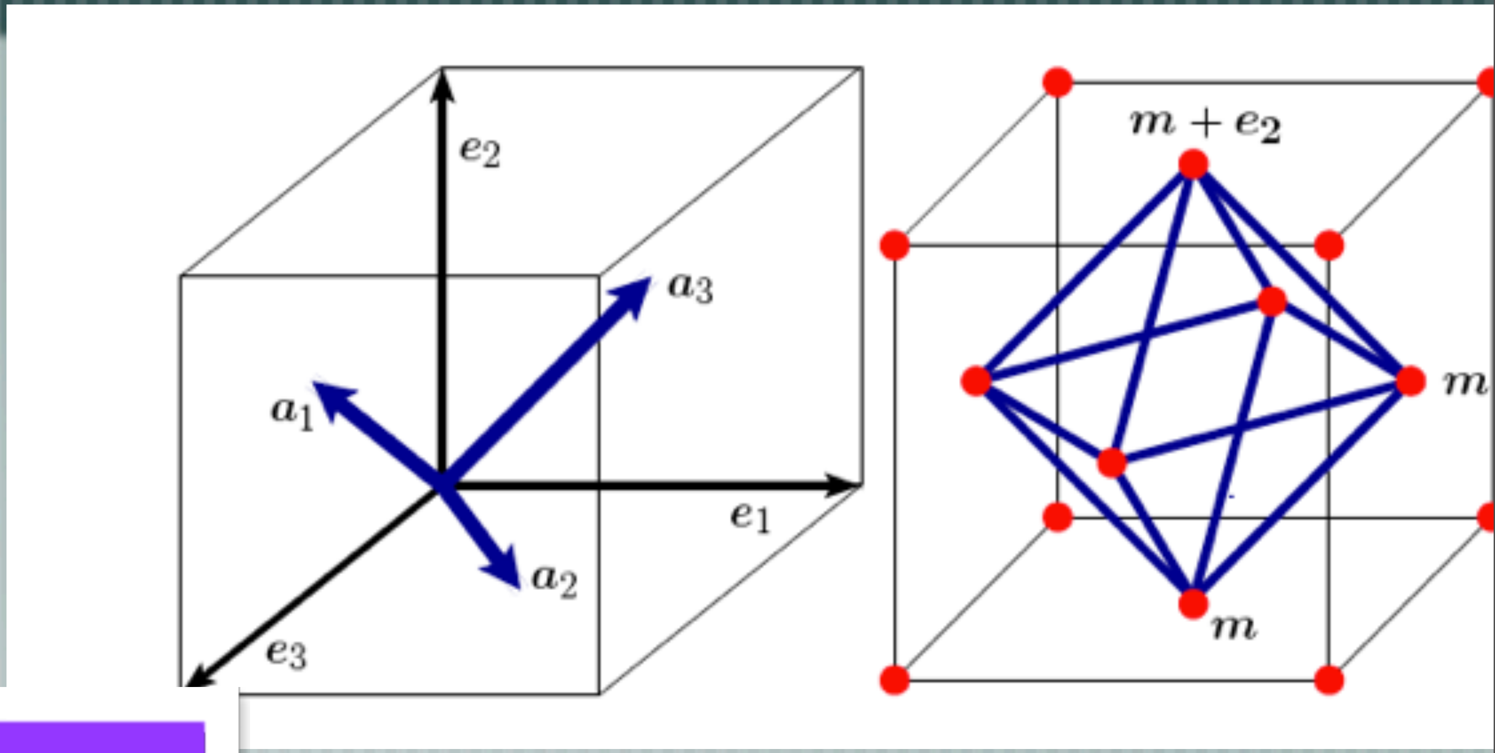


Three-dimensional system

$m \in \Lambda_{fcc}$

$4 \quad N_s/4$

Duality connecting the two theories

One-dimensional system (four decoupled Ising chains)

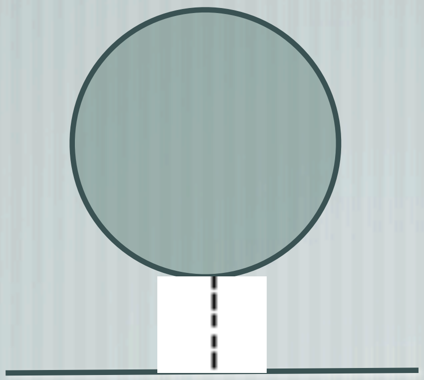
Exact Dimensional Reduction and holography in the large n limit

large n vector theories are trivial (by comparison to large n matrix models)

$$H_0 = \frac{1}{2}\sum_{x,y} J(x-y)\vec{\phi}(x)\cdot\vec{\phi}(y) = \frac{1}{2N_s}\sum_{\mathbf{k}} J(\mathbf{k})|\vec{\phi}(\mathbf{k})|^2,$$

$$H_1 = \sum_x (\vec{\phi}(x)\cdot\vec{\phi}(x))^2$$

Self-energy: $\Sigma^{(0)} =$



$$= \int \frac{d^D k}{(2\pi)^D}\frac{1}{J(\mathbf{k})+r}$$

Exact Dimensional Reduction and holography in the large n limit

If two systems share the same density of states

$$\rho(\epsilon) = \int \frac{d^D k}{(2\pi)^D} \delta^{(D)}(\epsilon - J(\mathbf{k}))$$

then they will have identical self-energies $\Sigma^{(0)} = \int d\epsilon \frac{\rho(\epsilon)}{\epsilon + r}$

This enables a universal reduction to a one dimensional system with a kernel $J_{eff}(\mathbf{k})$:

$$\int \frac{d^D k}{(2\pi)^D} \ \delta(\epsilon - J(\mathbf{k})) = \left| \frac{dJ_{eff}}{dk} \right|^{-1}_{J_{eff}(k)=\epsilon}$$

Uniform background gauge

$$\mathcal{L}_{\rm matter} = \frac{1}{2}|D_\mu \phi^\nu|^2 - \frac{M^2}{2}|\vec{\phi}|^2 + \frac{u}{4!}|\vec{\phi}|^4 + ...$$

$$D_\mu(x) = \partial_\mu - i\theta A_\mu(x)$$

For uniform non-Abelian $A_\mu(x)$

[e.g., emulating background curvature R of preferred orderings (Nelson and Sachdev)], the density of states at low energies can be be lower dimensional (ZN, Phys. Rev. B 69, 014208) and thus lower dimensional behavior appears. The low-energy entropy is also “holographic” (scaling with area).

Conclusions: Holography and dimensional reduction

In any system, there are inequalities that bound the correlation functions by those in lower dimensional systems.

These inequalities become most potent when there are "lower dimensional symmetries" and, e.g., afford bounds on auto-correlation times

These effective dimensional reductions due to matching symmetries can become exact when there are exact dualities.

Exact dualities can be derived by bond algebras that map two- and three-dimensional quantum systems to systems in lower dimensions

Universally, in the large n limit, exact dimensional reductions can be constructed by preserving the density of states

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Universally, in the large n limit, exact dimensional reductions can be constructed by preserving the density of states

Conclusions: There are exact spin-Majorana (and similar other) dualities, holography, and deconfinement

Further reading:

Z. Nussinov and G. Ortiz, “Autocorrelations and Thermal Fragility of Anyonic Loops in Topologically Quantum Ordered Systems”, Physical Review B 77, 064302 (2008)

Zohar Nussinov, Gerardo Ortiz “Orbital order driven quantum criticality”, Europhysics Letters 84 (2008) 36005

Z. Nussinov and G. Ortiz, “A symmetry principle for topological quantum order”, Annals of Physics 324, Issue 5, Pages 977-1057 (2009)

G. Ortiz, E. Cobanera, an Z. Nussinov, “Dualities and the phase diagram of the p- clock model”, Nuclear Physics B 854, 780 (2011)

Z. Nussinov, C. D. Batista, and E. Fradkin, “Intermediate symmetries in electronic systems: dimensional reduction, order out of disorder, dualities, and fractionalization”, International Journal of Modern Physics B 20, 5239 (2006)

Most directly related to this talk:

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E. Cobanera, G. Ortiz, and Z. Nussinov, **“Unified approach to classical and quantum dualities”,** Physical Review Letters 104, 020402 (2010)

E. Cobanera, G. Ortiz, and Z. Nussinov, **“The bond-algebraic approach to dualities”**, Advances in Physics 60, 679 (2011)

C. D. Batista and Z. Nussinov, ”**Generalized Elitzur’s theorem and dimensional reductions**”, Physical Review B 72, 045137 (2005)

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