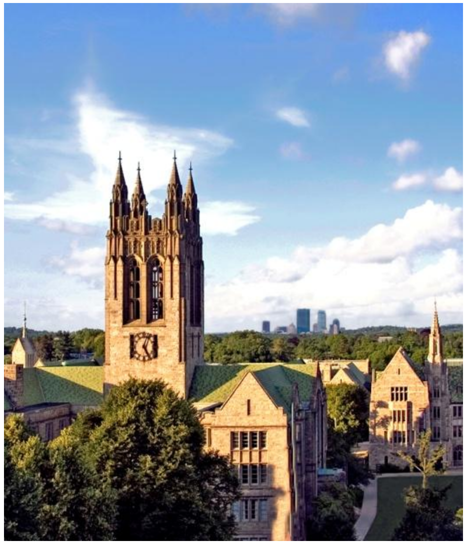
Dislocations in 3D topologically ordered phases



Ying Ran (Boston College)

Aug, 2012, KITP

Collaborators

Andrej Mesaros, Boston College

Yong-Baek Kim, Univ of Toronto

Motivation

- Quantum spin liquids --- the "non-trivial" Mott insulators
- In usual context, (gapped) quantum spin liquids feature fractionalized excitations and are topologically ordered

Questions behind this talk:

- Properties of topological defects in a topologically ordered phase?

(vortices, dislocations, domain walls….. Consequences of spontaneous symmetry breaking.)

- In general, how symmetry "acts" on topological order?

Why are these questions important?

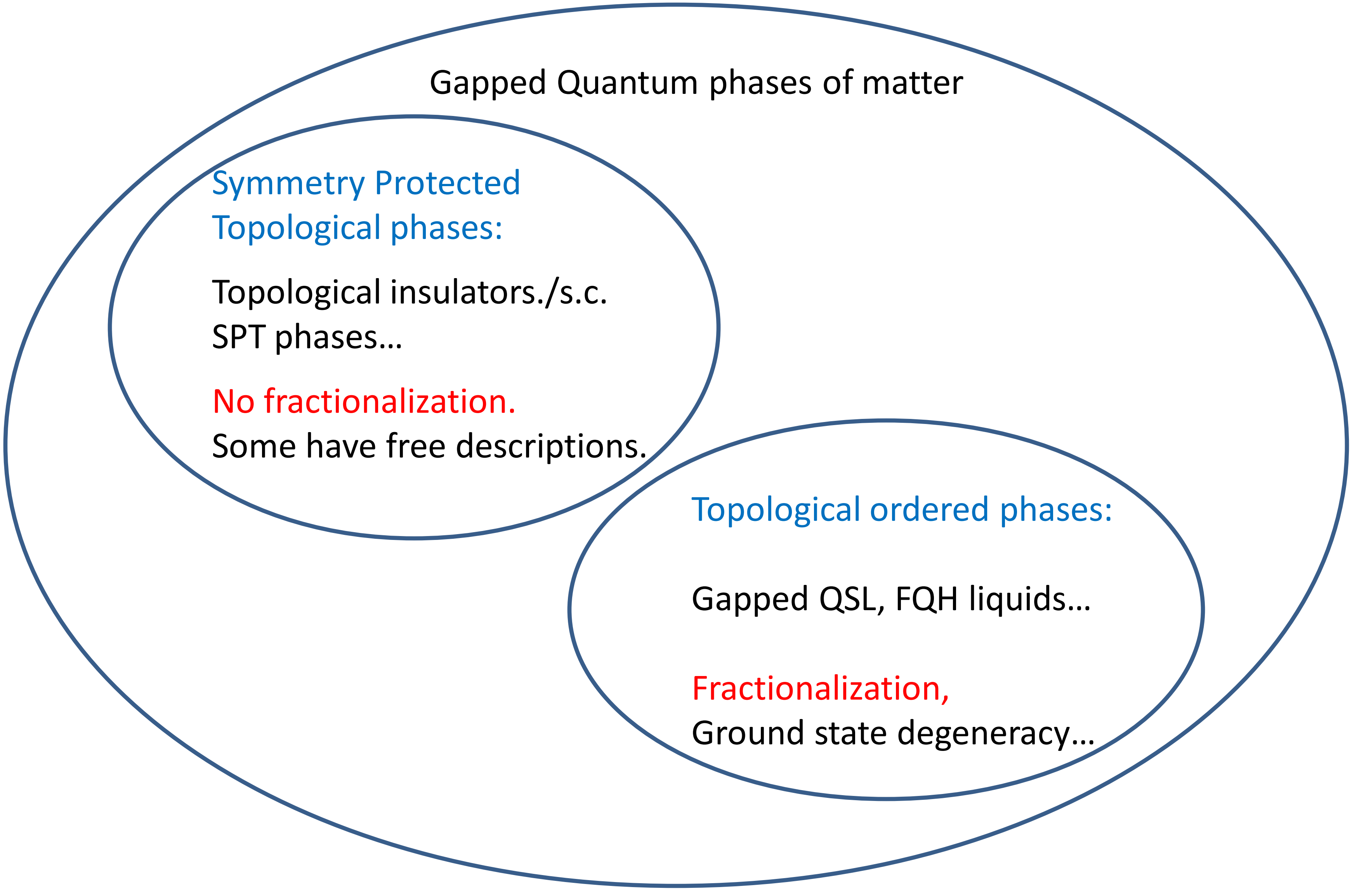
Questions behind this talk:
Properties of topological defects in a topologically ordered phase?
How symmetry "acts" on topological order?

- Classification of exotic phases of matter
- Qualitatively new properties of topological defects in a topological phase.

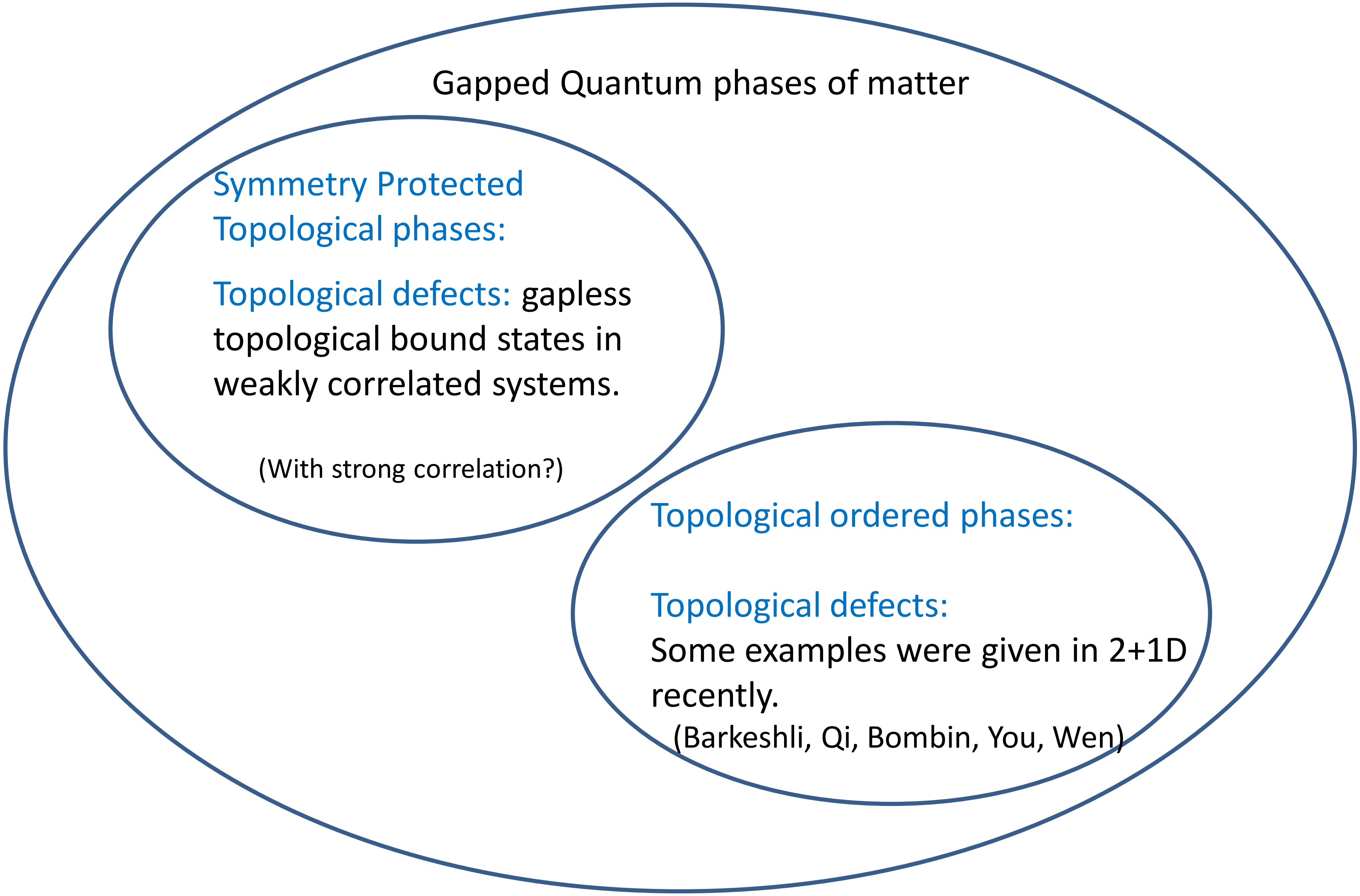
For example, vortices in topological superconductor are known to be interesting.

Here we will not provide general answers.
Instead we provide concrete examples where some new physics got realized.

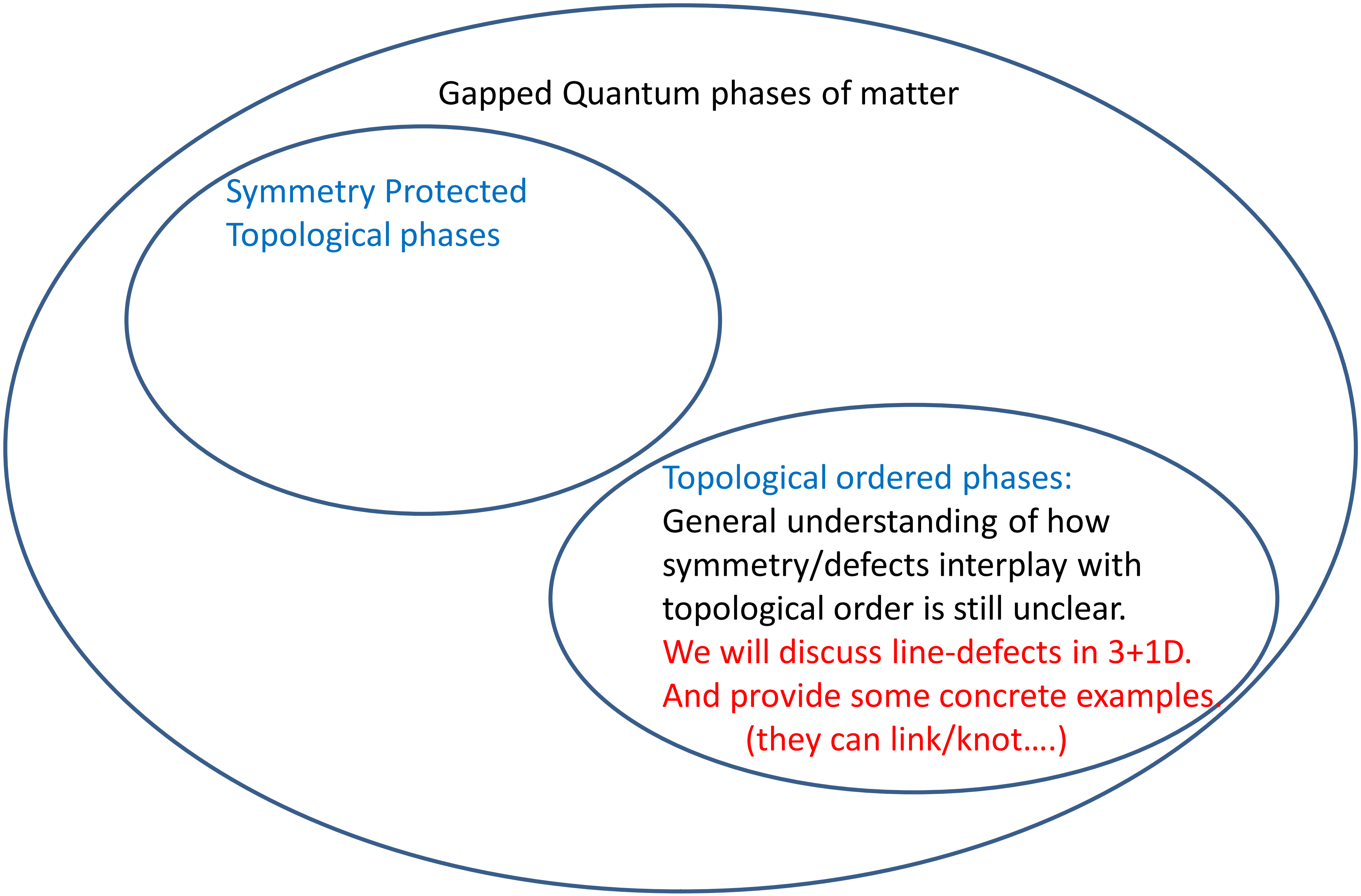
Topological phases



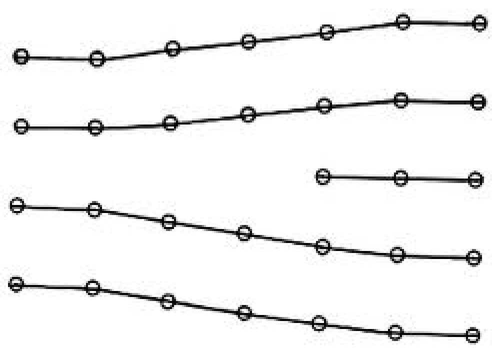
Topological phases



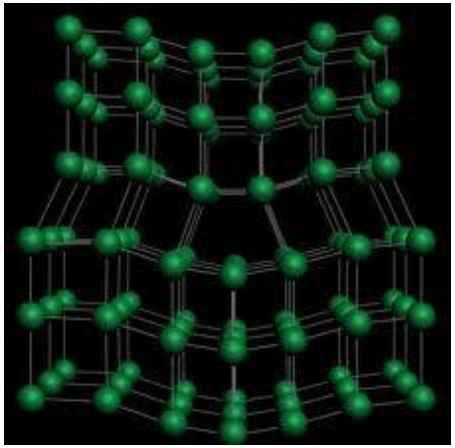
Topological phases



Dislocation:
topological defects in crystals



2D



3D

- We will focus on dislocations --- point defects in 2+1D and line defects in 3+1D.

Previous 2+1D examples

Dislocation points give extra topological ground state degeneracy.

- Bombin (2010), You &Wen (2012):

Kitaev-type exact solvable models.

Extra G.S.D = $2^{\#\text{dislocation pairs}-1}$

- Barkeshli & Qi (2012)

Fractional quantum hall states in bands with higher Chern number.

What happens in 3+1D, where dislocations can link/knot?

Because GSD is topological, it must be a knot/link invariant.

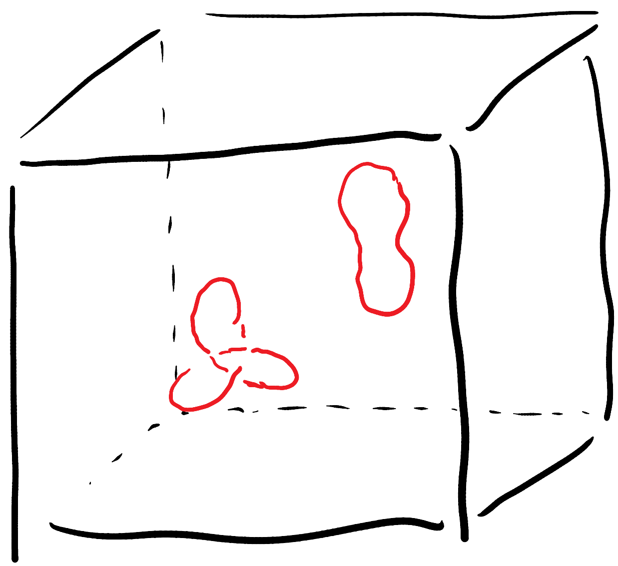
Main results

- We construct certain Kitaev-type exact solvable models in 3+1D, whose ground states are topologically ordered.

 (and characterized by Z_n x Z_n gauge theory)

- In these exact solvable models,

 we show dislocation loops give
 extra topological ground state
 degeneracies (cannot be lifted by
any local perturbations).



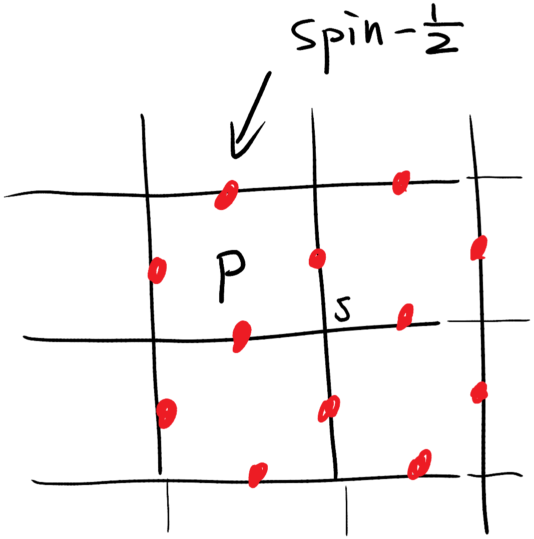
- In these models we can prove:

The extra ground state degeneracies= $n^{\#loops\ -1}$
independent of linking/knotting of the dislocation loops and only depend on the number of dislocation loops. (with periodic boundary conditions.)

(In fact, by examining the requirement of the proof, it suggests which models may have linking/knotting dependence.)

A warm-up: 2D doubled toric code model

- The Kitaev's toric code model:



$$H = -\sum_s A_s - \sum_p B_p$$

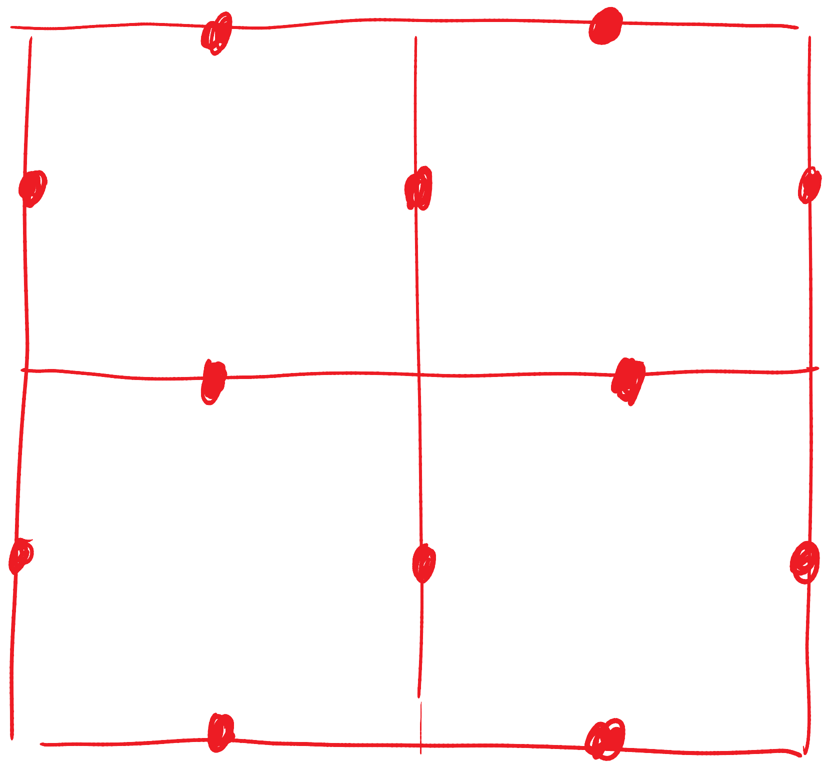
$$A_s = \sigma^z \sigma^z \sigma^z \sigma^z$$

$$B_p = \sigma^x \sigma^x \sigma^x \sigma^x$$

Topological ordered phase: Z_2 gauge theory.
Quasiparticles are created by string operators.

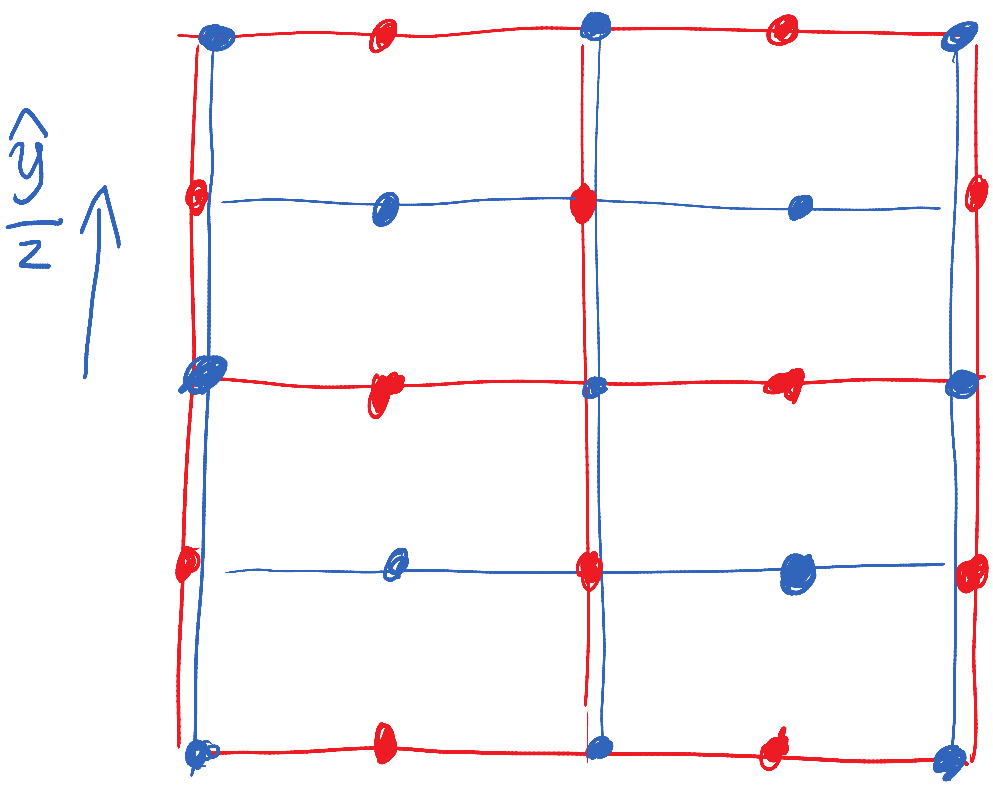
A warm-up: 2D doubled toric code model

- Doubled toric code model:



A warm-up: 2D doubled toric code model

- Doubled toric code model:



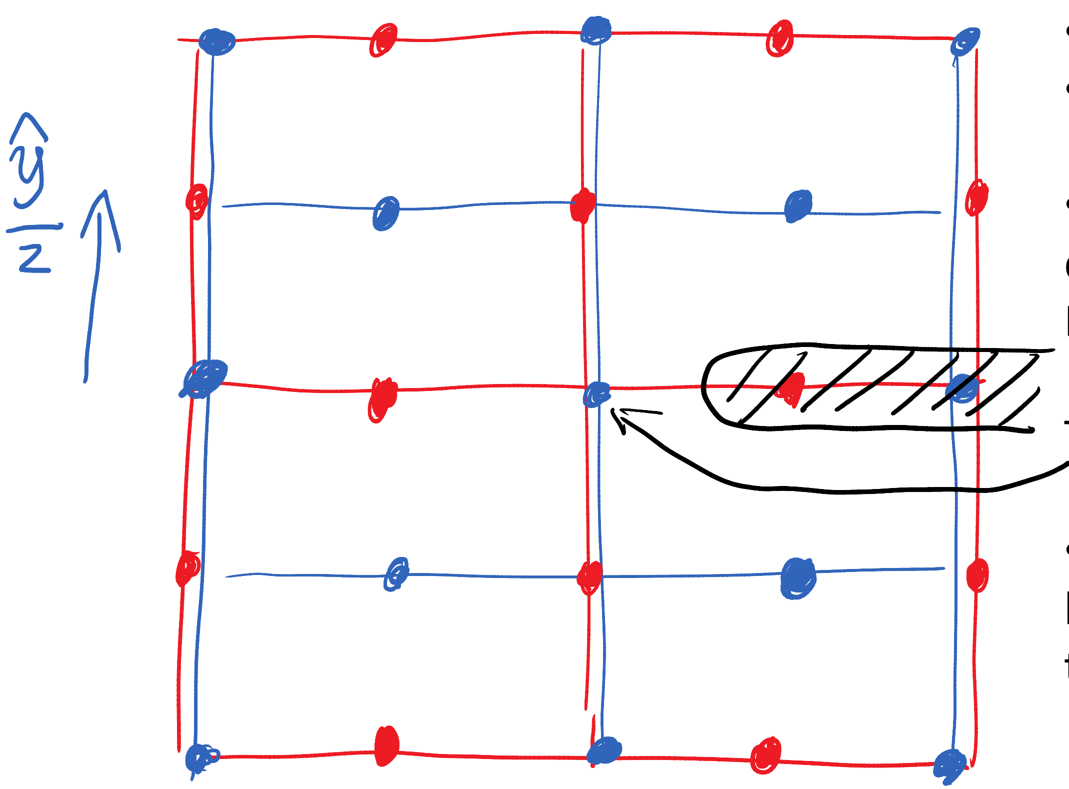
Two decoupled copies of the Toric Code, related by lattice translation symmetry: $\hat{y}/2$

Z2 x Z2 gauge theory.

4x4=16 fold GSD on torus.

A warm-up: 2D doubled toric code model

- Create dislocations



- Remove a chain of sites
- Glue the two sides together

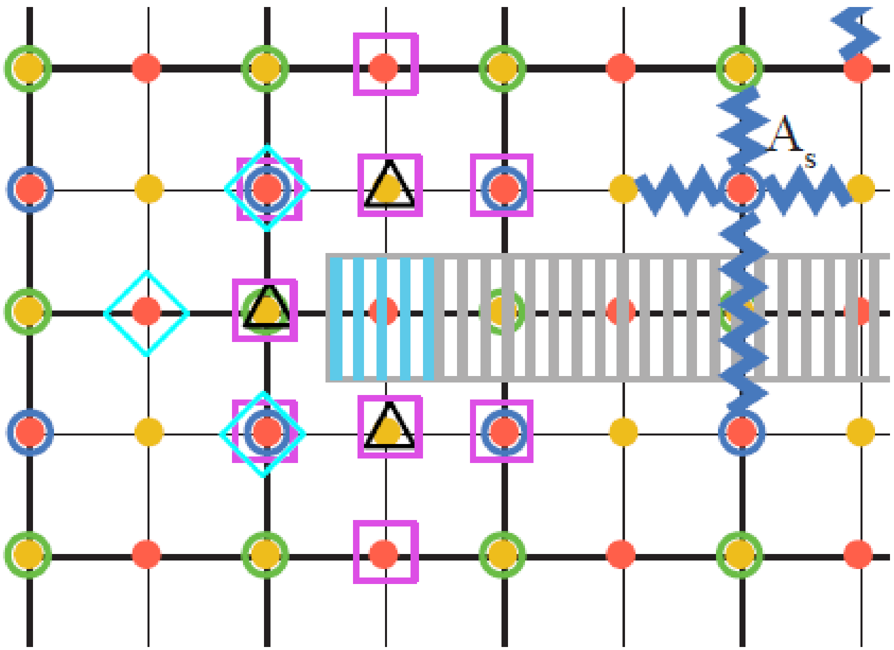
- Some stablizers close to the dislocation core got messed up. For example,

This star operator has three red spins.

- A red quasi-particle becomes a blue quasi-particle after moving around the dislocation.

A warm-up: 2D doubled toric code model

- Prescriptions for “bad” stabilizers:



Because some stabilizers close to dislocation are messed up, we need to redefine them so that:

(1) Stabilizers still commute with each other.
(2) They lift ground state degeneracy “as much as possible”.
(Will explain shortly.)

A warm-up: 2D doubled toric code model

- To count the ground state degeneracy in the presence of dislocations:

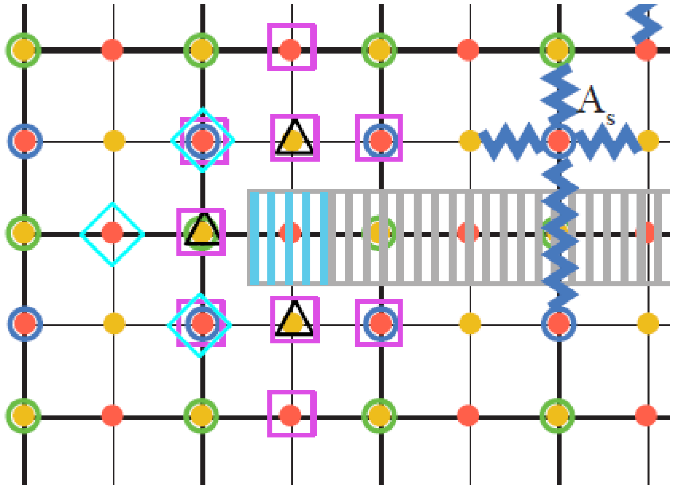
A theorem of general Z_2 toric code (Gottesman 1996):

GSD=2^(# of spins - # of independent stabilizers)

- Because we have a well-defined prescription for stabilizers around dislocations, we can simply count this GSD.

- Later we will show that the GSD with our prescription is topologically protected. (namely, cannot be lifted further by any local perturbations.)

Counting ground state degeneracy



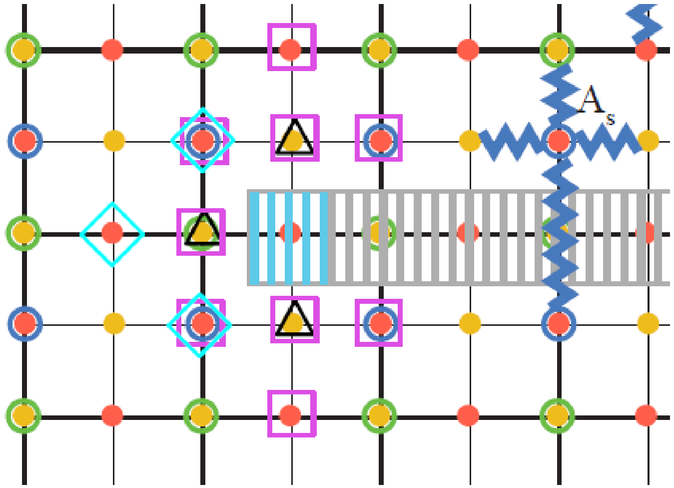
We always work with torus boundary condition.

For one pair of dislocations, let's count:

(1) Remove N spins, and N stabilizers along the line.
(2) Remove 3 stabilizers, replace them by 2 new stabilizers, at each end.
(3) Naively we will have 4 extra GSD. But because the dislocation mixed the two copies of toric codes, we also have 2 less dependence relations for stabilizers. (product of all stars=1, product of all plaquettes=1).
(4) Finally there is no extra GSD.

These can be explicitly checked by exact numerics ---computing the rank of a Z2 valued matrix tells the # of independent stabilizers. . And we did that.

Counting ground state degeneracy



We always work with torus boundary condition.

For two pair of dislocations, let's count:

(1) Remove 2N spins, and 2N stabilizers along the line.
(2) Remove 3 stabilizers, replace them by 2 new stabilizers, at each end.
(3) Still have 2 less dependence relations for stabilizers. (product of all stars=1, product of all plaquettes=1).
(4) Finally we have 2^2=4 fold extra GSD.

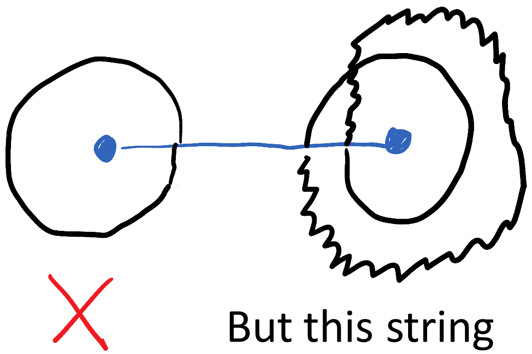
Clearly, for N pairs of dislocations, extra GSD=4^{N-1}.

These can be explicitly checked by exact numerics ---computing the rank of a Z2 valued matrix tells the # of independent stabilizers. And we did that.

Extra GSD is topological

- Now let's show the extra GSD cannot be further lifted.
- Construct independent string operators acting within the GS sector:

For a single pair of dislocations



But this string
is shrinkable---
a product of local stabilizers

NO new string operators

Because of well-defined prescription,
We know a string is shrinkable or now.

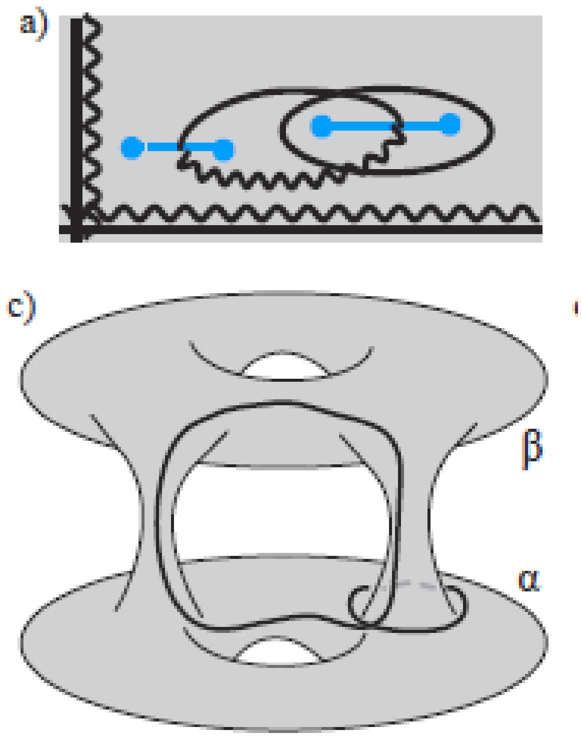
For two pairs of dislocations

Two new e-strings, Two new m-strings.
They are not shrinkable and independent.
Supporting the extra 4-fold topological GSD.
(Easily generalized to more pairs.)

Again, these can be checked by exact numerics.

A nice geometric interpretation

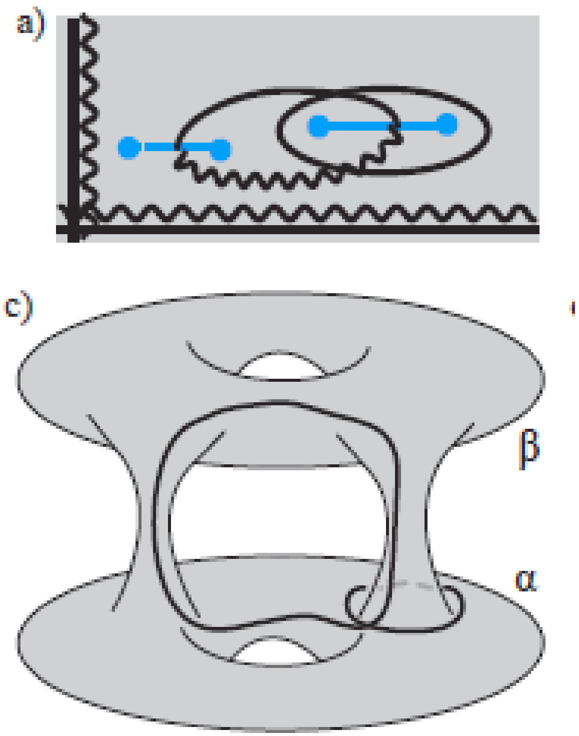
- Barkeshli&Qi 2012:



- Dislocation pairs can be interpreted as Adding “worm-holes” connecting the two copies of toric code.
- The first pair does not give more genus. The second pair gives one more genus:
- GSD=4^{N-1}

A nice geometric interpretation

- Barkeshli&Qi 2012:

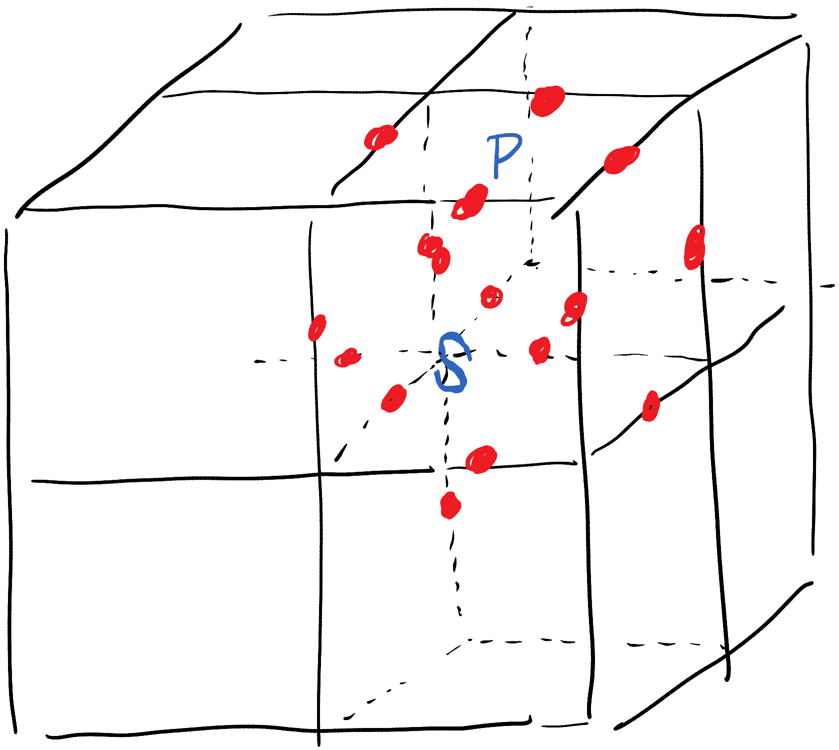


- Dislocation pairs can be interpreted as Adding “worm-holes” connecting the two copies of toric code.
- The first pair does not give more genus. The second pair gives one more genus:
- $GSD=4^{N-1}$

- Now I will go to 3+1D topological ordered phases, where dislocations also give extra GSD.
- It would be nice to have similar geometric interpretation. Unfortunately, I do not know too much about topology of 3-manifolds.
- In fact, this is why we work with exact solvable models --- because of lack of knowledge…

3D doubled toric code

- The 3D toric code:



$$H = -\sum_s A_s - \sum_p B_p$$

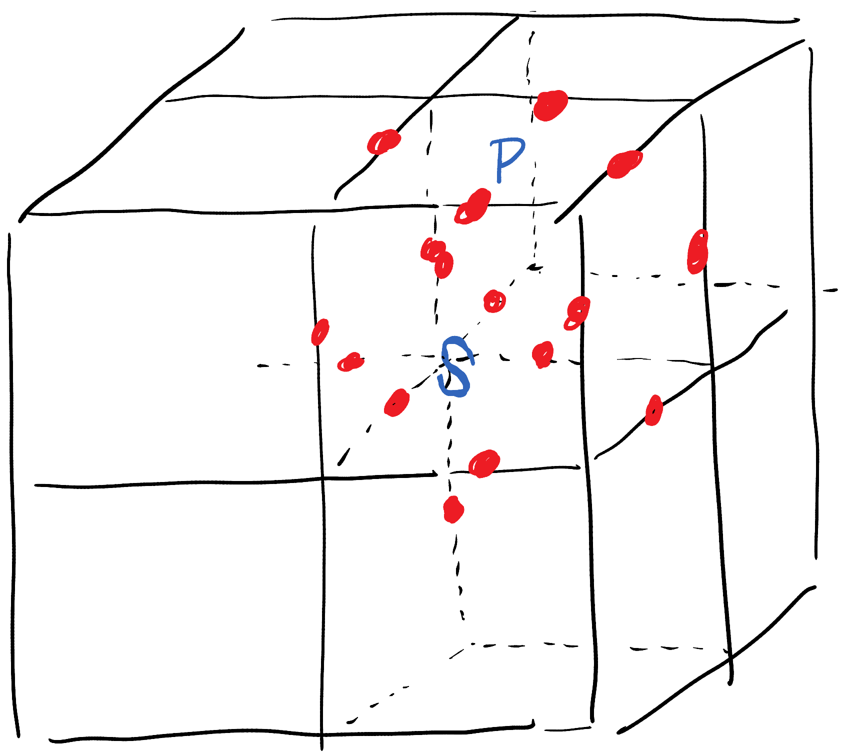
$$A_s = \sigma^z\sigma^z\sigma^z\sigma^z\sigma^z\sigma^z$$

$$B_p = \sigma^x\sigma^x\sigma^x\sigma^x$$

Ground state is topological ordered: 3+1D Z_2 gauge theory.
Excitations are created by string operators (electric charge) and membrane operators (magnetic flux loop).

3D doubled toric code

- The 3D toric code:



$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \sigma^z \sigma^z \sigma^z \sigma^z \sigma^z \sigma^z$$

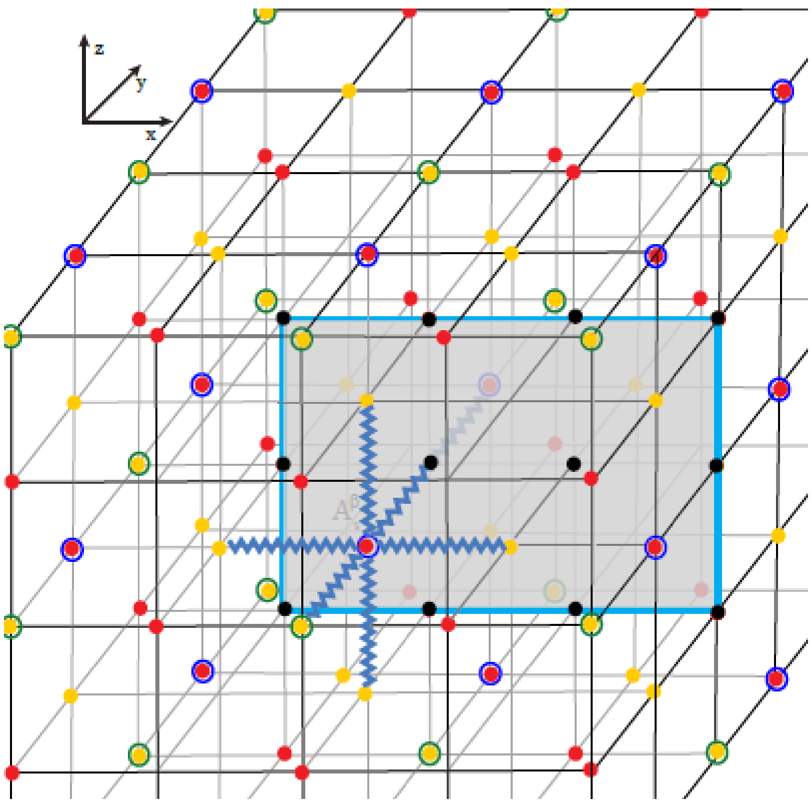
$$B_p = \sigma^x \sigma^x \sigma^x \sigma^x$$

2^3=8 GSD on 3-Torus.

(Note that different from 2D case, each cube has a local constraint:
Product of 6 Bp operators on the 6 faces =1)

3D doubled toric code

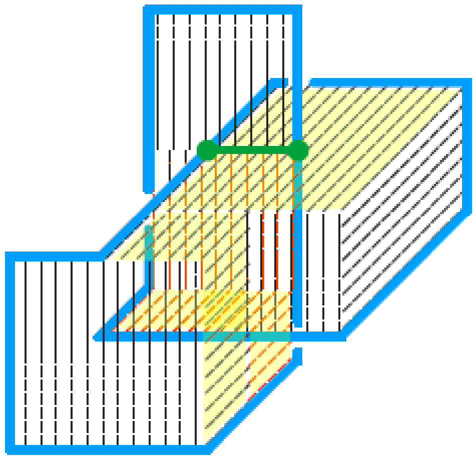
- Sorry for the figure.



- As 2D case, two decoupled copies of 3D toric codes related by lattice translation symmetry along y-direction.
- Can create dislocation loops, for instance, by removing a plane of sites.
- Glue two sides together.
- Some stabilizers close to the dislocation core got messed up. Need prescription to replace them.
- Then one can count the extra GSD by # of spins- # of independent stabilizers. Again, independence of stabilizers can be Checked by exact numerics.

We did all these and find: extra GSD=2^(#loops-1)

3D doubled toric code



We even computed the GSD
by exact numerics for the trefoil knot….

- As 2D case, two decoupled copies of 3D toric codes related by lattice translation symmetry along y-direction.
- Can create dislocation loops, for instance, by removing a plane of sites.
- Glue two sides together.
- Some stabilizers close to the dislocation core got messed up. Need prescription to replace them.
- Then one can count the extra GSD by # of spins- # of independent stabilizers. Again, independence of stabilizers can be Checked by exact numerics.

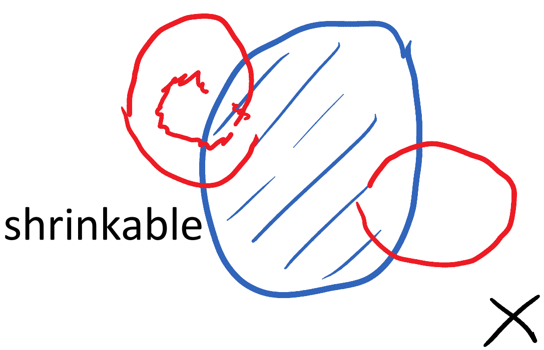
We did all these and find: extra GSD=2^(#loops-1), independent of linking/knotting.

Next, I will show you a proof for general knots.

First, the extra GSD is topological

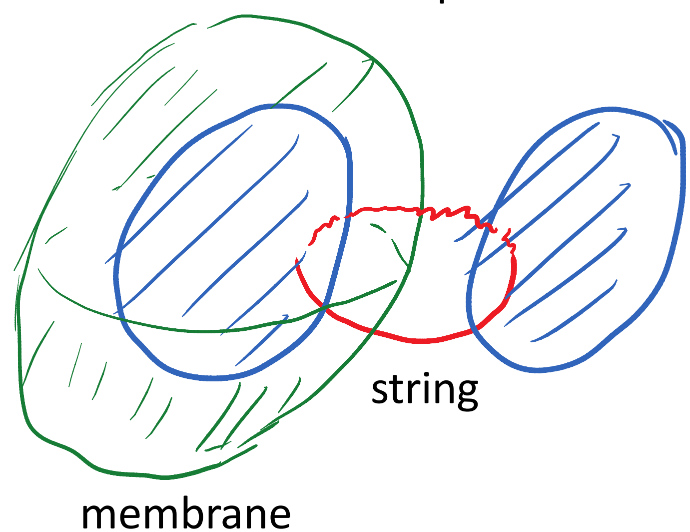
- In 3D, one can construct string and membrane operators supporting the topological GSD.
- Because we have a well-defined prescription, we know whether a string/membrane is shrinkable or not.

One dislocation loop



No extra GSD

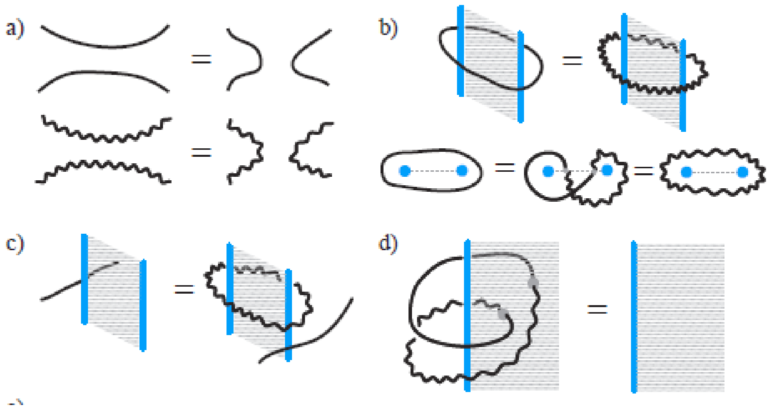
Two dislocation loops



2 fold extra GSD supported by the one string and one membrane operators.

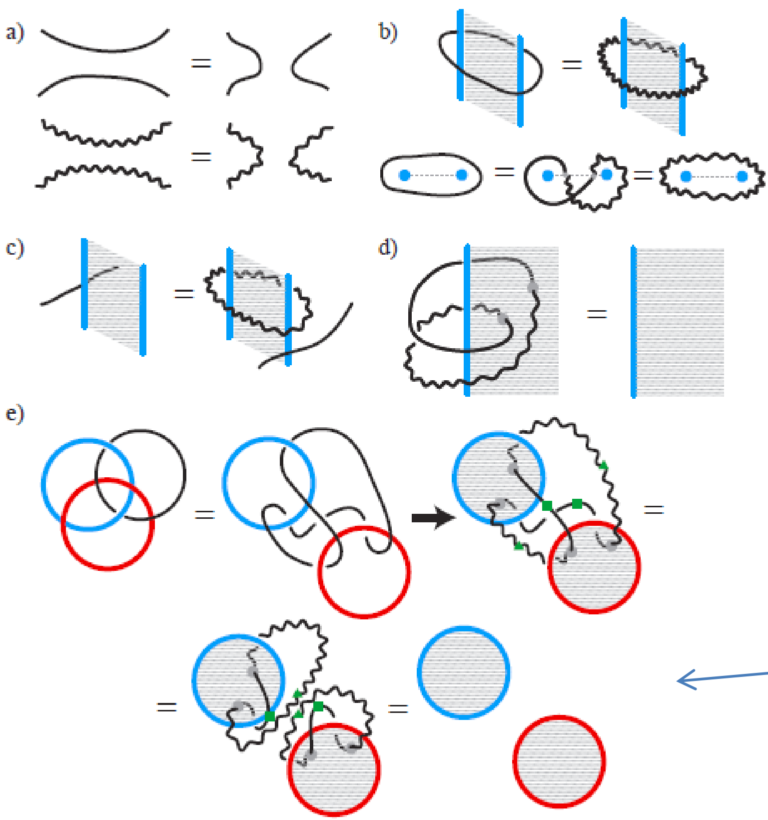
Surgery rules of string operator

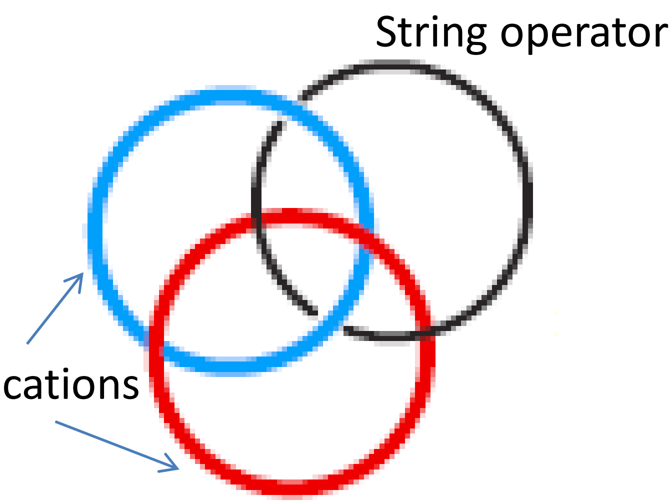
- These rules specify what are allowed motion of string operators. (by multiplying local stabilizers) Namely these tells whether a string operator is shrinkable, and whether two string operators are dependent.



Surgery rules of string operator

- An example on how to use these rules: Borromean rings.





This string must be shrinkable,
As shown here.

Proof of knotting independence

- Need to show: all string operators are shrinkable for a general knotted dislocation loop.

Trefoil

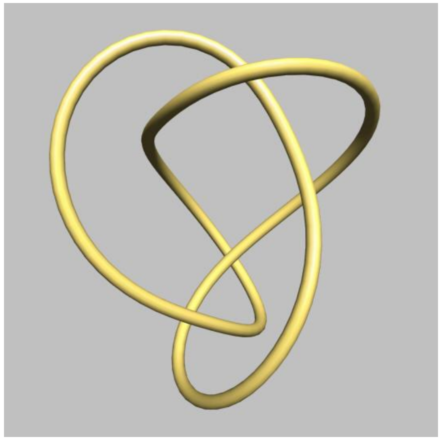


Figure-8

Proof of knotting independence

- Need to show: all string operators are shrinkable for a general knotted dislocation loop.

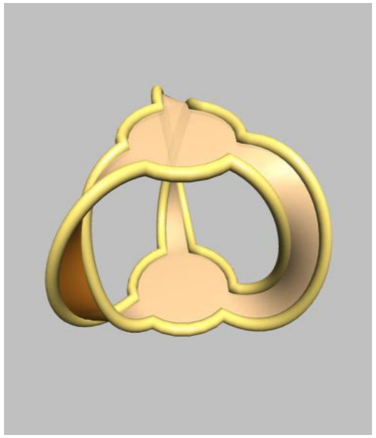
- To prove this, we need some general way to diagnose knots:

Seifert Surface (1934)

Any knot can be viewed as the single edge of an oriented connected surface.

Seifert surface is not unique.

In general they can be constructed as a set of discs connected by twisted ribbons:

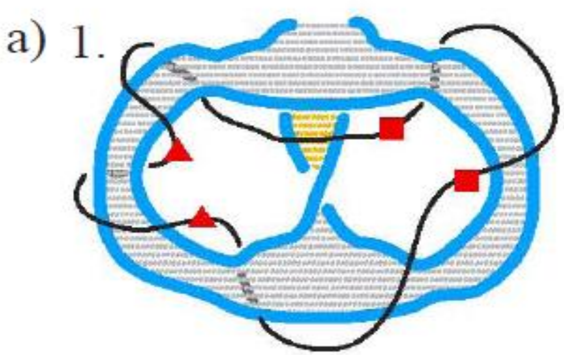


Trefoil knot

Figure-8 knot

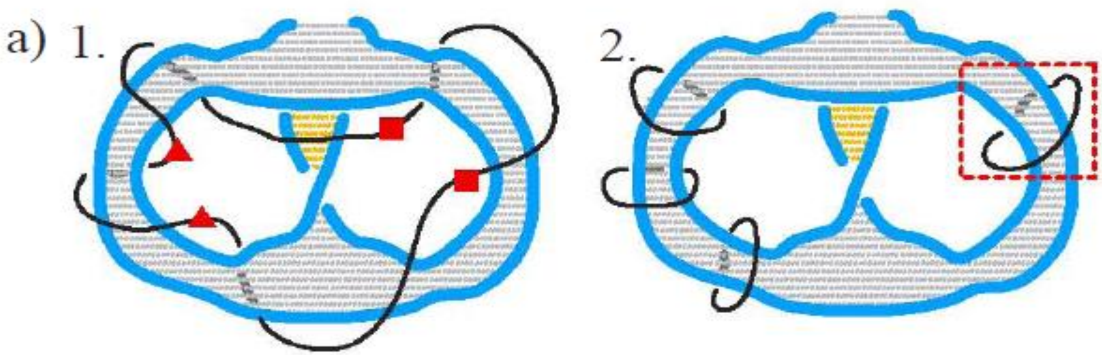
Proof of knotting independence

Given a general knot, using local surgery rules to shrink a string operator:



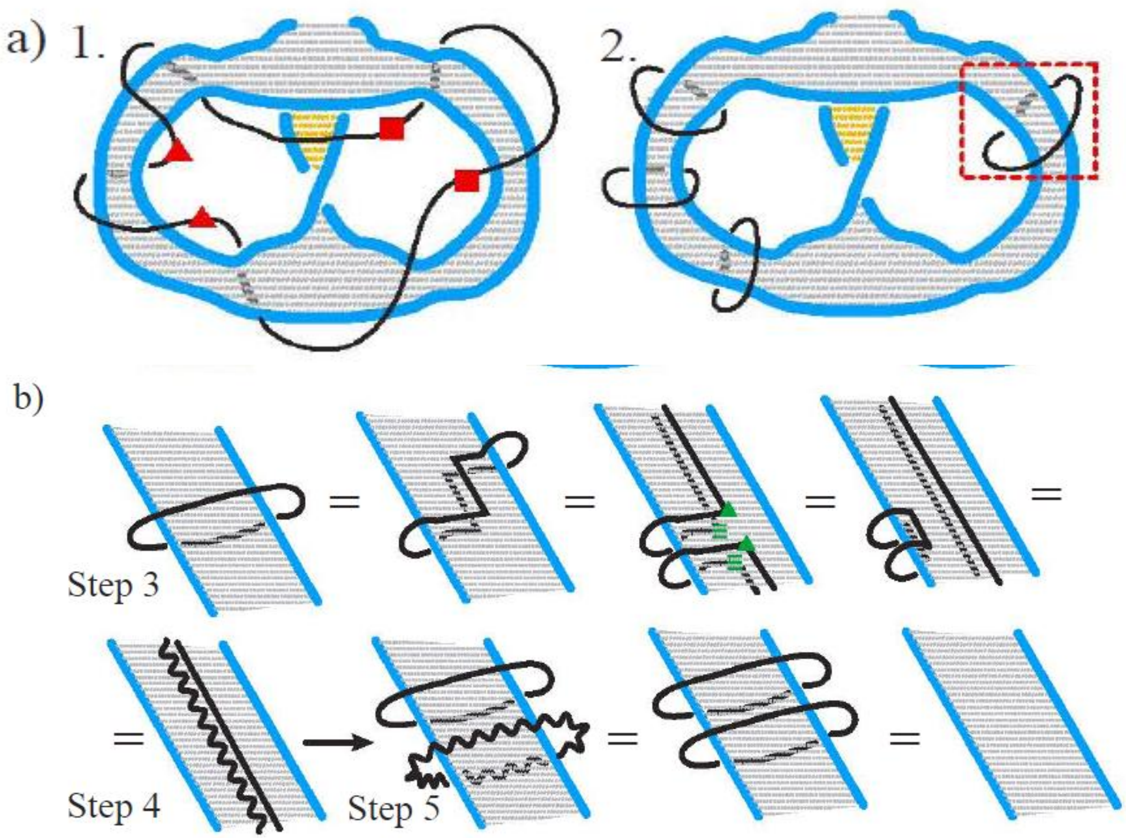
Proof of knotting independence

Given a general knot, using local surgery rules to shrink a string operator:



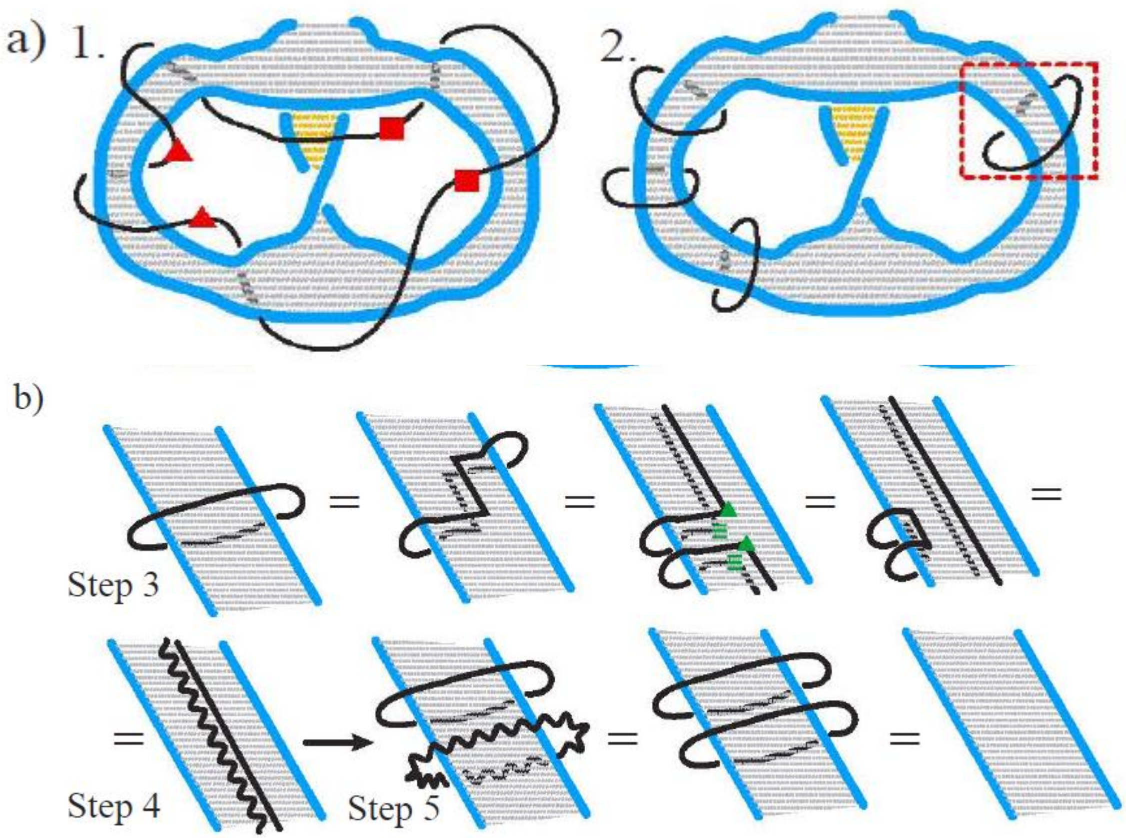
Proof of knotting independence

Given a general knot, using local surgery rules to shrink a string operator:



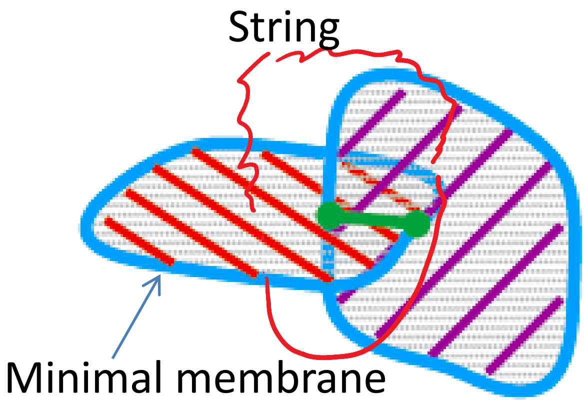
Proof of knotting independence

Given a general knot, using local surgery rules to shrink a string operator:



Discussion: This proof requires being able to annihilate two string operators --- maybe in some doubled non-abelian models, dislocation GSD can have knotting dependence?

linking independence



Here we do not provide general proof.

But for two linked dislocations, the extra non-shrinkable string is easy to construct.

The more non-trivial task is to construct the non-shrinkable membrane operator.

We constructed the non-shrinkable “minimal” membrane operator enclosing one dislocation--- flat thin double sheets.

This membrane operator is basically the same as the unlinked case, except for local surgery along the green line.

Discussion

- In 2D, GSD=4^(# of pairs-1), signaling a “square-root” theory may exist. Indeed Bombin (2010) showed a model with GSD=2^(# of pairs-1).

But in 3D, GSD=2^(# of loops-1). Probably the minimal model   where this occurs.

And one can easily generalize these for Zn x Zn double Kitaev models in 3D: GSD= n^(# of loops-1).

- What if one melt the lattice by double dislocations?

The single dislocations →dynamic excitations.

In both 2D and 3D, doubled toric code → D4 non-abelian gauge theory

Future directions

- General theoretical framework to understand/classify topological defects in topological ordered phases?
- Linking/knotting dependence of GSD in other 3D models?

Future directions

- General theoretical framework to understand/classify topological defects in topological ordered phases?

- Linking/knotting dependence of GSD in other 3D models?

- Thank you!