Topological Insulators and Majorana Fermions

Charles Kane, University of Pennsylvania

I. Introduction: Topological Band Theory

III. Majorana Fermions on Topological Insulators

IV. Generalized “Periodic Table” for topological defects in insulators and superconductors

IV. Non-Abelian statistics in 3D

Thanks to Gene Mele, Liang Fu, Jeffrey Teo

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And special thanks to Mike Freedman. Happy Birthday!

Topological Band Theory

Integer Quantum Hall Effect, TKNN invariant (Thouless et al. 1984)

$H(\mathbf{k})$: Brillouin zone (T^2) $\mapsto$ Bloch Hamiltonans with energy gap

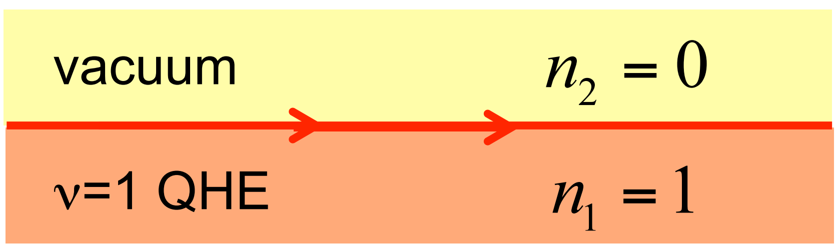
First Chern Number : topological invariant characterizing occupied bands

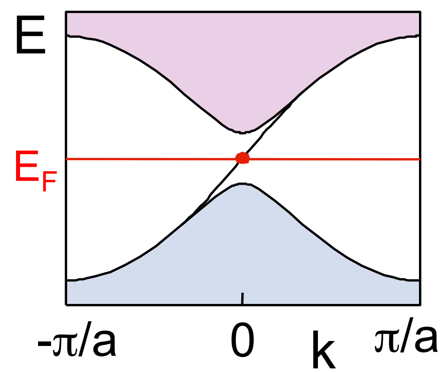
$$n = \frac{1}{2\pi}\int_{T^2} d^2\mathbf{k}\,\mathrm{Tr}\left[\mathbf{F}\right] \in \mathbb{Z}$$

(**F** = Berry curvature)

$$\sigma_{xy} = n\frac{e^2}{h}$$

Edge States





Chiral Dirac fermion edge mode

Bulk - Boundary Correspondence

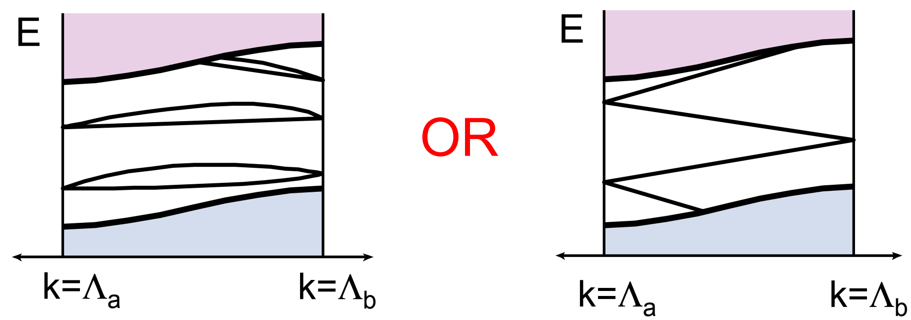
$$n_1 - n_2 = \text{\# Chiral Fermion modes}$$

Time Reversal Invariant $\mathbb{Z}_2$ Topological Insulator

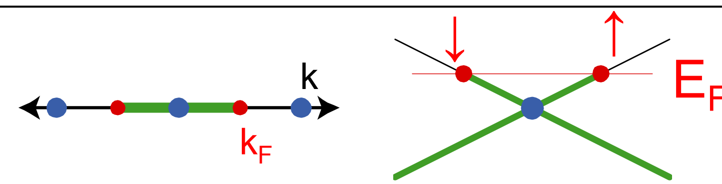
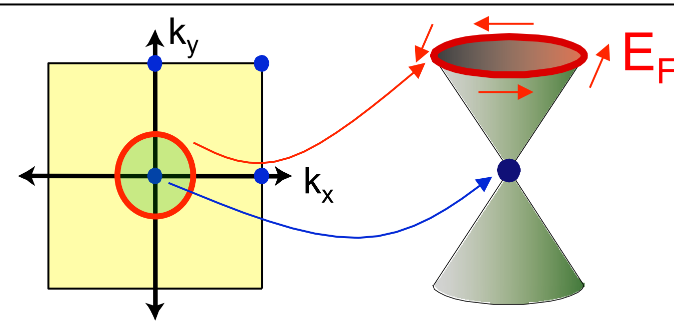
Time Reversal Symmetry : $\Theta H(\mathbf{k})\Theta^{-1} = H(-\mathbf{k})$ $\quad\Theta\psi = i\sigma^y\psi^*$

Kramers' Theorem : $\Theta^2 = -1 \quad \Rightarrow$ All states doubly degenerate

$\mathbb{Z}_2$: two ways to connect Kramers pairs on surface



Bulk - Boundary Correspondence

	Equivalence classes of $H(\mathbf{k}) \quad \mathbf{k}\in T^d$	surface/edge: even or odd number Dirac points enclosed by Fermi surface
d=2	$\mathbb{Z}_2$	
d=3	$\mathbb{Z}_2 \oplus 3\mathbb{Z}_2$ ↑ (weak Topo. Ins.)	

Topological Insulators

Two dimensions: Quantum Spin Hall Insulator

Graphene

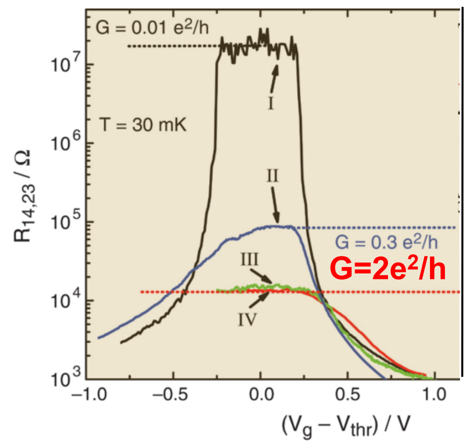
Kane, Mele '05

HgCdTe quantum well

Bernevig, Hughes, Zhang '06

Edge state transport experiments

Konig, et al. '07



Three dimensions: Strong Topological Insulator

Theory: Moore, Balents '06, Roy '06, Fu, Kane, Mele '06

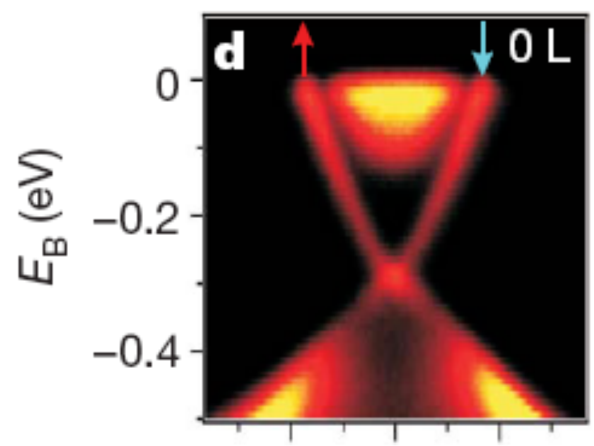
$Bi_{1-x}Sb_x$

Fu, Kane '07 (Th)
Hsieh, et al '07 (Exp)

Bi_2Se_3, Bi_2Te_3

Xia, et al '09 (Exp+Th)
Zhang, et al '09 (Th)
Hsieh, et al '09 (Exp)
Chen et al. '09 (Exp)

Surface States probed by ARPES:



Bi_2Se_3

D Hsieh, et al. Nature '09

Topological Superconductivity, Majorana Fermions

$$H = \sum_k \left(c_k^\dagger \quad c_{-k} \right) H_{BdG}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

Bogoliubov de Gennes Hamiltonian

$$H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix}$$

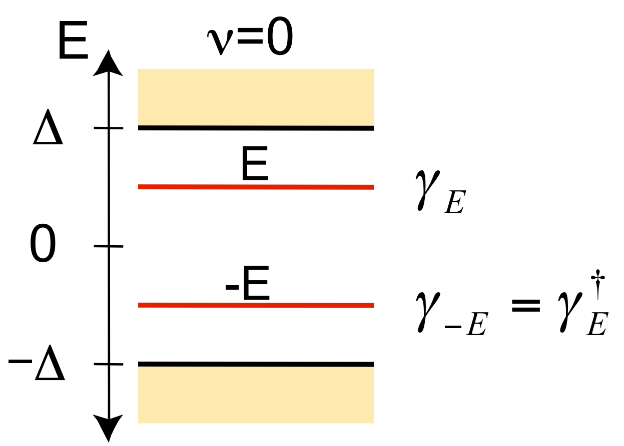
Particle-Hole symmetry : $\Xi H_{BdG}(k) \Xi^{-1} = -H_{BdG}(-k)$ $\quad \Xi\psi = \tau_x \psi^*$

Quasiparticle redundancy : $\psi_{-E} = \Xi \psi_E \;\Rightarrow\; \gamma_E^\dagger = \gamma_{-E}$

Simplest Example: 1D superconductor, spinless electrons (Kitaev '01)

$\mathbb{Z}_2$ Topological Superconductor : $\nu = \frac{1}{2\pi}\int dk \ \mathrm{Tr}[\mathbf{A}] \ \mathrm{mod}\ 2 = 0 \text{ or } 1$

Discrete end state spectrum :



Majorana Fermion bound state

$$\gamma_0^\dagger = \gamma_0$$

Periodic Table of Topological Insulators and Superconductors

Kitaev, 2008
Schnyder, Ryu, Furusaki, Ludwig 2008

Anti-Unitary Symmetries :

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})$; $\Theta^2 = \pm 1$
- Particle - Hole : $\Xi H(\mathbf{k})\Xi^{-1} = -H(-\mathbf{k})$; $\Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k})\Pi^{-1} = -H(\mathbf{k})$; $\Pi = \Theta\Xi$

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Altland-Zirnbauer Random Matrix Classes

Complex K-theory

Real K-theory

Bott Periodicity

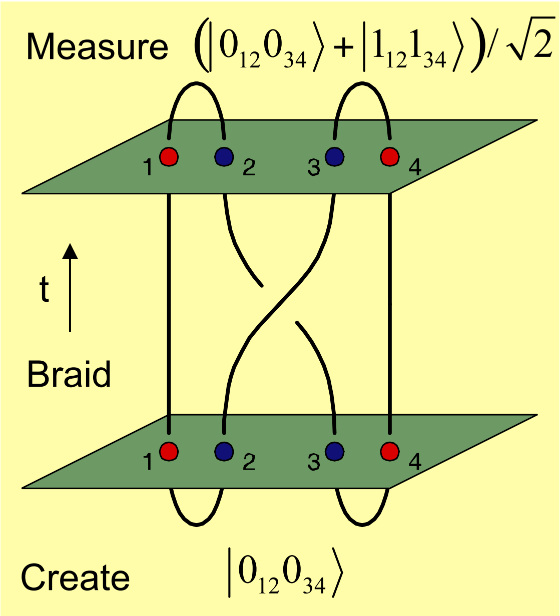
Majorana Fermion bound states

Potential hosts :

- Quasiparticle excitations of Moore Read FQHE state
- Prototype topological superconductors
  - vortex in 2D spinless p+ip SC — Read Green 00
  - end state in1D spinless p wave SC — Kitaev 01

Topological Quantum Computing Kitaev 03

- A topological protected quantum memory
- Non-Abelian braiding statistics
  → Quantum computation

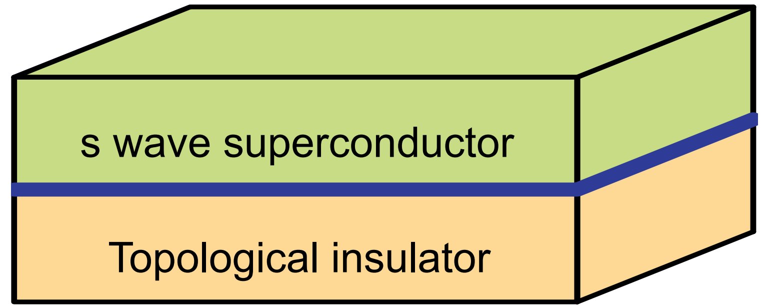


Engineering Topological Superconductivity with ordinary superconductors: Superconducting Proximity Effect

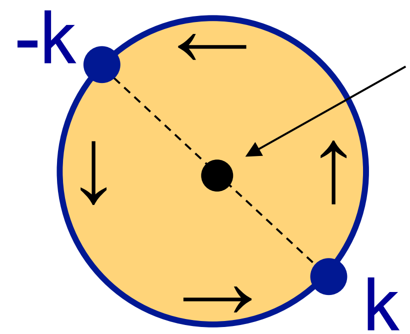
$$H = \psi^\dagger(-iv\vec{\sigma}\cdot\vec{\nabla} - \mu)\psi$$

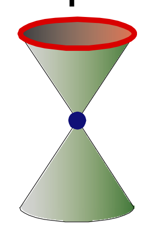
$$+\Delta_S \psi_\uparrow^\dagger \psi_\downarrow^\dagger + \Delta_S^* \psi_\downarrow \psi_\uparrow$$

proximity induced superconductivity at surface



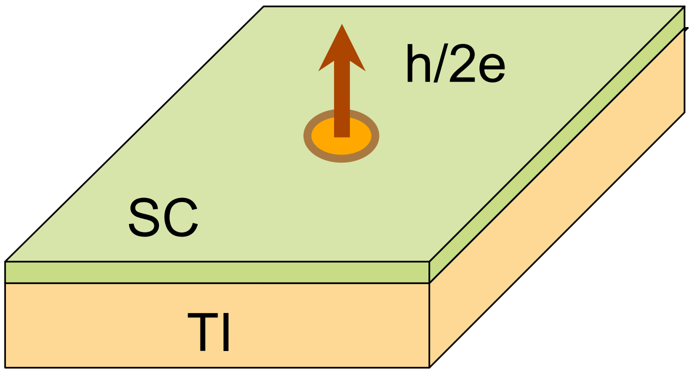
- Half an ordinary superconductor
- Similar to 2D spinless p_x+ip_y topological superconductor, except :
  - Does not violate time reversal symmetry
  - s-wave singlet superconductivity
  - Required minus sign is provided by π Berry's phase due to Dirac Point
- Nontrivial ground state supports Majorana fermion bound states at vortices

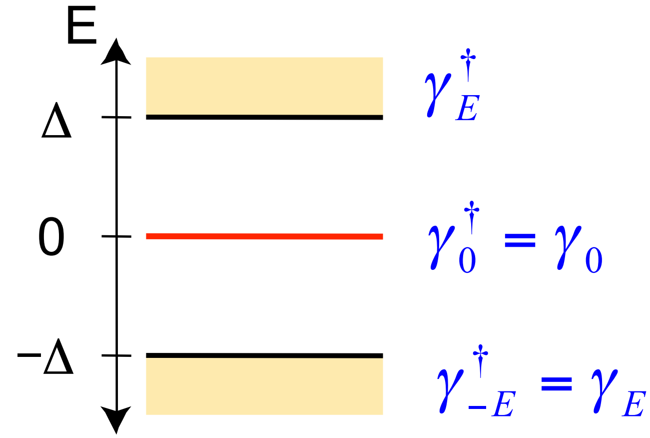




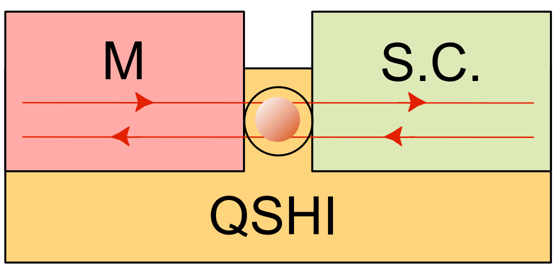
Majorana Bound States on Topological Insulators

1. h/2e vortex in 2D superconducting state

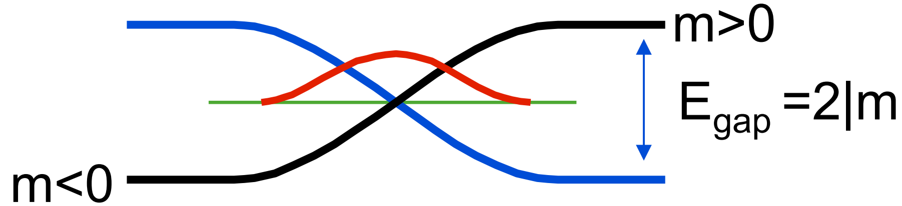




2. Superconductor-magnet interface at edge of 2D QSHI



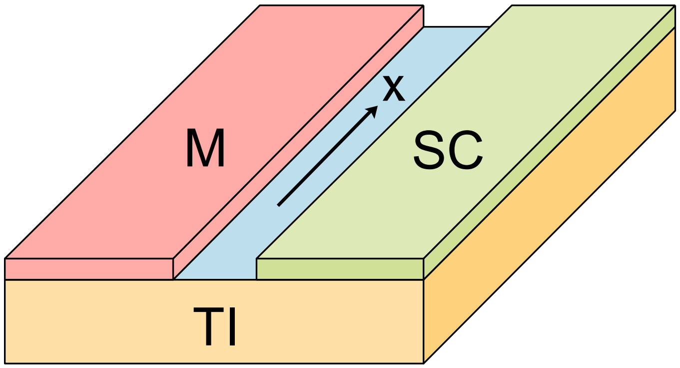
$$m = |\Delta_S| - |\Delta_M|$$

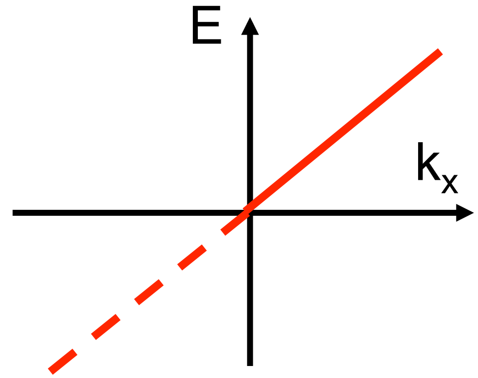


Domain wall bound state γ_0

1D Majorana Fermions on Topological Insulators

1. 1D Chiral Majorana mode at superconductor-magnet interface

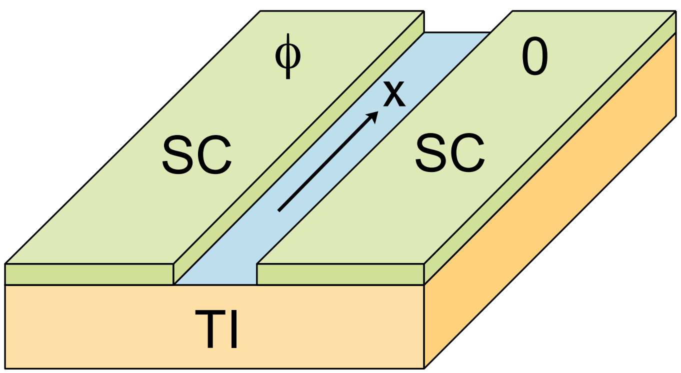


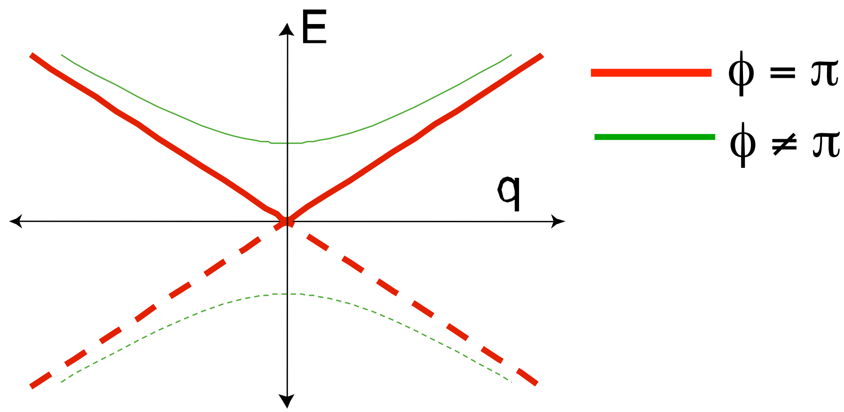


$\gamma_k = \gamma_{-k}^{\dagger}$: "Half" a 1D chiral Dirac fermion

$$H = -i\hbar v_F \ \gamma \partial_x \gamma$$

2. S-TI-S Josephson Junction





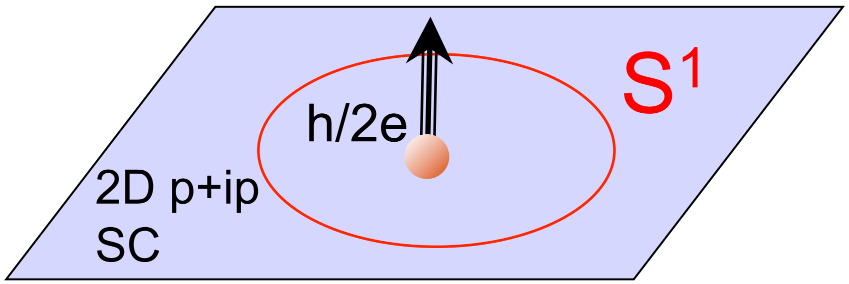
Gapless non-chiral Majorana fermion for phase difference $\phi = \pi$

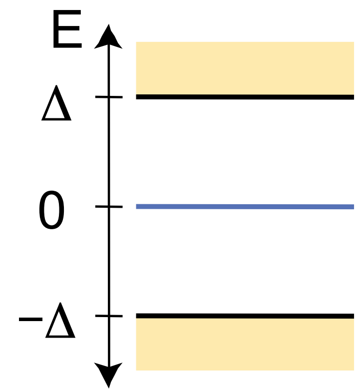
$$H = -i\hbar v_F \ \left(\gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R\right) + i\Delta \cos(\phi/2)\gamma_L \gamma_R$$

Protected Modes at Topological Defects

Example : Majorana zero mode vortex in p+ip superconductor

Read, Green '00





What property of H guarantees that a zero mode is present?

Expect presence of zero mode is "known" by the by the BdG Hamiltonian far from defect

Adiabatic approximation : topologically classify families of gapped BdG Hamiltonians

$$H(\mathbf{k}, s)$$

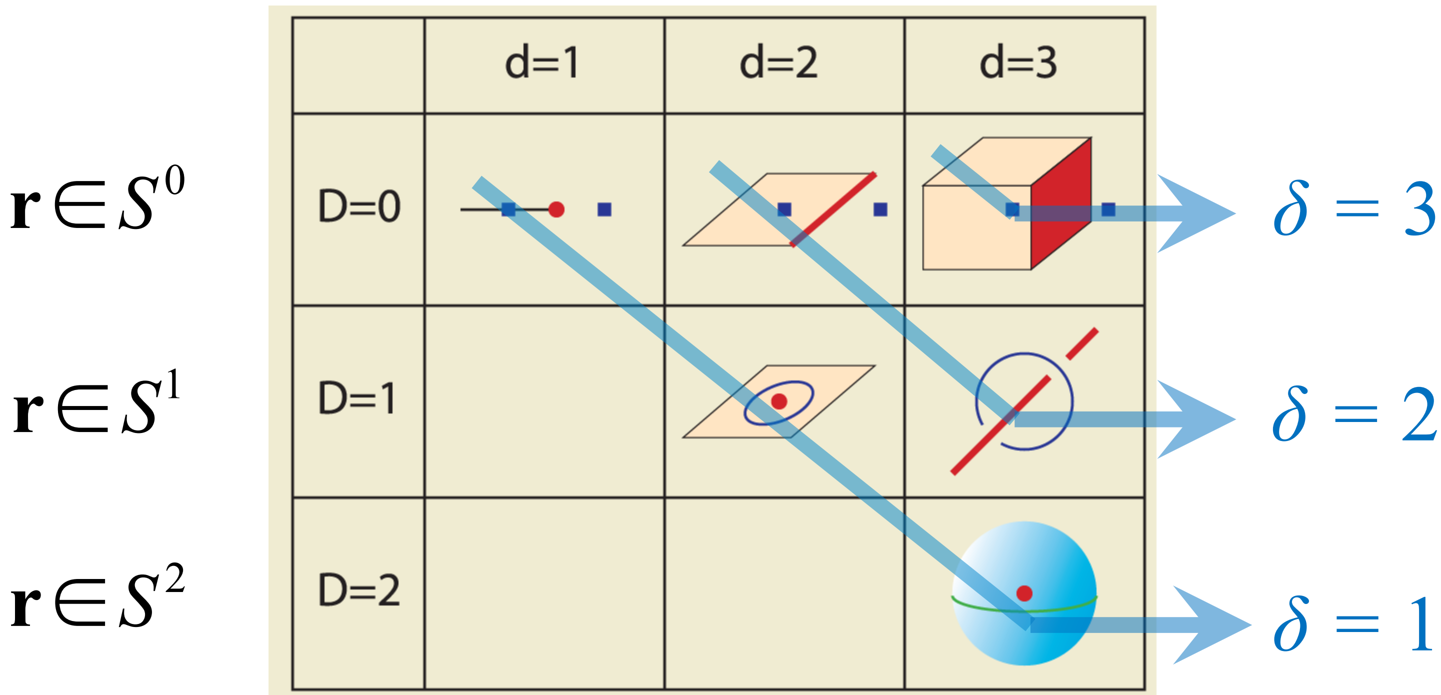
$\mathbf{k} \in T^2$: Brillouin zone

$s \in S^1$: Surrounding circle

Topological Defects

Classify families of Bloch-BdG Hamiltonians parameterized by **r** subject to time reversal and/or particle-hole symmetry constraints

$$H(\mathbf{k},\mathbf{r}) = \Theta H(-\mathbf{k},\mathbf{r})\Theta^{-1} \qquad H(\mathbf{k},\mathbf{r}) = -\Xi H(-\mathbf{k},\mathbf{r})\Xi^{-1}$$



Generalized bulk-boundary correspondence :

Topological classes of defect Hamiltonians are associated with protected gapless modes associated with the defect.

Generalized Periodic Table for Topological Defects

Topological classes depend only on the difference

$$\delta = d - D = \text{ Defect dimensionality } + 1$$

Symmetry					$\delta = d - D$							
s	AZ	Θ^2	Ξ^2	Π^2	0	1	2	3	4	5	6	7
0	A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
1	AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0	AI	1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
2	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
3	DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
4	AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
5	CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
6	C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
7	CI	1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- δ=2 : line defects
- δ=1 : point defects
- δ=0 : adiabatic temporal pumps

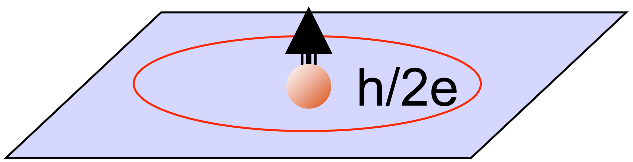
Teo, Kane '10
Freedman, et al. '10
Ryu, Schnyder, Furusaki, Ludwig '10

Majorana Bound States

Z_2 invariant, d=1,2,3 $$\nu = \frac{2}{d!(2\pi)^d} \int_{T^d \times S^{d-1}} d^d\mathbf{k}\, d^{d-1}\mathbf{r}\; Q_{2d-1} \text{ mod } 2$$

Chern Simons 2d-1 form

Vortex in 2D topological superconductor

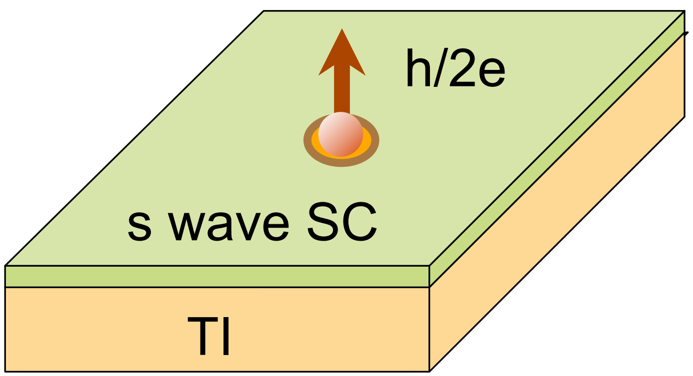


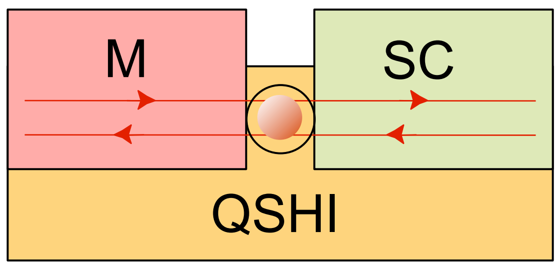
$$\nu = nm \text{ mod } 2$$

Chern number of topological superconductor

winding number of vortex

Topological Insulator Heterostructures





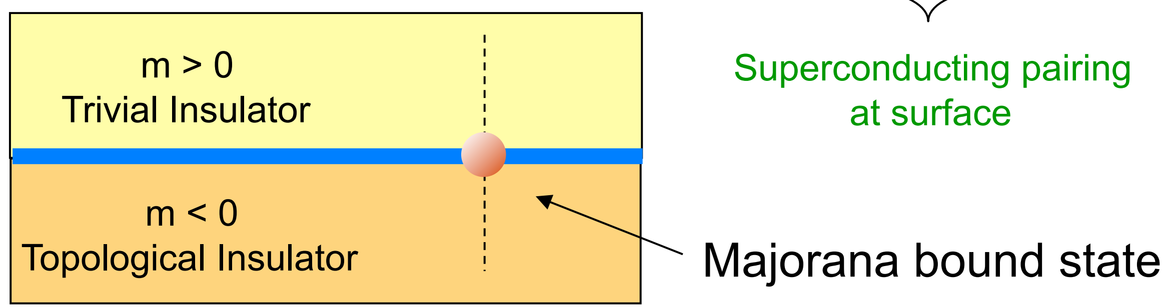
Majorana Fermions in Three Dimensions

Majorana bound states arise as solutions to three dimensional BdG theories

$$H = \tau_z \left[-i\mu_x \vec{\sigma} \cdot \vec{\nabla} + m(\mathbf{r})\mu_z \right] + \mathrm{Re}\big(\Delta(\mathbf{r})\big)\tau_x + \mathrm{Im}\big(\Delta(\mathbf{r})\big)\tau_y$$

Qi, Hughes, Zhang Model for edge of 3D topological insulator

Superconducting pairing at surface



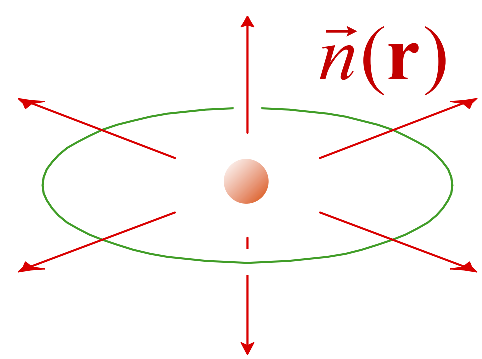
Minimal massive Dirac model:

$$H = -i\vec{\gamma} \cdot \vec{\nabla} + \vec{\Gamma} \cdot \vec{n}(\mathbf{r})$$

$(\gamma_1,\gamma_2,\gamma_3)$, $(\Gamma_1,\Gamma_2,\Gamma_3)$: 8x8 Dirac matrices

$$\vec{n}(\mathbf{r}) = \left(n_1, n_2, n_3\right) = \left(\mathrm{Re}(\Delta), \mathrm{Im}(\Delta), m\right)$$

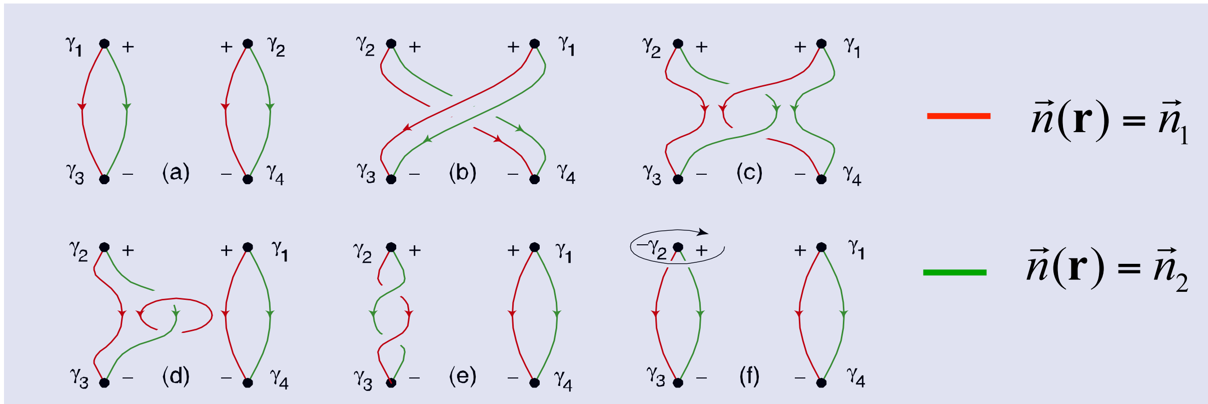
"hedgehog" configuration



Teo and Kane, PRL '10

Non-Abelian Exchange Statistics in 3D

Exchange a pair of hedgehogs: Teo and Kane '10



2π rotation : Wavefunction of Majorana bound state changes sign $\left(\pi_1\left[O(3)\right] = \mathbb{Z}_2\right)$

Interchange rule: Ising Anyons

$$\gamma_1 \to \gamma_2$$
$$\gamma_2 \to -\gamma_1$$

$$T_{12} = e^{\frac{\pi}{4}\gamma_1\gamma_2}$$

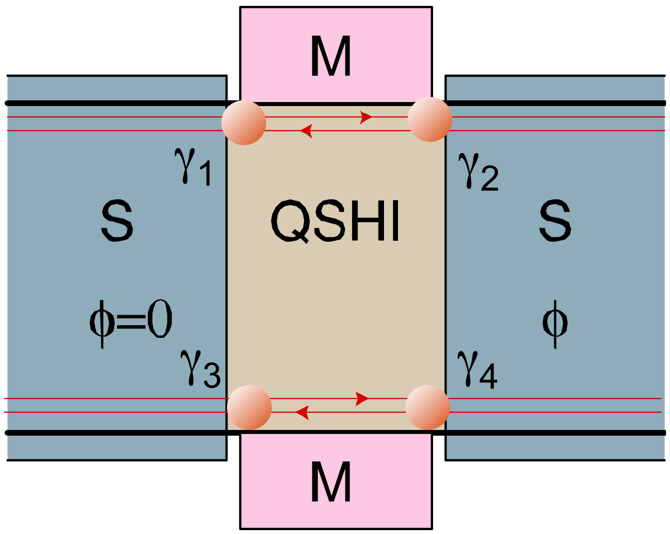
Nayak, Wilczek '96
Ivanov '01

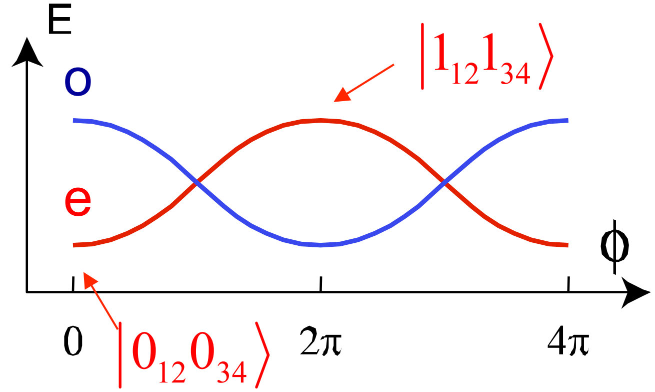
With mathematical rigor: Projective Ribbon Permutation Statistics

Freedman, Hastings, Nayak, Qi, Walker, Wang '10

Fractional Josephson Effect

Fu, Kane ’08
Kitaev ’01
Kwon, Sengupta, Yakovenko ‘04





- 4π perioidicity of E(ϕ) protected by local conservation of fermion parity.
- AC Josephson effect with half the usual frequency: f = eV/h

Conclusion

- The intersection of topology and condensed matter physics is both beautiful and physically important.

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- Majorana Fermions are cool.

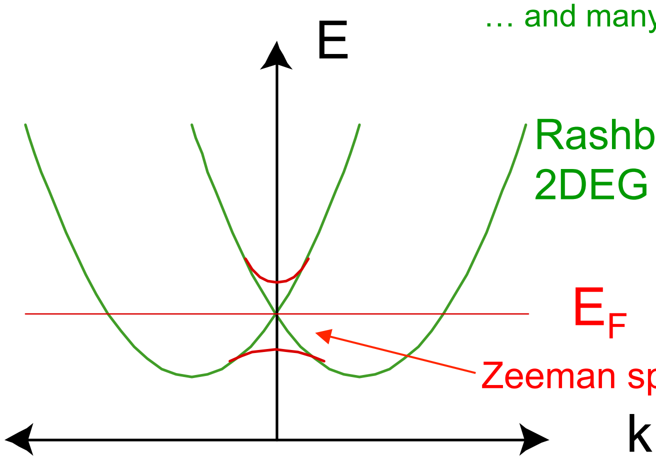
Conclusion

- The intersection of topology and condensed matter physics is both beautiful and physically important.
- Majorana Fermions are cool.
- Let us hope that at a future party we can give Mike a birthday present

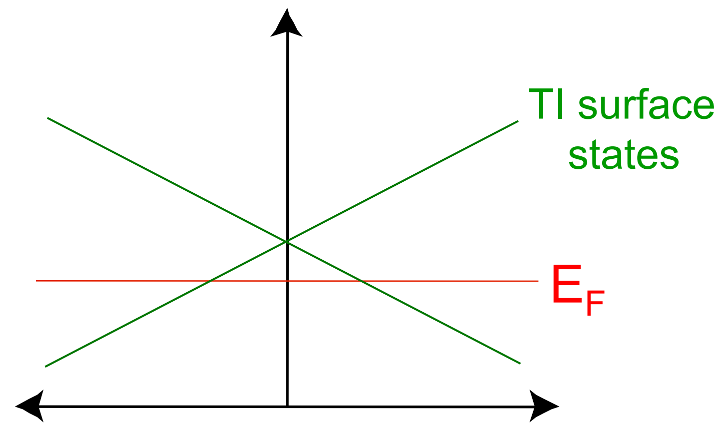
Two routes to 1D or 2D topological superconductivity :

Semiconductor - Magnet - Superconductor structure

Sau, Lutchyn, Tewari, Das Sarma '09 … and many others



Topological Insulator structure



Advantages of S-M-S structures

- Well studied materials ie InAs
- No poorly insulating substrate required induced gap

Disdvantages of S-TI structures

- TI materials have not been perfected

Disadvantages of S-M-S structures

- Requires fine tuning – especially when Rashba is weak
- Disorder (even in superconductor) strongly suppresses proximity induced gap

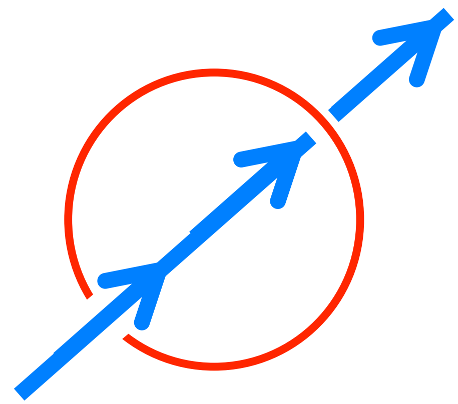
Advantages of S-TI structures

- The proximity induced gap can be as large as the bulk superconducting gap for strong coupling.
- Time reversal symmetry protects superconductivity from disorder (Anderson's thm)

(see e.g. A Potter, PA Lee arXiv:1103.2129)

Topological Invariant for a line defect

Class A (no symmetries), d=3, D=1 :



- 2nd Chern number

$$n = \frac{1}{8\pi^2}\int d^3\mathbf{k}\,ds\,\mathrm{Tr}\left[\mathbf{F}\wedge\mathbf{F}\right]$$

- Characterizes families of Bloch states $|u_n(\mathbf{k},s)\rangle$

- Specifies # Chiral Dirac Fermion modes ~ QH edge states

- Can be expressed as a winding number :

$$n = \frac{1}{2\pi}\oint ds\cdot\nabla\theta(s)$$

$$\theta(s) = \frac{1}{4\pi}\int d^3\mathbf{k}\ \mathrm{Tr}\left[\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right]$$

Qi, Hughes, Zhang (2008) formula for topological magnetoelectric coupling

$\theta = 0$: trivial insulator (T - invariant)
$\theta = \pi$: topological insulator (T - invariant)
$\theta \neq 0,\pi$: magnetic insulator (T breaking)

- It would be interesting to engineer chiral Dirac fermions using topological insulator and/or magnetic topological insulator structures.