Characterizing homomorphism functions and the limit theory of graphs

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Three questions

Mike Freedman: *Which graph parameters are partition functions?*

Jennifer Chayes: *What is the limit of a randomly growing graph sequence like the internet?*

Vera Sós: *How to characterize generalized quasirandom graphs?*

Homomorphism functions

Homomorphism: adjacency-preserving map

$\hom(F,G)$:= # of homomorphisms of F into G

$$t(F,G) = \frac{\hom(F,G)}{|V(G)|^{|V(F)|}}$$

Probability that random map $V(F) \to V(G)$ is a hom

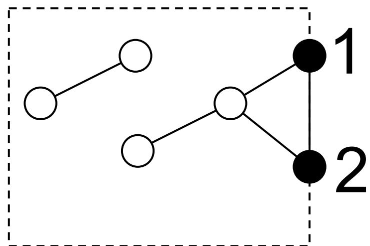
Weighted version:

$$H = (V, E, \alpha, \beta), \quad \alpha : V \to \mathbb{R}_+, \quad \beta : E \to \mathbb{R}$$

$$\hom(G,H) := \sum_{\varphi: V(G) \to V(H)} \prod_{i \in V(G)} \alpha_{\varphi(i)} \prod_{ij \in E(G)} \beta_{\varphi(i)\varphi(j)}$$

k-labeled graphs

k-labeled graph: k nodes labeled $1,\ldots,k$,
any number of unlabeled nodes



G_1, G_2 : k-labeled graphs

$G_1 G_2 := G_1 \cup G_2$, labeled nodes identified

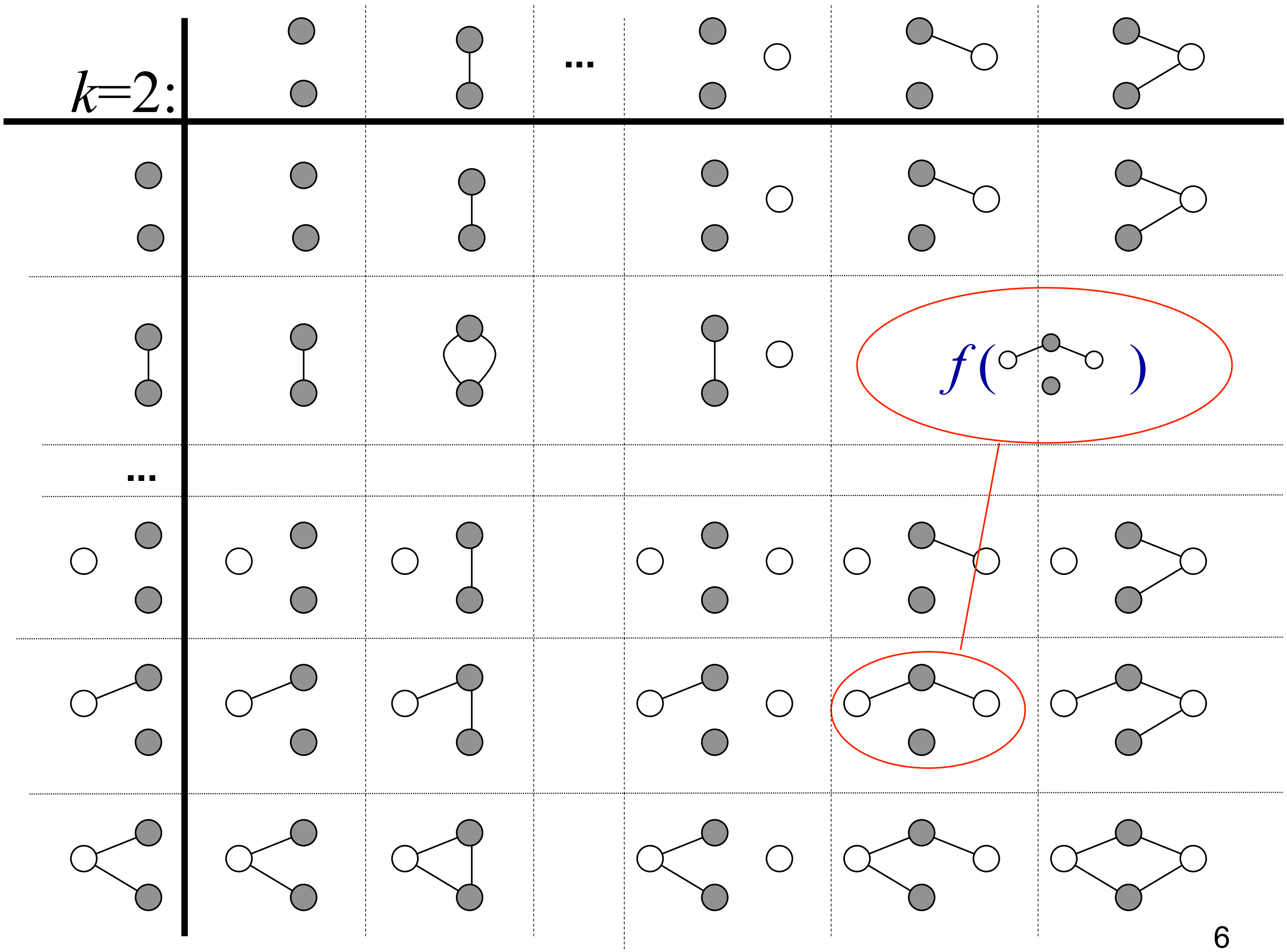
Graph parameter

Graph parameter: isomorphism-invariant function f on finite graphs

Example: hom(.,H)

Connection matrices

$M(f, k)$



Which parameters are homomorphism functions?

f=hom(.,H) for some weighted graph H

$\Updownarrow$

$M(f,k)$ is positive semidefinite for all k

and has rank $\le c^k$

Freedman - L - Schrijver

Difficult direction: ⇐

Easy but useful direction:

many applications in extremal graph theory

Generalizations, analogues and relaxations

Generalizations: directed graphs (Schrijver),
hypergraphs, semigroups (L-Schrijver)

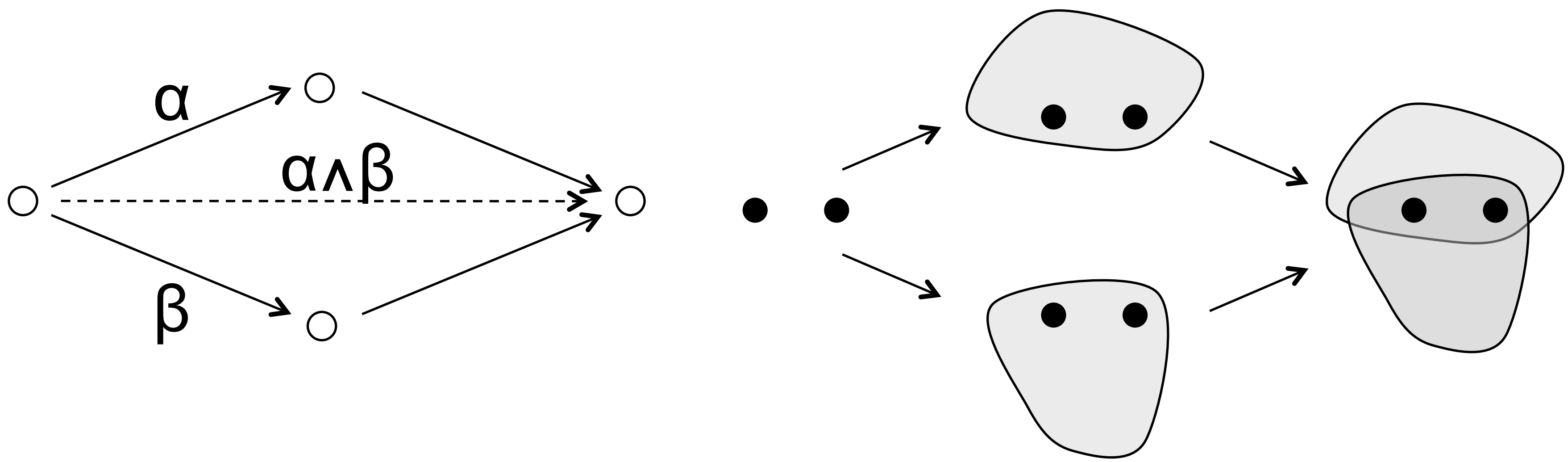
Analogues: edge coloring models (Szegedy),
unweighted graphs (L-Schrijver),
tensor algebras (Schrijver),
categories

Which graph parameters have connection matrices
- positive semidefinite?
- finite rank?

Locally finite categories

$\mathcal{K}$: category, $\mathcal{K}(a,b)$ finite for any two objects

(other conditions: zero object, generator object, pushouts, epi-mono decompositions)



Locally finite categories

f: objects $\to$ reals

$M(f,a)$: matrix indexed by morphisms from object a

$M(f,a)_{\alpha\beta} = f(\alpha \wedge \beta)$

$f = |\mathcal{K}(.,b)|$ for some object b

$\Leftrightarrow$

$M(f,a)$ is positive semidefinite for every object a.

L - Schrijver

Corollary for simple graphs

f =hom(.,G) for some s hardcore model G (with loops)

$M^*(f,k)$ is positive semidefinite for every k.

L - Schrijver

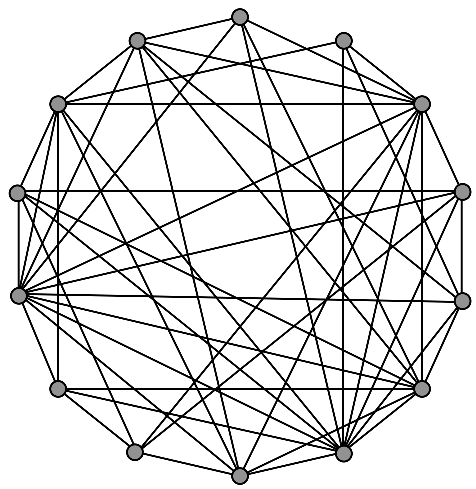
Allow a node to have more than one labels

Convergence and limit objects

$$\mathbf{W}_0 = \left\{ W : [0,1]^2 \text{ ® } [0,1] \quad \text{symmetric, measurable} \right\}$$

(graphons)

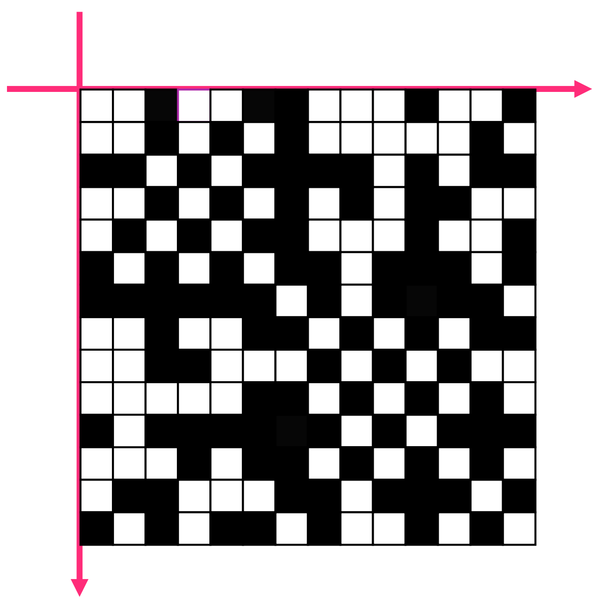
Graphs Graphons



G

0	0	1	0	0	1	1	0	0	0	1	0	0	1
0	0	1	0	1	0	1	0	0	0	0	0	1	0
1	1	0	1	0	1	1	1	1	0	1	0	1	1
0	0	1	0	1	0	1	0	1	0	1	1	0	0
0	1	0	1	0	1	1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1	0	1	1	1	0	1
1	1	1	1	1	1	0	1	0	1	1	1	1	0
0	0	1	0	0	1	1	0	1	0	1	0	1	1
0	0	1	1	0	0	0	1	1	1	0	1	0	0
0	0	0	0	0	1	1	0	1	0	1	0	1	0
1	0	1	1	1	1	1	1	0	1	0	1	1	1
0	0	0	1	0	1	1	0	1	0	1	0	1	0
0	1	1	0	0	0	1	1	0	1	1	1	0	1
1	0	1	0	1	1	0	1	0	0	1	0	1	0

A_G



W_G

Convergence and limit objects

$$\mathcal{W}_0 = \{W : [0,1]^2 \to [0,1] \text{ symmetric, measurable}\}$$

(graphons)

$$t(F,W) = \int_{[0,1]^{V(F)}} \prod_{ij \in E(F)} W(x_i, x_j)\, dx$$

$$t(F, W_G) = t(F, G)$$

Convergence and limit objects

$(G_1, G_2, \ldots)$ convergent: $\forall F \quad t(F, G_n)$ is convergent

Borgs, Chayes, L, Sós, Vesztergombi

$G_n \to W: \qquad \forall F \quad t(F, G_n) \to t(F, W)$

This is for dense graphs
$\Rightarrow$ mean field theory

Convergence and limit objects

For every convergent graph sequence (G_n) there is a graphon W such that $G_n \to W$.

L-Szegedy

For every graphon W there is a graph sequence (G_n) such that $G_n \to W$.

L-Szegedy

W is essentially unique (up to measure-preserving transformation).

Borgs-Chayes-Lovasz

Semidefinite connection matrices

f: graph parameter

f =*t*(.,*W*) for some graphon *W*

M(*f*,*k*) is positive semidefinite for all *k*, *f*( ) =1, and *f* is multiplicative over components.

L - Szegedy

Finite rank connection matrices

If $M(f,k)$ has finite rank, then $f(G)$ can be evaluated in polynomial time for graphs with tree-width at most k.

L- Welsh

Finite rank connection matrices

M(f,k) has finite rank for many parameters:
hom(.,H), hom(F,.), # of Hamilton cycles,
max no. of independent nodes, 1_{planar},
1_P for any minor-closed property P.

If f can be defined by a monadic second order counting problem, then $M(f,k)$ has finite rank.

Godlin-Kotek-Makowski

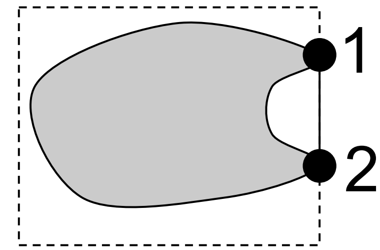
Happy Birthday Mike!

Computing with graphs

k-labeled quantum graph:

$$x = \sum_G x_G G$$

finite formal sum of *k*-labeled graphs



Q_k = {*k*-labeled quantum graphs}

infinite dimensional linear space

Computing with graphs

$$\left(\sum_{G} x_G G\right)\left(\sum_{G} y_G G\right) = \sum_{G_1,G_2} x_{G_1} y_{G_2} G_1 G_2$$

Q_k is a commutative algebra with unit element $\vdots$

Inner product: $\langle x,y \rangle = f(xy)$

Computing with graphs

$N_k(f) = \{x \in Q_k: <x,y> = 0 \ \forall y\}$

$Q_k / N_k(f)$ is a commutative algebra

$Q_k / N_k(f)$ is finite dimensional iff $M(f,k)$ has finite rank

$<.,.>$ is semidefinite iff $M(f,k)$ is positive semidefinite

Computing with graphs

$p_1,\ldots,p_q$: idempotent basis in Q_k nodes (states)

$□ = p_1 + \ldots + p_n$

$<□, p_i>$ nodeweights

$<□-□, p_i□p_j>$ edgeweights

> if no
> symmetries