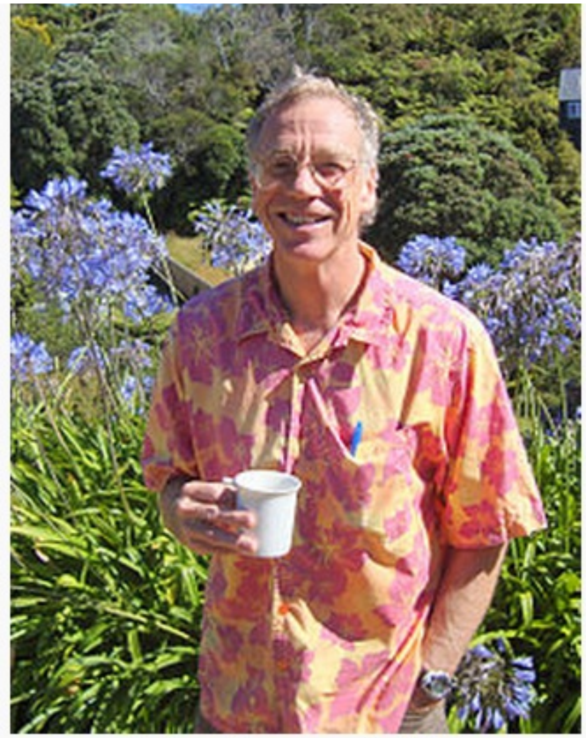
1986, he was awarded a Fields Medal for his work on the Poincaré conjecture. Freedman and Robion Kirby showed that an exotic $\mathbf{R}^4$ manifold exists.

Freedman was born in Los Angeles. His father, Benedict Freedman, was a aeronautical engineer, musician, and writer. His mother, Nancy Mars, performed as an actress and also trained as an artist. His parents cowrote a series of novels together. He entered the University of California, Berkeley, in 1968, and continued his studies at Princeton University where he received Ph.D. degree in 1973 for his doctoral dissertation titled *Codimension-Two Surgery*, written under the supervision of William Browder. After graduating, Freedman was appointed a lecturer in the Department of Mathematics at the University of California, Berkeley. He held this post from 1973 until 1975, when he became a member of the Institute for Advanced Study (IAS) at Princeton. In 1976 he was appointed assistant professor in the Department of Mathematics at the University of California, San Diego (UCSD). He spent the year 1980/81 at IAS, returning to UCSD, where in 1982 he was promoted to professor. He was appointed the Charles Lee Powell chair of mathematics at UCSD in 1985.

Freedman has received numerous other awards and honors



Michael Freedman in 2010

Born	April 21, 1951 (age 59) Los Angeles, California, U.S.
Nationality	United States
Fields	Mathematics and cond-mat physics
Institutions	Microsoft Station Q UC Santa Barbara

Happy Birthday, Mike!

My gift for Mike

Translation:

Happy birthday, Mike!

from Xiaoliang Qi,

dated: 辛卯年**=**

2011 AD mod 60

Interacting topological superconductors

Xiao-Liang Qi

Stanford University

Michael Freedman's Birthday Symposium

Station Q, UCSB, APR-16th-2011

Outline

- Brief review of topological superconductors
- Defining the interacting topological superconductors by observable topological effects
- Approach 1: gravitational response
- Approach 2: proximity with s-wave superconductor

Zhong Wang, Xiao-Liang Qi, Shou-Cheng Zhang, arXiv: 1011.0586

The first topological insulator: the quantum Hall state

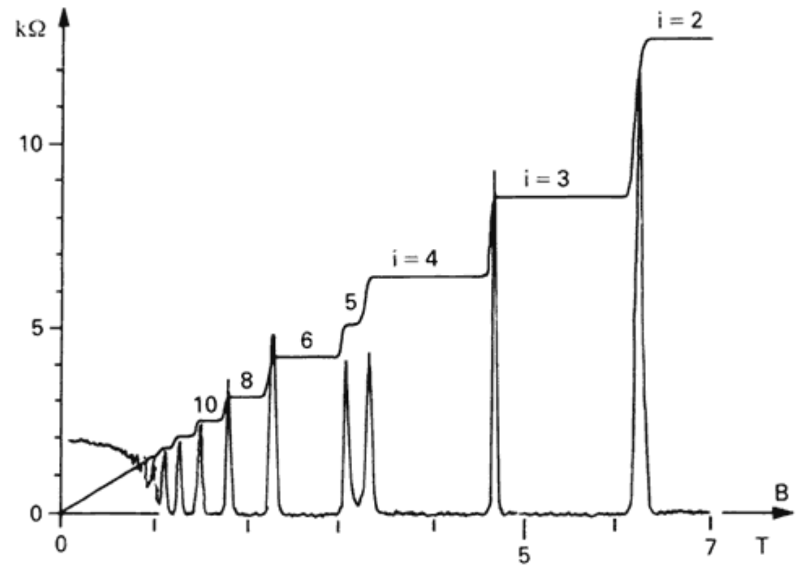
$$\sigma_{xy} = n\frac{e^2}{h}$$

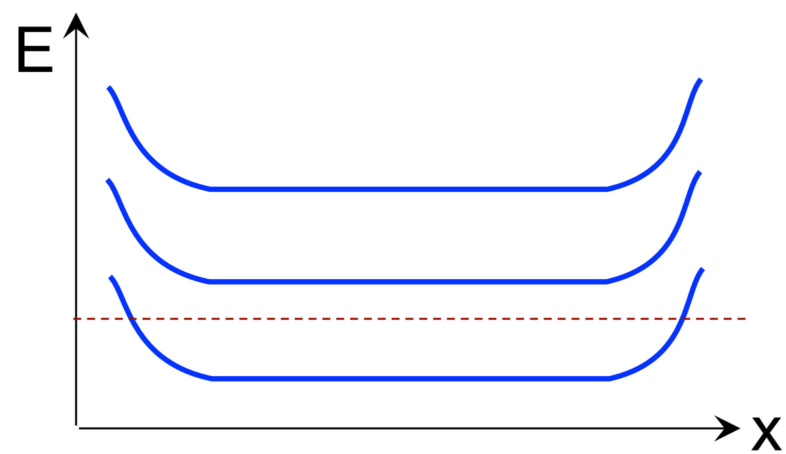
- Topological origin of the quantized Hall conductance:
- Bulk gap (Landau level gap)
- The first Chern number

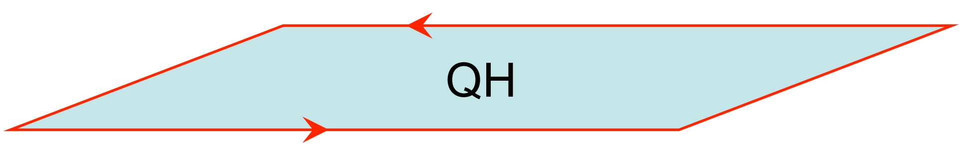
(Laughlin PRB 1981, Thouless, Kohmoto, Nightingale, den Nijs (TKNN), PRL 1982)

$$n = \int \frac{d^2k}{2\pi} f_{xy}(\mathbf{k})$$

- Chiral edge states on the boundary

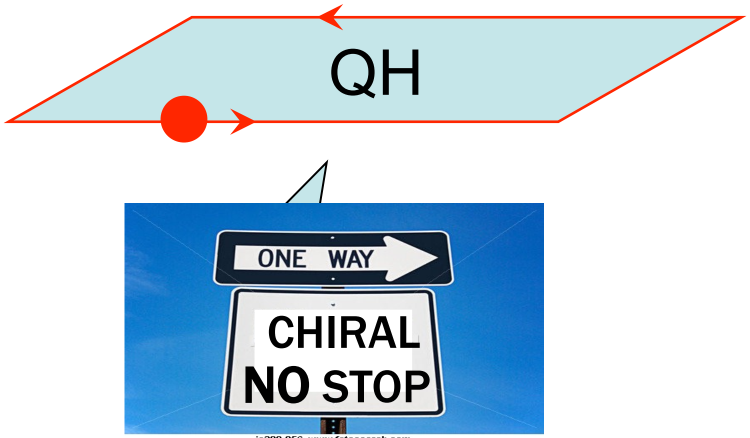


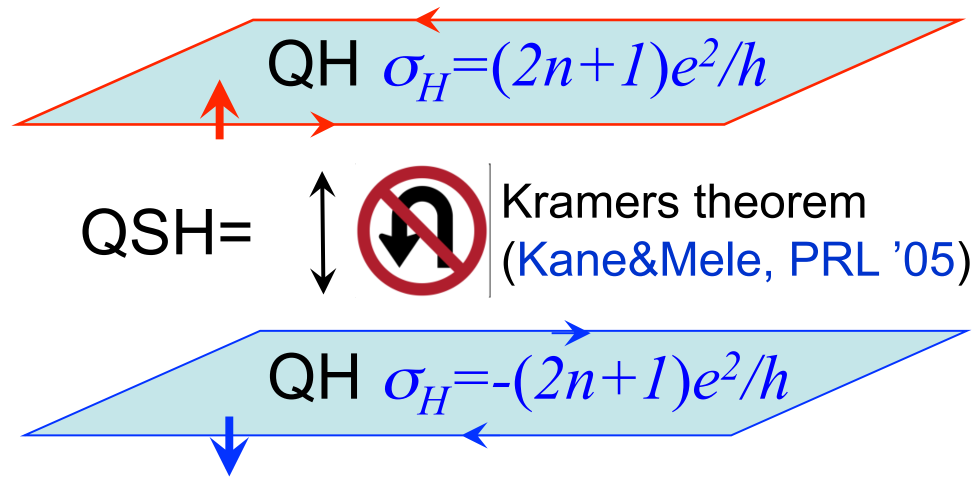




Topological insulators

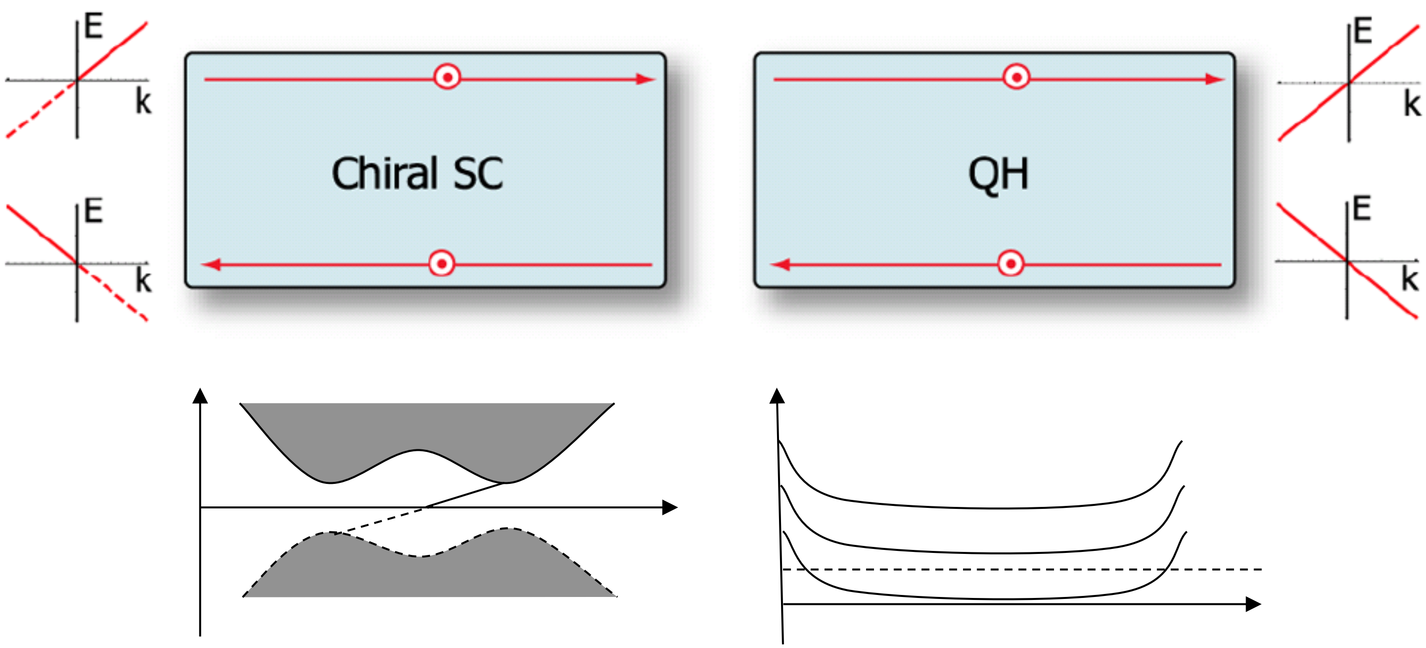
- Topological states are gapped (incompressible) states with protected gapless states on the boundary which are robust against arbitrary small perturbation respecting a given symmetry
- Topological states in different symmetry classes
- Example: Quantum Spin Hall state, a topological state with time-reversal symmetry (Kane&Mele, PRL '05, Bernevig&Zhang, PRL '06)





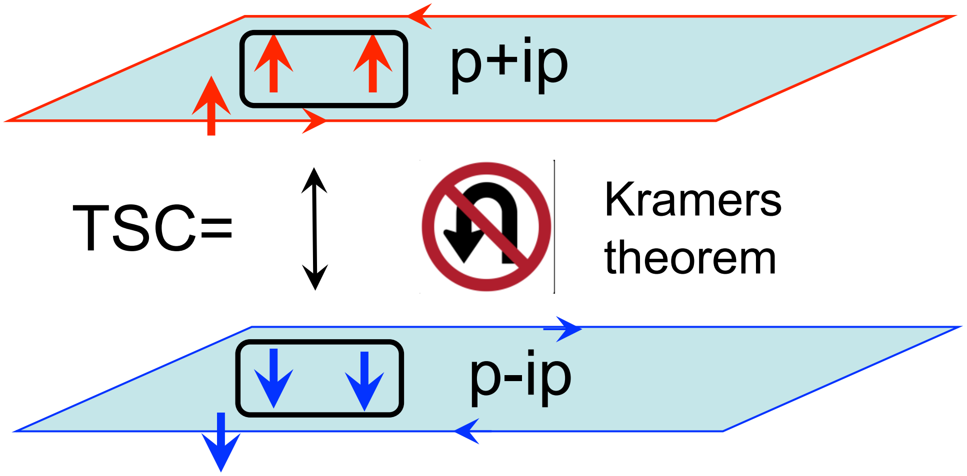
From topological insulators to topological superconductors

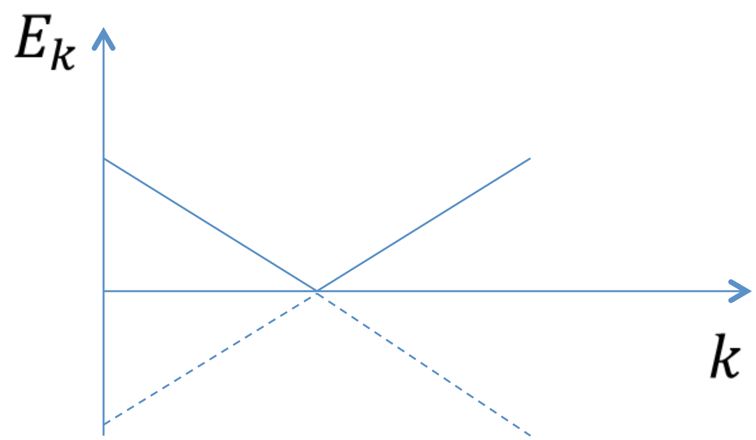
- Topological superconductors are fully gapped superconductors with protected surface states
- First example: (p+ip) superconductor with chiral Majorana edge states (Read&Green PRB 2000)



Time-reversal invariant topological superconductors

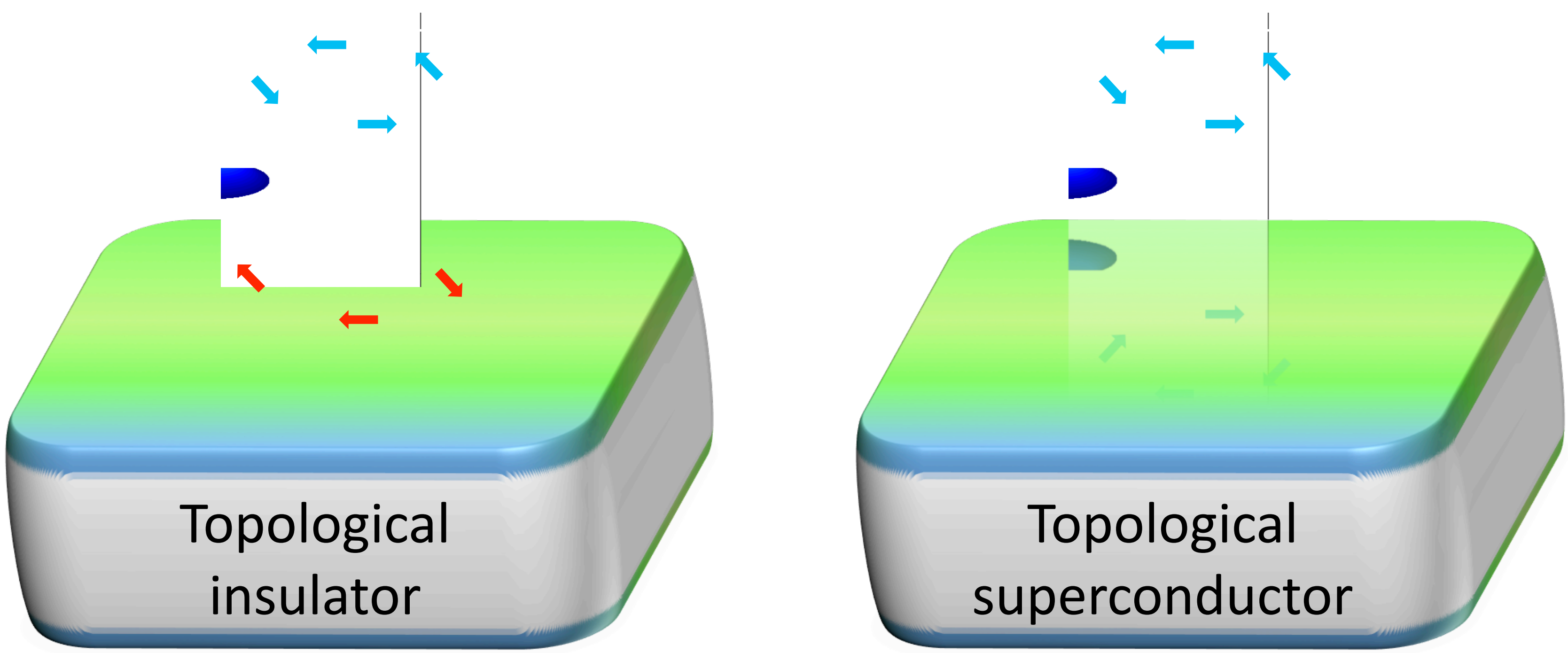
- Simplest 2d T-invariant topological superconductor: $(p+ip)\uparrow\uparrow/(p-ip)\downarrow\downarrow$
- Roy 2006, Qi et al PRB '08, Schnyder et al PRB '08
- Majorana fermion satisfies the same Kramers theorem as complex fermions➔Z_2 protection the same as in quantum spin Hall case (Kane&Mele, PRL '05)



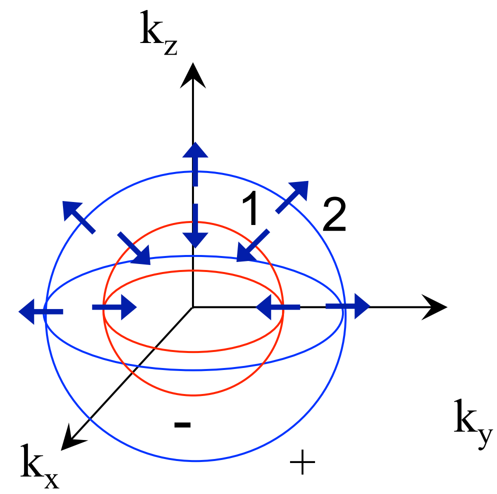


(3+1)d Time-reversal invariant topological superconductors

- Similar to 3d topological insulators, there are 3d T-invariant Topological superconductors
- 2d Majorana surface state
- Topological classification: Z rather than Z_2 (Schnyder et al PRB '08, Kitaev arxiv '09)



(3+1)d Time-reversal invariant topological superconductors



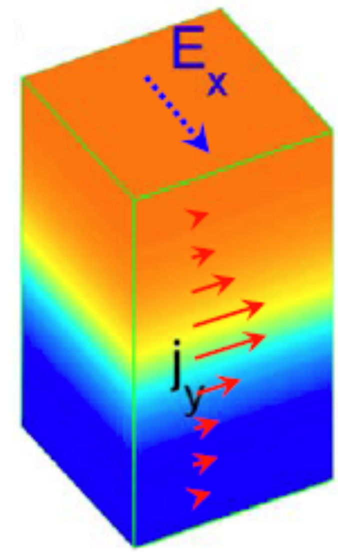
- Simplest example: He^3B phase. Triplet, opposite spin pairing
- Topological quantum number N=1
- In generally, the integer valued topological invariant can be calculated from bulk winding number (Schnyder et al PRB '08), or the sign of pairing and the Chern number of Fermi surfaces
  $N = \frac{1}{2}\sum_i \mathrm{sgn}(\Delta_i) C_{1i}$ (Qi et al PRB '10)

Generalization to interacting Topological superconductors

- Is the topological integer observable?
- ↔Is the Z classification stable with interaction?
- If a topological invariant corresponds to a physical observable which cannot change without breaking the symmetry, then the classification is stable
- Example: Quantized Hall conductance

Approach 1: gravitational response theory

- Comparison with 3d topological insulators.
- 3d time-reversal invariant topological insulators can be characterized by the topological response theory of electromagnetic field



- $e^{iS_{eff}[A_\mu]} = \int D\psi D\psi^* e^{iS[\psi,\psi^*,A_\mu]}$
- $S_{eff}\left[A_\mu\right] = \frac{\theta}{32\pi^2}\int d^4x\epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau} + \cdots$
- Time-reversal invariance requires
- $\theta = 0$ (trivial) or $\theta = \pi$ (topological)
- On a T-breaking boundary, θ changes by π, which leads to a half quantized Hall effect
- $S_{eff}[A_\mu] = \frac{1}{\ } \int d^4x\epsilon^{\mu\nu\sigma\tau}\frac{\partial_\mu\theta}{\ } A_\nu \partial_\sigma A_\tau$

Qi et al PRB 2008

Approach 1: gravitational response theory

- For topological superconductors, there is no charge conservation.
- A natural choice is to use the gravitational analogy of the θ term---the Pontryagin invariant

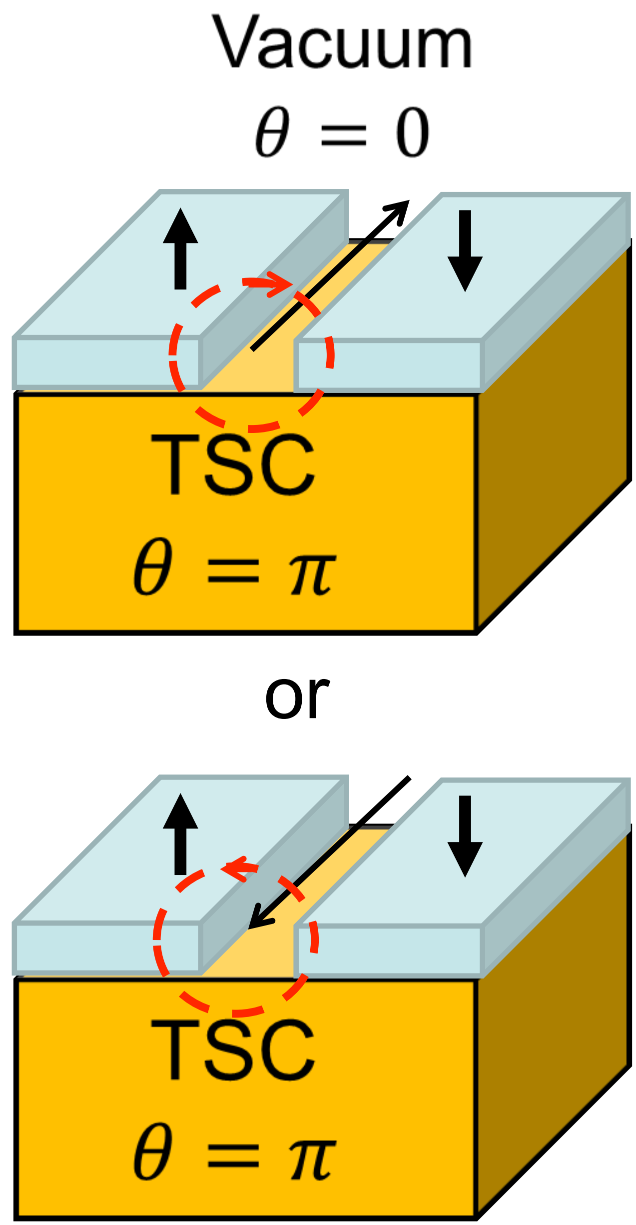
$$S_\theta = -\frac{\theta}{1536\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma} R^{\alpha}_{\beta\mu\nu} R^{\beta}_{\alpha\rho\sigma}$$

- Indeed, for a relativistic massive Majorana fermion (= He^3B!)
- $S = \int d^4x \bar{\psi}\left[\not{D} + me^{i\theta\gamma_5}\right]\psi$
- integrating out the fermion leads to such a topological term. (Eguchi&Frueund 1976, Fujikawa 1979)

(See also Ryu et al 1010.0936)

Approach 1: gravitational response theory

- Physical consequence:
- Gravitational Chern-Simons term on the T-breaking boundary (with level half of p+ip case in Read&Green 2000)
- If there are two boundary conditions which are time-reversal of each other, there is a vortex line of θ, leading to gravitational anomaly
- Consistent with the existence of chiral Majorana



Approach 2: s-wave proximity effect

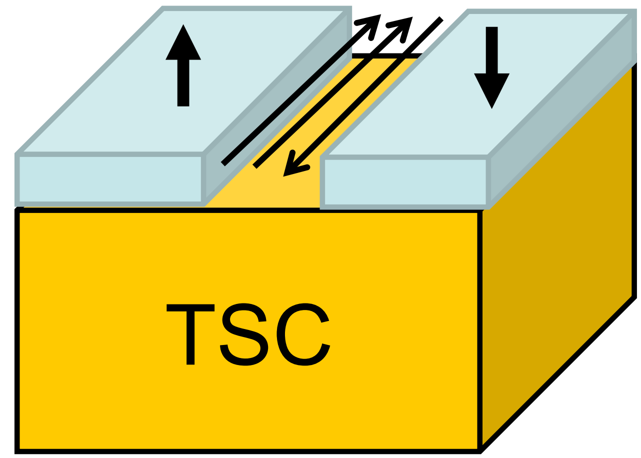
- However, this topological response theory approach gives a Z_2 classification rather than Z
- θ is only well-defined mod 2π
- If a system is $He^3B \otimes He^3B$, we have $\theta = 2\pi$ so that it's gravitational response is indistinguishable from vacuum
- Does that mean the Z classification in non-interacting systems is not stable?
- Try to find an alternative approach

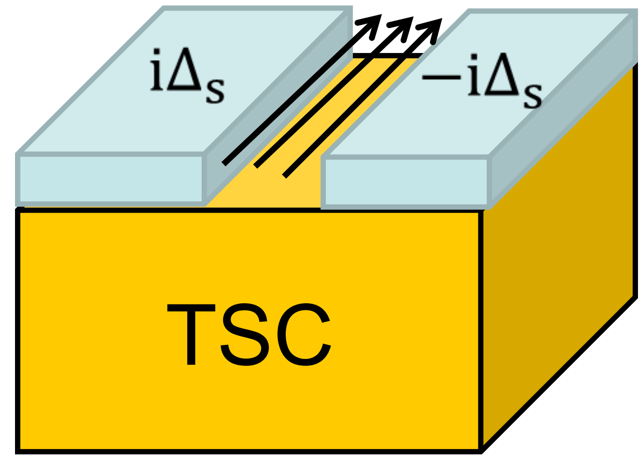
Surface state picture

- The integer classification for non-interacting system can be understood from the surface theory
- The surface theory is described by 2+1d massless Majorana fermion (Schnyder et al 2008)
- Different from Dirac fermion, there are two distinct types of massless Majorana fermions in 2+1d
- $h_L(\boldsymbol{k}) = k_x\sigma_z - k_y\sigma_x$, $h_R(\boldsymbol{k}) = k_x\sigma_z + k_y\sigma_x$
- σ_y is not allowed in the k linear terms, due to the condition $h(k)^T = h(-k)$
- A general topological superconductor with topological quantum number N has $N_{L(R)}$ surface Majorana fermions with $N_L - N_R = N$

Surface state picture

- Consider N "left" Majorana fermions
- In a generic T-breaking field, each Majorana fermion obtains different mass $m_i, i = 1,2,\ldots,N$
- #(domain wall chiral Majorana) = $\sum_i \mathrm{sgn}(m_i) = N$ mod 2
- However there is a special mass σ_y which gives the same mass for each Majorana



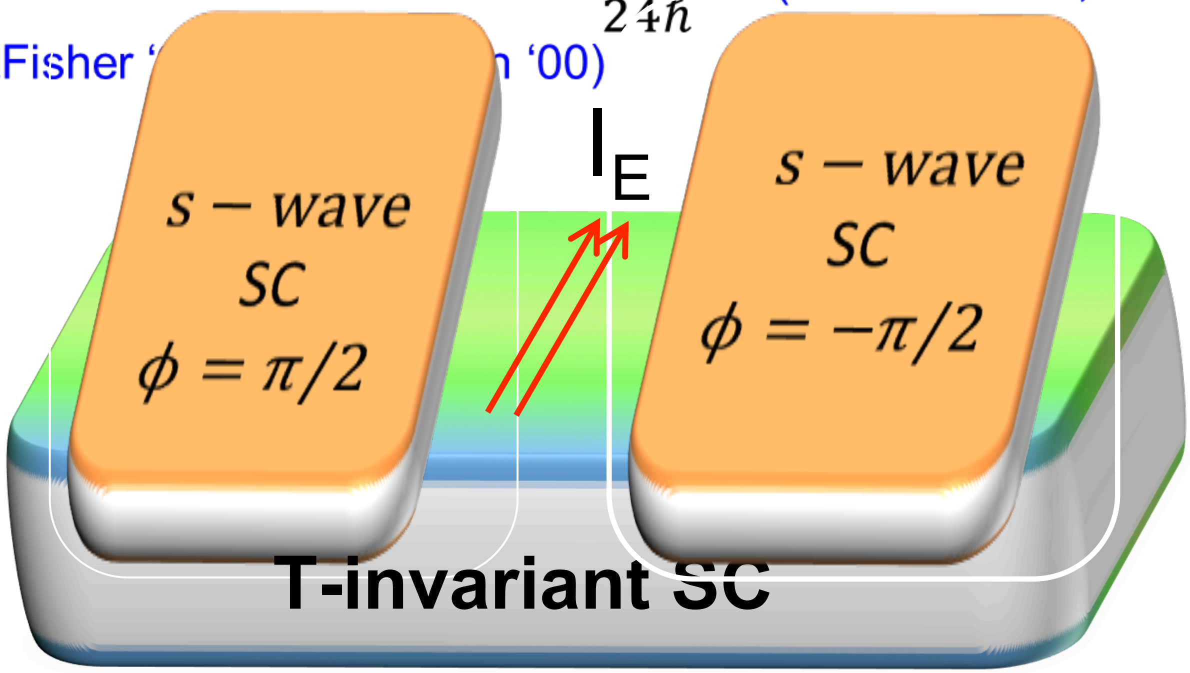


Approach 2: s-wave proximity effect

- Physical meaning of the σ_y mass:
- trivial s-wave pairing
- In He^3B, $H = \sum_k (\Delta \psi_k^+ i\sigma_y \psi_{-k}^+ + h.c.)$
- Generally $H = \sum_k (\Delta \psi_k^+ T(\psi_k^+) + h.c)$
- s-wave pairing is well-defined as long as time-reversal symmetry is defined
- Every state pairs with its time-reversal partner with the same pairing order-parameter

Measuring the topological integer by s-wave proximity effect

- Topological invariant N= #(chiral Majorana fermion on the junction)
- The topological invariant can be observed by chiral thermal current $I^E = \frac{N\pi k_B^2 T^2}{24\hbar}$ (Blote et al '86, Affleck '86, Kane&Fisher '... n '00)



Measuring the topological integer by s-wave proximity effect

- For interacting superconductors the number of chiral modes induced by s-wave pairing junction can be used as a general definition of the topological invariant
- Since the topological invariant is related to a physical observable, discrete and robust quantity, it is robust in the interacting system.
- Key point: to probe the Z invariant one needs to consider external field which breaks both time-reversal and charge conservation.

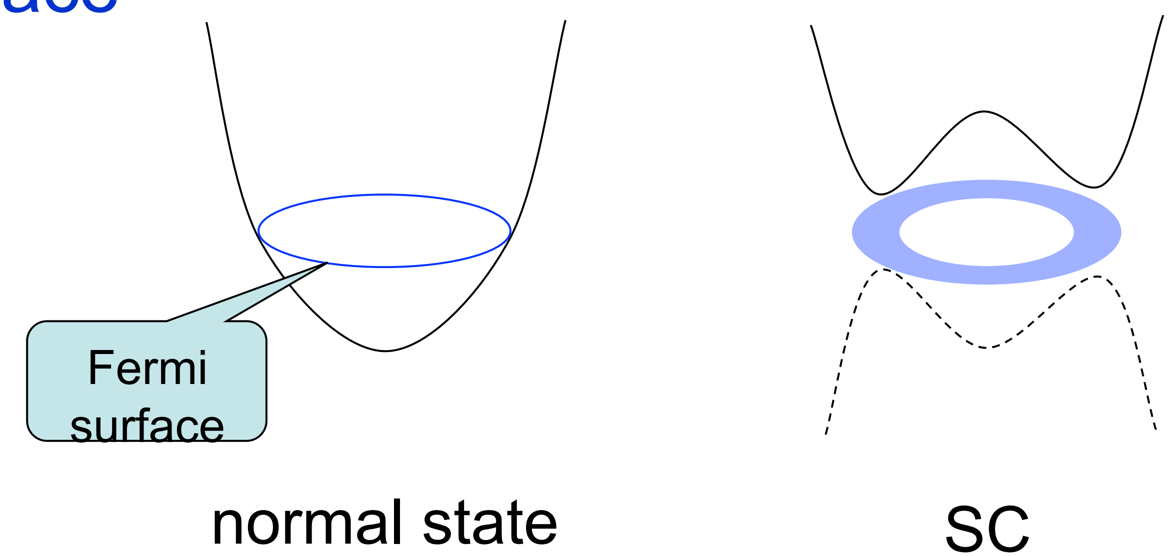
Summary

- Topological superconductors in 3d is classified by integer, which can be probed by a proximity effect with s-wave superfluid.
- The Z_2 part of the topological invariant leads to a topological gravitational response.

Happy birthday, Mike!

Fermi surface criteria of TRI topological SC

- T-invariant SC is classified by Z in 3d and Z_2 in 2d and 1d, with gapless non-chiral Majorana edge states (Qi et al, PRL '09; Roy, arxiv '08; Schnyder et al, PRB '08)
- Weak pairing limit: topological invariant of TSC is determined solely by the neighborhood of Fermi surface



Qi et al PRB 2010

Fermi surface criteria of TRI topological SC

- Topological invariant of 3d TSC (integer class)

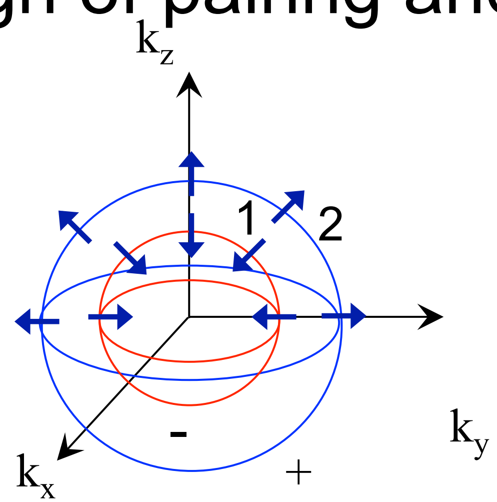
$$N_W = \frac{1}{2}\sum_s \text{sgn}(\delta_s) C_{1s}$$

Chern number of each Fermi surface

$$C_{1s} = \int_{\Sigma_s} d\mathbf{n} \cdot \nabla \times a_{\mathbf{k}}^{ss}$$

Sign of pairing at s-th Fermi surface

- Example: He^3B. Two fermi surfaces with opposite sign of pairing and opposite Chern number



General classification of TI and TSC

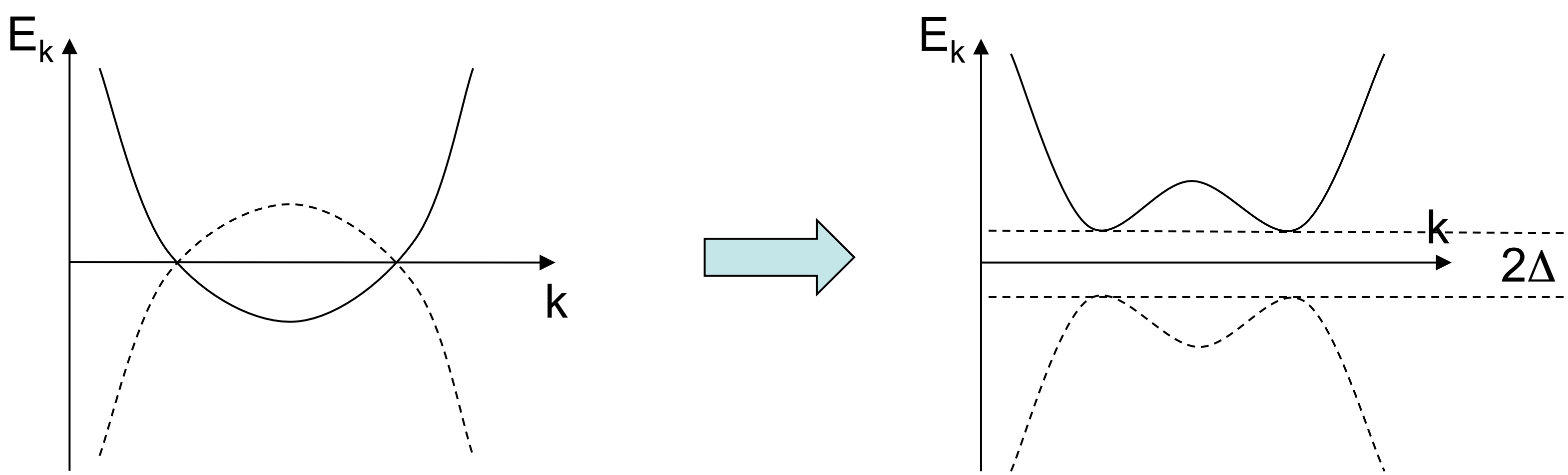
- Topological insulators and superconductors have been generalized to generic symmetry classes and dimensions (Qi et al PRB '08, Schnyder et al PRB '08, Kitaev arxiv '09)
- Kitaev's topological periodic table

Symmetry classes	Physical realizations	$d=1$	$d=2$	$d=3$
D	SC	$\mathbf{Z}_2$	$\mathbf{Z}$	0
DIII	TRI SC	Z_2	$\mathbf{Z}_2$	$\mathbf{Z}$
AII	TRI ins.	0	$\mathbf{Z}_2$	$\mathbf{Z}_2$
CII	Bipartite TRI ins.	$\mathbf{Z}$	0	Z_2
C	Singlet SC	0	$\mathbf{Z}$	0
CI	Singlet TRI SC	0	0	$\mathbf{Z}$
AI	TRI ins. wo/ SOC	0	0	0
BDI	Bipartite TRI ins. wo/ SOC	$\mathbf{Z}$	0	0

(from Freedman et al 2010)

From topological insulators to topological superconductors

- BCS Superconductors are similar to insulators
- Superconducting gap plays the role of insulating gap
- Some superconductors may be topological states of matter---topological superconductors



Fermi liquid (normal state)

Superconducting state