Some Phase Transitions Between Topological Phases

Steven H. Simon
w/ Fiona J. Burnell
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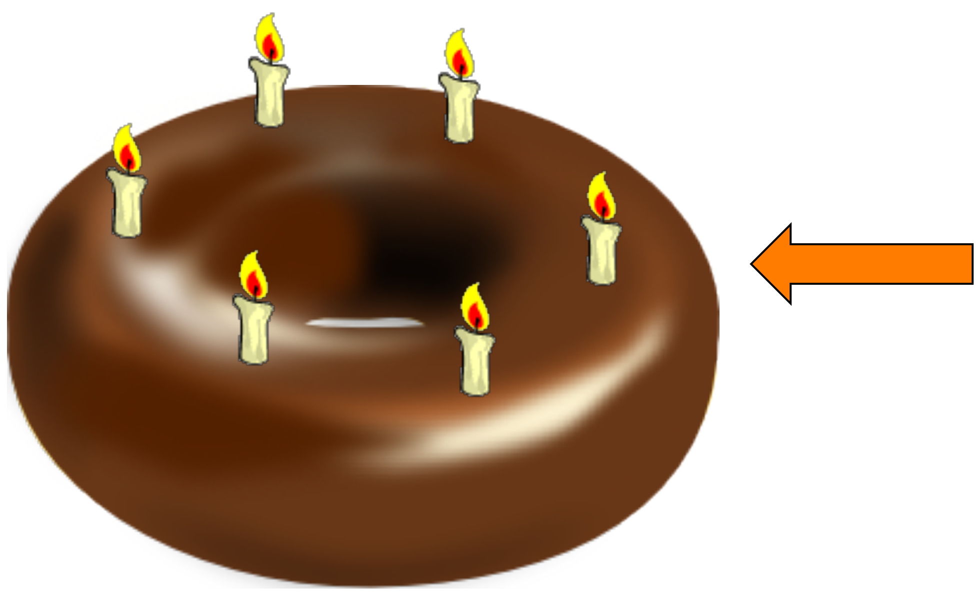
UNIVERSITY OF OXFORD

arXiv:1012.0317
arXiv:1104.1701

In Celebration of Mike Freedman's Birthday



Topologically Nontrivial Birthday Cake

The Landau Paradigm:

Phases, and Phase Transitions Described by Local Order Parameters; Local Broken Symmetries

Ex: Boson Condensation

Normal Phase

$$\langle b^\dagger \rangle = 0$$

Condensed Phase: Broken U(1)

$$\langle b^\dagger \rangle = |\phi| e^{i\theta}$$

Ex. $b^\dagger = c_\uparrow^\dagger c_\downarrow^\dagger$

Forcing Condensate Order

$$H = H_0 + \Delta\, b^\dagger + \Delta^*\, b$$

Boson Number Uncertainty in Condensed Phases

Transitions between *Topological Phases*?

Order is Nonlocal

- Can something condense to make a transition between two topological phases?

Algebraic structure of topological condensation transitions:

- de Wild Propitius (PhD Thesis '95); with Bais
- Bais and Slingerland PRB 79, 045316 ('09)
- Kitaev, Unpublished

…

Our Objective

- Realize these transitions in an (almost) solvable lattice model.
- Examine the critical theory.
- Relevance to "experiments" ?

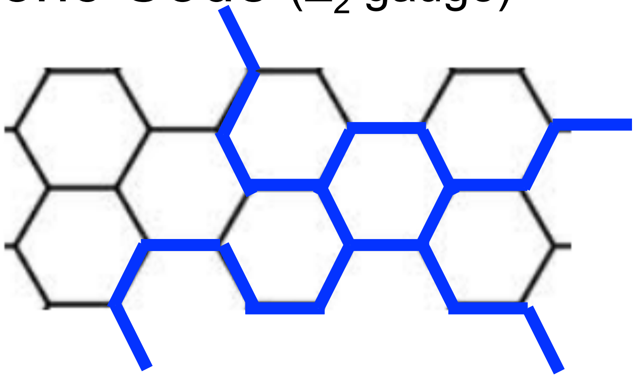
First Example: Phase Transition in Perturbed Toric Code

Prior Work By…

A. Kitaev, **Ann. Phys. 303** ('03) 2
Fradkin and Shenker **PRD 19**, 3682 ('79)
Trebst et. al. **PRL 98**, 070602 ('07)
C. Castelnovo and C. Chamon **PRB 78** 155120 ('08)
Vidal **PRB 79**, 033109 ('09)
etc.

Toric Code (Z_2 gauge)

$$H_{\text{Toric}} = -\sum_V \epsilon_e \prod_{\curlywedge} \sigma_z(i) - \sum_P \epsilon_m \prod_{\hexagon} \sigma_x(i)$$



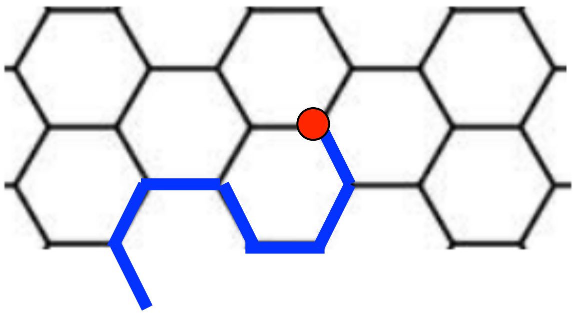
Edges: electric flux $\sigma_z = \pm 1$

enforces
$\nabla \cdot \mathrm{E} = 0$
closed loops of
$\sigma_z = -1$

enforces
$\nabla \times \mathrm{B} = 0$
flips all + ↔ –
around a plaquette

Ground state = equal superposition of all loop configurations

Toric Code (Z_2 gauge)



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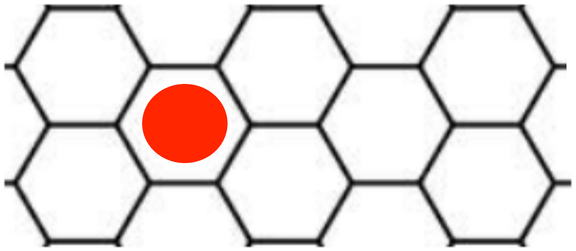

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violation is magnetic defect *m*

Ground state = equal superposition of all loop configurations

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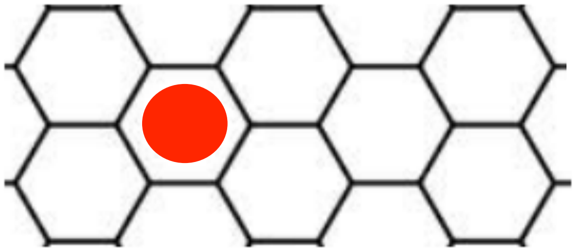

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violation is magnetic defect *m*

Edges: electric flux $\sigma_z = \pm 1$

m is a Z_2 boson : Can we condense it?
(trivial self braiding and *m* x *m* = 1)

Want:

$$H = H_{\text{Toric}} + \Delta\, m^\dagger + h.c.$$

but no local operator creates single *m*.

Toric Code (Z_2 gauge)

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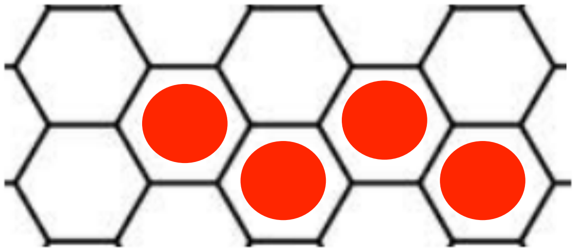

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(trivial self braiding and *m* x *m* = 1)

Want:

$$H = H_{\text{Toric}} + \Delta\, m^\dagger + h.c.$$

but no local operator creates single *m*.

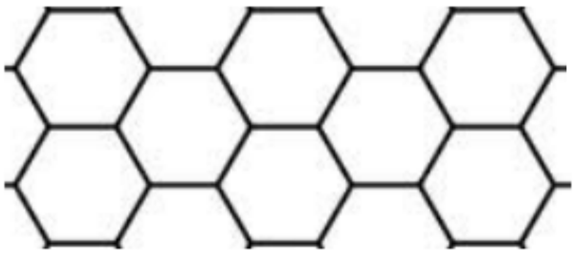
Instead

$$H = H_{\text{Toric}} - \underbrace{\Delta\, m_i^\dagger m_j^\dagger}\ + h.c.$$

Creates/annihilates/hops *m* bosons
(i,j neighbors)

Commutes with electric term. Never creates *e* defects
Can work in reduced Hilbert Space

Toric Code (Z$_2$ gauge)



$$H_{\text{Toric}} = -\sum_V \epsilon_e \prod_{\perp} \sigma_z(i) - \sum_P \epsilon_m \prod_{\hexagon} \sigma_x(i)$$

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flips all + ↔ −
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violation is
magnetic defect *m*

Edges: electric flux $\sigma_z = \pm 1$

Plaquettes: m violation vs none $\rightarrow \tau_z = \pm 1$

$$H_{\text{eff}} = \epsilon_m \sum_{i \in P} \tau_z^i - \Delta \sum_{<i,j> \in P} \tau_x^i \tau_x^j$$

Exact mapping to Transverse Field Ising Model

$$H = H_{\text{Toric}} + \underbrace{\Delta\, m_i^\dagger m_j^\dagger + h.c.}$$

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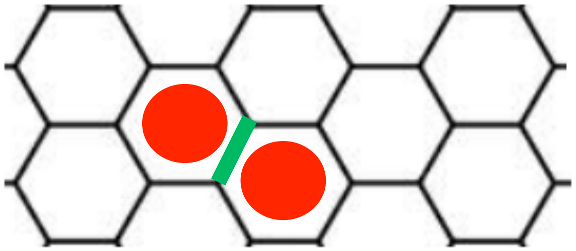

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Creates/annihilates/hops *m* bosons (i,j neighbors)

Operate with σ_z on edge creates *m* on two adjacent plaquettes

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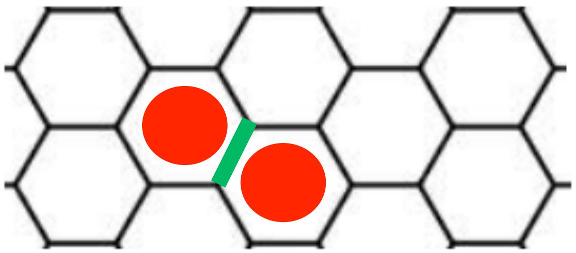

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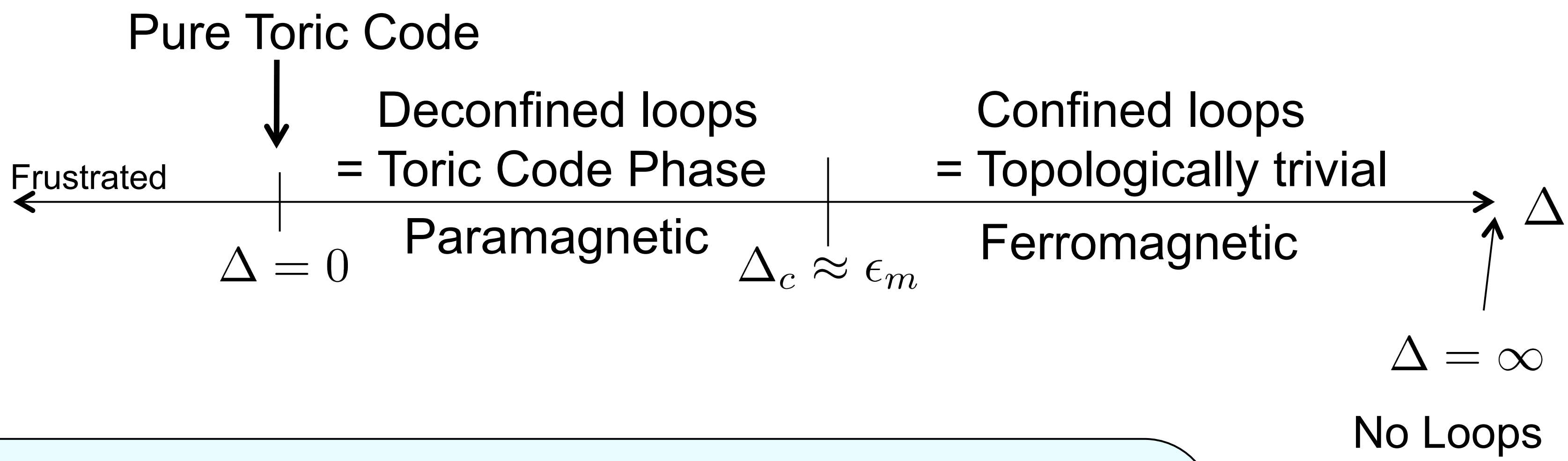
Exact mapping to Transverse Field Ising Model

$$H = H_{\text{Toric}} - \Delta \sum_{edge\, i} \sigma_z(i)$$

String tension for loops

Operate with σ_z on edge creates *m* on two adjacent plaquettes



Plaquettes: m violation vs none $\rightarrow \tau_z = \pm 1$

$$H_{\text{eff}} = \epsilon_m \sum_{i \in P} \tau_z^i - \Delta \sum_{<i,j> \in P} \tau_x^i \tau_x^j$$

Exact mapping to Transverse Field Ising Model

τ spins
in x direction
= (1 + m)
maximum
uncertainty
of # of bosons;

$$H = H_{\text{Toric}} - \Delta \sum_{edge\, i} \sigma_z(i)$$

String tension for loops

Operate with σ_z on edge creates *m* on two adjacent plaquettes

Transitions between two topologically nontrivial phases:

Much the same physics applies! (with some twists)

1. Start with Levin-Wen Lattice Model

   Levin and Wen PRB 71, 045110 ('05)
   Provides description of Uncondensed Phase

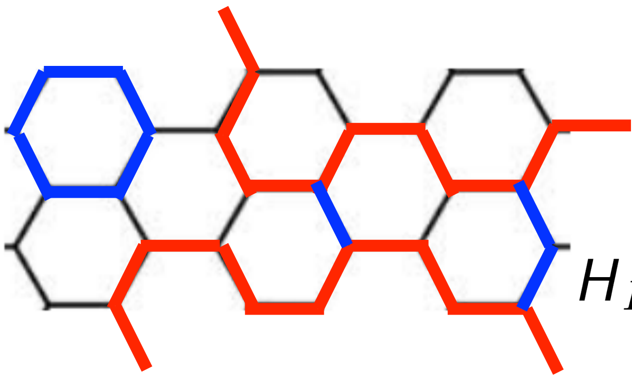
2. Find a Z_N "plaquette boson"
   (no vertex defect; trivial self braiding ; $\underbrace{b \times b \,..\, \times b}_{\text{N times}} = 1$)

3. Add condensation term to $H_{\text{Levin-Wen}}$

4. Exact mapping to "transverse field" Z_N spin model
   $\Rightarrow$ confinement transition for certain "loops"

Provides "solvable" realizations of (some) transitions proposed by
Bais and Slingerland PRB 79, 045316 ('09)

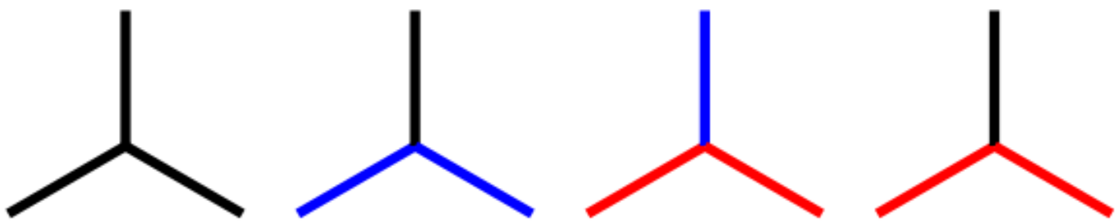
Levin and Wen model of $\text{Ising}_R \times \text{Ising}_L$



Hilbert space: Edge labels: $1, \sigma, \psi$

$$H_{LW} = -\sum_V \epsilon_V \, \delta_{E_1,E_2,E_3} - \sum_P \epsilon_m \mathrm{Proj}_0(P)$$

Enforces " $\nabla \cdot \mathrm{E} = 0$ " where allowed vertices are



$(1,1,1)$ $(1,\psi,\psi)$ (σ,σ,ψ) $(\sigma,\sigma,1)$

(note: σ forms closed loops)

enforces $\nabla \times \mathrm{B} = 0$

flips edge variables around a plaquette

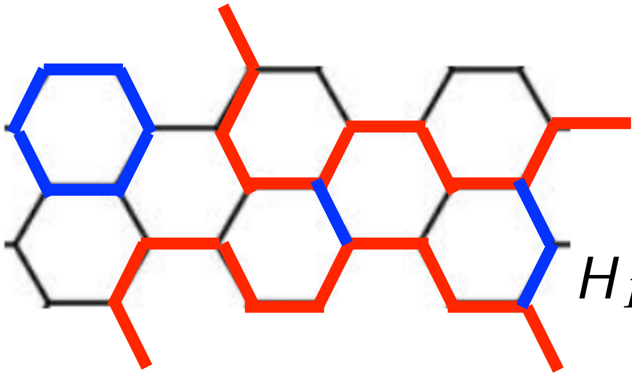
violation is "magnetic" defect

Ground state = weighted superposition of string net configurations

Spectrum: both vertex and plaquette defects = $\text{Ising}_R \times \text{Ising}_L$

Particles = $(1, \sigma, \psi)_R \times (1, \sigma, \psi)_L$

Levin and Wen model of $\text{Ising}_R \times \text{Ising}_L$



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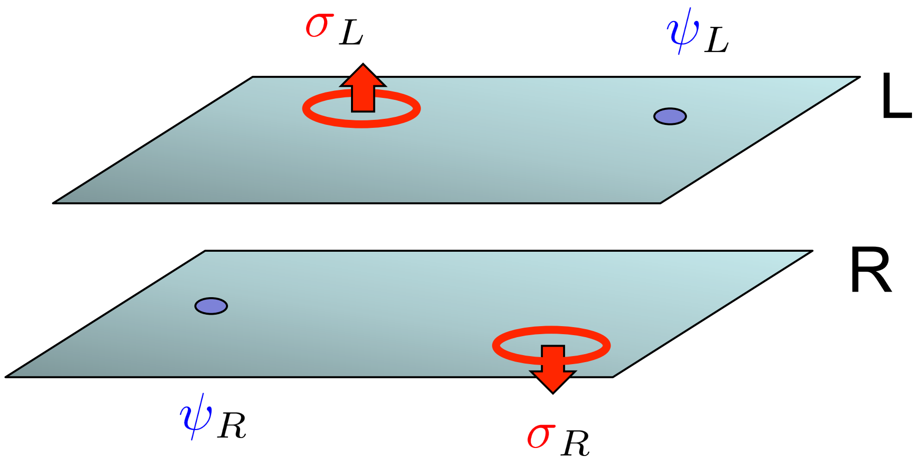
Particle Type	Statistics
1	Identity
ψ_R	$\mathbb{Z}_2$ Fermion
ψ_L	$\mathbb{Z}_2$ Fermion
σ_R	Chiral Anyon
σ_L	Chiral Anyon
$\psi_R \times \sigma_L$	Chiral Anyon
$\sigma_R \times \psi_L$	Chiral Anyon
$\sigma_R \times \sigma_L$	Boson
$b = \psi_R \times \psi_L$	$\mathbb{Z}_2$ Boson

← condense this boson *b*

Levin and Wen model of Ising$_R$ x Ising$_L$



Equivalent to bilayer

$$p_x + ip_y / p_x - ip_y$$

superconductor

Read and Green PRB 61, 10267 ('00)

Particle Type	Statistics
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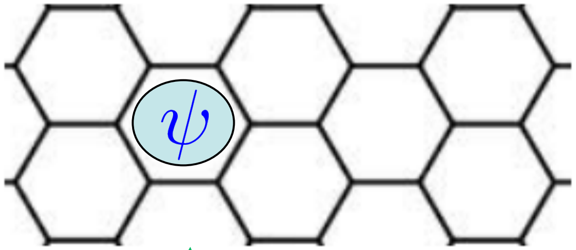

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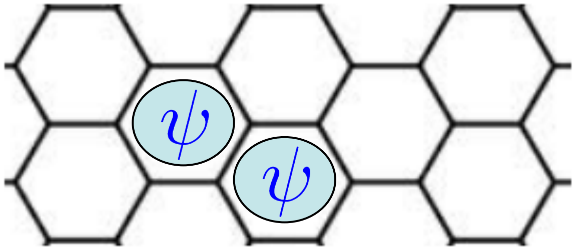
condense this boson *b*

b is equivalent to a ψ flux through a plaquette

Levin and Wen model of $\text{Ising}_R \times \text{Ising}_L$



$$H = H_{LW} \underbrace{- \Delta\, b_i^\dagger b_j^\dagger}_{} + h.c.$$

creates/annihilates/hops *b*'s

Creates only this one type of defect

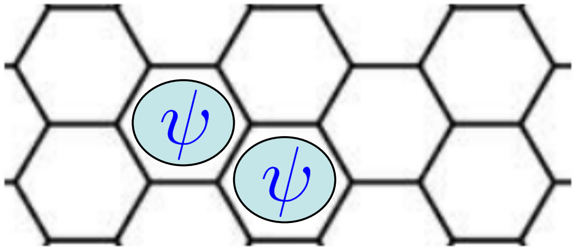
b is equivalent to a ψ flux through a plaquette

Plaquettes: b boson present or not $\rightarrow \tau_z = \pm 1$

$$H_{\text{eff}} = \epsilon_m \sum_{i \in P} \tau_z^i - \Delta \sum_{<i,j> \in P} \tau_x^i \tau_x^j$$

Exact mapping to Transverse Field Ising Model

Levin and Wen model of $\text{Ising}_R \times \text{Ising}_L$



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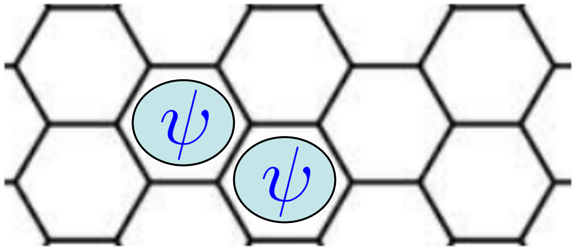

$$H = H_{LW} \underbrace{- \Delta\, b_i^\dagger b_j^\dagger}_{} + h.c.$$

$(-1)^{n_\sigma}$ on edge creates *b* on two adjacent plaquettes

String tension for σ loops!

Plaquettes: b boson present or not $\rightarrow \tau_z = \pm 1$

$$H_{\text{eff}} = \epsilon_m \sum_{i \in P} \tau_z^i - \Delta \sum_{<i,j> \in P} \tau_x^i \tau_x^j$$

Exact mapping to Transverse Field Ising Model

Pure Ising_R x Ising_L

Frustrated ← $\Delta = 0$ — Deconfined σ loops = Ising_R x Ising_L Phase — Paramagnetic — $\Delta_c \approx \epsilon_m$ — Confined σ loops — Ferromagnetic → Δ

$\Delta = \infty$

$$H = H_{LW} \underbrace{- \Delta\, b_i^\dagger b_j^\dagger}\ + h.c.$$

String tension for σ loops!

No σ loops
Edges labeled
With 1 or ψ only

$\Rightarrow$ *Toric Code*

Plaquettes: b boson present or not $\rightarrow \tau_z = \pm 1$

$$H_{\text{eff}} = \epsilon_m \sum_{i \in P} \tau_z^i - \Delta \sum_{<i,j> \in P} \tau_x^i \tau_x^j$$

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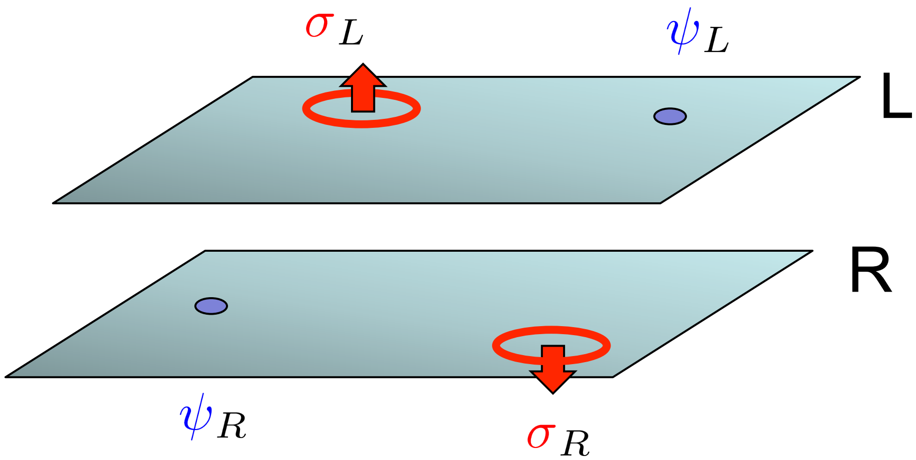

Equivalent to bilayer

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superconductor

Read and Green PRB 61, 10267 ('00)

In superconductor - language

$$\delta H = \psi_L^\dagger \psi_R^\dagger \; \psi_L \psi_R$$

Turns on interlayer s-pairing

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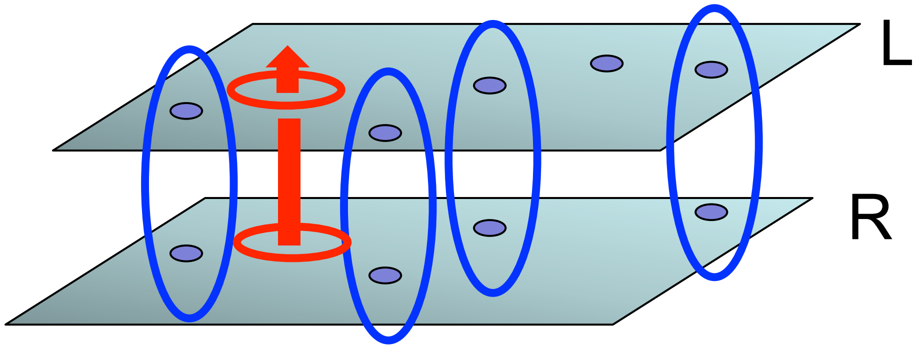

Equivalent to bilayer

$$p_x + ip_y / p_x - ip_y$$

superconductor

Read and Green PRB 61, 10267 ('00)

Excitations:

$\psi_L \equiv \psi_R$	Fermion
$(\sigma_L \times \sigma_R)_1$	Boson
$(\sigma_L \times \sigma_R)_\psi$	Boson

In superconductor - language

$$\delta H = \psi_L^\dagger \psi_R^\dagger \, \psi_L \psi_R$$

Turns on interlayer s-pairing

$b = \psi_R \times \psi_L$	$\mathbb{Z}_2$ Boson

← condense this boson *b*

General Structure:

- Start with (chiral) Chern-Simons theory (or any MTC)
- Choose a Z_N simple current ψ
- Double CS theory (Levin Wen Lattice Model)
- Add term to condense $b = \psi_L \times \psi_R$
- Maps to Z_N spin model
- Condensed phase is *Drinfeld Double* of the subcategory of particles from the original CS theory which braid trivially with ψ.

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