Topological order, long range quantum entanglement, and classification of gapped quantum phases

Xiao-Gang Wen, MIT

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Xie Chen

Z.-C. Gu

ZH Wang

A brief history of FQH theory: 1980s

- 1982: The discovery $\nu = 1/3, 2/5$. Tsui-Stormer-Gossard 1982; Stormer etal 1983
- 1983: The theory and fractional charge for $\nu = 1/3$. Laughlin 1983
  Theory for hierarchical FQH states $\nu = 2/5, \ldots$ Haldane 1983; Halperin 1984; Girvin 1984
- 1984: Fractional statistics. Arovas-Schrieffer-Wilczek 1984
- 1985:
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- 1987: Discovery of $\nu = 5/2$ FQH state Willett etal 1987
- 1988: Haldane-Rezayi spin-singlet state for $\nu = 5/2$ Haldane-Rezayi 1988

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  Off-diagonal long-range order for FQH states Girvin-MacDonald 1987
- 1988: Haldane-Rezayi spin-singlet state for $\nu = 5/2$ Haldane-Rezayi 1988
- 1989: Ginzburg-Landau-Chern-Simons effective theory Zhang-Hansson-Kivelson 1989; Read 1989
  Topological quantum field theory (TQFT), pure CS theory, and its connection to CFT Witten 1989

A brief history of FQH theory: 1990s

- 1990: Pure Chern-Simons effective theory with no order parameter Wen-Niu 1990; Blok-Wen 1990
  K-matrix classification of Abelian FHQ states Blok-Wen; Read; Wen-Zee
  Topological order in terms of topological ground state degeneracy and modular trans. Wen 1990
  Edge excitations of FQH state and CFT Wen 1990
- 1991: Non-Abelian (NAB) FQH states:
  $\nu = 1/2$ Pfaffian state from Ising CFT Moore-Read 1991
  $SU_k(N)$ CS effective theory from slave-particle ($SU_2(2)$ has $\nu = \frac{1}{2}$) Wen 1991
  $\nu = 1/2$ spin polarized state for soft-core Coulomb $\sim$ p-wave paired state Greiter-Wen-Wilczek 1991
- 1992:
- 1993: $\nu = 1/2$ spin polarized soft-core state = Pfaffian state (numerically). Edge of Pfaffian state is $c = 3/2$ CFT Wen 1993
- 1994:
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- 1996: NAB-statistics degeneracy in Pfaffian state Nayak-Wilczek; Read-Rezayi

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- 1997: Topological Quantum Computation using topological degeneracy of NAB statistics Kitaev 1997; Freedman 1998

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- 2000: The *p*-wave pair state is the Pfaffian state Read-Green 2000
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- 2000: The p-wave pair state is the Pfaffian state Read-Green 2000
- 2001:
- 2002: Most NAB states are universal quantum comp. Freedman-Larsen-Wang
- 2003:
- 2004: Design of topologically-protected qubits Das Sarma-Freedman-Nayak
- 2005: Founding of MS station Q. Support exp. on NAB FQH states. Interferometer of NAB particles (even-odd effect) Das Sarma-Freedman-Nayak 2005; Stern-Halperin 2006; Bonderson-Kitaev-Shtengel 2006; Fendley-Fisher-Nayak 2007
- 2006: Mach-Zehnder interferometer and shot noise Feldman-Kitaev 2006
- 2007: NAB hierarchy states Bonderson-Slingerland 2007; Levin-Halperin 2008
- 2008: Ultrahigh mobility and a clear $\nu = 5/2$ state Pan-Pfeiffer etal 2008
  Observed $e/4$ charge in NAB $\nu = 5/2$ state Dolev-Heiblum etal 2008
  Observed NAB CFT on NAB $\nu = 5/2$ edge Radu etal 2008
  Using pattern of zeros and Z_n simple current algebra, we can start to classify of NAB states Wen-Wang 2008; Lu -Wen-Wang-Wang 2009
- 2009: Observed the even-odd effect in $\nu = 5/2$ Willett-Pfeiffer-West 2009

Thanks, Mike!

- We would like to thank Mike for his vision, for his enthusiasm, and for his tremendous power of persuasion.
- He revived the field of FQH physics, energized the study of non-Abelian FQH states,
  formed a very active community,
  stimulated a lot of new research

I myself have benefited a lot from the new "family".

A hike with Mike

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- A highly entangled state
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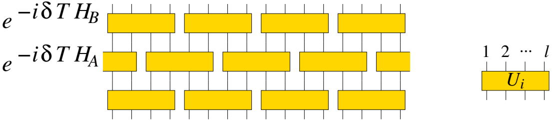
- Superfluid, as an exemplary quantum state of matter, is actually very classical and unquantum from entanglement point of view.

FQH states are highly and long-range entangled in any basis.
There is no classical picture for the order in FQH states.

Definition of quan. phases and long-range entanglements

- Two gapped states, $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$, belong to the same phase iff: Hastings-Wen 05; Bravyi-Hastings-Michalakis 10

$$|\Psi(1)\rangle = P\left(e^{-\mathrm{i}\,T\int_0^1 dg\; H(g)}\right)|\Psi(0)\rangle$$
$$= (\text{local unitary transformation})|\Psi(0)\rangle$$
$$= (\text{quantum circuit of finite depth})|\Psi(0)\rangle$$



- The local unitary transformations define an equivalence relation
A universality class of a gapped quantum phase is an equivalent class of the LU transformations

Topological order is a pattern of long range entanglement

Two kinds of gapped states with no symmetries:

- *The states that are equivalent to product state under LU transformations.* All those states belong to the same class (phase) $\to$ short-range entanglement and trivial topological order.
- *The states that are not equivalent to direct-product states.* Those states form many different equivalent classes (phases) $\to$ many patterns of long-range entanglements and many different topological orders.
- In absence of symmetry:

  Quantum phases of matter
  = patterns of long-range entanglement = topological orders
  = equivalence classes of the LU transformations
  Examples: FQH states, string-net condensed states

Symm. breaking orders and symm. protected topo. orders

- If the Hamiltonian H has some symmetries, its phases will correspond to equivalent classes of symmetric LU transformations: $|\Psi\rangle \sim P\Big(e^{-\mathrm{i}\int_0^1 dg\ \tilde{H}(g)}\Big)|\Psi\rangle$ where $\tilde{H}(g)$ has the same symmetries as H.

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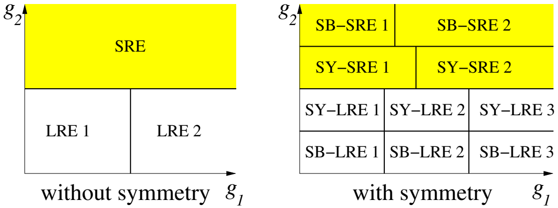

- SRE states with different symmetries
  $\to$ Landau's symmetry breaking orders.
- SRE states with the same symmetry can belong to different classes
  $\to$ **symmetry protected topological orders (SPT)** (symmetry protected trivial orders). Gu-Wen 09, Pollmann-Berg-Turner-Oshikawa 09
  Examples: Haldane phase and $S_z = 0$ phase of spin-1 XXZ chain.
  Band and topological insulators

Labeling and classifying topological orders

Topological order = pattern of long range entanglement
= equivalent class of LU transformations
How to label those equivalent classes?

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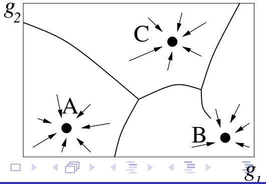
  Under the wave function renormalization generated by the LU transformation, the wave function flows to simpler one within the same equivalent class. Verstraete-Cirac-Latorre-Rico-Wolf 05; Vidal 07; Jordan-Orus-Vidal-Verstraete-Cirac 08; Jiang-Weng-Xiang 09; Gu-Levin-Wen 09

- Use the fixed-point wave function: Φ_{fix} to label topological order. Φ_{fix} may give us a one-to-one labeling of topological order, and a classification of topological order.



Classify 2D topological order (no symmetry)

The non-chiral 2D topological orders are classified by a set of tensors $N_{ijk}, F^{ijm,\alpha\beta}_{kln,\chi\delta}, P^{kj,\alpha\beta}_{i}, A^i$, that satisfy Levin-Wen 05; Chen-Gu-Wen 10

- ☞ $$\sum_m N_{jim^*} N_{kml^*} = \sum_n N_{kjn^*} N_{l^*ni},$$
- ☞ $$\sum_{t,\eta,\varphi,\kappa} F^{ijm,\alpha\beta}_{knt,\eta\varphi} F^{itn,\varphi\chi}_{lps,\kappa\gamma} F^{jkt,\eta\kappa}_{lsq,\delta\phi} = \sum_{\epsilon} F^{mkn,\beta\chi}_{lpq,\delta\epsilon} F^{ijm,\alpha\epsilon}_{qps,\phi\gamma},$$
- ☞ $$\sum_{\alpha=1}^{N_{kii^*}} \sum_{\beta=1}^{N_{j^*jk^*}} P^{kj,\alpha\beta}_{i} (P^{kj,\alpha\beta}_{i})^* = 1,$$
- ☞ $$P^{kj,\alpha\beta}_{i} = \sum_{m,\lambda,\gamma,l,\nu,\mu} F^{jj^*k,\beta\alpha}_{i^*i^*m^*,\lambda\gamma} F^{i^*mj^*,\lambda\gamma}_{m^*i^*l,\nu\mu} P^{lm,\mu\nu}_{i^*},$$
- ☞ $$P^{jp,\alpha\eta}_{i} \delta_{im}\delta_{\beta\delta} = \sum_{\chi} F^{ijm,\alpha\beta}_{klk,\chi\delta} P^{jp,\chi\eta}_{k^*} \quad \text{for all } k,i,l \text{ with } N_{kil^*} > 0.$$

......

- This a tensor category theory

Classify 2D fermionic topological order (no symmetry)

The non-chiral 2D **fermionic** topological orders are (partially?) classified by another set of tensors $N_{ijk}, N^f_{ijk}, F^{ijm,\alpha\beta,\pm}_{kln,\gamma\lambda,\pm}, O^{jk,\alpha\beta}_{i,\pm}, A^i$ that satisfy Gu-Wang-Wen 10

☞ $$\sum_{m=0}^{N} N_{jim^*} N_{kml^*} = \sum_{n=0}^{N} N_{kjn^*} N_{l^*ni},$$

☞ $$\sum_{m=0}^{N} (N^b_{jim^*} N^f_{kml^*} + N^f_{jim^*} N^b_{kml^*}) = \sum_{n=0}^{N} (N^b_{kjn^*} N^f_{l^*ni} + N^f_{kjn^*} N^b_{l^*ni}),$$

☞ $$\sum_{t} \sum_{\eta=1}^{N_{kjt^*}} \sum_{\varphi=1}^{N_{tin^*}} \sum_{\kappa=1}^{N_{lts^*}} F^{ijm,\alpha\beta,+}_{knt,\eta\varphi,-} F^{itn,\varphi\chi,+}_{lps,\kappa\gamma,-} F^{jkt,\eta\kappa,+}_{lsq,\delta\phi,-}$$

$$= (-)^{s_{jim^*}(\alpha) s_{lkq^*}(\delta)} \sum_{\epsilon=1}^{N_{qmp^*}} F^{mkn,\beta\chi,+}_{lpq,\delta\epsilon,-} F^{ijm,\alpha\epsilon,+}_{qps,\phi\gamma,-}$$

......

- This is a super tensor category theory.

No topological order in 1D, if there are no symmetries

In absence of symmetry, all 1D gapped states belong to the same phases. All 1D gapped states can be mapped to product states via LU transformations.

Verstraete-Cirac-Latorre-Rico-Wolf 2005; Chen-Gu-Wen 2010

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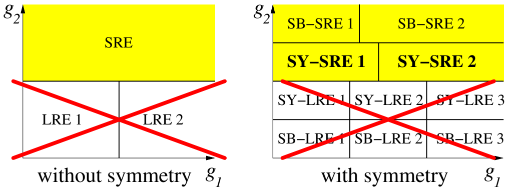
- But for systems with certain symmetries, we can only use the symmetric LU transformations to define states in the same phase. In this case, there are non-trivial phases: symmetry breaking phases and symmetry non-breaking phases – **Symmetry protected topological orders**



Classify 1D gapped states that do not break the symmetry

For 1D bosonic systems with only an on-site symmetry G which is realized by a linear representation, all the phases of gapped states that do not break the symmetries are classified by ω, where $\omega \in H^2(G, \mathbb{C})$ label different types of projective representations of G.

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A result/guess for higher dimensions Xie-Liu-Wen, in progress

For d-dimensional bosonic systems with only an on-site symmetry G which is realized by a linear representation, all phases of gapped short-range-entangled states that do not break the symmetries are classified by λ, where $\lambda \in H^{d+1}(G, \mathbb{C})$.

- 0-D: 1D representations $\alpha \in H^1(G, \mathbb{C})$
- 1-D: projective representations $\omega \in H^2(G, \mathbb{C})$
- 2-D: "3-cocycle representations" $\lambda \in H^3(G, \mathbb{C})$

Classify SRE states with translation and on-site symmetry

For 1D bosonic systems with only translation and an on-site symmetry G which is realized by a linear representation, all the phases of gapped states that do not break the two symmetries are classified by a pair (α, ω), where $\alpha \in H^1(G, \mathbb{C})$ label different 1D representations of G and $\omega \in H^2(G, \mathbb{C})$ label different types of projective representations of G.

For two dimensions Xie-Liu-Wen, in progress

For 2D bosonic systems with only translation and an on-site symmetry G which is realized by a linear representation, all phases of gapped short-range-entangled states that do not break the two symmetries are classified by $(\alpha, \omega_1, \omega_2, \lambda)$, where $\alpha \in H^1(G, \mathbb{C})$ $\omega_1, \omega_2 \in H^2(G, \mathbb{C})$ and $\lambda \in H^3(G, \mathbb{C})$.

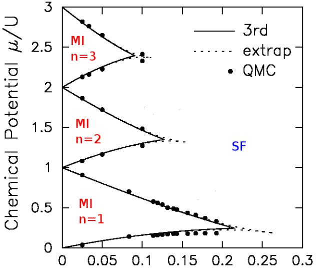
Examples: the meaning of the 1D rep. α in (ω, α)

Consider systems with translation of on-site $U(1)$ symmetry

- $H^2[U(1), \mathbb{C}] = 0$ → only one projective rep $\omega = 0$ which is of trivial-type
- 1D representations are label by $n \in \mathbb{Z}$: $\alpha_n(\theta) = e^{i n\theta}$.

For 1D bosonic systems with only translation and on-site $U(1)$ symmetry, there are infinity many 1D gapped phases labeled by $n \in \mathbb{Z}$ that do not break the symmetries.

Example: 1D boson Mott insulator with n boson per site.



A necessary condition of 1D gapped symmetric phases

In general, a symmetric state of L-sites satisfies

$$u(g) \otimes ... \otimes u(g)|\phi_L\rangle = \alpha_L(g)|\phi_L\rangle$$

Localization of 1D representation:For 1D bosonic systems of L sites with translation and an on-site symmetry G which is realized by a linear representation, a gapped state that do not break the two symmetries must transform as $u(g) \otimes ... \otimes u(g)|\phi_L\rangle = [\alpha(g)]^L|\phi_L\rangle$ for all large L.

This generalizes a result of Hastings (2003) from $U(1)$ to other groups

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- **Example:** a 1D state of conserved bosons with fractional bosons per site must be gapless, if the state does not break the translation symmetry.
- **Non example:** 2D conserved bosons with $1/2$ bosons per site can form $\nu = 1/2$ FQH Hall state that has a gap and do not break the $U(1)$ and the lattice translation.

Examples: the meaning of the proj. rep. ω in (ω, α)

Consider systems with translation of on-site $SO(3)$ symmetry

- $H^2[SO(3), \mathbb{C}] = \mathbb{Z}_2 \to$ Integer spins and half-integer spins
- Only one trivial 1D representation: $\alpha(g) = 1$.

For 1D bosonic systems with only translation and on-site $SO(3)$ symmetry, there are two 1D gapped phases labeled by $\omega = 0, 1$ that do not break the symmetries.

The boundary states at chain ends form a projective representation of the symmetry group $\to$ the symmetry is fractionalized.

Degeneracy in entanglement spectrum is also related to the projective representation of the symmetry group.

Pollmann-Berg-Turner-Oshikawa 2010

Parity and time reversal protected topological orders

For 1D spin systems with only translation and parity symmetry, there are four phases for gapped states that do not break the two symmetries.

For 1D systems with only translation and time reversal symmetry T, there are two phases for gapped states that do not break the two symmetries, if on each site the time reversal transformation satisfies $T^2 = I$.

- Haldane/AKLT phase can be stable without spin rotation symmetry. We only need parity or time reversal symmetry.

Gu-Wen 09; Pollmann-Berg-Turner-Oshikawa 09

Zoo of 1D gapped symmetric phases

Symmetry	No. of Different Phases
None	1
Trans. $+U(1)$	∞_α
Trans. $+SO(3)$	2_ω
Trans. $+D_2$	$2_\omega \times 4_\alpha = 8$
Trans. + P	4
Trans. + T	2
Trans. + P+T	8
Trans. $+SO(3)+$ P	8
Trans. $+D_2+$P	128
Trans. $+SO(3)+$ T	4
Trans. $+D_2+$T	64
Trans. $+SO(3)+$ P+T	8
Trans. $+D_2+$P+T	1024

Zoo of 1D gapped symmetric phases (no trans. symm.)

Symmetry	No. of Different Phases
None	1
$SO(3)$	2
D_2	2
T	2
$SO(3)+$ T	4
D_2+T	16

All classified by the projective representations of the symmetry group.
All characterized by end states.

Gapless states

A Generalization of Lieb-Schultz-Mattis theorem

For an 1D spin system with translation and an on-site symmetry G which is realized by a non-trivial projective representation, the system must be gapless if it does not break the two symmetries.

Gapless states

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For an 1D spin system with translation and an on-site symmetry G which is realized by a non-trivial projective representation, the system must be gapless if it does not break the two symmetries.

- An example: $SO(3)$ spin rotation and translation symmetric half-integer spin chain is gapless if it does not break the two symmetries.

For 1D systems with translation and time reversal symmetry T, the system is gapless if it does not break the two symmetries, provided that on each site the time reversal transformation satisfies $T^2 = -I$.

Topological order and entanglement – a rich world

- We can classify all 1D gapped quantum phases using symmetric LU transformation, MPS, and projective representation.
- One can also partially classify 2D gapped quantum phases using LU transformation, string-nets, and TPS.