# Spin Waves in a 1D Spinor Bose Gases

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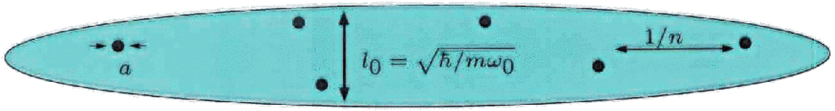
## 1D Bose Gas




one-dimensional regime $nl_0 \ll 1$

short-range potential $a \ll l_0$

interactions $g = 2\hbar^2 a/ml_0^2 > 0$

*(M. Olshanii '98)*

$$\boxed{a \ll l_0 \ll 1/n}$$

**The Hamiltonian**

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\sum_j \frac{\partial^2}{\partial x_j^2} + g\sum_{i<j}\delta(x_i - x_j) \\ &= \int dx\ \frac{\hbar^2}{2m}\,\partial_x\Psi^\dagger\partial_x\Psi + \frac{g}{2}\,\Psi^\dagger\Psi^\dagger\Psi\Psi \end{aligned}$$

## Quantum regimes at $T = 0$

Dimensionless Coupling Constant $\boxed{\gamma = mg/\hbar^2 n}$

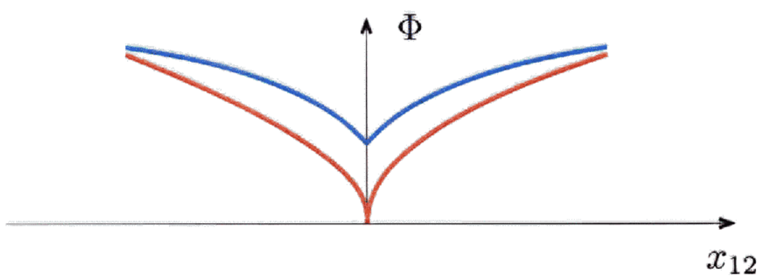
$\gamma \ll 1$ **weak coupling, Gross-Pitaevskii regime**

quasi-condensate *(Popov's book, D. Petrov et.al. '00)*

$l_c = \hbar/\sqrt{mgn}$ correlation length

$\gamma \gg 1$ **strong coupling, Tonks-Girardeau regime**

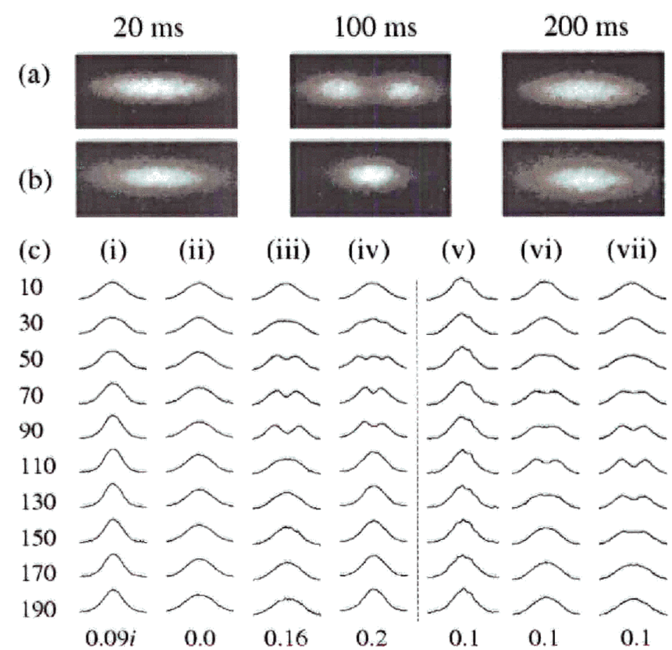
fermionization *(M. Girardeau '60)*






## Spin Waves in Ultra Cold Gases

2 Internal states $|(a)\rangle$, $|(b)\rangle$ <=> (iso) spin $\frac{1}{2}$
*Lewandowski et. al, PRL 2001*



## Spin-independent interactions

$$a_{aa} = a_{bb} = a_{ab} = a$$

$$H = -\frac{\hbar^2}{2m}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + g\sum_{i<j}\delta(x_i - x_j)$$

$$g = \frac{4\pi\hbar^2 a}{m}$$

Ground State fully polarized

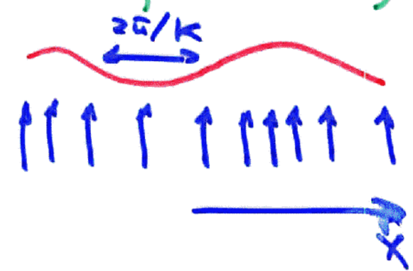
$$S_{tot} = \frac{N}{2}$$

(Eisenberg & Lieb '02)

## Excitations

Phonons (density waves)

$$\Delta S_{tot} = 0$$




$$E(k) = ck$$

c – sound velocity

Spinons (spin waves)

$$E(k) = \frac{k^2}{2m^*}$$

$m^*$ – effective mass

Quasicondensate $m^* \simeq m$

# Bethe Ansatz

Lieb & Liniger '63
C.N. Yang '67

$0 < x_{Q_1} < x_{Q_2} < \dots < x_{Q_N} < L$


+ periodic boundary cond.

$$\Psi(x_1, x_2, \dots x_N) \sim \sum_P A[P,Q]\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q_j}}$$

Scattering matrix $\hat{S}$

$$A[P_{ij} P, Q] = \hat{S}_{ij} A[P, P_{ij} Q]$$

$$A = A(\{k_j\}, \{\Lambda_1 \dots \Lambda_M\})$$

$S_{tot} = \frac{N}{2} - M$ - total spin

Quantum numbers $k_1 \dots k_N$

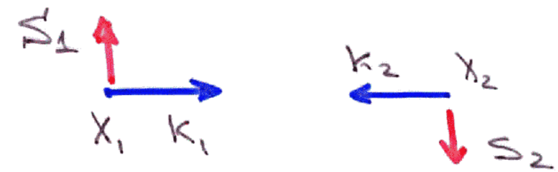
$\Lambda_1 \dots \Lambda_M$

# Scattering with spin

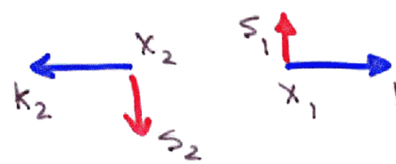
$$A[\dots P_j P_{j+1} \dots, Q] = \hat{S}(k_{P_j} - k_{P_{j+1}}) A[\dots P_{j+1} P_j \dots, Q]$$

## 2-body scattering matrix

$$\hat{S}(k_1 - k_2) = \frac{i\gamma n}{k_1 - k_2 + i\gamma n} - \frac{k_1 - k_2}{k_1 - k_2 + i\gamma n} \hat{P}_{12}$$

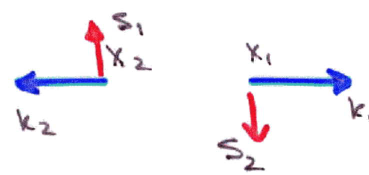



Direct




$\times \frac{i\gamma n}{k_1 - k_2 + i\gamma n}$

Exchange




$\times - \frac{k_1 - k_2}{k_1 - k_2 + i\gamma n}$

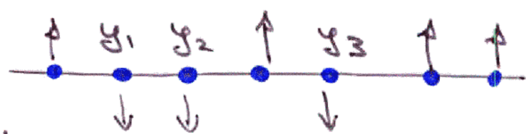
# Irreducible representations of $S_N$

1) No spin  Bosons $P_{12} = 1$

$$S_B(k_1-k_2) = \frac{i\gamma n - (k_1-k_2)}{i\gamma n + (k_1-k_2)}$$

Fermions $P_{12} = -1$  $S_F = -1$

2) Spin $\frac{1}{2}$ $\left(S_{tot} = \frac{N}{2} - M\right)$




A has $\dfrac{N!}{M!(N-M)!}$ components

$$A = \sum_{P_M}{}' B_P \, F(\Lambda_{P_1}, y_1) F(\Lambda_{P_2}, y_2) \ldots F(\Lambda_{P_M}, y_M)$$

$$y_1 < y_2 < \ldots < y_M$$

$$F(\Lambda, y) = \prod_{j=1}^{y-1} \frac{i k_j - i\Lambda + \gamma n/2}{i k_{j+1} - i\Lambda + \gamma n/2}$$

# BA equations

Periodicity:

$$\boxed{e^{i k_j L} = S_{j+1,j} S_{j+2,j} \ldots S_{N,j} S_{1,j} \ldots S_{j-1,j}}$$

$$L k_j = 2\pi I_j - 2\sum_{\ell=1}^{N} \tan^{-1}\left(\frac{k_j - k_\ell}{\gamma n}\right) + 2\sum_{\nu=1}^{M} \tan^{-1}\left(\frac{k_j - \Lambda_\nu}{\gamma n/2}\right)$$

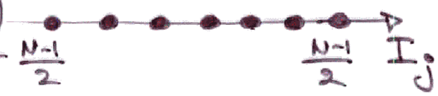
$$2\pi J_\mu = 2\sum_{\ell=1}^{N} \tan^{-1}\left(\frac{\Lambda_\mu - k_\ell}{\gamma n/2}\right) - 2\sum_{\nu=1}^{M} \tan^{-1}\left(\frac{\Lambda_\mu - \Lambda_\nu}{\gamma n}\right)$$

Energy $E = \sum_{j=1}^{N} k_j^2$;  Momentum $P = \frac{2\pi}{L}\left(\sum I_j - \sum J_\mu\right)$

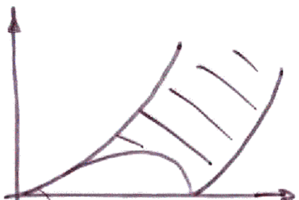
$I_j$, $J_\mu$ - integers

## Ground State $M = 0$

$$L k_j = 2\pi I_j^0 - 2\sum_{\ell=1}^{N} \tan^{-1}\left(\frac{k_j - k_\ell}{\gamma n}\right)$$




## Phonons (particle - Hole) $M=0$

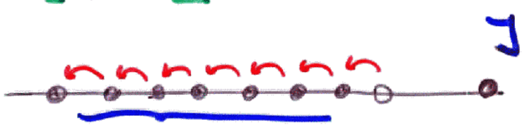


$E(k) = ck$

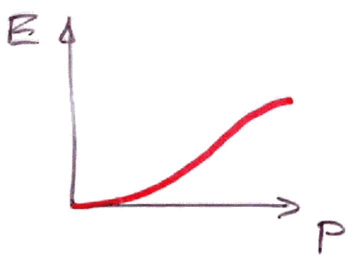
$$c = \begin{cases} \sqrt{\frac{gn}{m}} & \text{GP} \\ v_F = \frac{\hbar\pi n}{m} & \text{TG} \end{cases}$$

## Spinons $M = 1$

$$p = \frac{2\pi}{L}\left(\frac{N}{2} - J\right)$$




$$I_j = I_j^0 - \frac{1}{2}$$




$$E(p) \simeq \frac{p^2}{2m^*} \qquad p \to 0$$

## Weak Coupling

$$k_j = \sqrt{\gamma}\, n\, x_j \quad \text{- rescaling}$$

GROUND State

$$x_j = \frac{2}{N}\sum_{\ell \neq j} \frac{1}{x_j - x_\ell}$$

Density of solutions

$$\rho(x) = \frac{1}{N}\sum_j \delta(x - x_j) = \frac{1}{\pi}\,\mathrm{Im}\, G(x - i\varepsilon)$$

Resolvent $\quad G(z) = \int \frac{dx\, \rho(x)}{z - x}\ ; \quad z \in \mathbb{Z}$

$$G(z)_{z\to\infty} = \frac{1}{z}\ ; \quad \int \rho(x)\,dx = 1$$

$$\Rightarrow G(z) = \frac{z}{2} - \frac{1}{2}\sqrt{z^2 - 4}$$

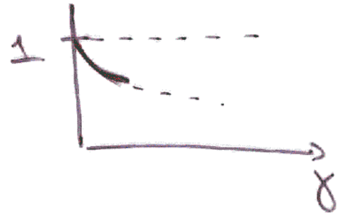
$\vdots$

$$\frac{E_0(\gamma)}{N n^2} = \gamma - \frac{4}{3\pi}\gamma^{3/2}$$

= Bogoliubov's result

## Weak coupling (continued)

$$\frac{m}{m^*(\gamma)} = 1 - \frac{2}{3\pi}\sqrt{\gamma}$$




## Bogoliubov Perturbation Theory

Bosonic field operator (spinor)

$$\hat{\Psi} = \begin{pmatrix} \Phi + \delta\Psi_{Bog} \\ \delta\Psi_{spin} \end{pmatrix}$$

$$\langle \delta\Psi^\dagger \delta\Psi \rangle = \frac{1}{E - k^2 - \Sigma(k,E)}$$

$\Sigma \sim$


## Strong Coupling

### Ground State

$$k_j^0 L = 2\pi I_j^0 \left(1 - \frac{2}{\gamma}\right)$$

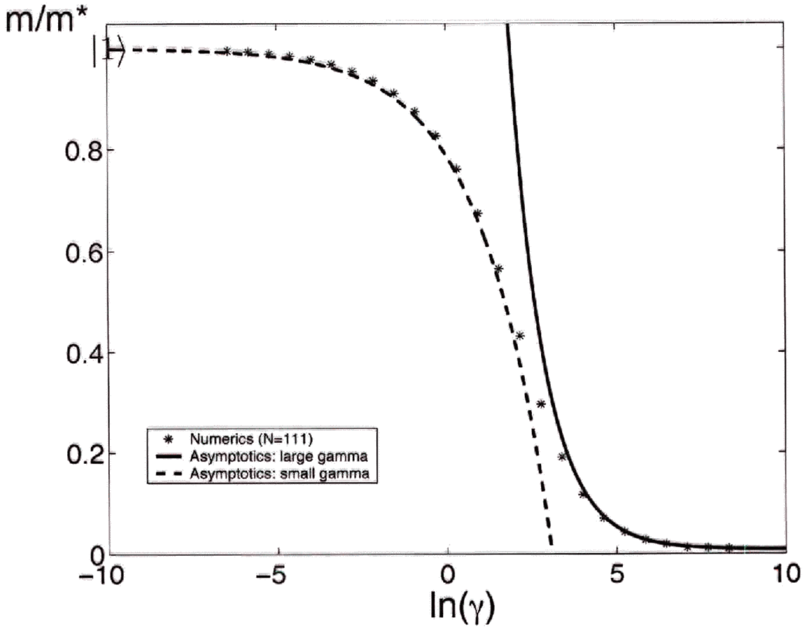
$$E_0(\gamma) = \sum (k_j^0)^2 = \left(1 - \frac{4}{\gamma}\right) E_0(\infty)$$

### Excited State

$$\begin{cases} \Delta K_j L = (k_j - k_j^0) L = \dfrac{\gamma n}{2\lambda}\left(1 + \dfrac{k_j^0}{2\lambda}\right) + \dfrac{2}{\gamma}\left(\dfrac{p}{n} - \pi\right) \\ \lambda = \dfrac{\gamma n^2}{2p} \end{cases}$$

$$E(p) = p^2 \left(\frac{1}{N} + \frac{2\pi^2}{3\gamma}\right)$$

$$\boxed{\frac{m}{m^*} = \frac{1}{N} + \frac{2\pi^2}{3}\frac{1}{\gamma}}$$




$\gamma \to 0$ - impurity, weakly interacting with (quasi) condensate

$\gamma \to \infty$ - impenetrable particles, $m^* \sim N$

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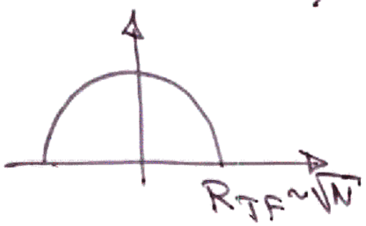
# Trapped System $\gamma \to \infty$

## Bosonic field

$$\hat{\Psi} = \sqrt{n(x)}\, e^{i\varphi(x)} \begin{pmatrix} \cos\frac{\theta(x)}{2} e^{i\chi(x)} \\ \sin\frac{\theta(x)}{2} e^{-i\chi(x)} \end{pmatrix}$$

## TF density profile

$$n(x) = n_0 \sqrt{1 - \left(\frac{x}{R_{TF}}\right)^2}$$




## Eq. of motion

$$\begin{cases} \dot{\theta} = -\frac{\hbar}{n} \partial_x \frac{n}{m^*} \partial_x \chi \\ \dot{\chi} = \frac{\hbar}{4n} \partial_x \frac{n}{m^*} \partial_x \theta \end{cases}$$

## WKB

$$\Omega_n = \frac{\pi^5}{48\,\Gamma^4\left(\frac{3}{4}\right)} \left(n + \frac{1}{2}\right)^2 \frac{\omega}{\gamma_0 N}$$

very small $\sim 1/\gamma_0 N$

# Conclusions

* Large effective mass

  $m^* \sim \gamma \quad \gamma \to \infty$

  due to strong interactions

  (beyond mean-field)

* Ultra slow excitations
  for trapped systems

* Experimental detection
  of strongly interacting
  (Tonks-Girardeau) regime