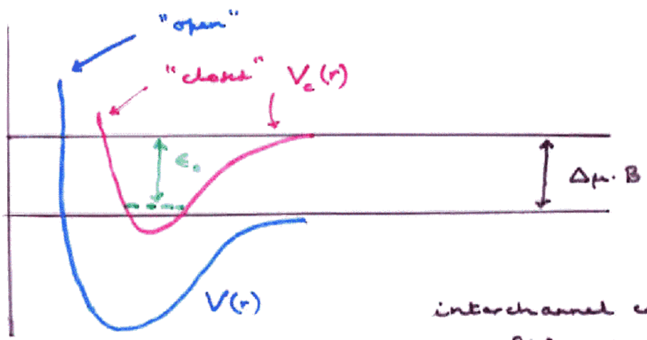
K'1

Many-Fermion Systems near a Feshbach Resonance



r_0

interchannel coupling

$g f(r)$, $f(r) \lesssim 1$.

range of $f(r) \sim l_0 \lesssim r_0$

Assume: $r_0 \ll k_F^{-1}$ (~ interparticle spacing)

a_{bg} (background open-channel sc. length) $\sim r_0$

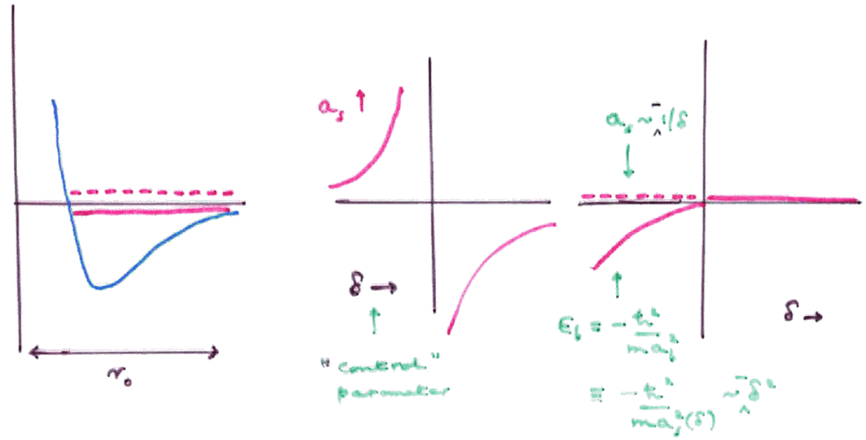
$g \ll |\epsilon_0|$.

Qn: What does the "naive" ansatz for the many-body wave function look like (qualitatively)?

1. Single-channel case, 2-body problem
2. " " many-body "
3. FR case, 2-body problem
4. " many-body "

K'2

1. Single-Channel Case: 2-body problem near resonance



Renormalization of potential: $\psi(r) \equiv \sum_k c_k e^{i\mathbf{k}\cdot\mathbf{r}}$

$$\langle H \rangle = \sum_k 2\epsilon_k |c_k|^2 + \frac{1}{2\Omega}\sum_{kk'} V_{kk'} c_k c^*_{k'} \qquad \sum_k |c_k|^2 = 1$$

introduce q_c s.t. $a_s^{-1} \ll q_c \ll r_0^{-1}$, and:

$$\hat{U} \equiv (1 + \hat{P}_2(\hat{H}_0 - E)^{-1}\hat{V}\hat{P}_2)^{-1}\hat{V}$$

approx. ind. of E for $E \ll \hbar^2 q_c^2/m$

for $|k|, |k'| < q_c$, $U_{kk'} = \text{const.} \equiv U(q_c)$

$$\langle H \rangle = \sum_k{}' 2\epsilon_k |c_k|^2 + \frac{1}{2\Omega} U(q_c) \sum_{kk'} c_k c^*_{k'} \qquad \sum_k |c_k|^2 = 1$$

$\Rightarrow$ for $E = 0$ (scattering state)

ind. of q_c

$$\frac{c_k}{c_0} = -\frac{U(q_c)/2\epsilon_k}{1 + \left(\frac{m}{2\pi^2\hbar^2}\right) q_c U(q_c)} \equiv -U_0/2\epsilon_k$$

$$U_0 = 4\pi\hbar^2 a_s/m$$

K'3

An explicit formula for a_s in case $\delta \equiv V_c(r) + \delta V(r)$

[$\psi(r)$ always means radial w.f. with factor r^{-1} extracted: so $\int |\psi(r)|^2 dr \equiv 1$ for normaliz^n]

generally, (also for bound state) for $r_0 \ll r \ll a_s$

$$\psi(r) = \left(1 - \frac{r}{a_s} + \cdots\right)$$

If $V_c(r)$ is critical value of $V(r)$, then if χ_0 is zero-energy scattering state,

$$-\frac{\hbar^2}{2m}\frac{d^2\chi_0}{dr^2} + V_c(r)\chi_0(r) = 0 \qquad (A)$$

Let $V(r) = V_c(r) + \delta V(r)$, then $E = 0$ solution obeys

$$-\frac{\hbar^2}{2m}\frac{d^2\chi}{dr^2} + (V_c(r) + \delta V(r))\chi(r) = 0 \qquad (B)$$

Multiply (A) by $\chi(r)$ and (B) by $\chi_0(r)$, subtract, use Green's theorem and fact that both χ_0 and χ must $\to 0$ for $r \to 0$:

$$-\frac{2m}{\hbar^2}\int_0^r \delta V(r')\chi_0(r')\chi(r')\,dr' = [\chi_0'\chi - \chi'\chi_0]_r = a_s^{-1}$$

but since $\chi(r) \approx \chi_0(r)$,

$$a_s^{-1} = -\int \delta V(r)|\chi_0(r)|^2 dr \sim \delta$$

K'4

2. Single-Channel Case: Many-Body Problem

Naive ansatz:

$$\Psi_N = \prod_k (u_k + v_k a_{k\uparrow}^+ a_{-k\downarrow}^+)|vac\rangle$$

$$F_k \equiv u_k v_k, \qquad \tilde{F}(\underset{\sim}{r}) \equiv \langle \psi_\uparrow^\dagger(0)\psi_\downarrow^\dagger(r)\rangle = \sum_k F_k e^{ik\cdot r}$$

$$\langle E\rangle = \langle T\rangle + \langle V\rangle$$

$$\langle V\rangle = \frac{1}{2\Omega}\sum_{k} V_{kk'} F_k F_{k'}^* \quad (+\text{ Hartree})$$

$$\langle T\rangle = \sum_k \epsilon_k\left(1 - \sqrt{1 - 4F_k^2}\right) = \sum_k 2\epsilon_k\{|F_k|^2 + |F_k|^4 + \cdots\}$$

Provided all F_k for $k > q_c$ are $\ll 1$, renormalization procedure goes through as in 2-body problem:

$$\langle E\rangle = \sum_k{}' \epsilon_k\left(1 - \sqrt{1 - 4F_k^2}\right) + \frac{U(q_c)}{2\Omega}\sum_{kk'} F_k F_{k'}^*$$

With standard notation $F_k \equiv 2\Delta_k/E_k$, $E_k \equiv \sqrt{(\epsilon_k - \mu)^2 + |\Delta_k|^2}$, get BCS gap eqn. with $\Delta_k \to \Delta$

$$\sum_k{}' \frac{1}{2E_k} = -U^{-1}(q_c) \qquad \left(E_k \equiv \sqrt{(\epsilon_k-\mu)^2 + |\Delta|^2}\right)$$

which can be rewritten, eliminating $U(q_c)$ in favor of a_s, as

$$\sum_k{}' (\epsilon_k^{-1} - E_k^{-1}) = \frac{m}{2\pi\hbar^2 a_s}$$

$\equiv r\tilde{F}(\underset{\sim}{r})$

Coordinate-space form of $F(r)$ for $r_0 \ll r \ll a_s; \xi$ etc.

$$\boxed{F(r) = \frac{m\Delta}{2\pi\hbar^2}\left(1 - \frac{r}{a_s}\right)}$$

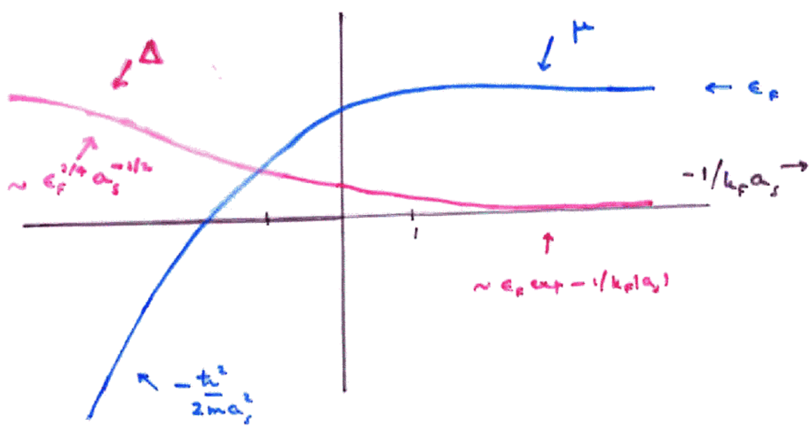
— i.e. similar to 2-body $\psi(r)$ except for normaliz^n

K'4a

In many-body problem, have two unknowns, μ and Δ. These are determined by the gap eqn. and the number conservation equation:

$$\sum_k \left(\epsilon_k^{-1} - E_k^{-1}(\mu,\Delta)\right) = \frac{m}{2\pi\hbar^2 a_s}$$

$$\sum_k \left(1 - \frac{(\epsilon_k - \mu)}{E_k(\mu,\Delta)}\right) = N \quad (= k_F^3/3\pi^2)$$



K'5

3. Feshbach Resonance: 2-body problem

$\chi(r)$ now always denotes open-channel amplitude ($\times r$)

Elimination of closed-channel amplitude gives for $\chi(r)$

$$\left(-\frac{\hbar^2}{m}\frac{d^2}{dr^2} + V(r) - E\right)\chi(r) + \frac{g^2}{E-\delta}\int_0^\infty K(rr')\,\chi(r')\,dr'$$

$$K(rr') \equiv \phi_0^*(r)\,\varphi_0(r')\,f(r)\,f(r')$$

closed-channel eigenfn

$\Delta\mu \cdot B - \epsilon_0$

$O(g^2/(\hbar^2/ma_s)\ell_0)$

There exists some value δ_0 of δ s.t. for $E=0$, $a_s(\delta) \to \infty$. Let χ_0 be the corresponding eigenfunction. Then by procedure similar to 2-body case, any eigenfunction $\chi(r)$ associated with eigenvalue E must satisfy

drop bec. $\sim (r/a_s)$ rel^n to terms kept

$$[\chi_0\chi' - \chi_0'\chi]_r = \frac{m}{\hbar^2}\Big\{-E\int^r \chi_0(r')\chi(r')\,dr'$$

$$+ \frac{\delta - E}{\delta_0(\delta_0+\delta-E)}\cdot g^2 \cdot \int_0^r dr \int_0^\infty dr'\,\chi_0(r)\,K(rr')\,\chi(r')$$

(normalizn.:

$\chi(r) \sim 1 - \frac{r}{a}$, $r_0 \ll r \ll a_s$)

not nec.y a_s!

$$\approx \underbrace{\int_0^\infty dr \int_0^\infty dr'\,\chi(r)\,K(rr')\,\chi(r')}_{\equiv\, \ell_0}$$

$$\Rightarrow \quad \boxed{\frac{1}{a(E)} = \frac{m}{\hbar^2}(E-\delta)\frac{g^2}{\delta_0^2}\ell_0} \qquad \left(\delta_0 \sim \frac{g^2}{\hbar^2/m\ell_0^2}\right)$$

in particular for $E=0$ (scattering state)

$$a_s = -\frac{\eta}{\delta}, \qquad \eta \equiv \frac{\delta_0^2\hbar^2}{mg^2\ell_0}$$

K'6

Consider now a bound state ($E<0$). It satisfies

$$\frac{1}{a_s(E)} = \frac{m}{\hbar^2}(E-\delta)\frac{g^2}{\delta_0^2}l_0 \qquad (1)$$

But since for $r > r_0$ $V(r)=0$, it must also satisfy

$$E = -\frac{\hbar^2}{m a_s^2(E)} \qquad (2)$$

Combining (1) and (2) and defining

$$\boxed{\delta_c \equiv \tfrac{1}{2}(\delta_0^4/g^4)(\hbar^2/m l_0^2)}$$

we find

$$(E-\delta)^2 + 2\delta_c E = 0$$

i.e.

$$|E| = \delta_c - \delta - \sqrt{\delta_c^2 - 2\delta\delta_c} \qquad (\delta<0)$$

So:

for $\delta \gg \delta_c$, $|E| = |\delta|$

for $\delta \ll \delta_c$, $|E| = \delta^2/2\delta_c$

Physical significance of δ_c:

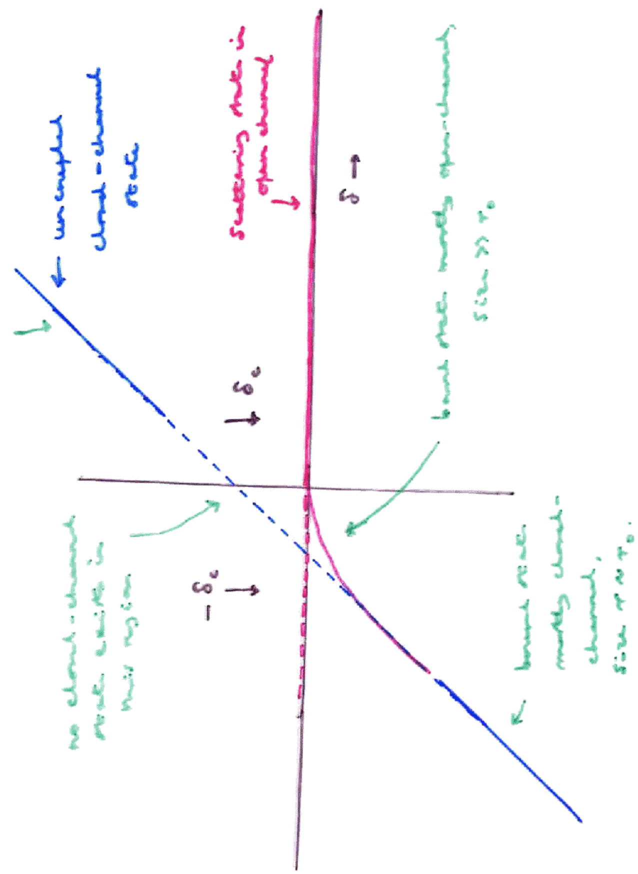
$\lambda \equiv$ prob. of closed channel / prob. of open channel in bound state

$$\lambda = |\delta - E|/2\delta_c \quad = \sqrt{1+2|\delta|/\delta_c} - 1$$

Hence, δ_c marks (to an o. of m.) the value of detuning at which the nature of the bound state switches over from mostly open-channel to mostly closed-channel. Note that at the crossover the value of a_s (i.e. "size" of open-channel compt.) is $\sim \xi_c \equiv (2m\delta_c/\hbar^2)^{-1/2}$ which is $\gg r_0$ (important)

K'7

QUALITATIVE PICTURE OF FESHBACH RESONANCE (2-body problem):



Note: In the region $|\delta| \lesssim \delta_c$, no well-defined state exists in closed channel!

K'8

4. Feshbach Resonance : Many-Body Problem

Distinguish 2 cases:

(a) Both atomic states are different in the two channels

(b) One atomic state is common to both channels

[Most exciting cases are (b)].

Case A

Ansatz: $\Psi_N = \prod_k (u_k^o + v_k^o a^+_{k\alpha} a^+_{-k\beta})(u_k^c + v_k^c a^+_{k\gamma} a^+_{-k\delta})|vac\rangle$

Df. $F_k^o \equiv u_k^o v_k^o$, $F_k^c \equiv u_k^c v_k^c$

Get coupled gap eqns. for Δ_o and Δ_c (i.e. F_k^o and F_k^c)

However, provided all F_k^c are $\ll 1$, can eliminate closed-channel amplitude exactly as in 2-body problem, and use standard to obtain eqn. for a_s with $E \to 2\mu$, i.e.

$$\boxed{\frac{1}{a_s(\mu)} = \frac{m(2\mu - \delta)}{\hbar^2 \delta_o^2} g^2 \ell_o}$$

However, 2μ is in general not simply $\hbar^2/m a_s^2(\mu)$, but is given by the open-channel gap eqn.

$$\sum_k \left(\epsilon_k^{-1} - E_k^{-1}(\mu, \Delta)\right) = \frac{m}{2\pi\hbar^2 a_s(\mu)}$$

This must be solved simultaneously with the number conservation eqn.:

$$\boxed{\sum_k \left(1 - \frac{\epsilon_k - \mu}{E_k(\mu, \Delta)}\right) = N_{open} \equiv N - N_c}$$

K'9

Now, from the solution of the closed-channel Schr. eqn. we obtain the value of N_c:

$$N_c / N_{open} = \alpha (k_F \xi_c) \left(\frac{\Delta}{\epsilon_F}\right)^2 . \qquad (\alpha = \text{num'l const } O(1))$$

$(2m\delta_c/\hbar^2)^{-1/2}$ $(\sim r^*)$

and so

$$N_{open} = N . \left(1 + \alpha k_F \xi_c \left(\frac{\Delta}{\epsilon_F}\right)^2\right)^{-1} \equiv (k_F^{eff})^3 / 3\pi^2$$

Hence, the complete effect of the closed channel can be incorporated in the replacement

$$k_F \to k_F^{(eff)} \equiv \frac{k_F}{\left(1 + \alpha k_F \xi_c (\Delta/\epsilon_F)^2\right)^{1/3}}$$

Case I. "Broad" resonance, $k_F \xi_c \ll 1$: $(\epsilon_F \ll \delta_c)$

$k_F^{eff} \cong k_F$ until $\Delta \gg \epsilon_F$ (when k_F becomes irrelevant)

All formulae identical to single-channel case provided $a_s(\delta)$ correctly given. (in Xover region: later, (for larger ... $\delta, \sim \delta_c$, BEC molecules become mostly closed-channel).

Case II: "Narrow" resonance, $k_F \xi_c \gg 1$ $(\epsilon_F \gg \delta_c)$

Qualitatively, $k_F \to k_F^{2/3} \xi_c^{-1/3} \times (\Delta/\epsilon_F)^{-2/3}$

At any given value of μ, (e.g. $\mu = 0$) value of Δ/ϵ_F is fixed by gap eqn. Hence, e.g. at "unitarity" $(a_s \to \infty)$ the characteristic energy scale is $k_F^{4/3} \xi_c^{-2/3}$ (rather than k_F^2 as in the broad-resonance case)

↑: Hartree terms!

LK'10

CASE B (one atomic state common to open + closed channels)

Must now write

$$\Psi_N = \prod_k \left(u_k + v_k^o a^+_{k\alpha} a^+_{-k\beta} + v_k^c a^+_{k\alpha} a^+_{-k\gamma}\right)|vac\rangle$$

with

$$u_k^2 + |v_k^o|^2 + |v_k^c|^2 = 1$$

The closed-channel KE is now of the form

$$\langle T \rangle = \sum_k \epsilon_k^{(c)}\left(1 - \sqrt{1 - 4(F_k^o)^2 - 4(F_k^c)^2}\right)$$

This physically reflects the fact that if the common state α is used to form a pair in the open channel, it is not available for the closed channel.

We can thus eliminate the closed channel if and only if not only all $F_k^c \ll 1$, but also all $F_k^o \ll 1$. The first condition (which is all that is required in case A) requires only $nr_0^3 \ll 1$, but the second requires the much more stringent condition (order of mag.) $na_s^3 \ll 1$, which is never satisfied near unitarity.

However, since the number of states in the closed channel which are affected are only $\sim (r_0/a_s)^3$ $(\sim nr_0^3)$ of the whole, we would expect the relative shift in E to be of this order* relative to ϵ_0. The condition for ~~this~~ the absolute shift to be comparable to the smallest char. en. (δ_c) is crudely

$$\epsilon_F/\delta_c \gtrsim \left(\frac{\epsilon_0}{\delta_c}\right)^{1/3}$$

which is not obviously unattainable.

* Also true in case (A), but only in "cc-dominated" regime.