Phonon Excitations of a Bose Gas within an Optical Lattice

Eugene Zaremba
Department of Physics, Queen's University, Canada

Work done in collaboration with Ed Taylor [Phys. Rev. A68, 053611 (2003)]

Why of interest?

- observing solid state effects in a dilute Bose gas
- novel superfluid states
- superfluid-Mott insulator transition
- dissipation and breakdown of superfluidity

<u>Experimental Background</u>

* 1. Burger et al., PRL <u>86</u>, 4447 (2001)

- observation of dissipative flow ⇒ breakdown of superfluidity: Landau instability?

2. Cataliotti et al., New J. Phys <u>5</u>, 71.1–71.7 (2003)

- similar to Burger et al., but with deeper potentials → tight-binding regime
- qualitative agreement with DNSE results of Smerzi et al. ⇒ dynamical instability

3. Ferlaino et al., PRA <u>66</u>, 011604 (2002)

- effects of thermal component

4. Fallani et al., cond-mat/0404045

- adiabatic loading into a moving optical lattice.

Theoretical Work

1. Wu + Niu, PRA 64, 061603 (2001)

   - 1D optical lattice → Bogoliubov excitations
   - emphasize distinction between energetic (Landau) and dynamic instabilities
   - suggest that dynamic instability is origin of dissipation in Burger et al. expt.

2. Machholm et al, PRA 67, 053613 (2003)

   - detailed numerical study of 1D optical lattices
     - band structure (swallowtails)
     - excitations → dynamic + energetic instabilities

3. Krämer et al., Eur. Phys. J. D 27, 247 (2003)

   - numerical study of 1D optical lattices in 'weak' interaction limit (no swallowtails)
   - excitations – emphasize long wavelength behaviour.

4. Modugno et al., cond-mat/0405653

   - excitation spectrum for cylindrical geometry
   - generate stability diagram (similar to Wu + Niu)
   - first full 3D simulation of Burger et al. expt.
     → dynamical instabilities

S. Burger, et. al., PRL 86, 4447 (2001)

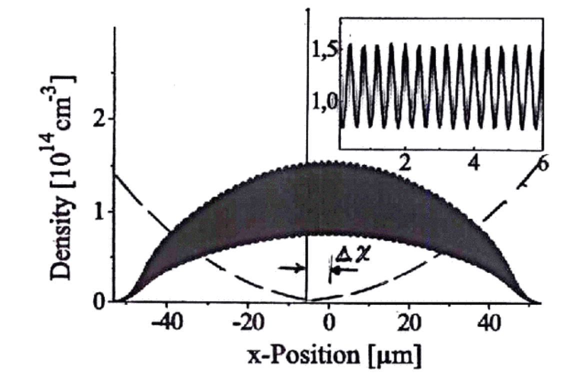


FIG. 1. Density distribution of a BEC in a harmonic trap with a superimposed optical lattice, from a numerical simulation of the 3D GPE for $N = 3 \times 10^5$ and $V_0/k_B = 270$ nK. The inset shows an enlargement of the central region of the BEC. The envelope of the modulated density distribution follows the parabolic distribution in the harmonic trap.

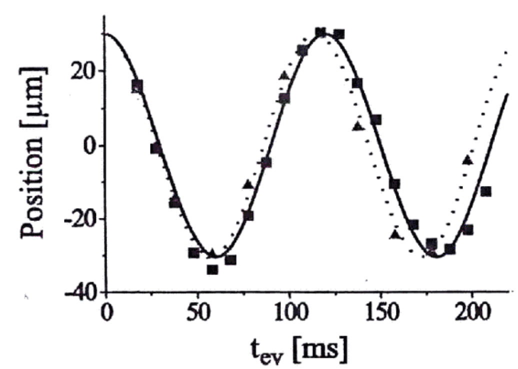


FIG. 2. Superfluid oscillations of a BEC in the presence of an optical lattice potential of height $V_0/k_B \simeq 270$ nK (squares) and in a purely magnetic trap (triangles), for initial displacement $\Delta x = (31 \pm 3)$ µm. The lines give results from a numerical simulation of the 1D GPE at the experimental parameters.

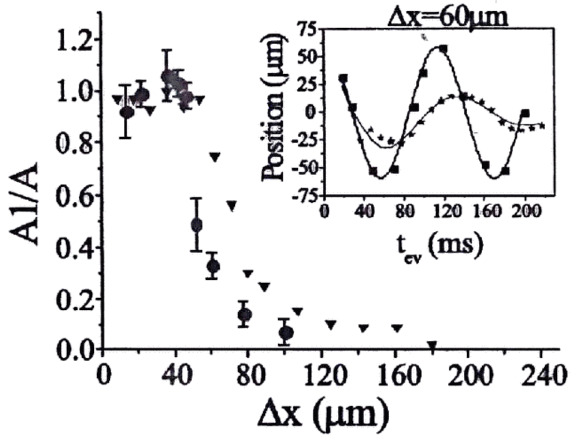


FIG. 3. Ratio of the first-peak amplitude of the oscillation of the ensemble to the free-oscillation amplitude, A_1/A, as a function of initial displacement Δx, for the potential $V_0/k_B \simeq$ 270 nK and atom number $N \simeq 3 \times 10^5$.

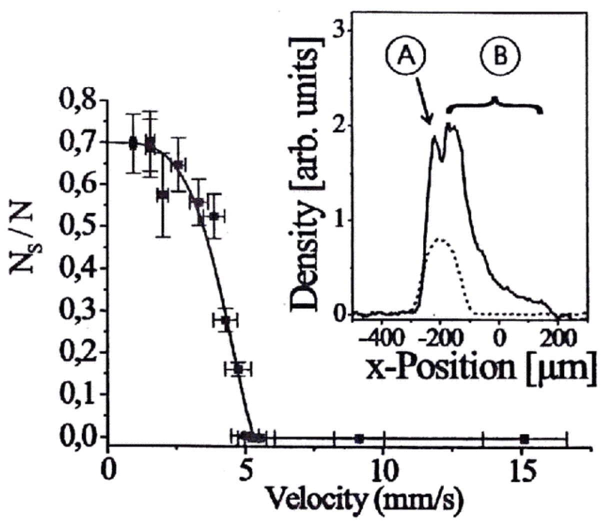


FIG. 4. The fraction of atoms remaining in the undistorted part of the BEC, N_s/N, as a function of the velocity reached during the evolution in the periodic potential.

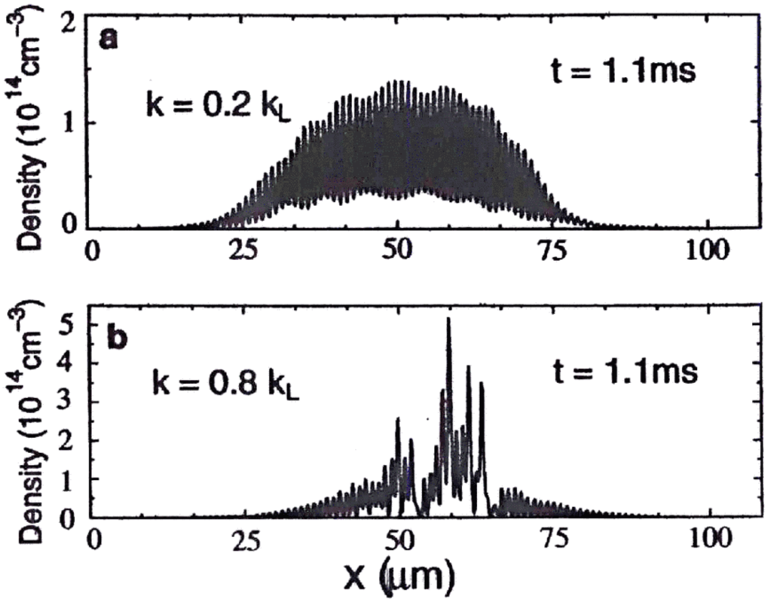


FIG. 1. Inhomogeneous BEC Bloch waves after evolving $t =$ 1.1 ms. (a) Bloch wave number $k = 0.2k_L$; (b) $k = 0.8k_L$. The distorted and fragmented wave function signals the onset of dynamical instability.

B. Wu + Q. Niu, PRL (Comment) 89, 088901 (2002)

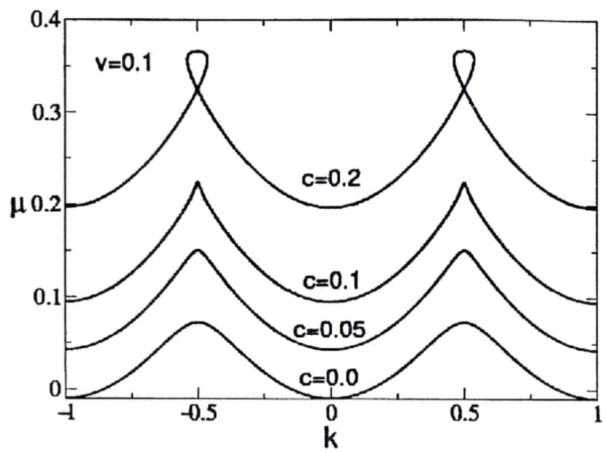


FIG. 1: Lowest Bloch bands at $v = 0.1$ for $c = 0.0$, $c = 0.05$, $c = 0.1$, and $c = 0.2$ (from bottom to top). As c increases, the tip of the Bloch band turns from round to sharp at the critical value $c = v$, followed by the emergence of a loop.

B. Wu + Q. Niu, cond-mat/0306411

Machholm, Pethick + Smith,
PHYSICAL REVIEW A 67, 053613 (2003)

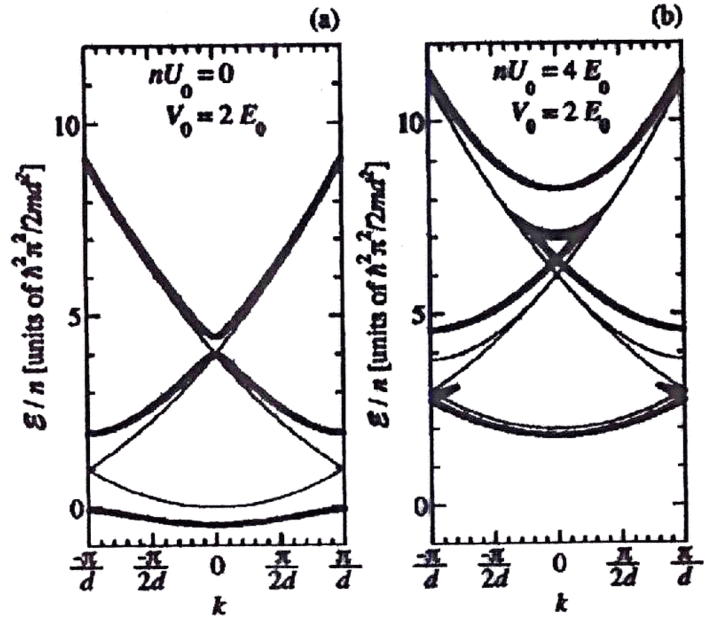


FIG. 3. Energy per particle in the first Brillouin zone as in Fig. 2. The results are obtained by a variational method with the trial function given in Eq. (10). (a) In the absence of interaction the band structure (bold curves) exhibits the usual band gaps at $k=0$ and $k=\pi/d$. The band gap is V_0 at $k=\pi/d$ and $V_0^2/8E_0$ at $k=0$ for small V_0. The thin curves show the energies for $V_0\to 0$, i.e., for the free noninteracting system. (b) In the presence of interaction the swallow tails appear for U_0 larger than a critical value, which depends on V_0 and is different for the two band gaps (bold curves). The thin curves illustrate the limit $V_0\to 0$.

Condensate Dynamics

- at low temperatures, the dynamics of the condensate is governed by the *time-dependent Gross-Pitaevskii (GP) equation*

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + V_{\rm opt} + g|\Psi|^2\right)\Psi$$

  where $g|\Phi|^2$ is a mean-field interaction

- the GP equation provides a description of the possible *collective excitations* of the condensate
- we consider an *extended* condensate in a 3D optical potential: $V_{\rm opt}(\mathbf{r}+\mathbf{R}) = V_{\rm opt}(\mathbf{r})$
- of particular interest are the long wavelength *phonon* excitations of the optical lattice

Stationary States

$$\Psi(\mathbf{r},t) = \Phi_0(\mathbf{r})e^{-i\mu t/\hbar}, \qquad \int_V d^3r|\Phi_0|^2 = N$$

$$-\frac{\hbar^2}{2m}\nabla^2\Phi_0 + V_{\rm opt}\Phi_0 + g|\Phi_0|^2\Phi_0 = \mu\Phi_0$$

- because of lattice periodicity, the GP equation admits Bloch state solutions of the form

$$\Phi_0(\mathbf{r}) = \sqrt{n}e^{i\mathbf{k}\cdot\mathbf{r}}w(\mathbf{r})$$

  where $w(\mathbf{r}+\mathbf{R}) = w(\mathbf{r})$

- for $\mathbf{k} \neq 0$, the Bloch state is an excited state in which the condensate has the superfluid current density

$$\mathbf{j}_s(\mathbf{r}) = \frac{\hbar}{2mi}\left[\Phi_0^*\nabla\Phi_0 - (\nabla\Phi_0)^*\Phi_0\right]$$

- average current density

$$\langle \mathbf{j}_s \rangle = \frac{1}{V} \int d^3r\, \mathbf{j}_s(\mathbf{r}) = \bar{n}\mathbf{v}$$

where $\mathbf{v} = \nabla_{\mathbf{k}} \tilde{\epsilon}(\bar{n}, \mathbf{k})/\hbar$

- $\tilde{\epsilon}(\bar{n}, \mathbf{k}) \equiv E_{\rm tot}/N$ is the energy per particle:

$$E_{\rm tot} = \int_V d^3r \Phi_0^* \left(-\frac{\hbar^2 \nabla^2}{2m} + V_{\rm opt} \right) \Phi_0 + \frac{g}{2} \int_V d^3r |\Phi_0|^4$$

- the chemical potential for the current-carrying state is

$$\mu(\bar{n}, \mathbf{k}) = \frac{\partial E_{\rm tot}}{\partial N} = \frac{\partial}{\partial \bar{n}} (\bar{n}\tilde{\epsilon})$$

with

$$\tilde{\epsilon}(\bar{n}, \mathbf{k}) = \mu(\bar{n}, \mathbf{k}) - \frac{g}{2N} \int_V d^3r\, |\Phi_0|^4$$

Collective Excitations

- the condensate supports small amplitude oscillations about the stationary state:

$$\Psi(\mathbf{r}, t) = [\Phi_0(\mathbf{r}) + \delta\Phi(\mathbf{r}, t)] e^{-i\mu t/\hbar}$$

- the fluctuation $\delta\Phi(\mathbf{r}, t)$ is a solution of the linearized TDGP equation and has the form

$$\delta\Phi(\mathbf{r}, t) = u(\mathbf{r}) e^{-iEt/\hbar} - v^*(\mathbf{r}) e^{iEt/\hbar}$$

- the quasiparticle amplitudes (u, v) and the quasiparticle energy E are determined by the Bogoliubov equations

$$\begin{aligned} \hat{L}u(\mathbf{r}) - g\Phi_0^2(\mathbf{r})v(\mathbf{r}) &= Eu(\mathbf{r}) \\ \hat{L}v(\mathbf{r}) - g\Phi_0^{*2}(\mathbf{r})u(\mathbf{r}) &= -Ev(\mathbf{r}) \end{aligned}$$

where

$$\hat{L} \equiv -\frac{\hbar^2 \nabla^2}{2m} + V_{\rm opt} + 2g|\Phi_0|^2 - \mu$$

- the Bogoliubov equations admit Bloch-like solutions

$$u(\mathbf{r}) = e^{i(\mathbf{q}+\mathbf{k})\cdot\mathbf{r}}\bar{u}(\mathbf{r})$$
$$v(\mathbf{r}) = e^{i(\mathbf{q}-\mathbf{k})\cdot\mathbf{r}}\bar{v}(\mathbf{r})$$

where $\bar{u}(\mathbf{r}+\mathbf{R}) = \bar{u}(\mathbf{r})$, etc. $\mathbf{k}$ is the Bloch wave vector of the stationary state, while $\mathbf{q}$ is the Bloch wave vector of the excitation.

- the excitations form *bands*, labelled by the band index m and the wave vector $\mathbf{q}$
- of particular interest is the lowest band $(m = 0)$ in the long wavelength limit $(\mathbf{q} \to 0)$
- solutions of the Bogoliubov equations can be developed by means of a systematic expansion in q

Phonon Excitations $(\mathbf{k} = 0)$

- for the lowest band,

$$E(q) = \hbar s q + \cdots$$

- the sound speed is given by

$$s = \sqrt{\frac{\bar{n}}{m_0}\frac{\partial \mu_0}{\partial \bar{n}}}$$

- μ_0 is the GP eigenvalue of

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V_{\rm opt} + g\bar{n}w_0^2\right) w_0 = \mu_0 w_0$$

- w_0 and μ_0 depend parametrically on $\bar{n}$, the mean density

- the effective mass m_0 of the lowest band is determined by the equation

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V_{\rm opt} + g\bar{n}w_0^2\right)\phi_{\bf q} = \varepsilon_{\bf q}\phi_{\bf q}$$

where

$$\phi_{\bf q} = e^{i{\bf q}\cdot{\bf r}}w_{\bf q}, \qquad \varepsilon_{\bf q} = \mu_0 + \frac{\hbar^2 q^2}{2m_0} + \cdots$$

- NB: one does *not* have to solve the GP equation self-consistently to determine $\varepsilon_{\bf q}$ and hence m_0

- m_0 increases with the strength of the lattice potential and the sound speed decreases from the uniform gas limit $s_0 = \sqrt{g\bar{n}/m}$

Phonon Excitations $({\bf k} \neq 0)$

- for the case of a current-carrying condensate

$$E({\bf q}) = e_{,\bar{n}i}q_i + \sqrt{e_{,\bar{n}\bar{n}}e_{,ij}q_iq_j}$$

- $e(\bar{n},{\bf k}) \equiv \bar{n}\tilde{\epsilon}(\bar{n},{\bf k})$ is the mean energy density $E_{\rm tot}/V$ and

$$e_{,\bar{n}\bar{n}} = \frac{\partial^2 e}{\partial\bar{n}^2}, \qquad e_{,\bar{n}i} = \frac{\partial^2 e}{\partial\bar{n}\partial k_i}, \qquad e_{,ij} = \frac{\partial^2 e}{\partial k_i\partial k_j}$$

- the above result can be derived from the pair of hydrodynamic equations (*Machholm et al., 2003*)

$$\frac{\partial\hbar{\bf k}}{\partial t} = -\nabla\mu, \qquad \frac{\partial\bar{n}}{\partial t} + \nabla\cdot{\bf j}_s = 0$$

with

$$\mu = \frac{\partial e}{\partial\bar{n}}, \qquad {\bf j}_s = \frac{1}{\hbar}\nabla_{\bf k}e$$

- for small $\mathbf{k}$,

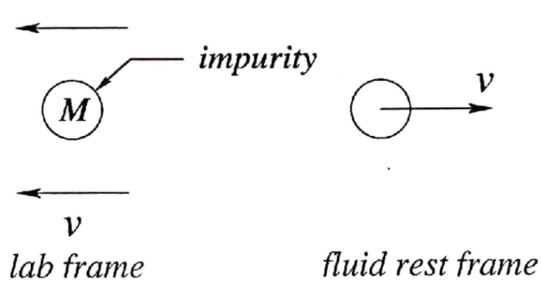
$$E(\mathbf{q}) = \hbar^2 \mathbf{k} \cdot \mathbf{q} \frac{\partial}{\partial \bar{n}} \left(\frac{\bar{n}}{m_0} \right) + \hbar s q$$

- for $V_{\rm opt} \to 0$, $\mathbf{q} = \pm q \hat{\mathbf{k}}$,

$$E(q) = \hbar (s_0 \pm v) q$$

where $v = \hbar k / m$

- if $v > s_0$, E becomes negative for $\mathbf{q} = -q\hat{\mathbf{k}}$; this signals an *energetic instability* given by the *Landau criterion*



Energetic Instability: Landau Criterion

- the GP Φ_0 state is a stationary point of the functional

$$G[\Phi] = E[\Phi] - \mu N$$

where $E[\Phi]$ is the GP energy

- for the variation $\Phi_0 \to \Phi_0 + \delta\phi$

$$\delta G = \frac{1}{2} \int d^3 r \delta \Phi^\dagger \hat{A} \delta \Phi$$

where

$$\delta \Phi = \begin{pmatrix} \delta\phi \\ \delta\phi^* \end{pmatrix}, \qquad \hat{A} = \begin{pmatrix} \hat{L} & g\Phi_0^2 \\ g\Phi_0^{*2} & \hat{L} \end{pmatrix}$$

- if the operator $\hat{A}$ has negative eigenvalues, then the state Φ_0 is energetically unstable; for a homogeneous system, a zero eigenvalue occurs when $\mathbf{v} \cdot \mathbf{q} = \pm s_0 q$, the *Landau criterion* for the spontaneous emission of phonons

Energetic Instability

$$\delta G = \frac{1}{2}\int d^3r\, \delta\Phi^{+} \hat{A}\, \delta\Phi$$

$$\hat{A} = \begin{pmatrix} \hat{\mathcal{L}} & g\Phi_0^2 \\ g\Phi_0^{*2} & \hat{\mathcal{L}} \end{pmatrix}$$

$$\hat{A}\,\Phi_\lambda = \lambda\,\Phi_\lambda$$

$\lambda > 0 \Rightarrow G$ is a minimum

$\lambda = 0$ signals an energetic instability

Bogoliubov Equations

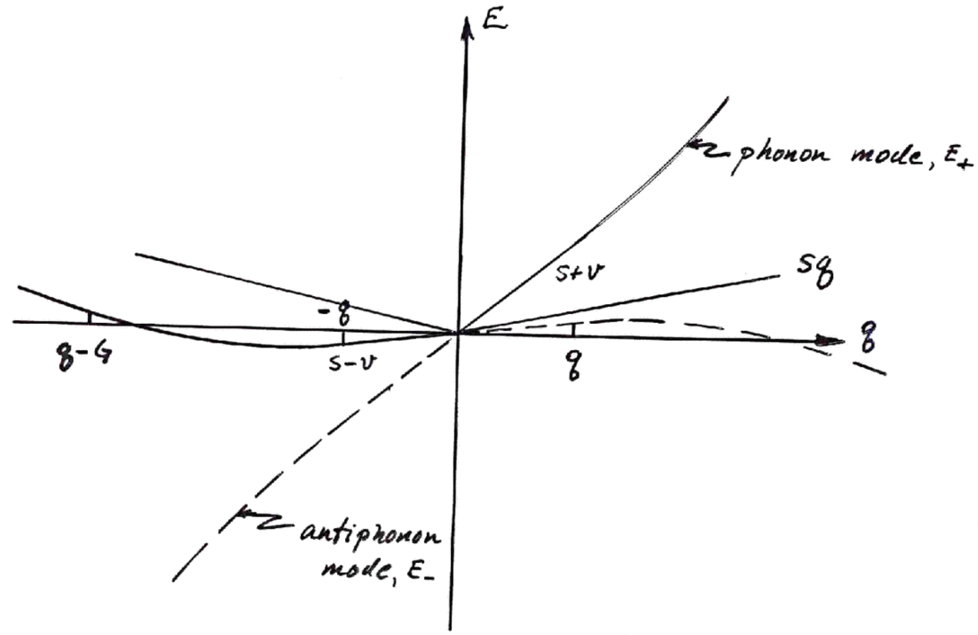
$$\begin{pmatrix} \hat{\mathcal{L}} & g\Phi_0^2 \\ g\Phi_0^{*2} & \hat{\mathcal{L}} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = E\,\sigma_3 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\hat{\sigma}_3 \hat{A} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

$\lambda = 0$ eigenvector of $\hat{A}$ is also a solution of this equation with $E = 0$.

$\lambda < 0 \Rightarrow$ a Bogoliubov eigenvalue becomes negative

Dynamic Instabilities in the Weak Potential Limit



$$E_\pm(q) = \frac{\hbar^2 k q}{m} \pm \sqrt{\frac{g\bar{n}\hbar^2 q^2}{m} + \frac{\hbar^4 q^4}{4m^2}}$$

$$E_+(-q) = -E_-(q)$$

Phonon-antiphonon resonance: $E_-(q) = E_+(q-G)$

OR

Two-phonon emission: $E_+(-q) + E_+(q-G) = 0$

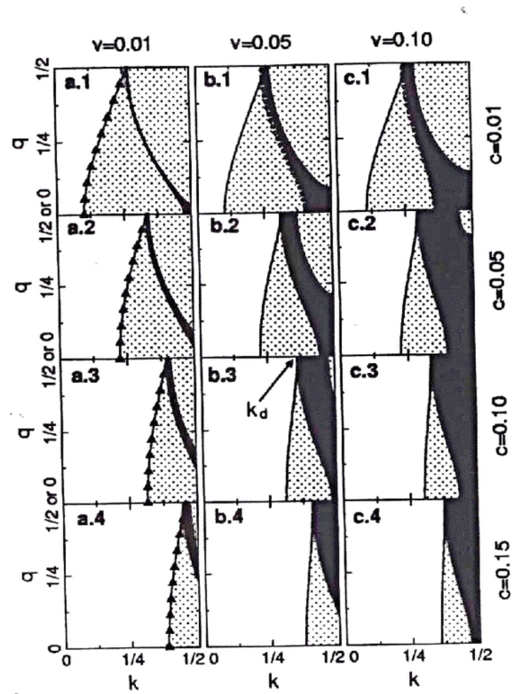


FIG. 5: Stability phase diagram of BEC Bloch waves. k is the wave number of BEC Bloch waves; q denotes the wave number of perturbation modes. In the shaded (light or dark) area, the perturbation mode has negative excitation energy; in the dark shaded area, the mode grows or decays exponentially in time. The triangles in (a.1-a.4) represent the boundary, $q^2/4 + c = k^2$, of saddle point regions at $v = 0$. The solid dots in the first column are from the analytical results of Eq.(5.13). The circles in (b.1) and (c.1) are based on the analytical expression (5.14). The dashed lines indicate the most unstable modes for each Bloch wave ϕ_k.

B. Wu + Q. Niu, Phys. Rev. A64, 061603 (2001)