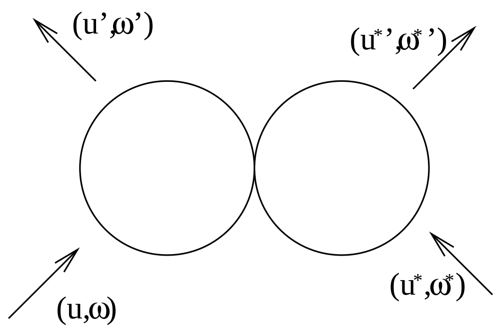
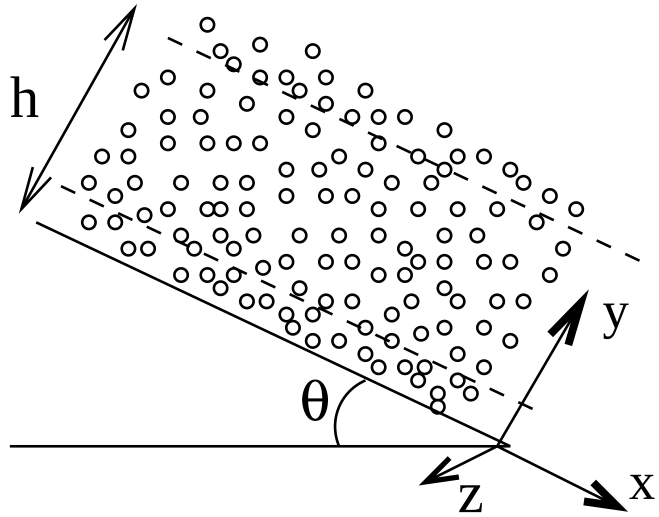
Dense granular flows:
from particle dynamics to hydrodynamics.



$\Rightarrow$



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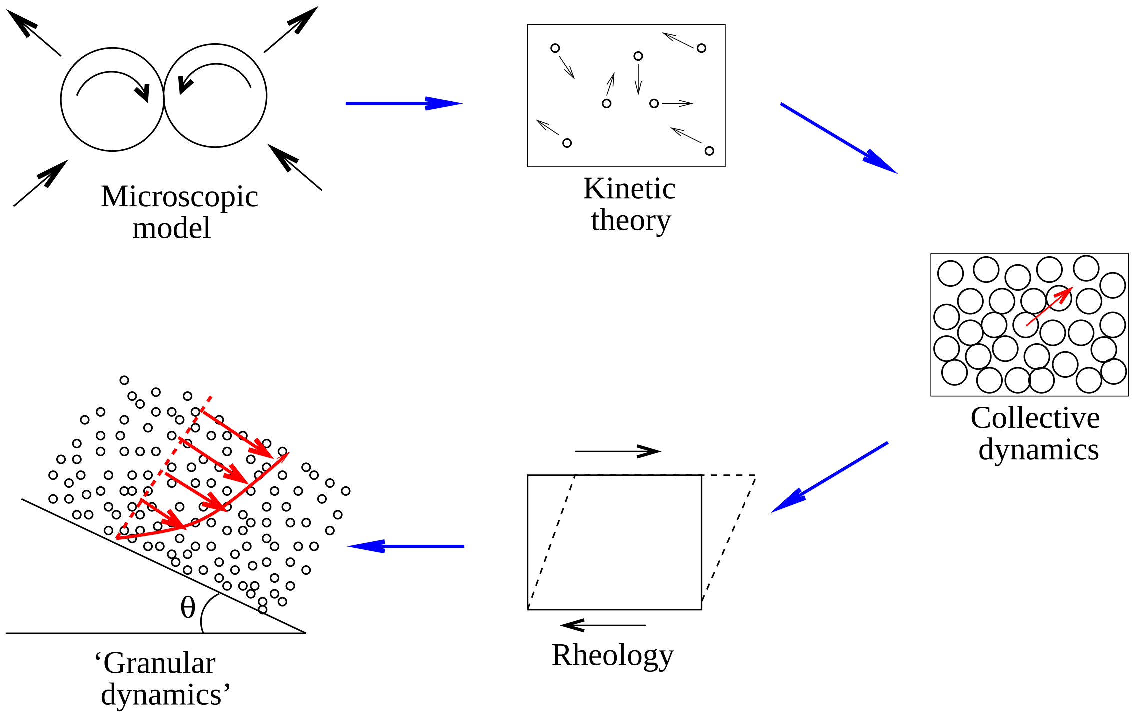
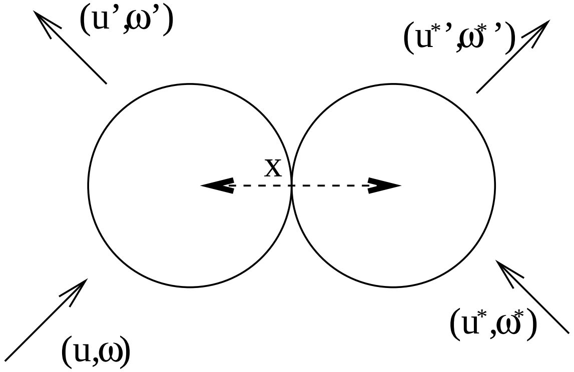
Granular materials:

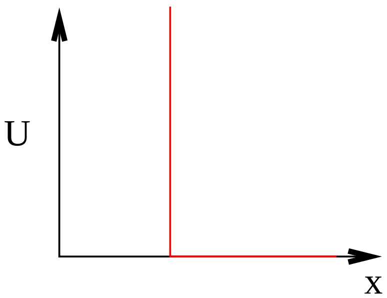
- Thermal velocity $(3k_BT/m)^{1/2} \sim 7.5 \times 10^{-12} m/s$
- Fluid forces negligible.
- Energy dissipation due to particle interactions.
- Steady flow requires energy input to 'fluidise' the particles.
- Energy input from boundaries or through distributed forcing (mean shear).

Outline:

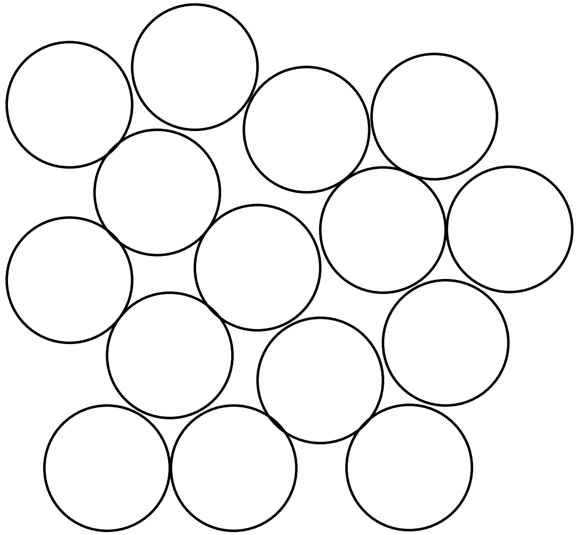

Microscopic model
Kinetic theory
Collective dynamics
Rheology
'Granular dynamics'
θ

The hard-particle model:



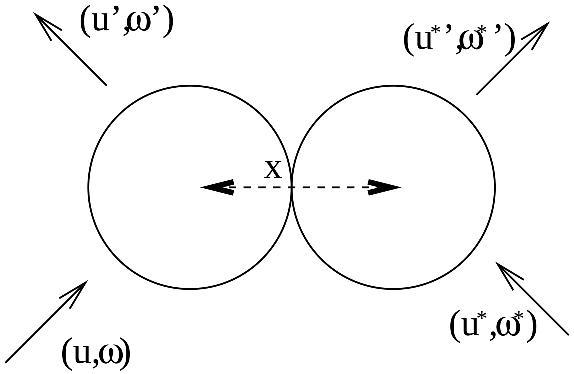


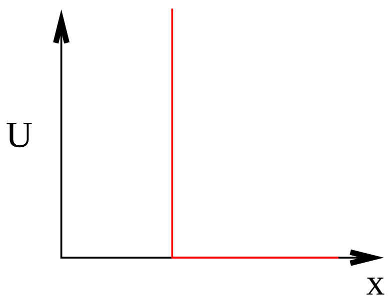
Static state:



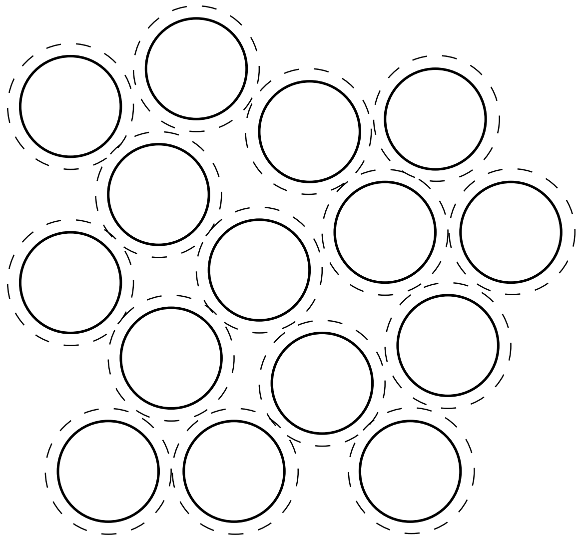
Isostatic packings: (D+1) contacts per particle for frictional packings. (Apart from rattlers).

The hard-particle model:





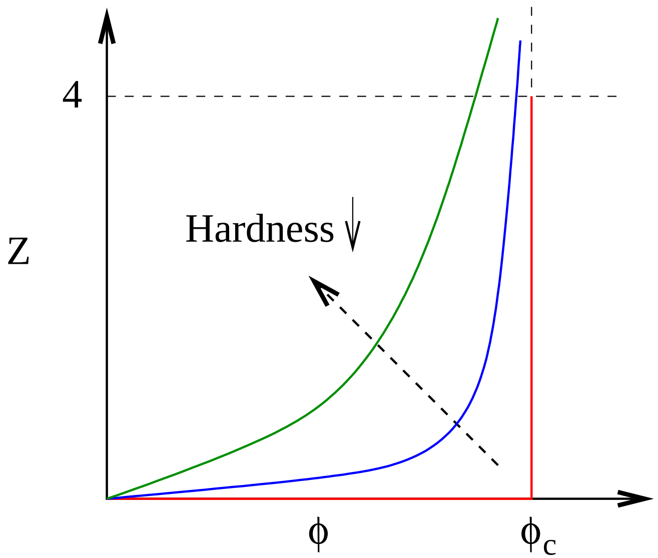
Decrease volume fraction little bit:
(Decrease particles size a little at constant volume).



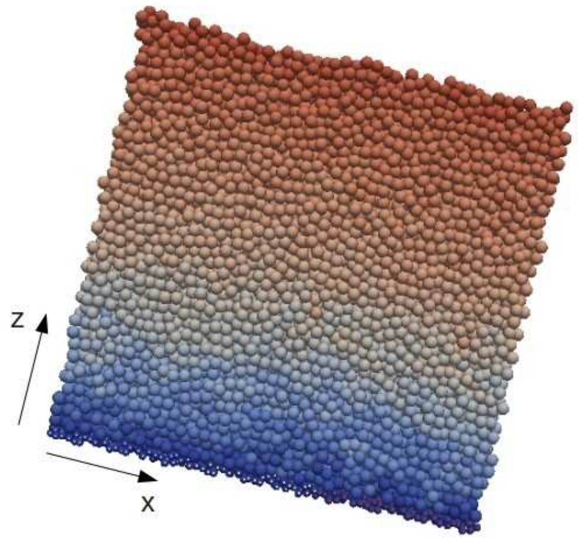
Duration of interaction $\tau_c = 0$.

Flowing hard particles:Realistic?

- Collisions instantaneous.
- Av. co-ordination number $Z \ll 1$.



Inclined plane flow

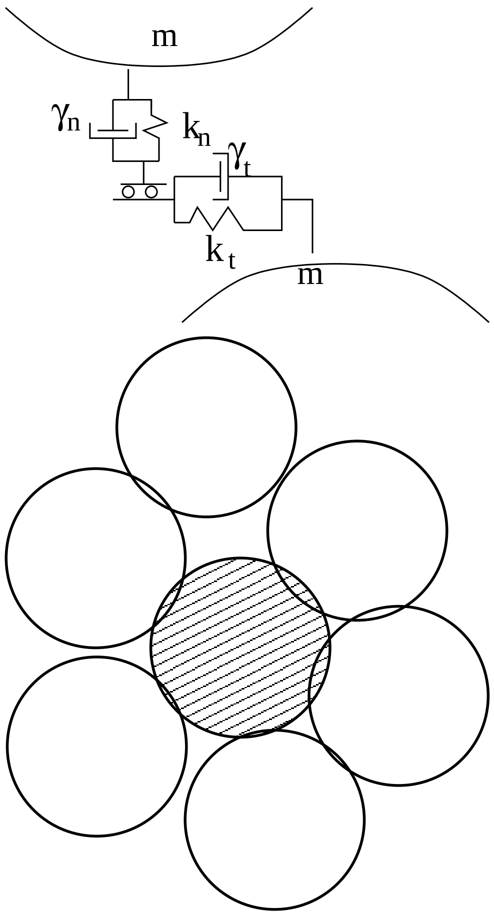


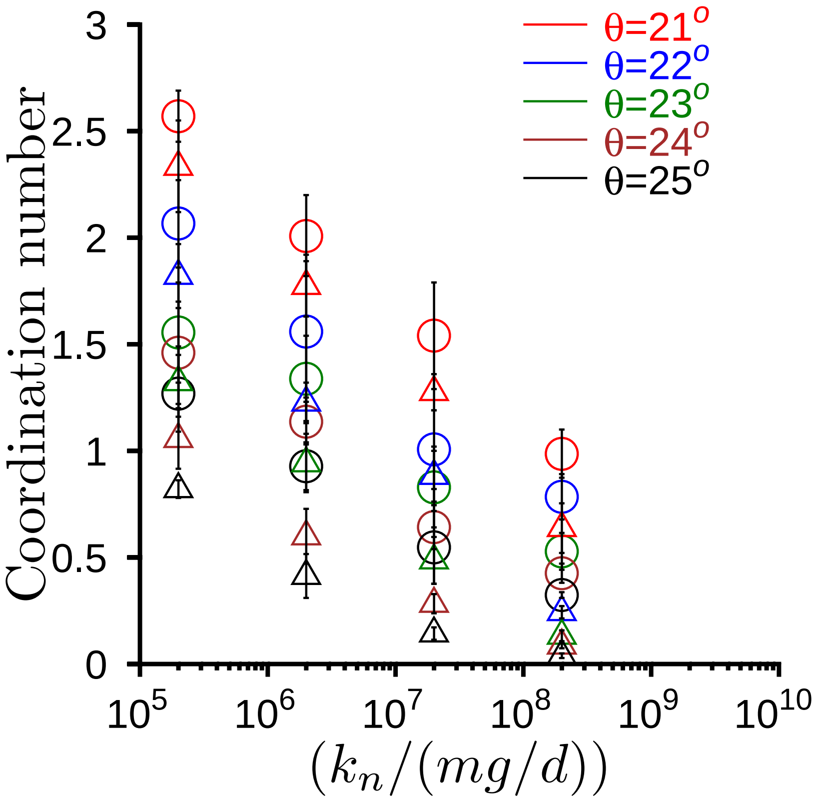
Flow starts at $\theta = 21^o$,
Stable flow till $\theta = 25^o$.

Is it useful? Qualitatively? Quantitatively?

Flowing hard particles: Realistic?

Average co-ordination number $e_t = e_n = 0.9$:



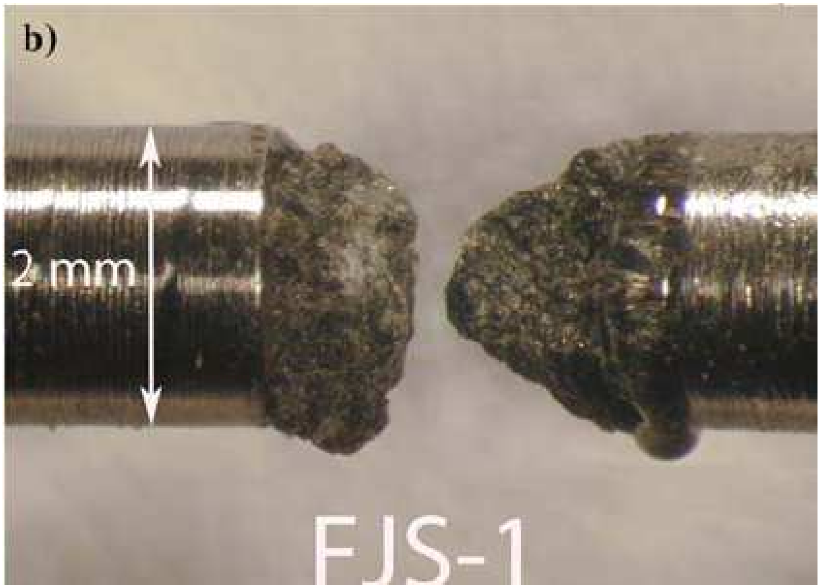


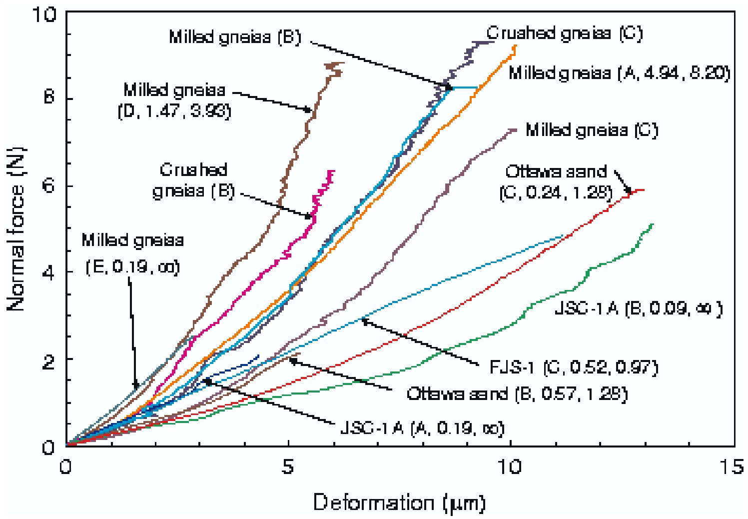
$\circ e_n = 0.5$; $\triangle e_n = 0.9$.

Flowing hard particles: Realistic?

Contacts between real particles: Experiments.

Cole & Peters (*Gran. Matt.* **10**, 171, 2008; **9**, 309, 2007).





Rough sand particles linear contact law due to asperities,
$d = 0.2 - 2mm$ $k_n = 10^6 N/m$.

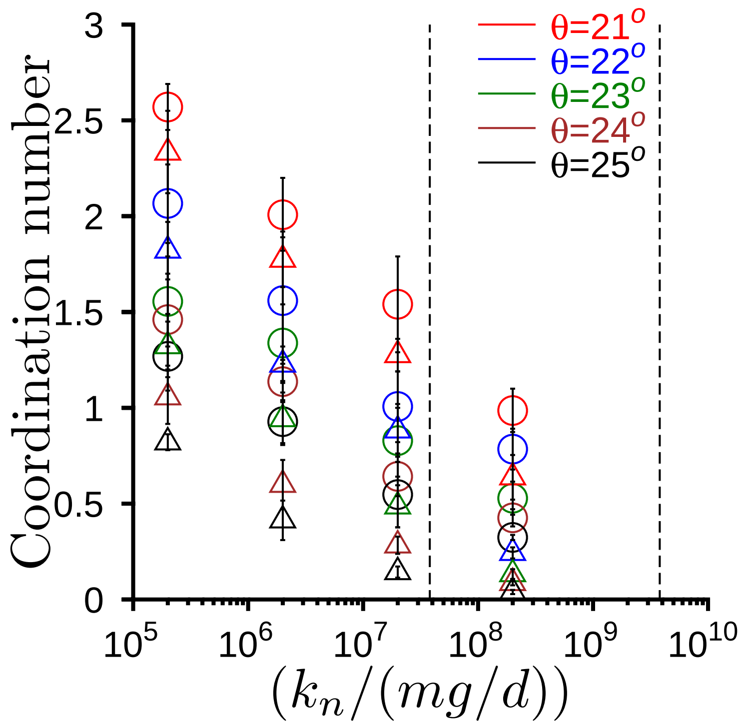
Smooth particles — Hertzian contact law,

$k_n \approx 0.8$ times Mindlin-Deresiewicz prediction, $k_n \sim Ed^{1/2}$.

Flowing hard particles: Contact model.

Rough particles:

- Linear contact law
  $F = k_n \delta$
  due to compression of asperities.
- $k_n = 10^6 N/m$ for particles in 0.2-2mm size.
- Scaled spring constant
  $k_n/(mg/d^{3/2}) \sim 7.6 \times 10^7 - 7.6 \times 10^9$.
  for $d = 1 - 0.1mm$.

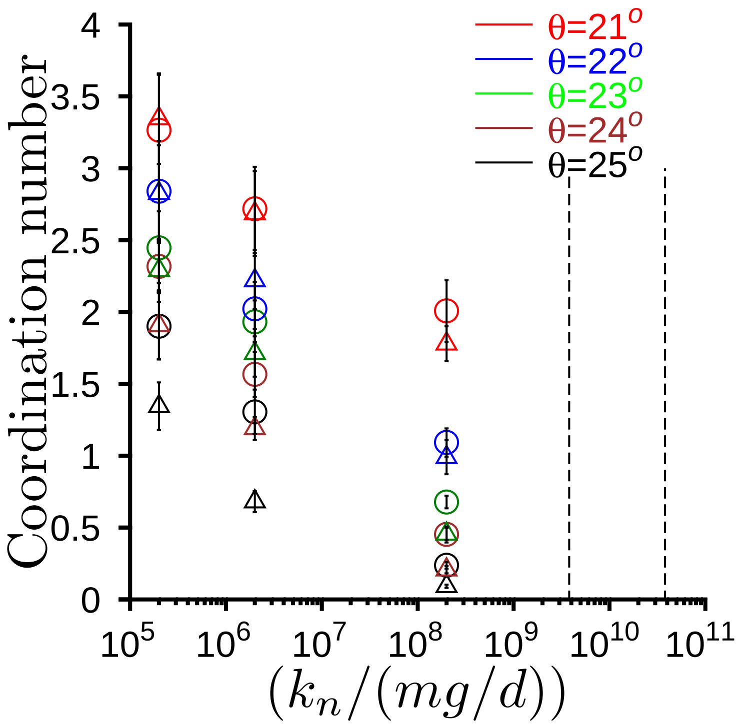


$\circ e_n = 0.9$; $\triangle e_n = 0.5$.

Flowing hard particles: Contact model.

Smooth particles:

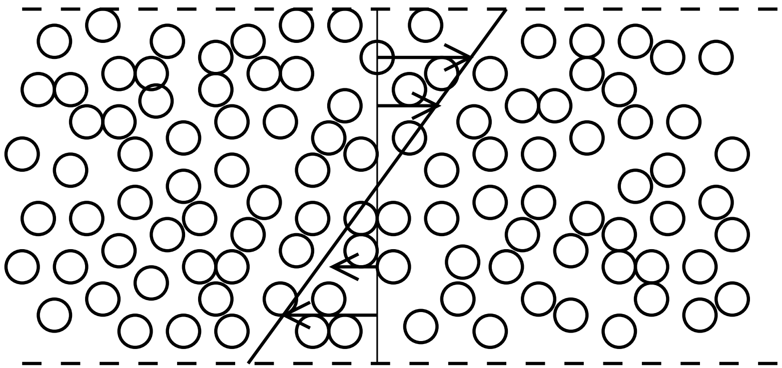
- Hertzian contact law
  $F = k_n \delta^{3/2}$
- Value of k_n from dimensional analysis.
  $k_n \sim E d^{1/2}$.
- Sand, glass, $E \sim 10^{11} N/m^2$.
- Hertzian spring constant $k_n = 10^7 - 10^8 N/m^{3/2}$.
- Scaled spring constant
  $k_n/(mg/d^{1/2}) \sim 7.6 \times 10^9 - 7.6 \times 10^{10}$
  for $d = 1 - 0.1mm$.



$\circ$ — $\gamma_n = 375, 12000.$

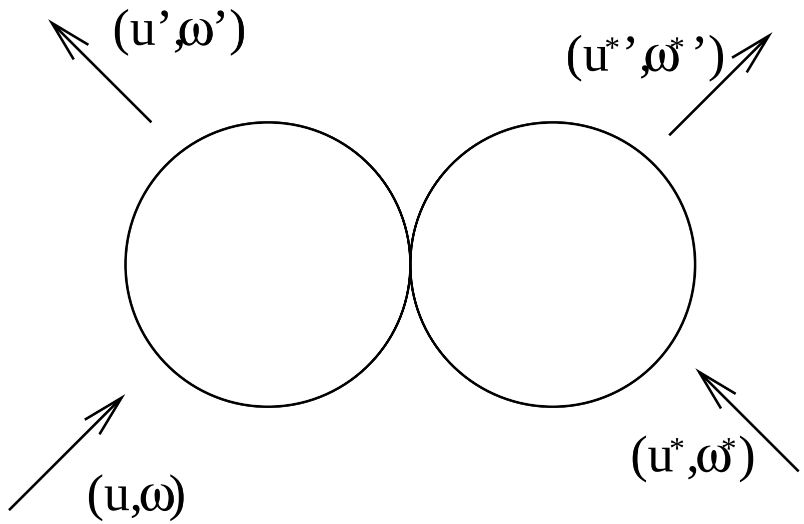
$\triangle$ — $\gamma_n = 55, 1850.$

Hard-particle model:



Dimensional variables

- Particle mass m,
- Particle diameter d,
- Strain rate G_{xy}.

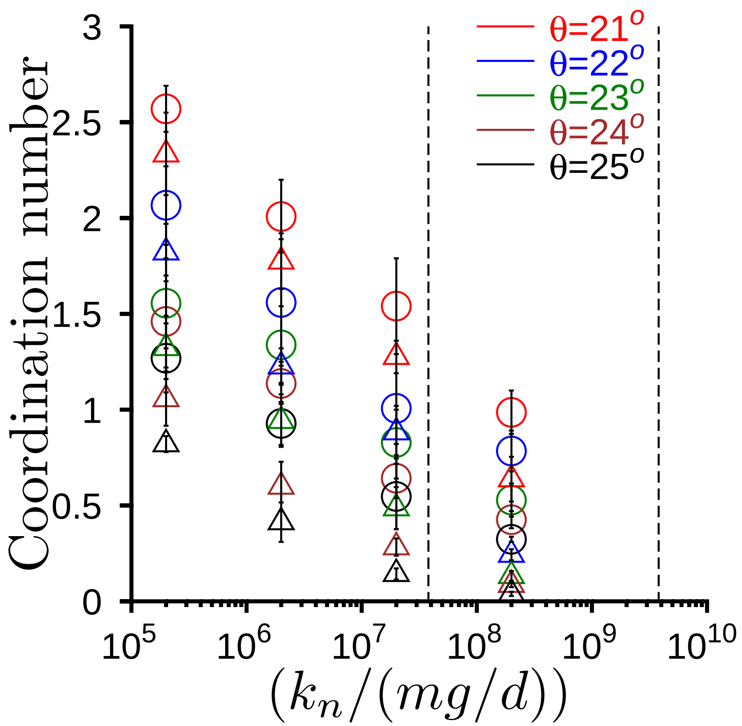


Dimensionless variables

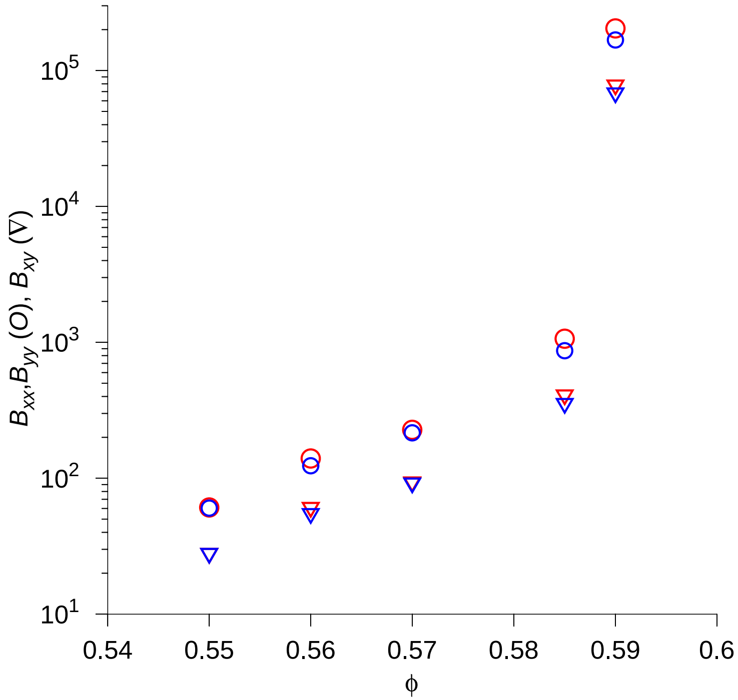
- Volume fraction ϕ,
- Coefficients of restitution e_n, e_t.

Constitutive relation $\sigma_{ij} = G_{xy}^2 B_{ij}(\phi, e)$.

Flowing hard particles: Contact regime.

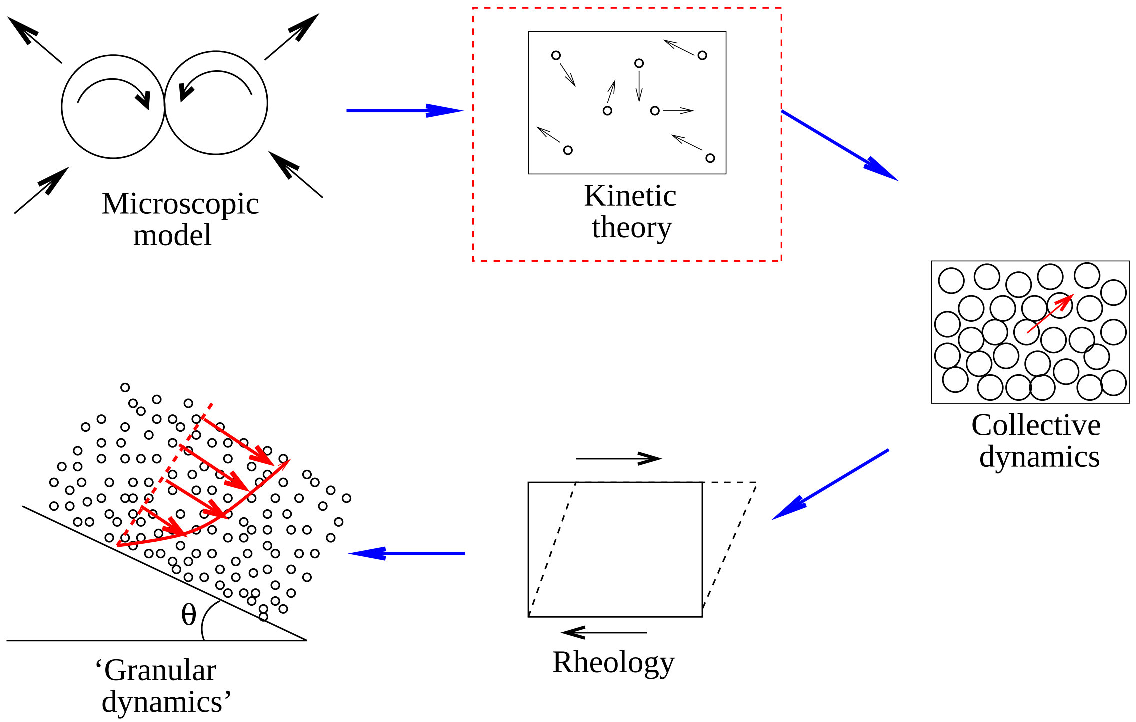


$\circ e_n = 0.9$; $\triangle e_n = 0.5$.



$(k_n/(mg/d)) = 2 \times 10^5$,
$(k_n/(mg/d)) = 2 \times 10^8$.

Statistical mechanics of dense granular flows:



Conservation equations:

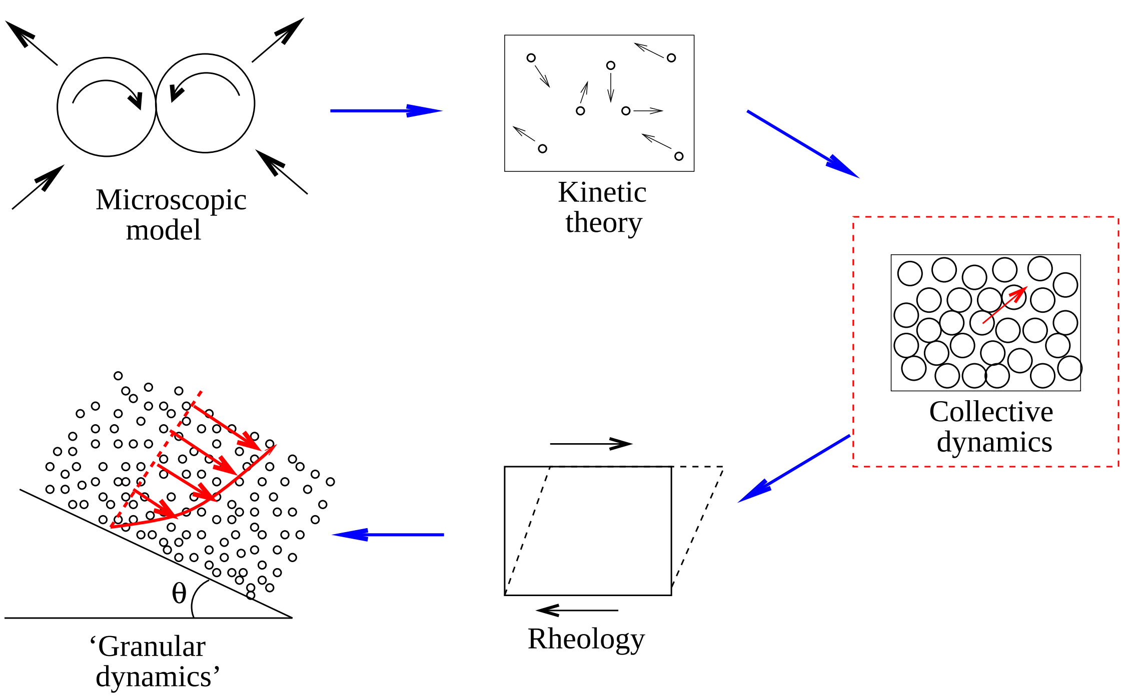
$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} \right) = -\nabla.\sigma$$

$$\rho C_v \left(\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T \right) = -\nabla.\mathbf{q} + \sigma :(\nabla \mathbf{u}) - D$$

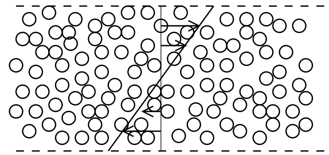
$$\sigma = -p_\phi T \mathbf{I} + \mu_\phi T^{1/2} \mathbf{S} + B_1 \mathbf{S}.\mathbf{S} + B_2(\mathbf{S}.\mathbf{A} - \mathbf{A}.\mathbf{S}) + B_3 \mathbf{A}.\mathbf{A}$$

Statistical mechanics of dense granular flows:

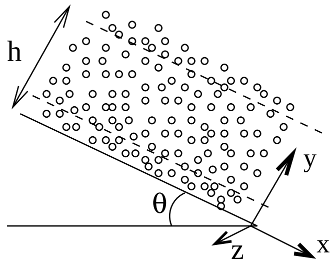


Linear shear:

(Lees-Edwards)

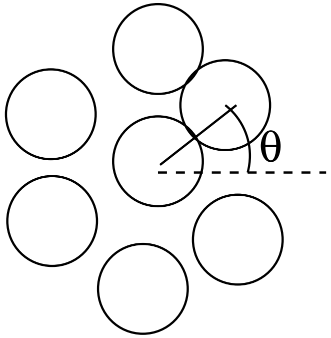


Surface flow:



Sheared inelastic hard-particles: an unusual fluid.

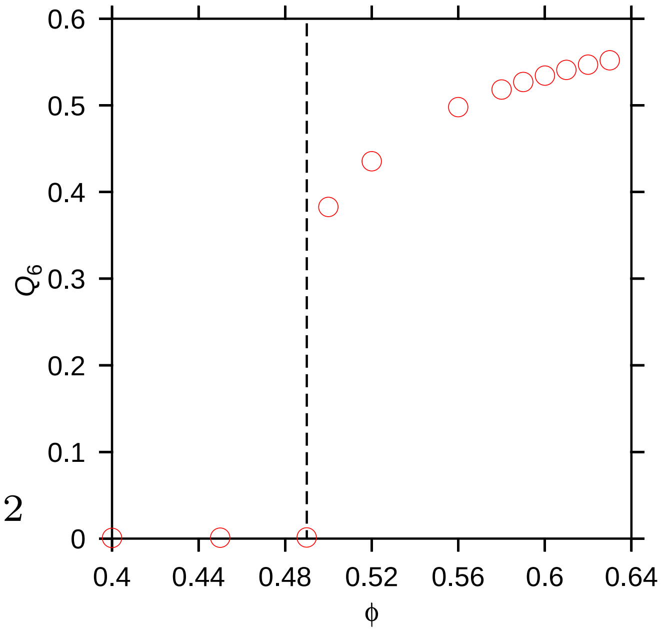
Structure:



- 2D: $q_6 = \sum_{i=1}^{N} \exp(6\imath\theta)$
  $q_6 = 1$ for hexagonal packing.
- 3D Icosahedral order parameter:

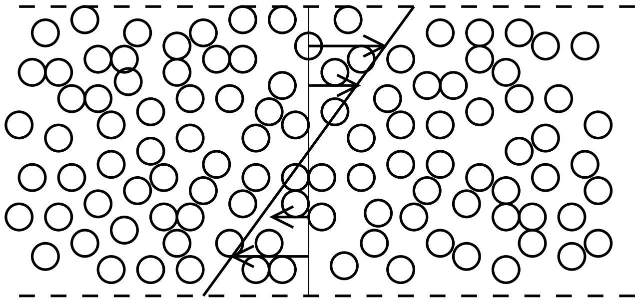
$$Q_l = \left(\frac{2l+1}{4\pi} \sum_{m=-l}^{l} |\langle Y_{lm}(\theta,\phi)\rangle|^2 \right)^{1/2}$$

$Q_6 = 0.6$ for FCC/HCP.



Structure:

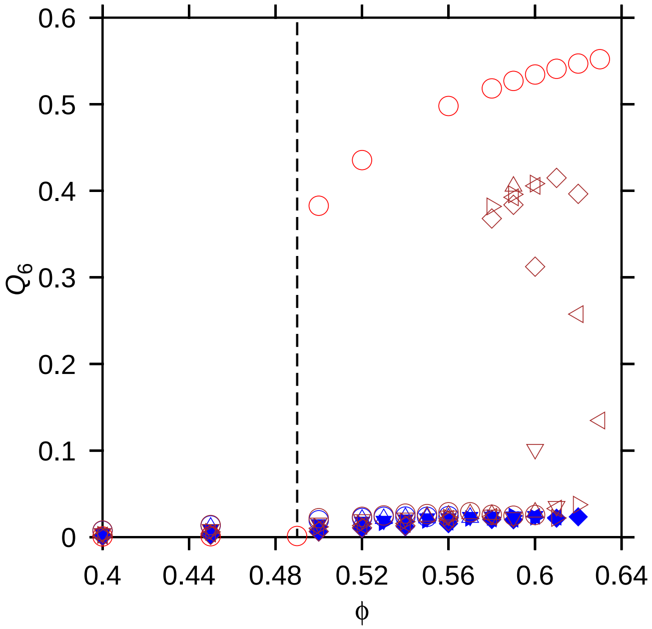
Sheared inelastic fluid:



Balance between shear production and inelastic dissipation.

$$\mu\dot{\gamma}^2 = D$$

$$T = (d\dot{\gamma})^2 F(\phi, e_n, e_t)$$

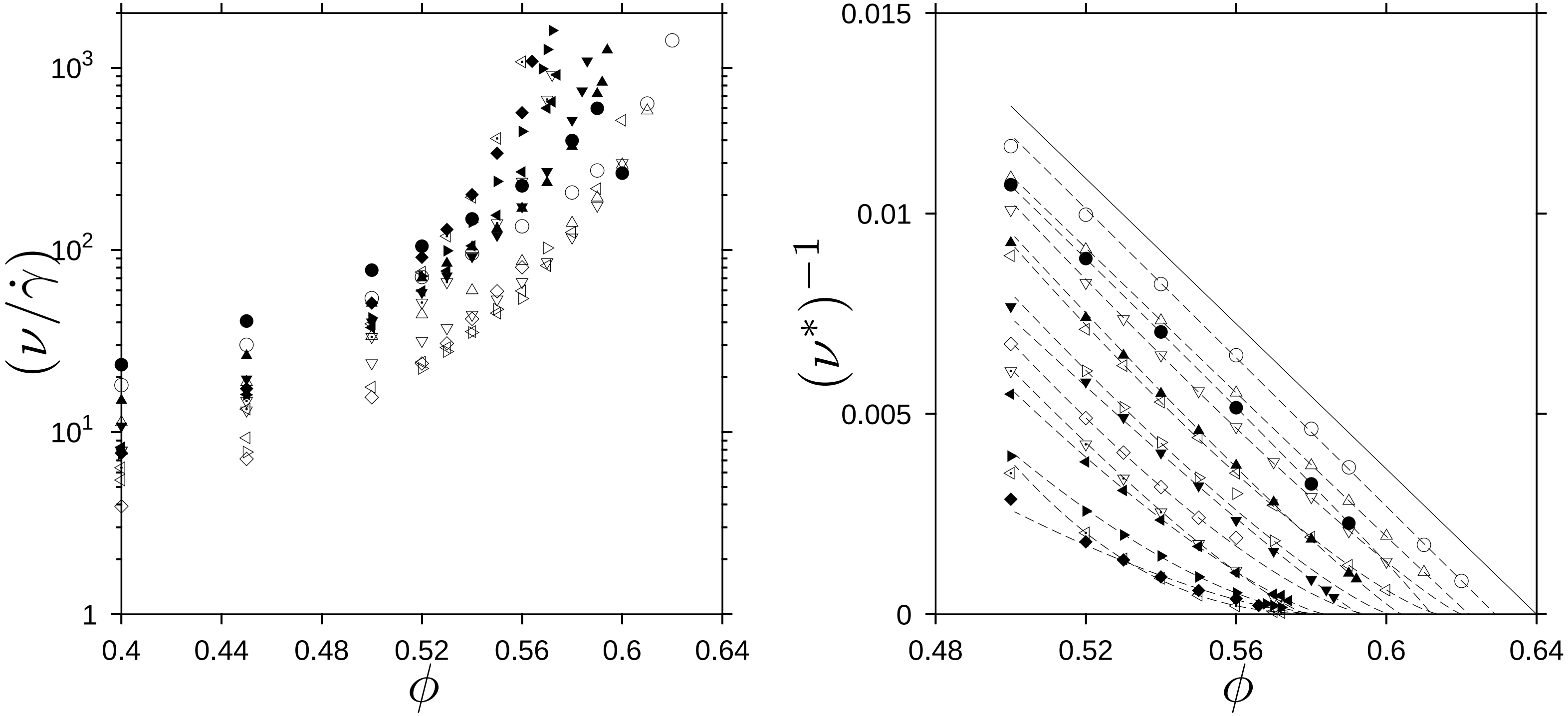


∇ $e_n = 0.8$; $\triangleright$ $e_n = 0.9$; $\triangleleft$ $e_n = 0.95$; $\diamond$ $e_n = 0.98$; Box $e_n = 1.0$.

System size: $n = 256$, $n = 500$.

Collective effect:

Collision freq. diverges at lower volume fraction than RCP (0.64).



$e_n = 0.98$ ($\circ$), $e_n = 0.95$ ($\triangle$), $e_n = 0.9$ (∇), $e_n = 0.8$ ($\triangleleft$), $e_n = 0.7$ ($\triangleright$), $e_n = 0.6$ ($\diamond$),

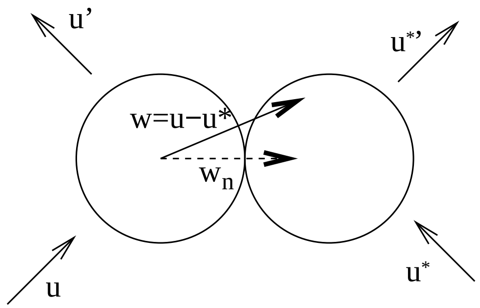
Smooth particles (open symbols), rough particles (filled symbols).

Need to redefine ϕ_{RCP} for a sheared state.

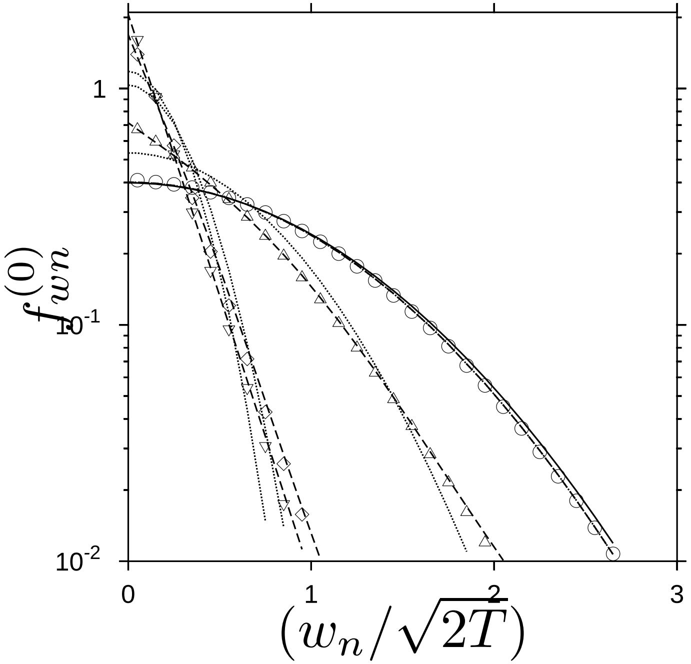
Important correlation effect

Distribution of relative velocities $w = u - u^*$

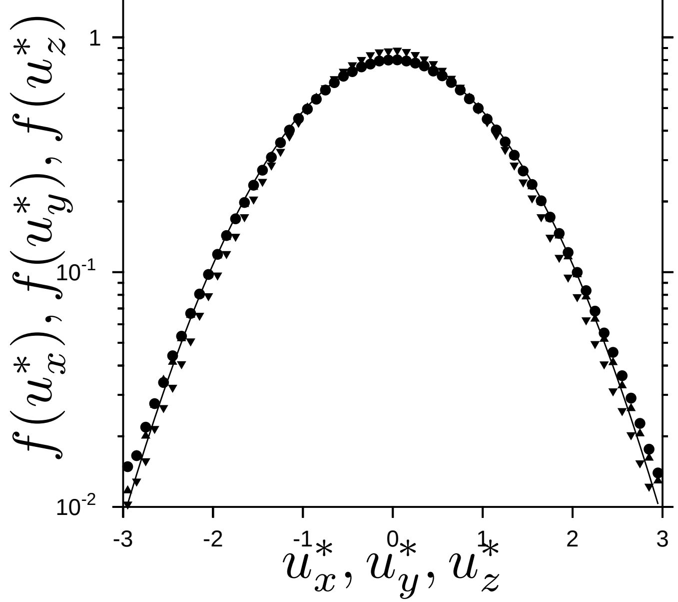
Not a Gaussian!



Relative velocity distribution:



Single-particle distribution



Rough particles, $\phi = 0.56$, $e_n = 0.98$ ($\circ$), $e_n = 0.8$ ($\triangle$), $e_n = 0.6$ (Box).

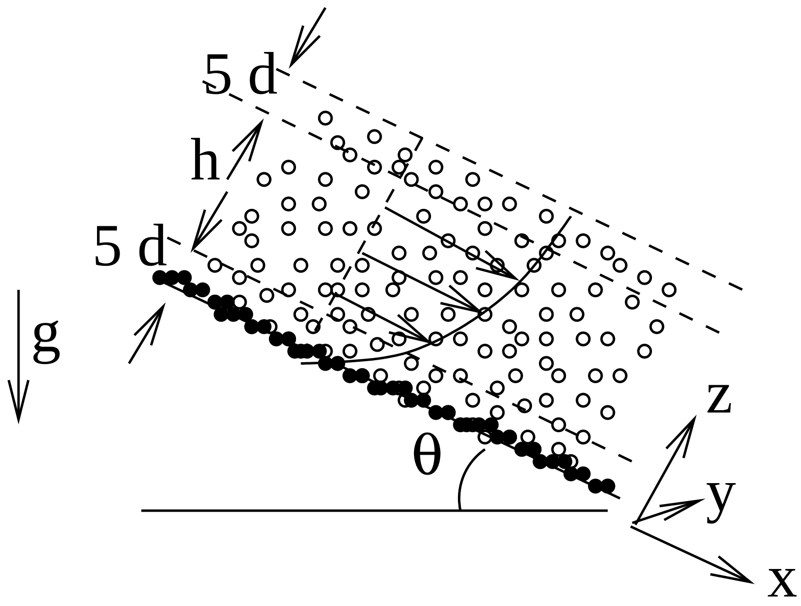
Effect of correlations:

Dynamical arrest in sheared hard-particle fluids.

- No crystallisation transition in the presence of shear!
- Volume fraction $\phi_{da} < \phi_{rcp}$, function of e.
- Motion diffusive, fast decay of autocorrelation function.
- Strong correlation effect on relative velocity distribution of colliding particles.

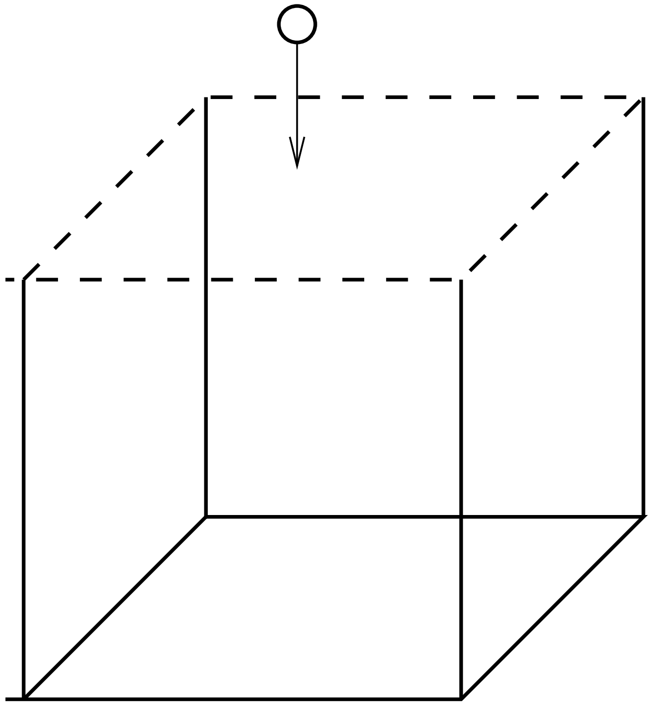
Flow down inclined plane: Configuration.

- Simulation box $x : y = 40d : 20d$; $80d : 40d$; $160d : 80d$.
- Periodic streamwise (x) and spanwise (y).
- Bottom rough base, top free surface.
- Two system sizes $32000, 64000$ particles ($\approx 35d, 70d$ height)
- Random base configuration.



Flow down inclined plane: Initial state.

- Vertical box of desired base dimensions.
- Sequentially drop particles into box to desired height.



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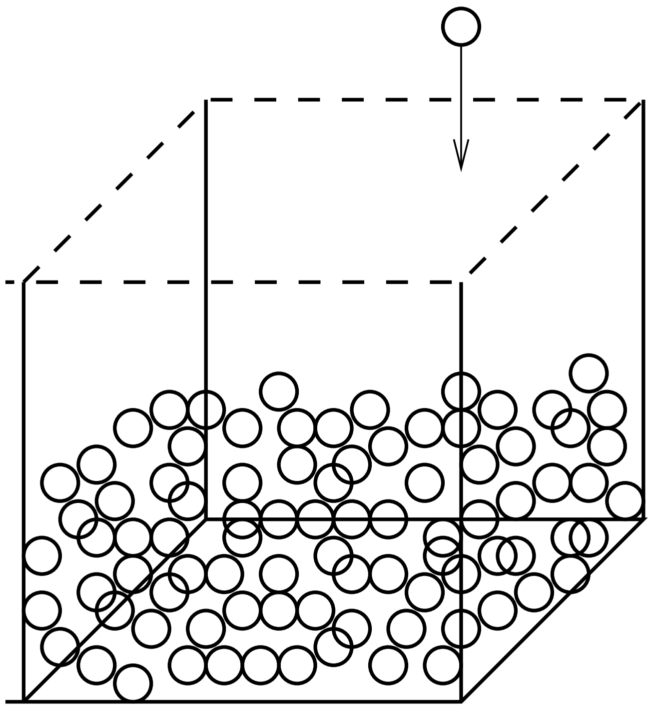
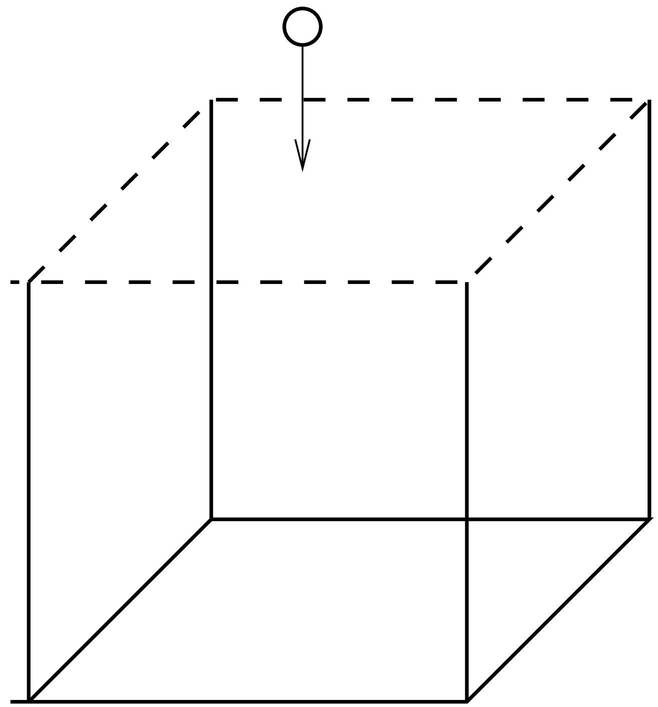


Flow down inclined plane: Base particle roughness.

- Vertical box of desired base dimensions.
- Sequentially drop particles into box.
- Select a layer of particles with centers within height d_b.
- Use this as frozen base.
- Vary ratio of diameters of base and moving particles, (d_b/d_f).



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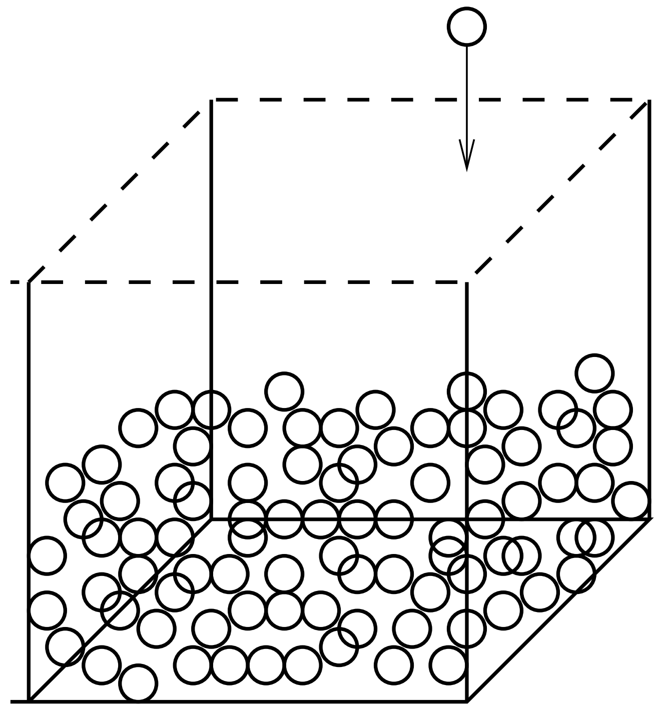

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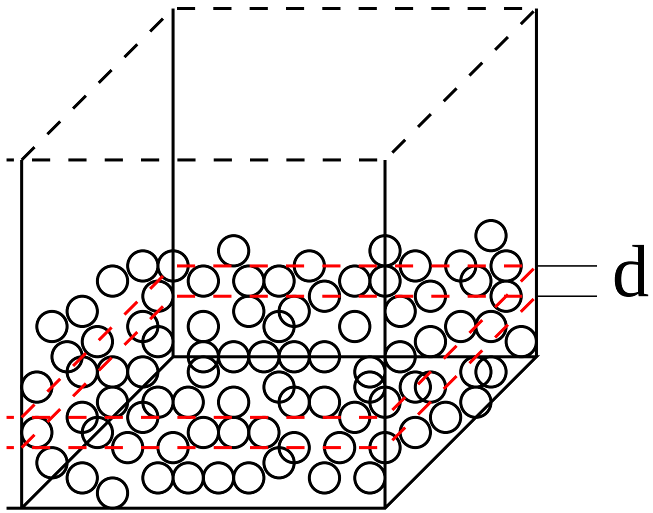

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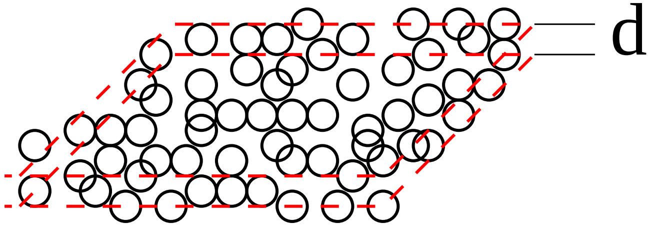

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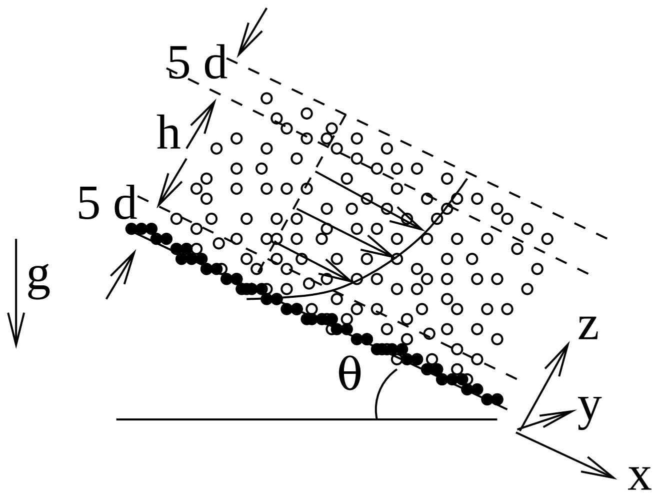
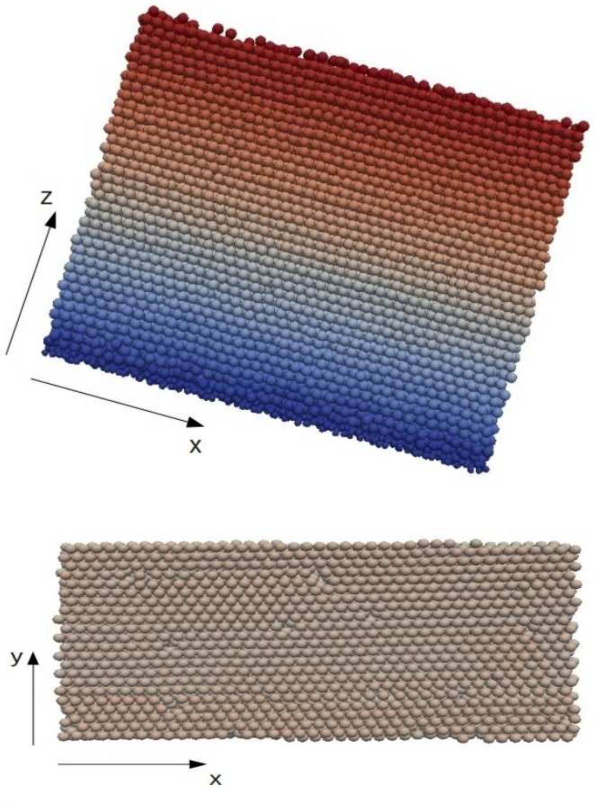
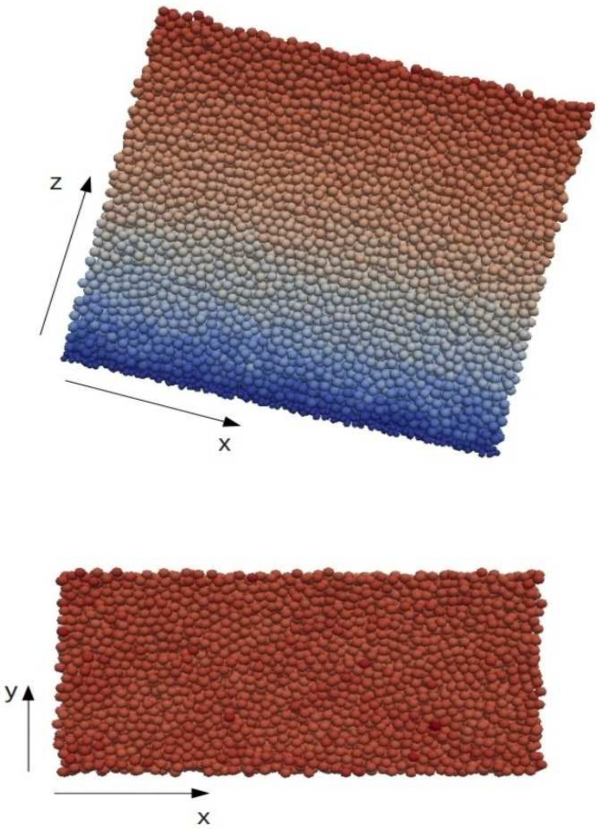


Flow down inclined plane: Flow regimes.



$(d_b/d_f) = 0.61$



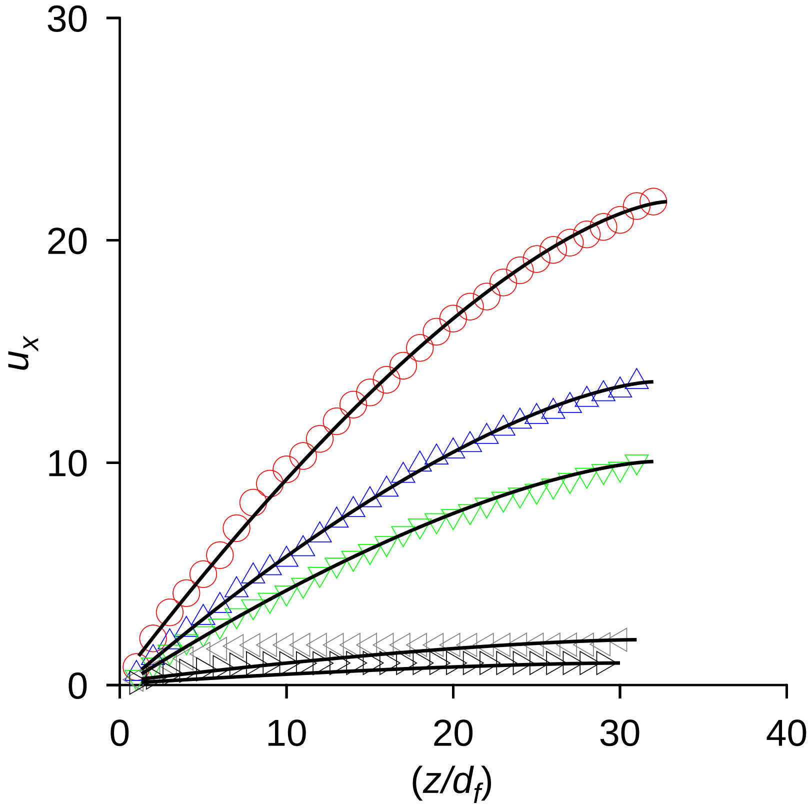
$(d_b/d_f) = 0.62$

Flow down inclined plane: Order

Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous

Flow down inclined plane: Inclination for flow

Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous



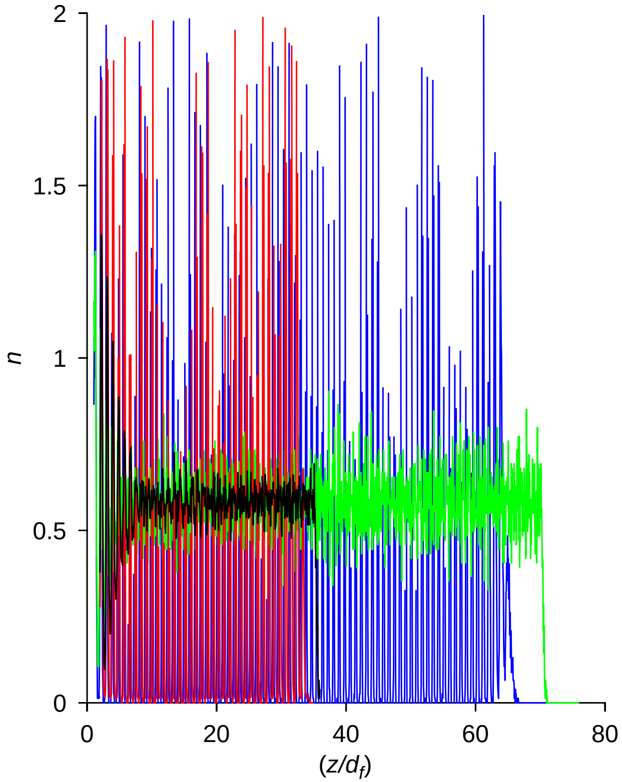
$(d_b/d) = 0.5$

$\theta = 18^o, 16^o, 15^o, 14^o, 13^o.$

Flow down inclined plane: Layering

Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous

$n(y)A\Delta y$ = (# of particles with centers in Δy).

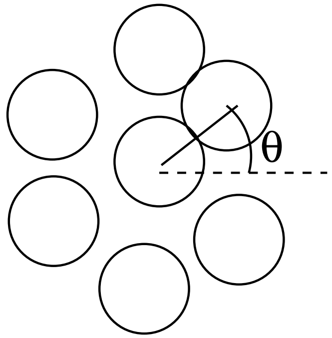


$(d_b/d) = 0.61$, (32000); 0.62, (32000)

$(d_b/d) = 0.62$, (64000) 0.63, 64000

Flow down inclined plane: In-layer order

Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous



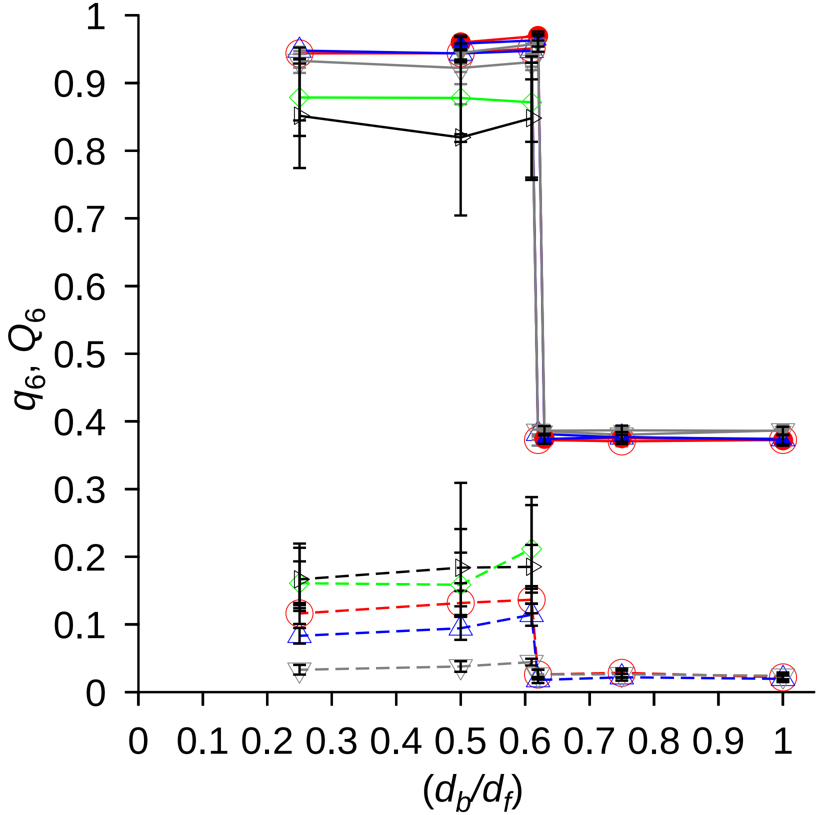
- 2D: $q_6 = \sum_{i=1}^{N} \exp\left(6\imath\theta\right)$
- $q_6 = 1$ hexagonal packing.
- 3D Icosahedral order parameter:

$$Q_l = \left(\frac{2l+1}{4\pi} \sum_{m=-l}^{l} |\langle Y_{lm}(\theta, \phi) \rangle|^2 \right)^{1/2}$$

- $Q_6 = 0.6$ for FCC/HCP.

Flow down inclined plane: In-layer order

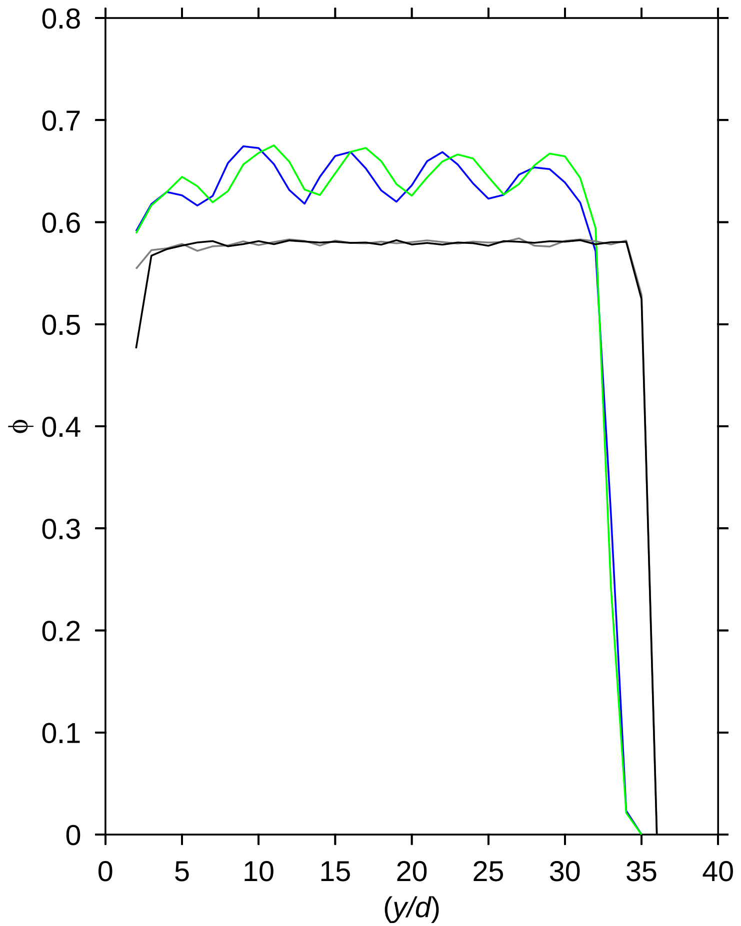
Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous



Angle $= 15^o$, 18^o, 20^o, 22^o, 25^o.

Flow down inclined plane: Volume fraction

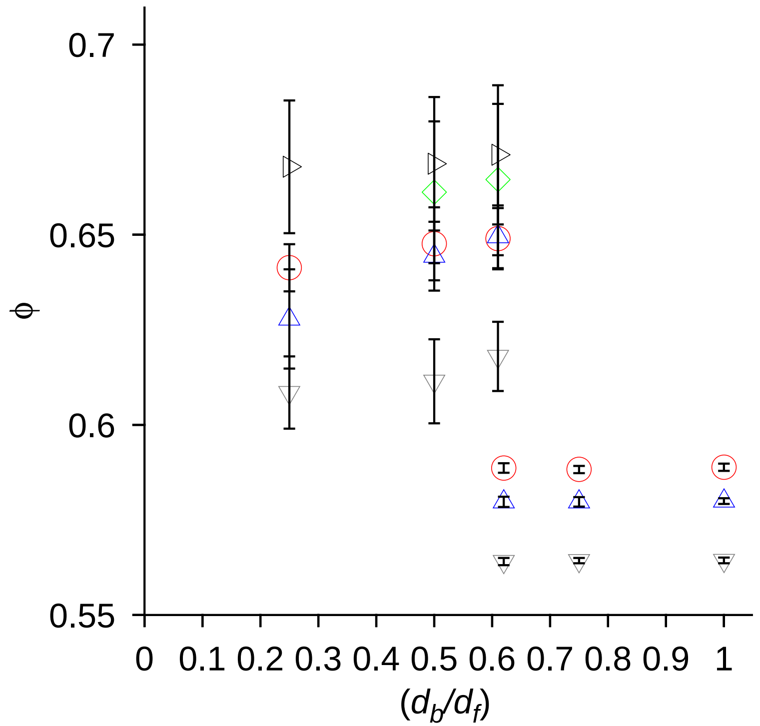
Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous



Angle $= 22^o$ $h \approx 35$,
$(d_b/d) = 0.5, 0.61, 0.62, 1.00.$

Flow down inclined plane: Volume fraction

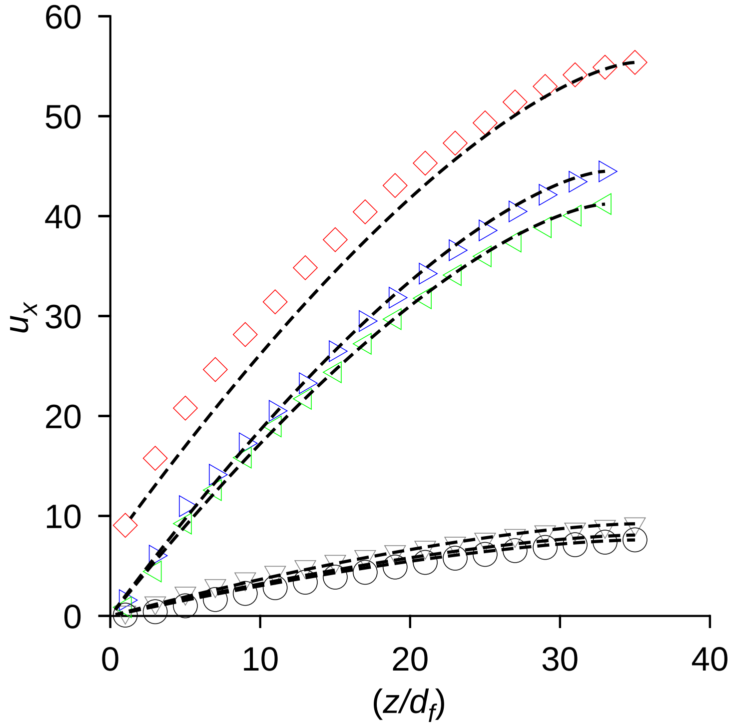
Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous



Angle $= 15^o, 18^o, 20^o, 22^o, 25^o$.

Flow down inclined plane: Bagnold law

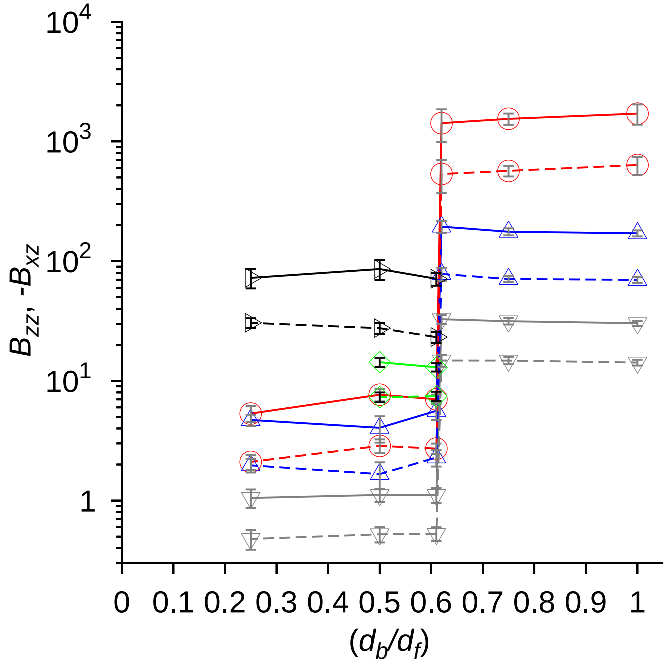
Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous



$(d_b/d) = 0.25, 0.5, 0.61, 0.62, 1.00.$

Flow down inclined plane: Bagnold coefficients

Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous



B_{yy} —, $-B_{xy}$ - - -

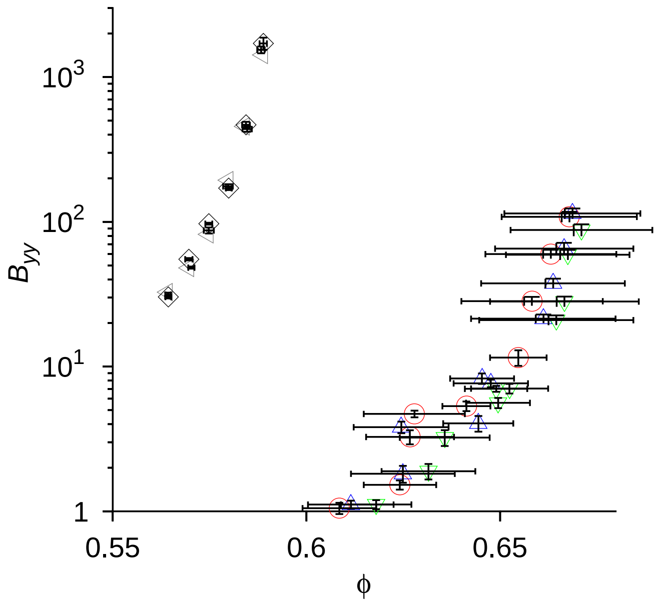
Angle $= 15^o, 18^o, 20^o, 22^o, 25^o$.

0.61, 0.62, 1.00.

Flow down inclined plane: Bagnold coefficients

Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous

Bagnold coeff vs. ϕ

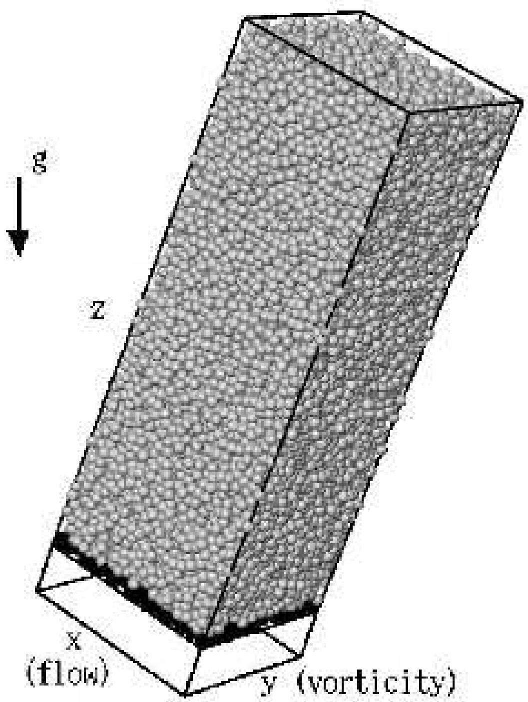


$(d_b/d) = 0.25, 0.5, 0.61, 0.62, 1.00.$

Flow down inclined plane: Unsteady flow

Ordered	Disordered
$(d_b/d) < d_c$	$(d_b/d) > d_c$
Flow $\theta \geq 14^o$	Flow $\theta \geq 21^o$
Layering	No layering
In-layer order	No order
More dense	Less dense
Bagnold law valid	
Faster	Slower
Polydispersity	
Unsteady flow	
Plug	Homogeneous

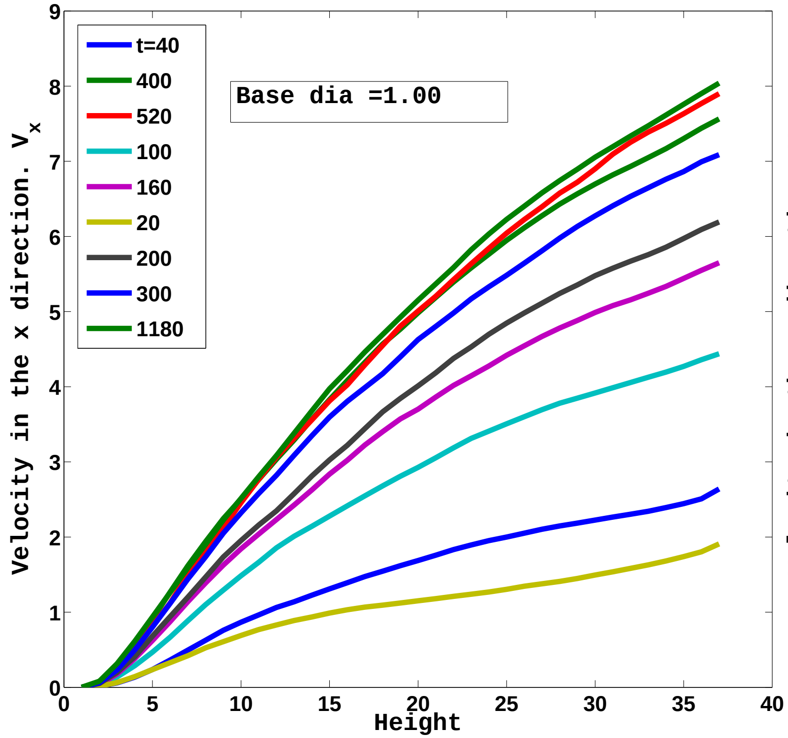
Velocity evolution
(Zero initial velocity)

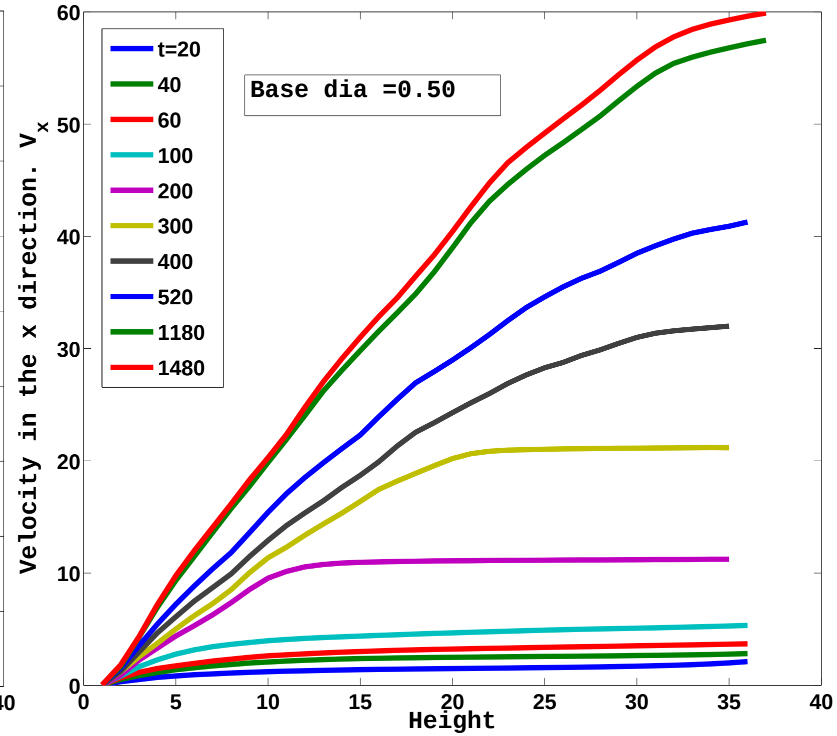


Flow down inclined plane: Unsteady flow

Disordered flow

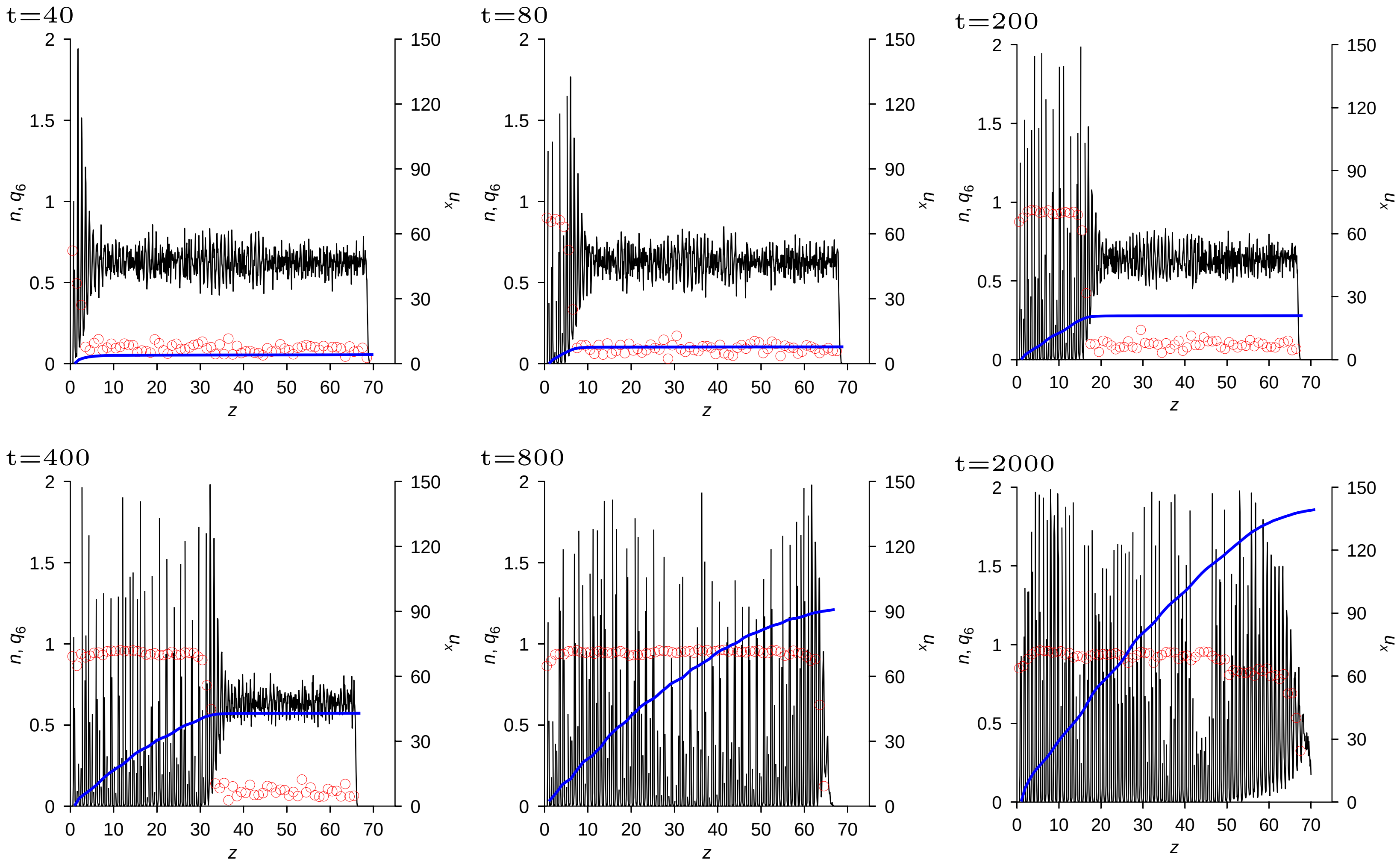
Ordered flow



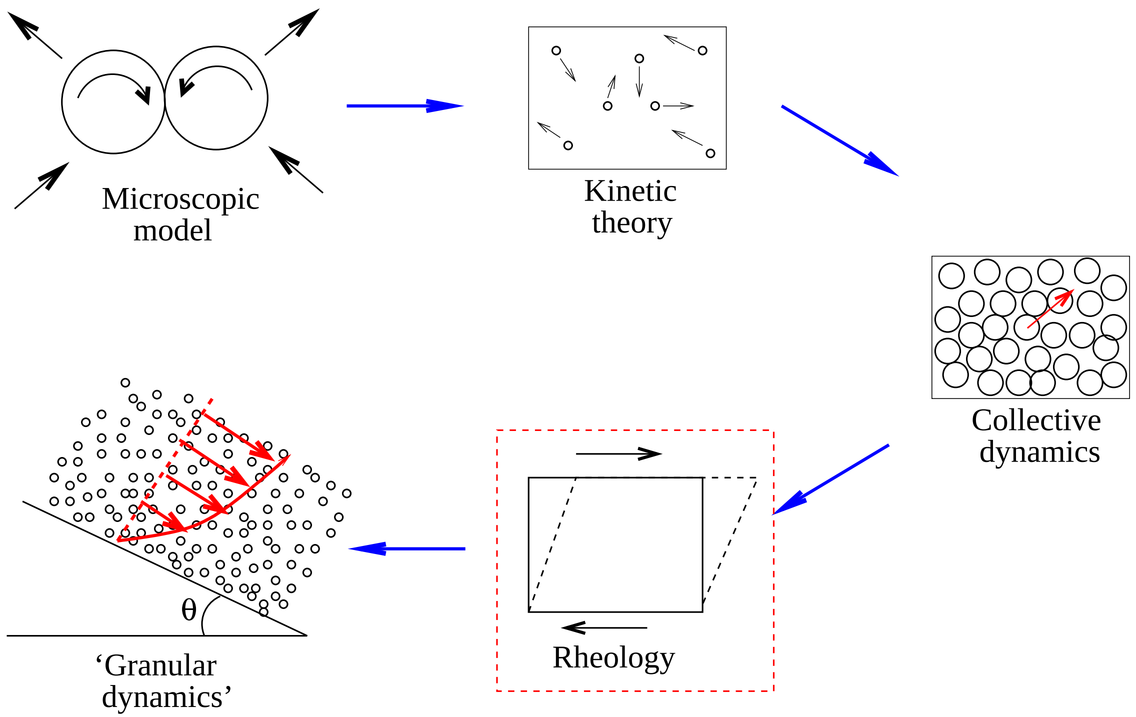


Angle $= 22^o$.

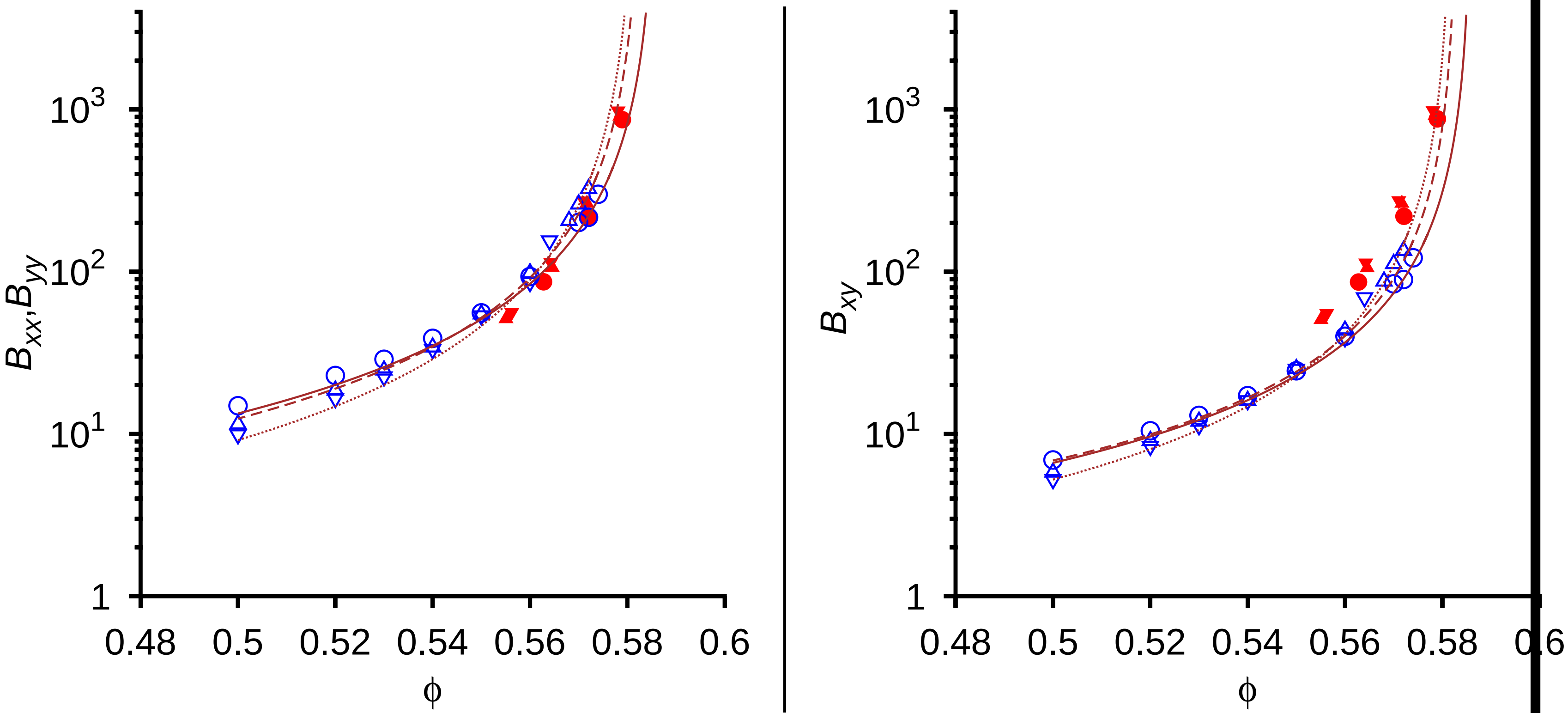
Flow down inclined plane: Ordered flow: n, q_6 u_x



Statistical mechanics of dense granular flows:



Flow down inclined plane: Bagnold coefficients

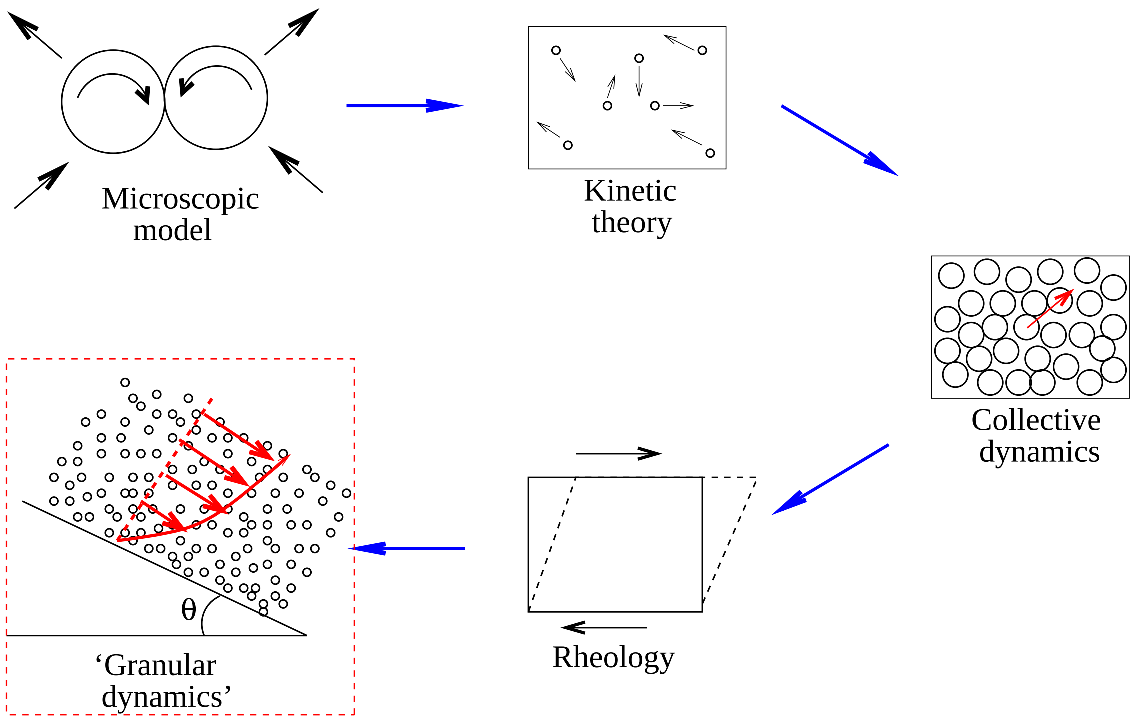


DEM simulations $(k_n/(mg/d)) = 2 \times 10^8$, Hard-particle simulations,

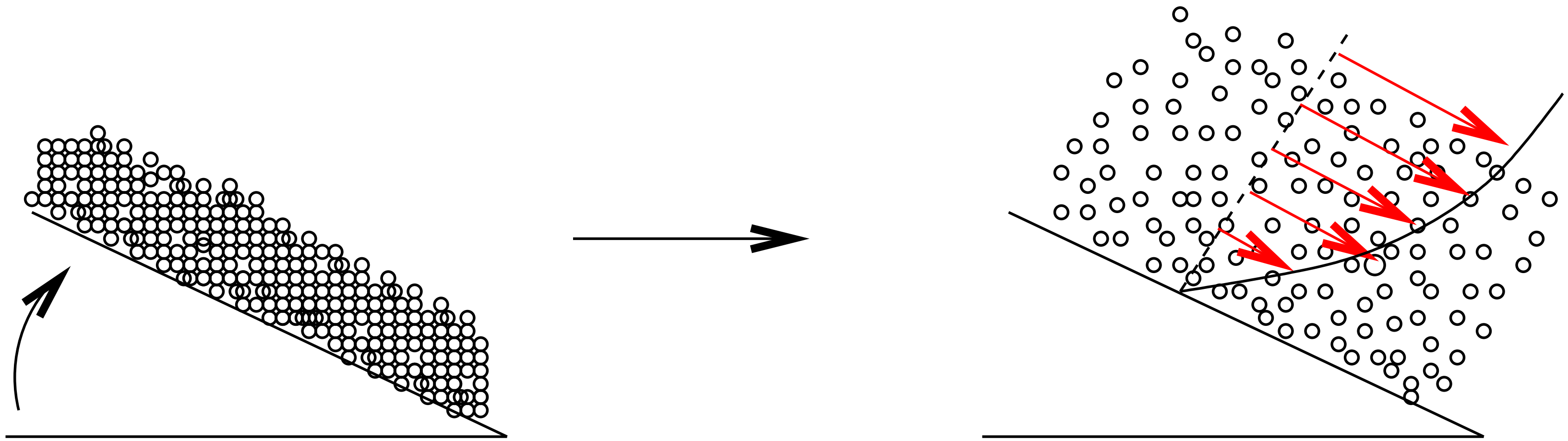
Hard-particle model with collision frequency from ED simulations.

$\circ e_n = 0.8$, $\triangle e_n = 0.7, \nabla e_n = 0.6$.

Statistical mechanics of dense granular flows:

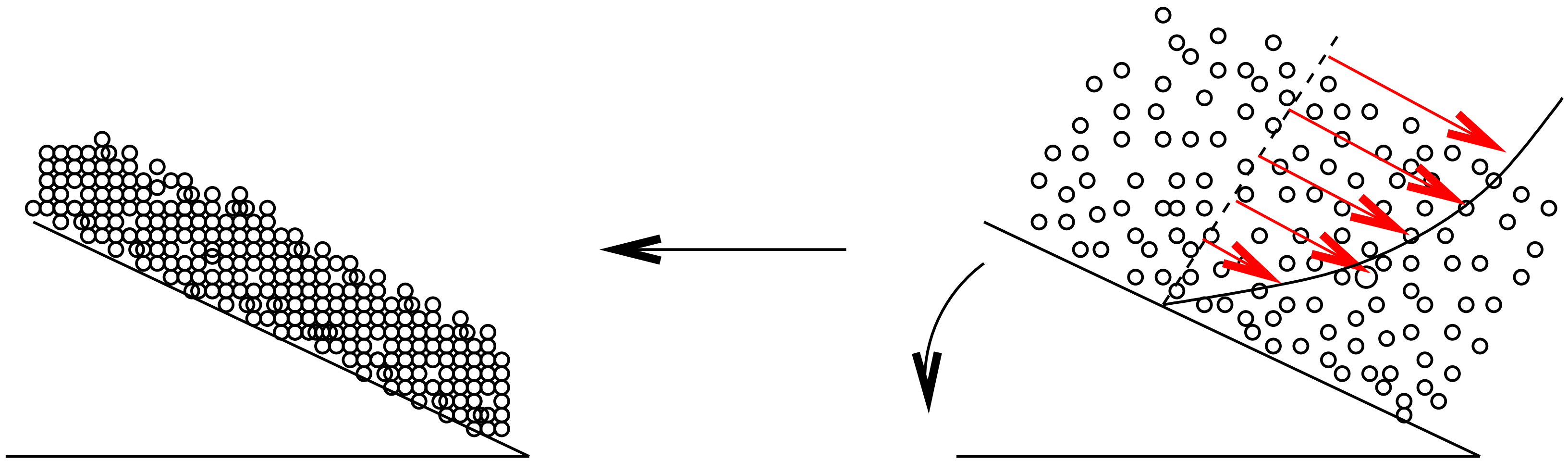


Flow down inclined plane:



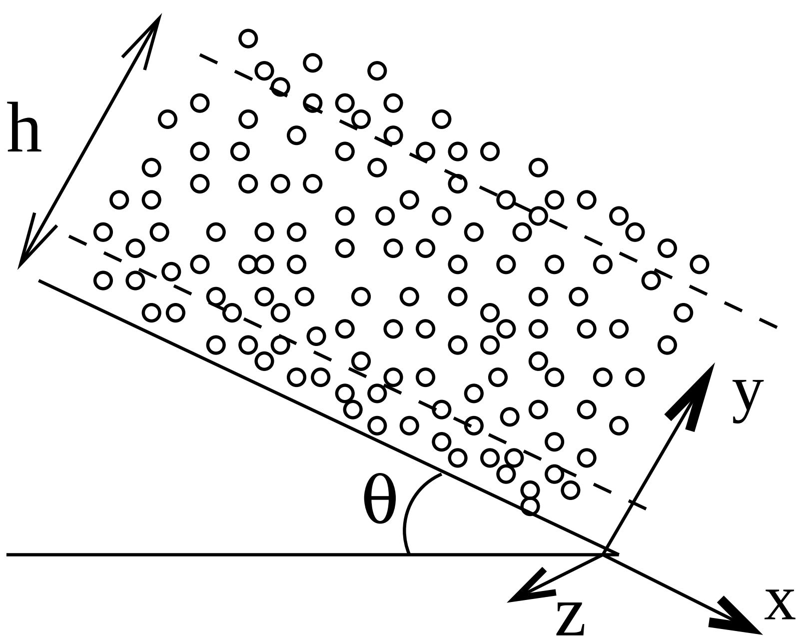
$$\sigma_{ij} = \sigma_{ij}^{(y)} + \sigma_{ij}^{(k)} ??$$

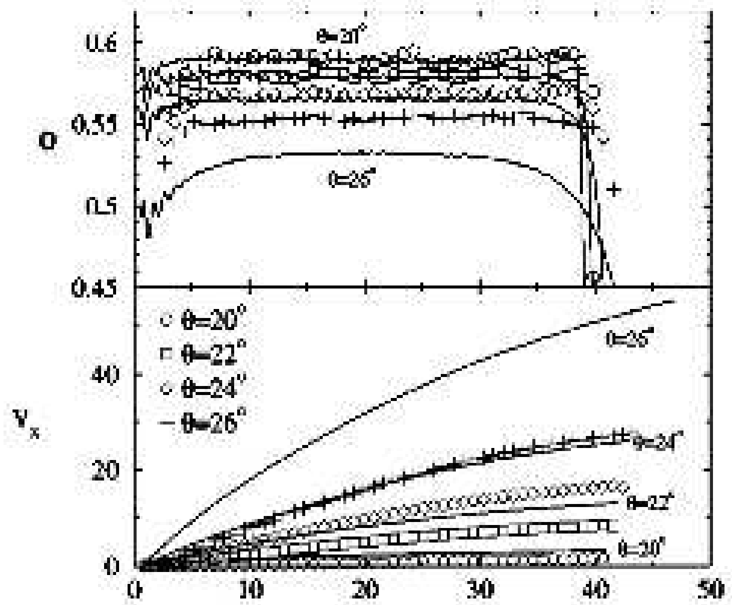
Flow down inclined plane:



$$\sigma_{ij} = \sigma_{ij}^{(k)}!!$$

Flow down inclined plane:

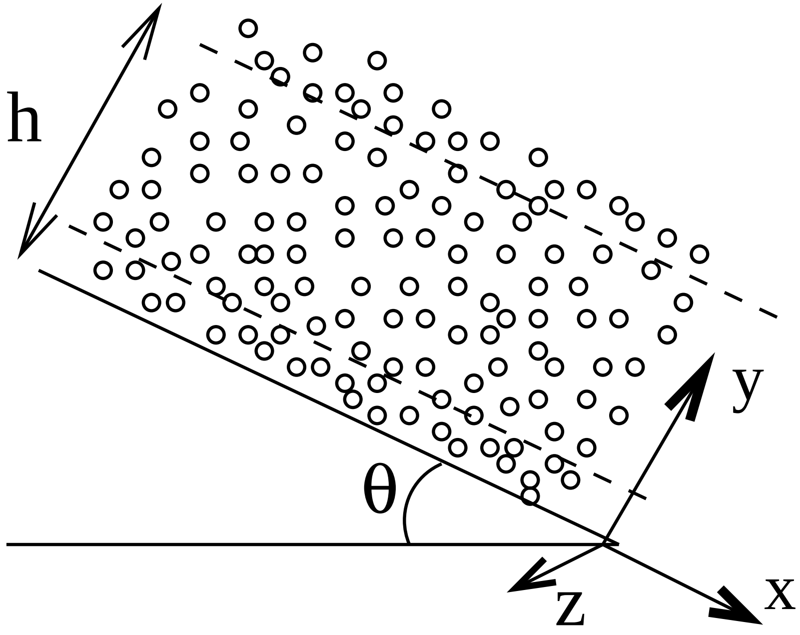




Silbert et al PRE 2001.

- Density independent of height.
- Stress follows Bagnold law $\sigma_{ij} \propto \dot{\gamma}^2$.

Granular dynamics:



Momentum equations:

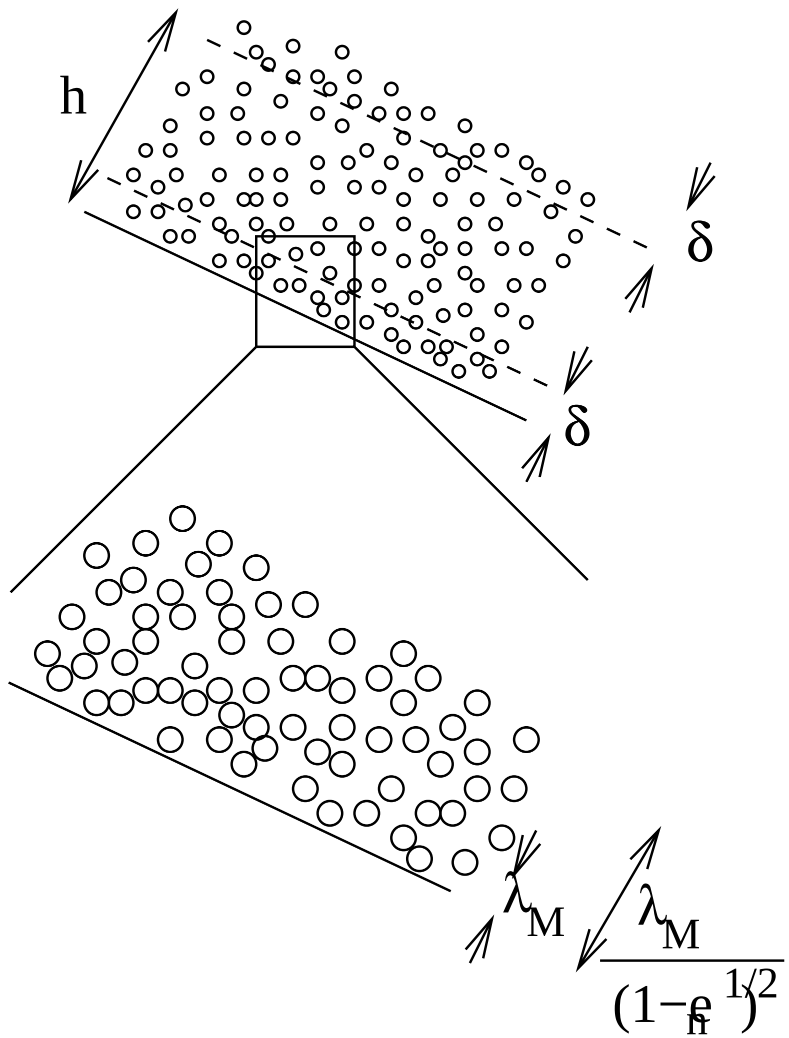
-
$$
\begin{aligned}
(d\sigma_{xy}/dy) &= -\rho g \sin(\theta) \\
(d\sigma_{yy}/dy) &= \rho g \cos(\theta)
\end{aligned}
$$

- Ratio $(\sigma_{xy}/\sigma_{yy}) = -\tan(\theta)$

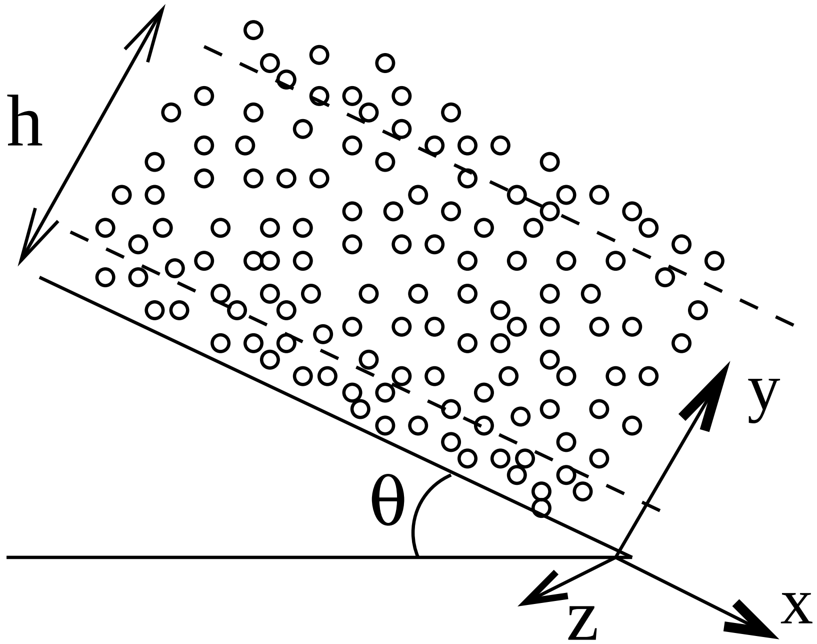
- (Conduction/Dissipation) $\sim d^2/(1-e_n)h^2) \sim (\delta^2/h^2)$.

Granular dynamics: Energy conservation.

- Energy *not conserved.*
- Source of energy.
- Rate of conduction $\nabla.(k\nabla T) \sim (dT^{3/2}/L^2)$.
- Rate of dissipation $((1-e)T^{3/2}/d)$.
- Conduction length $L \sim \delta = (d/(1-e)^{1/2}$.
- Energy conserved $h \ll \delta$.
- *Adiabatic approx.* $h \gg \delta$. Local balance between source and dissipation.

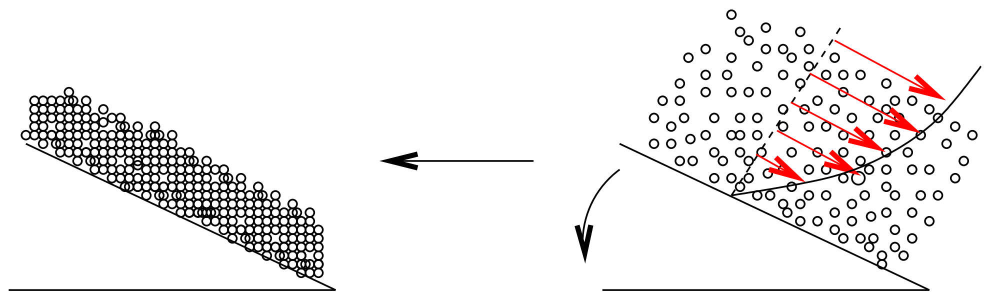


Granular dynamics



- Bulk of flow: $(\delta/h) \ll 1$.
- Energy balance:
  $\mu\dot{\gamma}^2 - D = 0$
  $T \propto (\dot{\gamma}^2\delta^2)$.
- Dimensional analysis:
  $\sigma_{ij} = B_{ij}(\phi)\dot{\gamma}^2$
  $\tan(\theta) = -(B_{xy}(\phi)/B_{yy}(\phi))$
- ϕ is independent of height in bulk.
- $\phi = \text{function}(\tan(\theta))$.

Granular dynamics:



$$\sigma_{ij} = \sigma_{ij}^{k}!!$$

$$\sigma_{xy} = B_{xy}\dot{\gamma}^2 = \rho g \sin(\theta).$$

For $\phi \to \phi_c$:

$B_{xy} \to \infty$;

$\dot{\gamma} \to 0$.

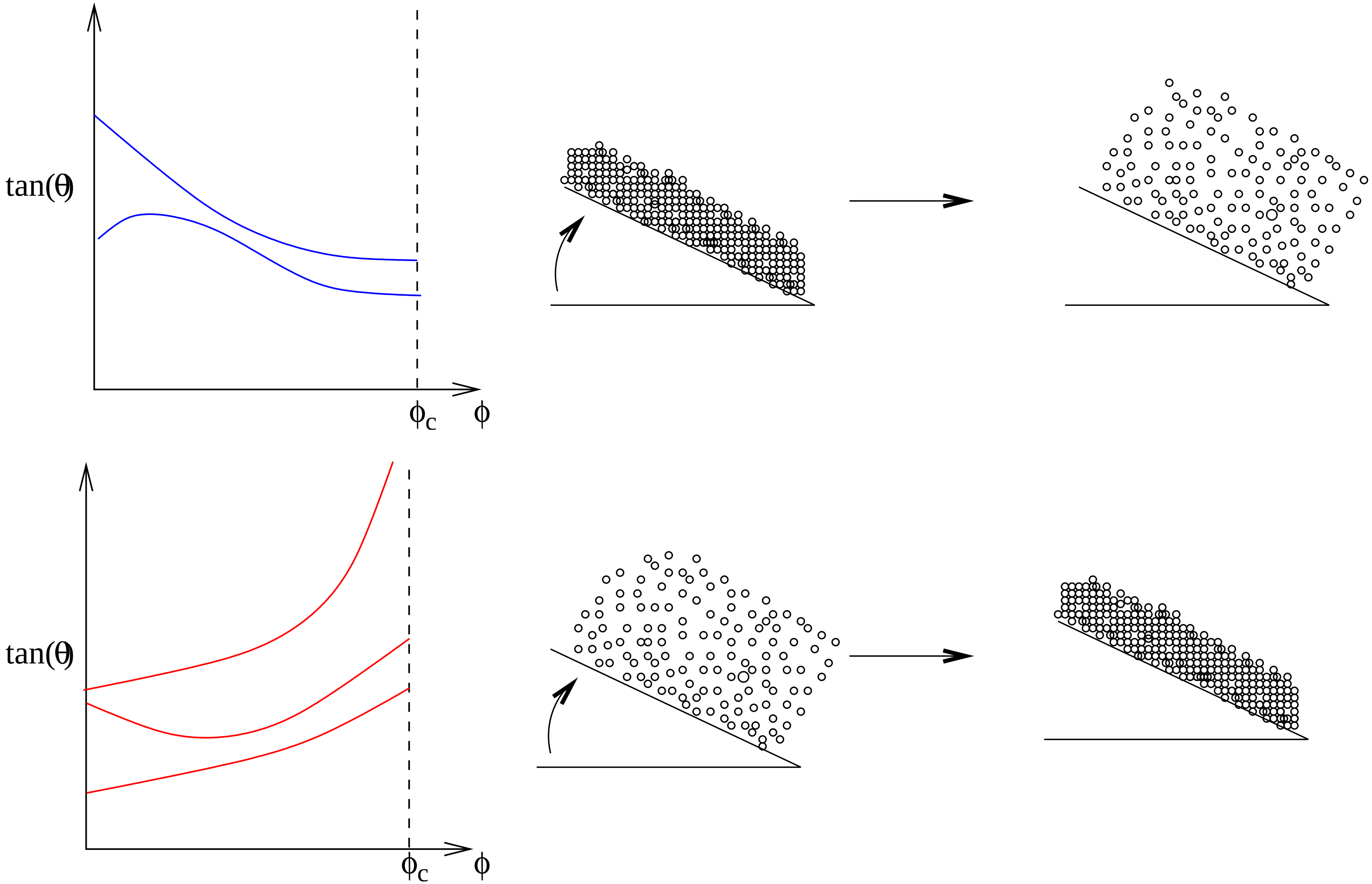
$$\frac{\sigma_{xy}}{\sigma_{yy}} = \tan(\theta) = \frac{B_{xy}}{B_{yy}}$$

For $\phi \to \phi_c$:

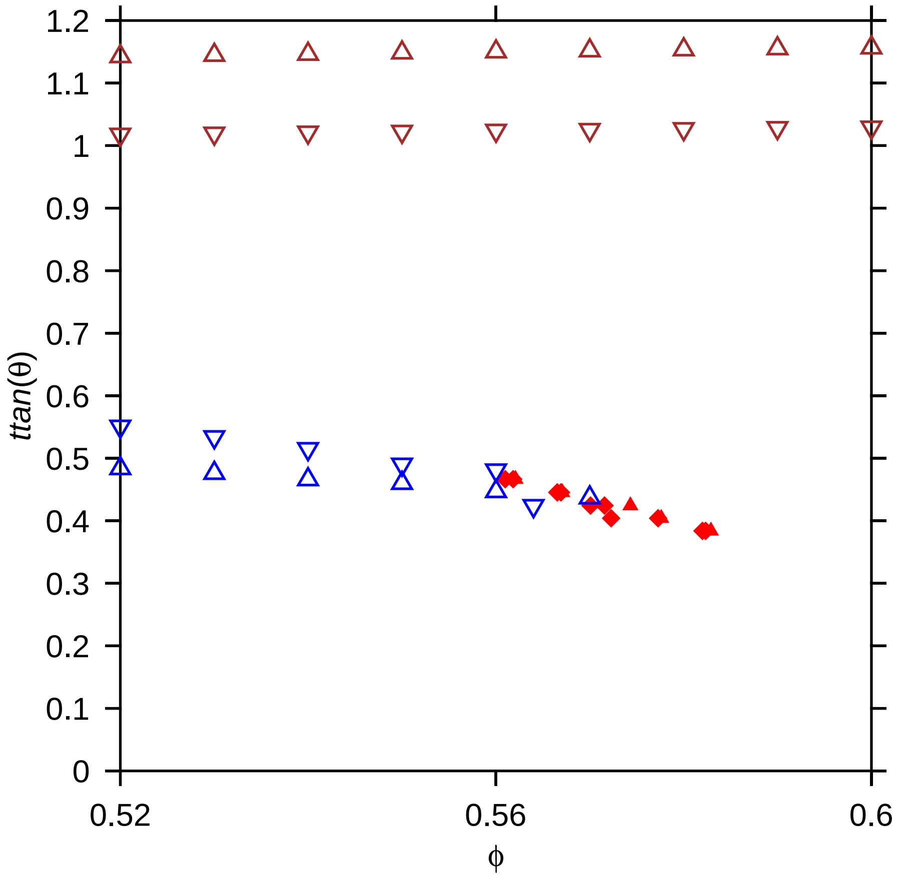
$B_{xy}, B_{yy} \to \infty$.

$\tan(\theta) \to \tan(\theta_c)$.

Flow down inclined plane:



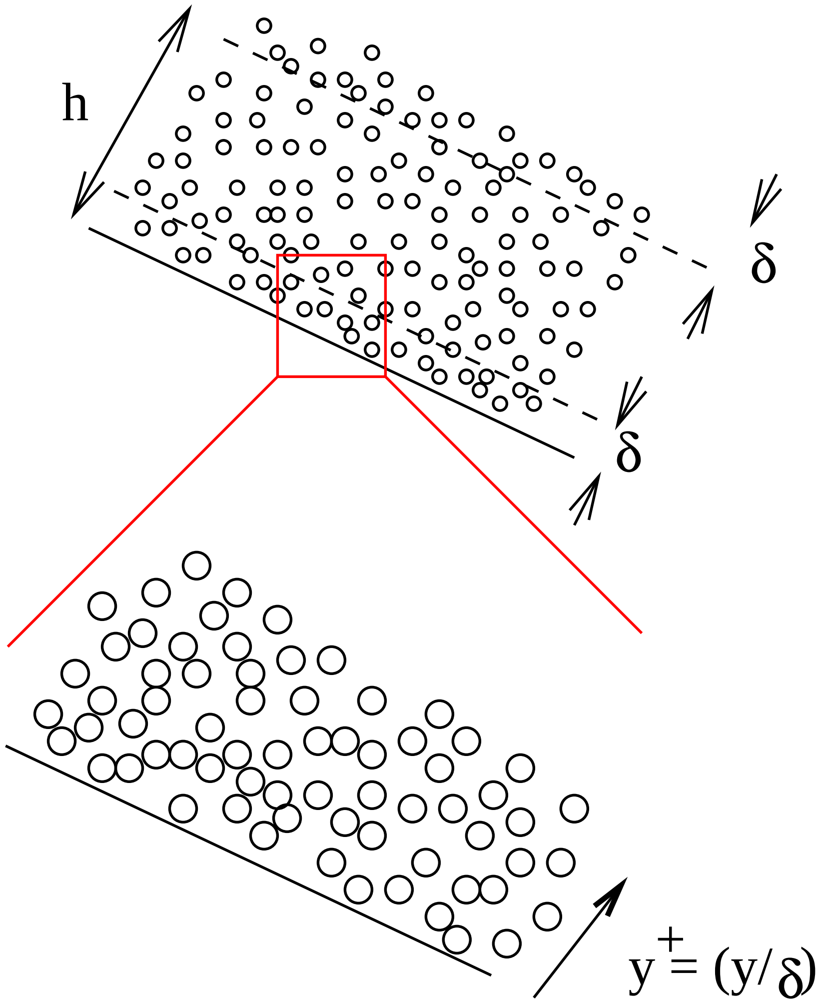
Flow down inclined plane:



$\circ e_n = 0.8$,
$\triangle e_n = 0.7$,
$\nabla e_n = 0.6$

DEM simulations $(k_n/(mg/d)) = 2 \times 10^8$, Kinetic theory with collision frequency from ED simulations, Theory including correlations in relative velocity distribution.

‘Boundary layer analysis’:



Boundary layer $y \sim \delta$:

Rescale $y^{\dagger} = (y/\delta)$.

$$\frac{1}{\delta^2}\frac{d}{dy^{\dagger}}\left(K\frac{dT}{dy^{\dagger}}\right) + \mu\dot{\gamma}^2 - D = 0$$

Boundary condition $\frac{dT}{dy} = \beta T$.

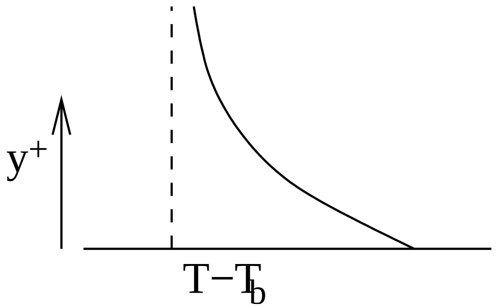
‘Boundary layer analysis’:

Boundary layer equation for deviation from bulk temperature T_b:

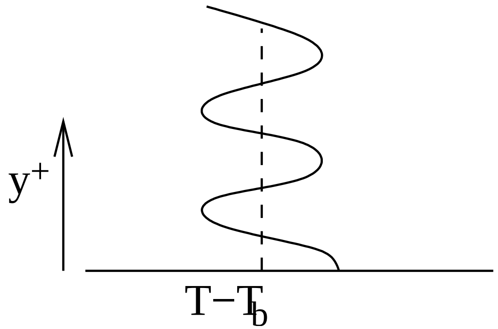
$$\frac{d^2(T - T_b)}{dy^{\dagger 2}} + \ldots - \alpha(T - T_b) = 0$$

$\alpha = (R_c D_c / 2K_c \chi_b)$ where $R_c = \frac{d}{d\phi}(\mu\dot{\gamma}^2/D - 1)\Big|_{\phi=\phi_b}$.

Boundary layer $\alpha > 0$:
$(T - T_b) \propto \exp(-\sqrt{\alpha y^{\dagger}})$.



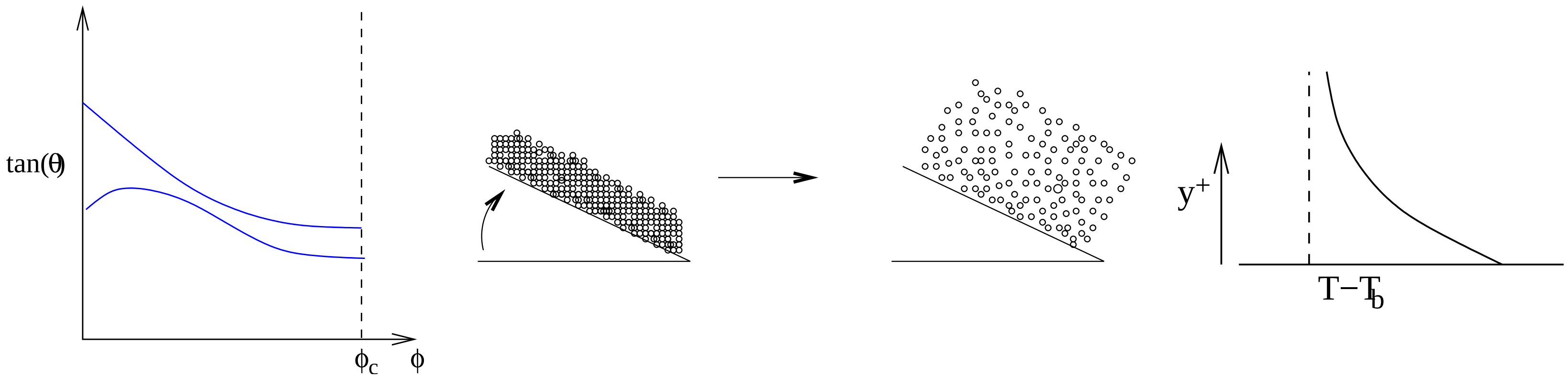
No Boundary layer $\alpha < 0$:
$(T - T_b) \propto \exp(-\imath\sqrt{|\alpha| y^{\dagger}})$.



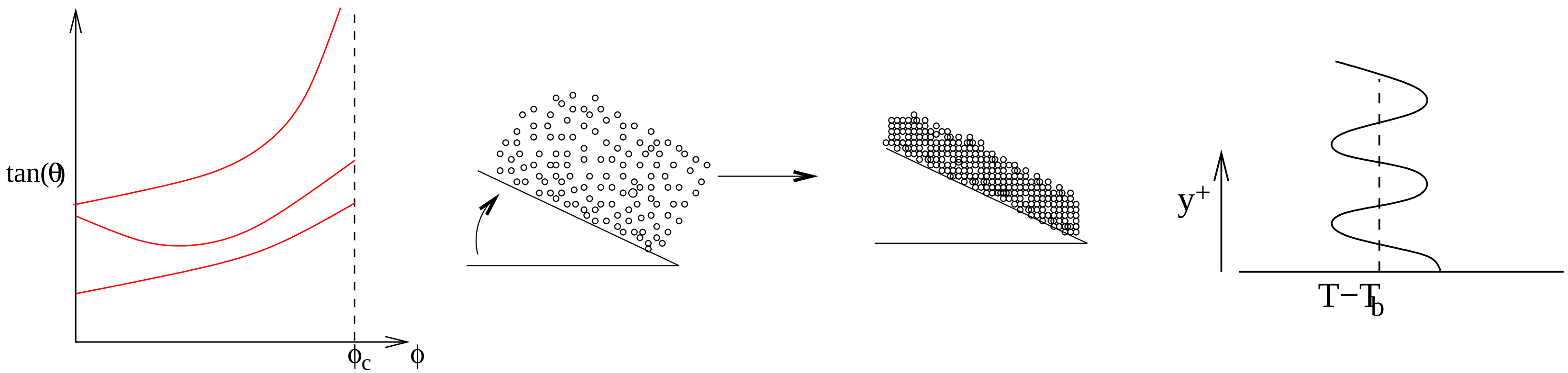
Boundary condition $\frac{dT}{dy} = \beta T$.

Granular dynamics: Exact result.

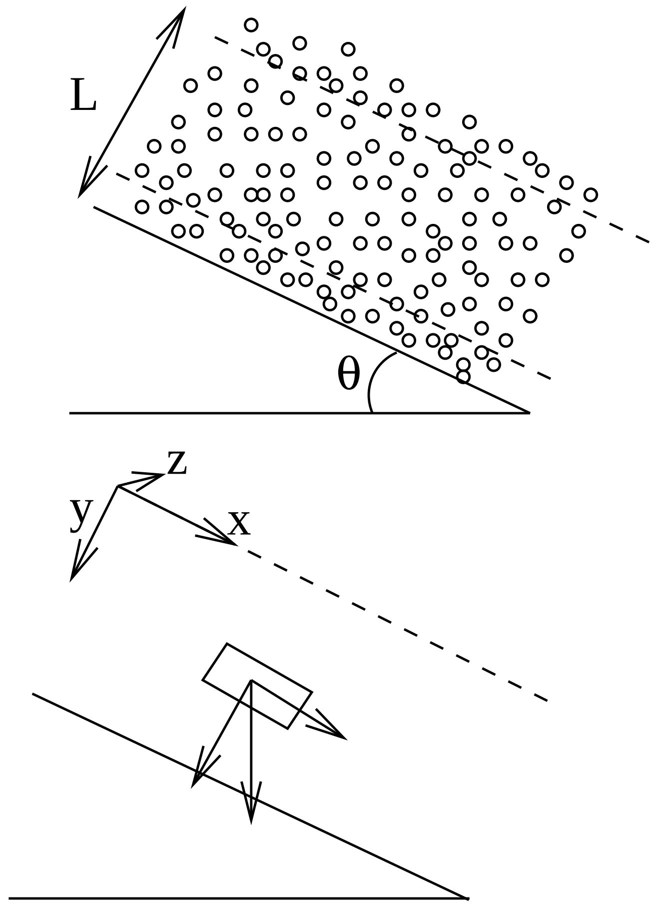
Physical: Boundary layer solutions exist $\alpha > 0$.



Unphysical: No boundary layer solutions $\alpha < 0$.



Granular dynamics:



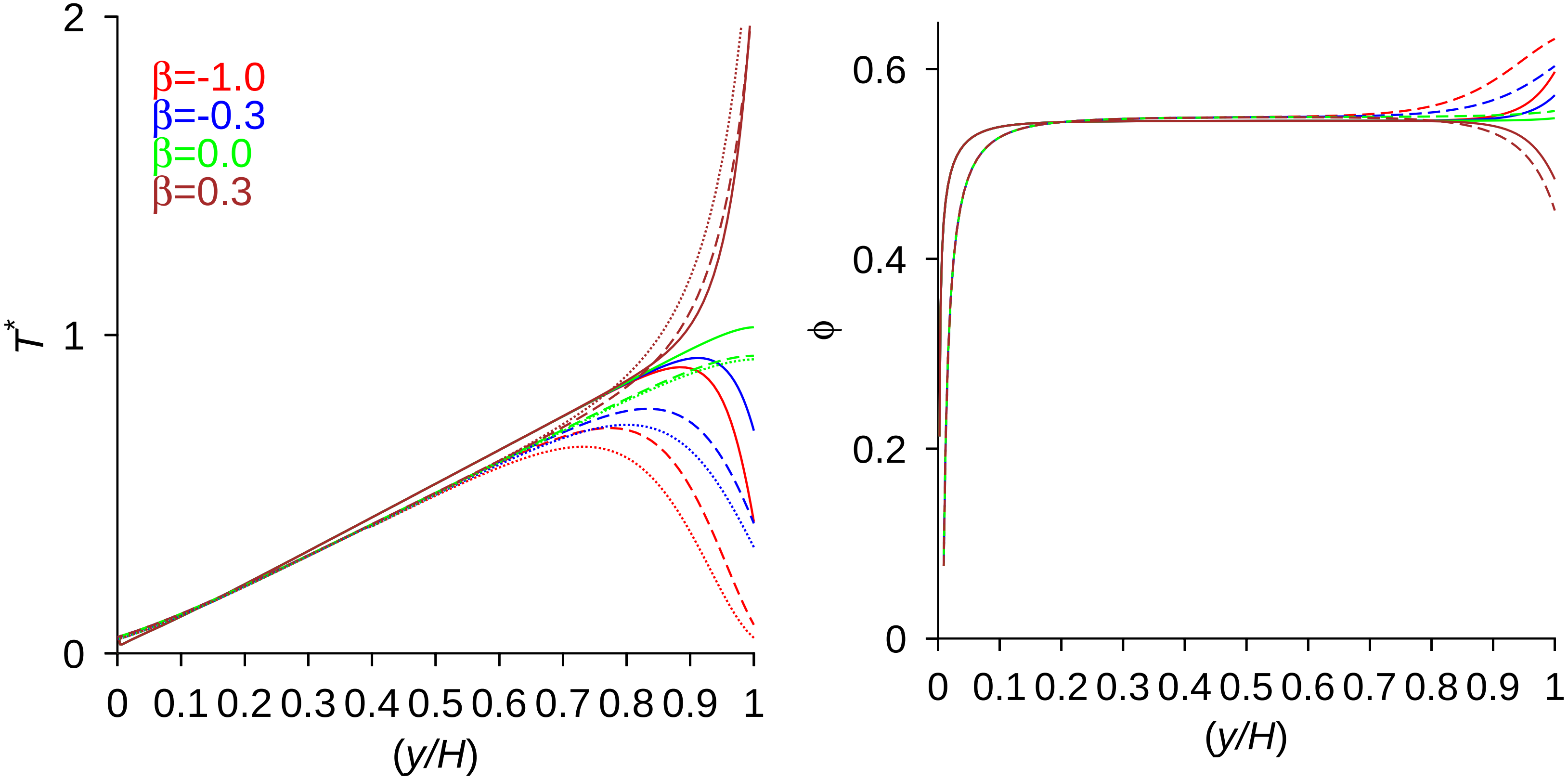
$$(\sigma_{xy}/\sigma_{yy}) = -\tan(\theta)$$

$$\frac{d\sigma_{yy}}{dy} = \rho g \cos(\theta)$$

$$\frac{d}{dy} K \frac{dT}{dy} + \mu \dot{\gamma}^2 - D = 0$$

Numerical (—) approximate (- - -) asymptotic (· · ·)

$H = 40, e_n = 0.8, e_t = 1, \phi_o = 0.55$:



Flow down inclined plane: Unsteady flow

Momentum equations:

$$\rho \frac{\partial u_x}{\partial t} = \frac{\partial \sigma_{xy}}{\partial y} + \rho g_x$$

$$0 = \frac{\partial \sigma_{yy}}{\partial y} + \rho g_y$$

Bagnold law:

$$\sigma_{xy} = B_{xy}(\phi) \left(\frac{du_x}{dy} \right)^2$$

$$\sigma_{yy} = B_{yy}(\phi) \left(\frac{du_x}{dy} \right)^2$$

Steady solution:

$\bar{u}_x = (5\bar{u}/3) \left(1 - (1 - (y/h))^{3/2}\right)$

$$\bar{u} = \frac{2h^{3/2}}{5} \left(\frac{\rho g_x}{B_{xy}} \right)^{1/2}$$

Unsteady flow:

$u_x = \bar{u}_x(y) + u'_x(y,t)$

$\phi = \bar{\phi}(y) + \phi'(y,t)$

$B_{ij}(\phi) = \bar{B}_{ij} + B_{\phi ij} \phi'(y,t)$

Linearise in u'_x, ϕ'.

Flow down inclined plane: Disordered flow

$$\rho \frac{\partial u_x'}{\partial t} = \frac{\partial}{\partial y}\left(B_{\phi xy}\phi'\left(\frac{\partial \bar{u}_x}{\partial y}\right)^2 + \bar{B}_{xy}\left(2\frac{\partial \bar{u}_x}{\partial y}\frac{\partial u_x'}{\partial y} + \left(\frac{\partial u_x'}{\partial y}\right)^2\right)\right)$$

$$0 = \frac{\partial}{\partial y}\left(B_{\phi yy}\phi'\left(\frac{\partial \bar{u}_x}{\partial y}\right)^2 + \bar{B}_{yy}\left(2\frac{\partial \bar{u}_x}{\partial y}\frac{\partial u_x'}{\partial y} + \left(\frac{\partial u_x'}{\partial y}\right)^2\right)\right)$$

Combine to give:

$$\tau \frac{\partial u_x'}{\partial t} = h^2 \frac{\partial}{\partial y}\left(\sqrt{1-(y/h)}\frac{\partial u_x'}{\partial y}\right) \quad \Bigg| \quad \tau = \left(\frac{5\bar{u}}{\rho h^3}\left(\bar{B}_{xy} - \frac{B_{\phi xy}\bar{B}_{yy}}{B_{\phi yy}}\right)\right)^{-1}$$

Can show $\tau > 0$ if $\phi \downarrow$ as $\theta \uparrow$.

Flow down inclined plane: Disordered flow

Solution for u'_x:

$$u'_x = \sum C_n e^{(-\alpha_n t/\tau)}(1-(y/h))^{1/4}$$
$$\times J_{-1/3}((4\sqrt{\alpha_n}(1-(y/h))^{3/4}/3)$$

Discrete eigenvalues α_n:

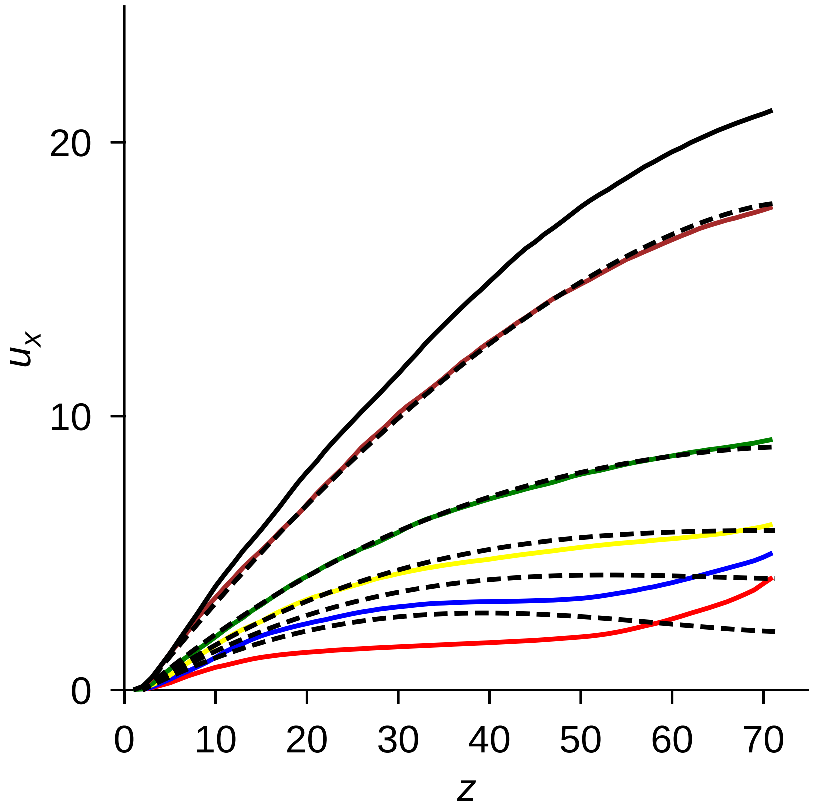
$$u'_x = 0 \text{ at } y = 0$$
$$\frac{du'_x}{dy} = 0 \text{ at } y = h$$

Orthogonality relation C_n from initial condition $u'_x = -\bar{u}_x$.

n	1	2	3	4
α_n	1.95934	13.9943	37.1371	71.363
$(C_n/\bar{u})$	-2.32322	0.200244	0.059188	0.026166

Flow down inclined plane: Disordered flow

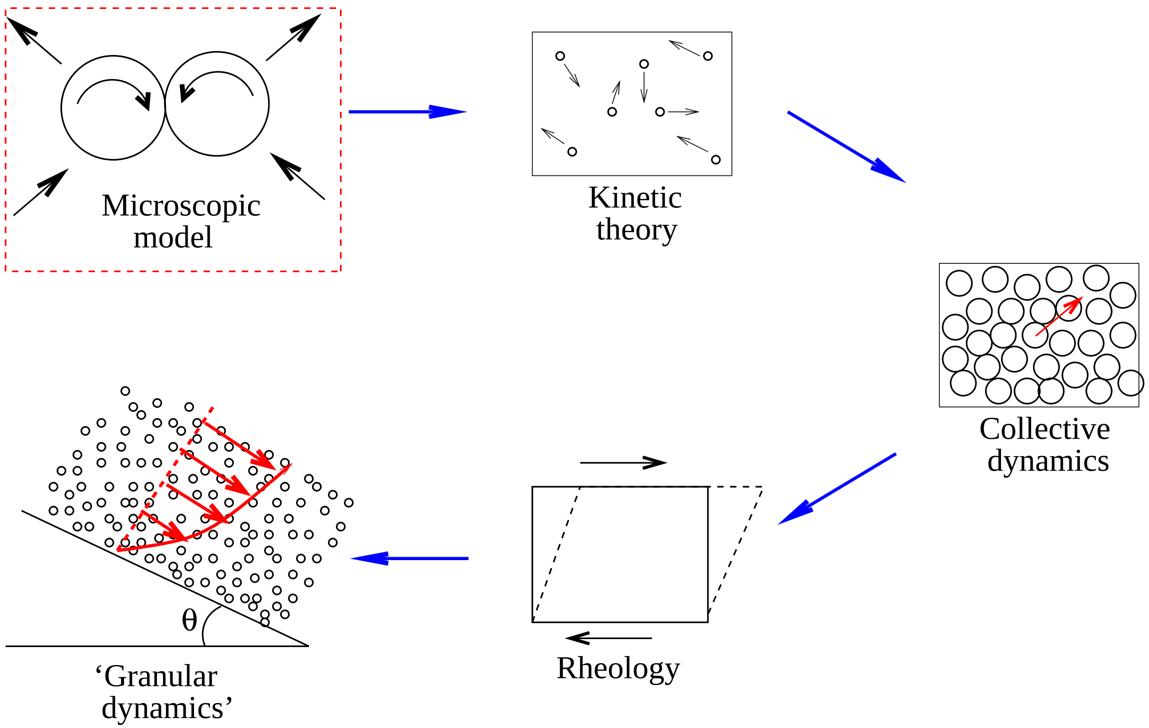
Rough base, 22^o, $d_b = 1$:



- Unsteady momentum equation; Bagnold law.
- Linearise & use separation of variables.
- Retain first term in the expansion.

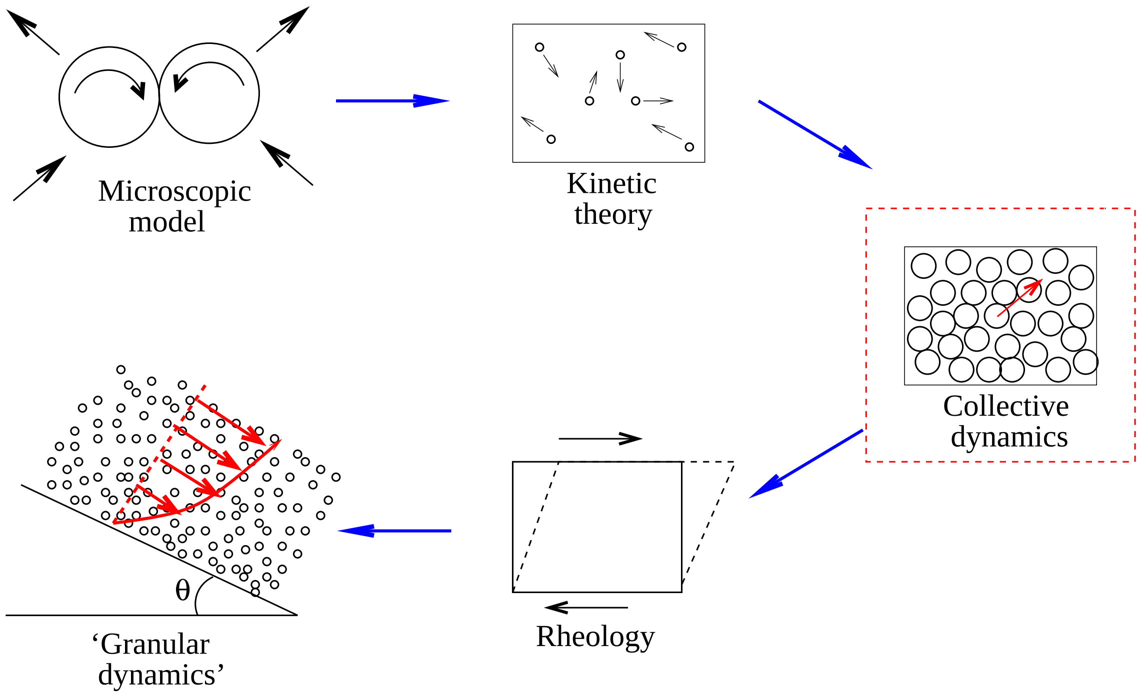
— t=40; — t=80; — t=120; — t=200; — t=600; — t=1000.

Conclusions:



Real granular flows may be well represented as inelastic hard-particle fluid in some cases. Rheology surprisingly insensitive to details of contact regime.

Conclusions



Sheared inelastic hard particles are unusual fluids:

Conclusions: Sheared inelastic hard particles are unusual fluids:

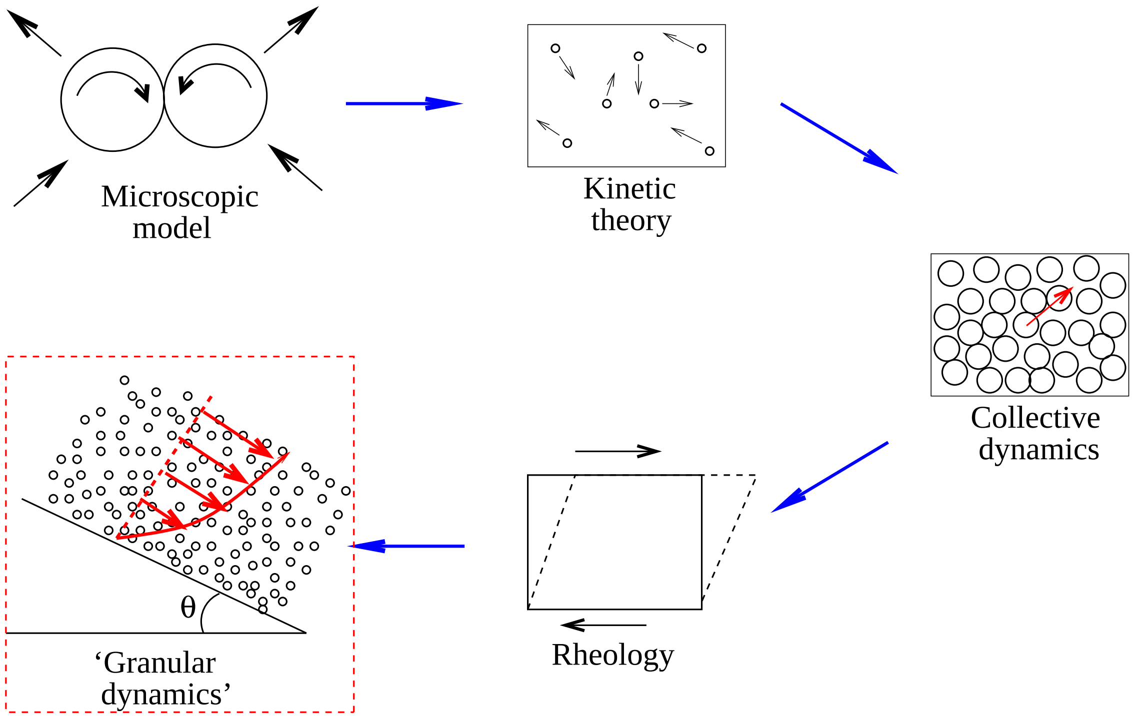
Homogeneous shear:

- Fluid-like structure & diffusion.
- Fast decay of autocorrelation function $\rightarrow$ no divergences in transport coefficients.
- Freq. diverges at volume fraction less than elastic fluid RCP.
- Strong correlation effects in relative velocity distribution.

Wall-bounded flow:

- Distinct flow regimes with universal properties in each regime.
- Discontinuous transition at specific base roughness.
- Transient disordered flow well described by Bagnold law.
- Ordered flow: shearing layer & plug flow for smooth base during flow development!

Conclusions:



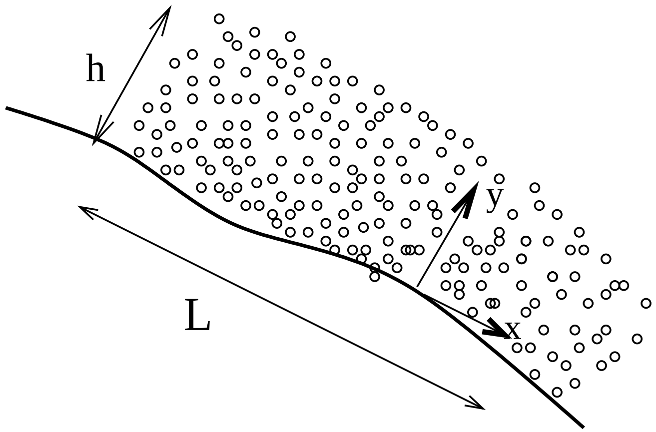
Conclusions:

Model predicts features of flow down inclined plane.

- Initiation of flow at finite angle.
- Constant density in bulk, Bagnold law.
- Numerical values of Bagnold coefficients.
- Density and temperature in temperature boundary layers (subject to boundary conditions).
- Unsteady flow.

Can be used to develop boundary layer equations for shallow flows.

Boundary layer equations:

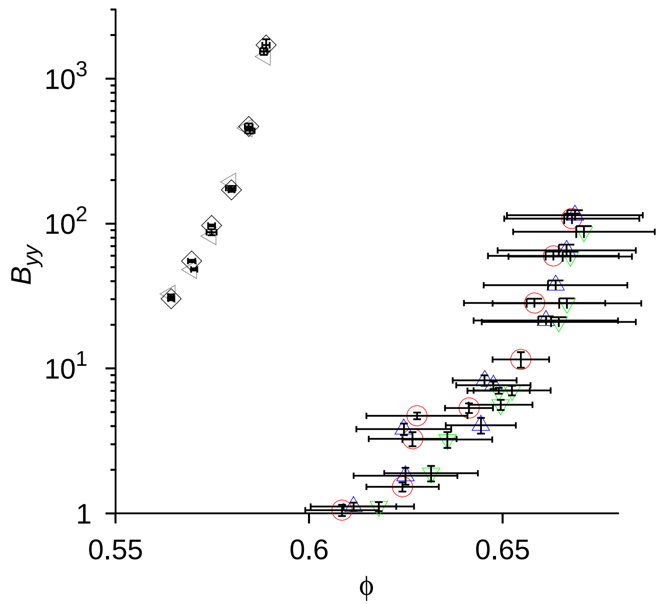


Mach number

$$\mathrm{Ma} \sim (\dot{\gamma} h/(\partial p/\partial \rho)^{1/2})$$
$$\sim (B_{yy}/(dB_{yy}/d\phi))$$

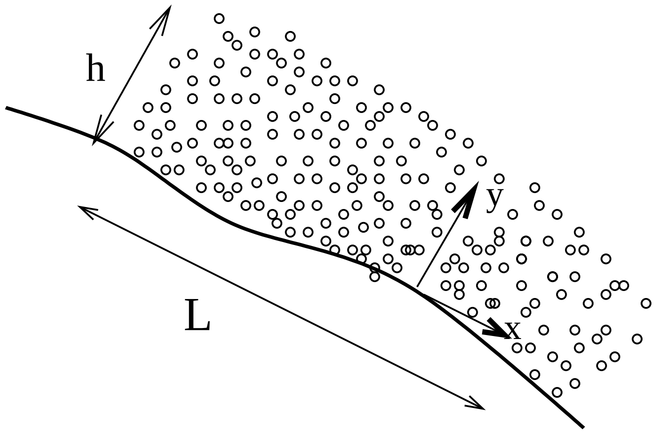
Reynolds number

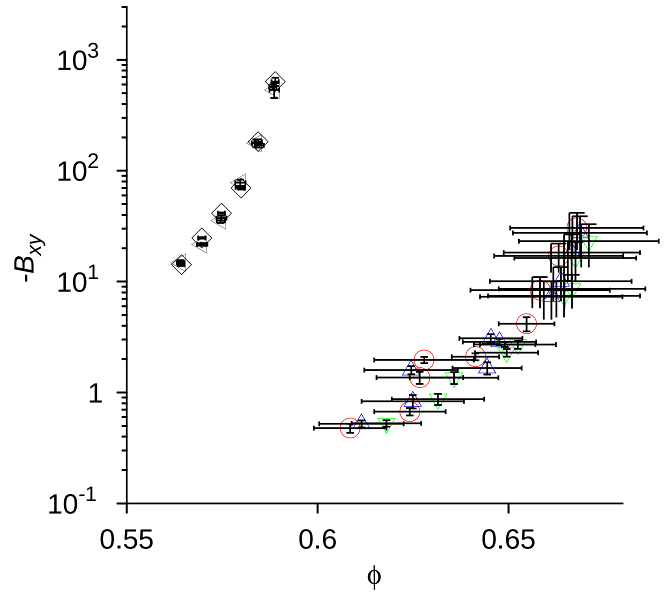
$$\mathrm{Re} \sim (h^2/d^2)(T/p)$$
$$\sim (h^2/d^2)(1/B_{yy})$$



$$B_{yy} \sim (\phi_c - \phi)^{-\alpha}$$
$$(B_{yy}/(dB_{yy}/d\phi)) \sin(\phi_c - \phi)^{-1}$$
$$\mathrm{Ma} \ll 1.$$

Boundary layer equations:





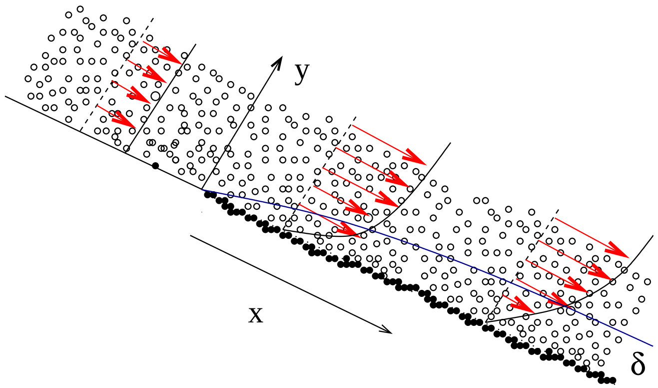
$$\nabla.\mathbf{u} = 0$$

$$\rho\left(u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = \frac{\partial}{\partial y}\left(B_{xy}(\phi)\left(\frac{\partial u_x}{\partial y}\right)\left|\frac{\partial u_x}{\partial y}\right|\right) + \rho g_x$$

$$0 = -\frac{\partial}{\partial x}\left(B_{yy}(\phi)\left(\frac{\partial u_x}{\partial y}\right)\left|\frac{\partial u_x}{\partial y}\right|\right) + \rho g_y$$

Flow variation length: $\frac{L}{h} = \frac{\rho d^3}{dB_{xy}}\frac{h^2}{d^2}\left(1 - \frac{B_{xy}(dB_{yy}/d\phi)}{B_{yy}(dB_{xy}/d\phi)}\right) \sim \frac{h^2}{d^2 B_{xy}}$

Boundary layer equations:



Boundary layer solutions for $U \propto x^{\alpha}$.

$$\delta \sim \left(\frac{2((B_{xy}/d\phi)B_{yy} - B_{xy}(dB_{yy}/d\phi))}{B_{yy}\rho} x \right)^{1/3}$$

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