Simulations of Fluvial Landscapes and Optimal Transport

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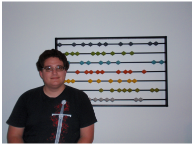

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Outline

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What should a mathematical model of erosion produce?

- The emergence of channelized drainage patterns from unchanneled surfaces
- The development of relatively stable surfaces characterized by branching patterns of ridges and valleys
- The decline of the surfaces and the dissipation of the forms
- The variability of landforms under varying environmental conditions

A Typical Surface Simulated by David

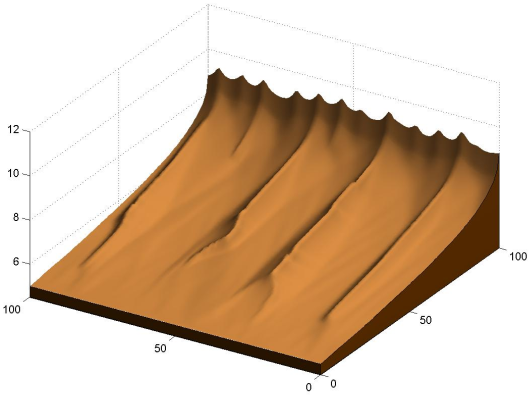


Figure: A Pattern of Ridges and Valleys

The B.-Bretherton-Smith Equations

- Let $H = z + h$ be the *height of the free water surface*, where z is the height of the land surface and h is the water depth.

$$\eta^2 \frac{\partial h}{\partial t} = \nabla \cdot \left[h^{3/2} \, |\nabla H|^{1/2} \mathbf{u} \right] + R, \qquad (1)$$

$$\frac{\partial H}{\partial t} = \nabla \cdot \left[h^{10/3} |\nabla H|^3 \mathbf{u} \right] - \delta h^{3/2} |\nabla H|. \qquad (2)$$

- $\mathbf{u} = \frac{\nabla H}{|\nabla H|}$ is the unit normal down the gradient of the water surface, R is the rainfall rate and η is small.
- The second term in Equation (2) models erosion and is inspired by Kramer and Marder 1992.

Initial Ridge

Starting with a a linear ridge extending uniformly in the lateral (x)-direction and defined over a rectangular domain of length L and width W,

$$D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq L, \ 0 \leq y \leq W\},$$

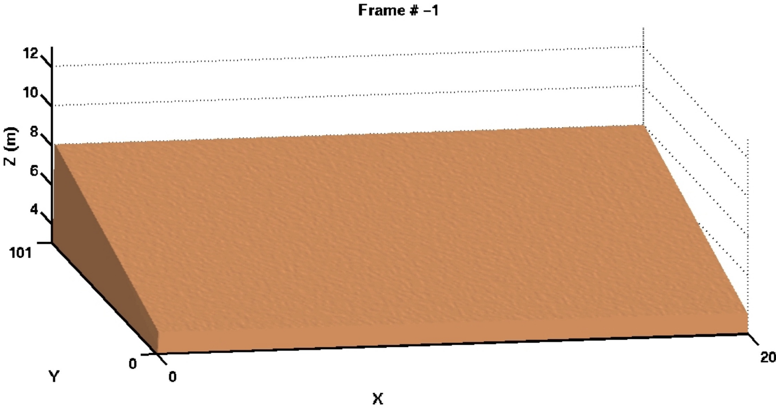
with initial conditions

$$\begin{aligned} h(x, y, 0) &= d(y), \ \ d(0) = h_o, \ \ d(W) = 0, \\ H(x, y, 0) &= cy + h_o, \quad 0 \leq y \leq W \end{aligned} \tag{3}$$

and boundary conditions

$$\begin{aligned} h(x, W, t) &= 0, \\ H(x, 0, t) = h_0 &= h(x, 0, t) \end{aligned} \tag{4}$$

Initial Surface



Instabilities

Linearize the PDEs around the initial surface we get two instabilities:

- If the PDE (2) has no erosion term, the dispersion relation becomes

$$\omega = \frac{5}{3} d^{\frac{2}{3}} c^{\frac{1}{2}} [(2-d)k_1^2 + (\frac{1}{2} - 3d)k_2^2],$$

  where d is small. It shows that all the spatial frequencies are unstable and that the highest frequencies grow the fastest.

- If the erosion term is included we get an additional instability

$$\omega = \frac{3}{2}\delta - k_1^2.$$

  This instability gives rise to river channels.

Outline

Numerical Methods

- If the smallest frequencies grow the fastest, we have a real problem numerically.
- In nature there is a natural (lower) cutoff, when the scale of the grain size is reached.
- Nonlinearities also saturate the exponential growth of the instabilities.
- How does one capture this numerically?
- Answer: Implicit methods work, explicit methods do not capture the small scales.

Crank-Nickolson/Upwind versus Predictor Corrector Numerical Schemes

- The predictor-corrector is an explicit scheme
- At time $n+\frac{1}{2}$ predictor step (backward difference), at time $n+\frac{1}{2}$ corrector step (forward difference).
- At time $n+1$ predictor step (forward difference), at time $n+1$ corrector step (backward difference).
- Upwind (water flow) is explicit, but the Crank-Nickolson (sediment flow) is an implicit scheme

$$\frac{H_{ij}^{n+1}-H_{ij}^{n}}{\Delta t} = \frac{C_1}{2}\left[\delta_x\left\{\left(h_{ij}^{n+1}\right)^B H_{ij,x}^{n+1}\left(\left(H_{ij,x}^{n+1}\right)^2+\left(H_{ij,y}^{n+1}\right)^2\right)^{\frac{C}{2}}\right\}+\delta_y\left\{\left(h_{ij}^{n+1}\right)^B H_{ij,y}^{n+1}\left(\left(H_{ij,x}^{n+1}\right)^2+\left(H_{ij,y}^{n+1}\right)^2\right)^{\frac{C}{2}}\right\}\right]+\frac{C_1}{2}\left[\delta_x\left\{\left(h_{ij}^{n}\right)^B H_{ij,x}^{n}\left(\left(H_{ij,x}^{n}\right)^2+\left(H_{ij,y}^{n}\right)^2\right)^{\frac{C}{2}}\right\}+\delta_y\left\{\left(h_{ij}^{n}\right)^B H_{ij,y}^{n}\left(\left(H_{ij,x}^{n}\right)^2+\left(H_{ij,y}^{n}\right)^2\right)^{\frac{C}{2}}\right\}\right]+O(\Delta x^2,\Delta t^2) \quad (5)$$

Comparison of the Numerical Scheme

- Artificial viscosity must be added to the Predictor-Corrector scheme to keep it stable.
- Small viscosity is build into the Crank-Nickolson/Upwind scheme in a very controlled way. It is small and decreases with the discretization size.
- Both schemes capture the large scale features of the landscape. They are the same when, water depth, land elevation, water and sediment flow are averaged at fixed (y) upslope cross sections.
- The number of ridges and the number of valleys are the same and the half-width of the valleys.

Scaling of the Variogram

- The variogram

$$V_f(\mathbf{x}, t) = < |f(\mathbf{x} + \mathbf{y}, t) - f(\mathbf{y}, t)|^2 >^{\frac{1}{2}} \tag{6}$$

  is the root mean square of the elevation differences as a function of distances of separation (or lag) $\mid \mathbf{x} \mid$.
- This function, known as the *variogram, height-height correlation function*, *roughness function*, or *width function*, characterizes the roughness of the surface.
- The variogram is just the second structure function from turbulence.
- Crank-Nickolson/Upwind produces the scaling exponents $1/2$ for h and $3/4$ for H, see B., Smith and Merchant (2001).

Hack's Law

- The length ℓ of the main river in a river basin scales with the area A of the river basin as $\ell \sim A^{0.58}$

Figure: The Amazon River Basin

- Predictor-Corrector produces the same scaling exponent, dependent on the viscosity, for h and H.

The computation of Hack's exponent

$$A \sim \ell^D$$

- The avalanche dimension is $D = 1 + \chi, \ell$ being the length of the main river. Then the width of the basin in the direction perpendicular to the main river, is ℓ^χ, $\chi = 3/4 = 0.75$, whereas along the main river it is ℓ, hence

$$\ell \sim A^{\frac{1}{1+\chi}}$$

$$\approx A^{0.58}$$

Outline

Optimal Transport

In 1781 Monge asked the question:

- What is the least expensive way to transport mounds of dirt in order to fill holes?
- If one has a collection of mounds of dirt and one also has several holes to fill, and if the amount of dirt in the mounds is precisely the amount needed to fill the holes, how should one move the dirt from the mounds to the holes with the least amount of work?
- As a geometer, Monge recognized that the direction of transport should be along straight lines that would be orthogonal to a family of surfaces.
- Erosion is nature's process of "moving dirt", thus not unreasonable to expect connections between erosion and optimal transport.

Outline

Model Equations with a Steady Water Depth

Consider the equations on $\Omega = \mathbb{T}^1 \times [0, W]$,

$$\nabla \cdot \left[h^{3/2} \, |\nabla H|^{1/2} \mathbf{u} \right] + R = 0, \tag{7}$$

$$\frac{\partial H}{\partial t} = \nabla \cdot \left[h^{10/3} |\nabla H|^3 \mathbf{u} \right] - \delta h^{3/2} |\nabla H|. \tag{8}$$

Assume that (7) has a solution; the equlibrium water depth that satisfies the conditions: $h \geq 0$, on Ω, $h > 0$, a.e. on Ω, and

$$\sup \left(\frac{1}{|B|} \int_B h^{10/3} dx \right) \left(\frac{1}{|B|} \int_B h^{-10/9} dx \right)^3 < \infty. \tag{9}$$

$|B|$ is the volume of the ball B, and the supremum is taken over all balls $B \subset \mathbb{R}^2$. This condition implies that $h^{10/3}$ is in Muckenhoupt's A_4 class. (For example, $h \sim x^\alpha, \alpha < \frac{9}{10}$).

The existence and uniqueness of entropy solutions

The condition (9) means that smooth functions are dense in the weighted Sobolev space $W_h^{1,4}$ with norm

$$\|u\|_{W_h^{1,4}} = \left(\int_\Omega |u|^4 + |\nabla u|^4 h^{10/3})dx\right)^{1/4},$$

see F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo (2011). Consider the domain, B. and J. Rowlett (2013),

$$\Omega = \{(x,y) \in \mathbb{R}^2 |\ 0 \le x \le W,\ 0 \le y \le L\}$$

Theorem (1)

Let h be a given function satisfying the assumptions (9) above. Then for any $H|_{t=0} = H^0 \in W^{1,1}(\Omega)$, and $T > 0$, there exists unique entropy solution to the erosion PDE (8) for the water surface H, with the same boundary conditions as above.

Entropy Solutions

Consider $W^{1,1}(\Omega) := \{u \in L^1(\Omega) \text{ such that } \nabla u \in L^1(\Omega)\}$

Definition

A function $H \in W^{1,1}(0, T; L^1(\Omega))$ is an *entropy solution* of (8) on $(0, T)$ with initial data $H_0 \in W^{1,1}(\Omega)$, if $H(0) = H_0$, and for all $k > 0$, and

$$T_k(H(t,x,y)) := \sup\{\inf\{H(t,x,y),k\}, -k\} \in W_h^{1,4}(\Omega),$$

$$\int_\Omega (H'(t) T_k(H(t) - \phi) + h^{10/3} |\nabla H(t)|^2 \nabla H(t) \times \nabla (T_k(H(t) - \phi)) d\mathbf{x} \le 0; \tag{10}$$

where the last equation holds for all $\phi \in W_h^{1,4}(\Omega) \cap L^\infty(\Omega)$.

$T_k(H)$ is equal to H if the value of H lies in $[-k, k]$, and otherwise is equal to $-k$ if $H < -k$ or k if $H > k$.

Weak Solutions

We do not know if weak solutions of (8) exist. However, with one more assumption: the equilibrium water depth h is continuous on $\Omega/h^{-1}(0)$ and $h^{-1}(0)$ consists of a finite union of piecewise smooth curves. The following holds:

Lemma

Let H be a weak solution of (8), then the L^2 norm $|H|_2$ and the energy functional $K(H) = \int_\Omega \frac{\nabla H|^4}{4} h^{10/3} dx$ are both decreasing functions of $t \in [0, T]$.

Theorem (2)

Weak solutions are unique.

Lemma

Weak solutions are also entropy solutions.

The Optimal Transport Problem

- Let μ and ν be non-negative Radon measures with (respectively) compact supports $U, V \subset \mathbb{R}^n$ satisfying,

$$\int_U d\mu = \int_V d\nu. \tag{11}$$

- A map $s : U \to V$ pushes μ onto ν, and we write $s_\#(\mu) = \nu$ if s is Borel measurable and for any Borel set $E \subset V$,

$$\int_{s^{-1}(E)} d\mu = \int_E d\nu. \tag{12}$$

- Associated to the optimal transport problem is a cost function which is typically given by

$$C(s) := \int_U c(\mathbf{x}, \mathbf{s}(\mathbf{x}))\mathbf{d}\mu(\mathbf{x}), \qquad \mathbf{c}(\mathbf{x}, \mathbf{y}) := \frac{|\mathbf{x} - \mathbf{y}|^{\mathbf{p}}}{\mathbf{p}}, \tag{13}$$

What do we want to know?

- A general optimal transport problem is

  Does there exist $s : U \to V$ *which minimizes* C

  with $s_{\#}(\mu) = \nu$?

  If it exists, such a map s is called an "optimal mass reallocation plan," or an "optimal mass transport plan."
- In the context of erosion, we pose the following natural question

  Is sediment "optimally transported" according to (8)?

- Some immediate difficulties arise. Monge's problem does not depend on time; erosion does. Moreover, the mass of the sediment is not preserved over time since it flows out of the region Ω.

An optimal transport problem for the sediment flow

- Define the measures μ and ν with support on Ω,

$$d\mu := -\frac{\partial H}{\partial t}(\mathbf{x}, t)d\mathbf{x}, \quad d\nu := -Fd\mathbf{x} \tag{14}$$

$$F := \bar{F}_\Omega / |\Omega|,$$

$$\bar{F}_\Omega := \int_\Omega \frac{\partial H}{\partial t} d\mathbf{x} = \int_0^L \nabla H |\nabla H|^2 h^{10/3}(W, y, t) \cdot \hat{n} dy < 0 \tag{15}$$

  and $|\Omega|$ denotes the area of Ω. This follows L. C. Evans (1999).

- We make the assumption that the landsurface is eroding:

$$\frac{\partial H}{\partial t} \leq 0 \quad \text{a. e. on } \Omega. \tag{16}$$

- Under these assumptions, both the measures are non-negative.

The Sediment Flux is Optimally Transported

Theorem (4)

Assume that for a given function h satisfying (7) and $H_0 \in W^{1,4}$, H is a weak solution of (8) initial data H_0. Assume that at $t \in (0, T)$ (15) and (16) are satisfied, and let μ and ν be the measures supported on Ω and defined by (14). Then, there exists an optimal mass reallocation plan $s : \Omega \to \Omega$, which solves (14), and there exists a function u so that s and u satisfy the equation

$$\frac{s(\mathbf{x}) - \mathbf{x}}{|s(\mathbf{x}) - \mathbf{x}|} = -\nabla u. \tag{17}$$

When Erosion implements Optimal Transport

Corollary

If ∇H is defined a.e. on Ω and satisfies

$$\nabla \times (\nabla H / |\nabla H|) = 0, \tag{18}$$

at time t, at a.e. points where ∇H is defined and non-zero, then at these points $\nabla u = \frac{\nabla H}{|\nabla H|}$. In this case, the sediment flow implements the optimal transport.

Does this ever happen? For what kind of surfaces is (18) satisfied?

Outline

Mountains and Ridges

It turns out that there exist weak solutions satisfying the condition $\nabla \times (\nabla H / |\nabla H|) = 0$. These are called mountains and ridges. $H(x, y, t) = H_o(x, y)T(t), \;\; T(t) = (a - 2\lambda t)^{-1/2}$.

Figure: A Mountain and a Ridge using separable solutions.

$$
\begin{aligned}
h_o(x, y) &= h_1(H_1^{1/c} + a(x - x_0) + b(y - y_0))^d \\
H_o(x, y) &= (H_1^{1/c} + a(x - x_0) + b(y - y_0))^c
\end{aligned}
\tag{19}
$$

Collapsing Hills

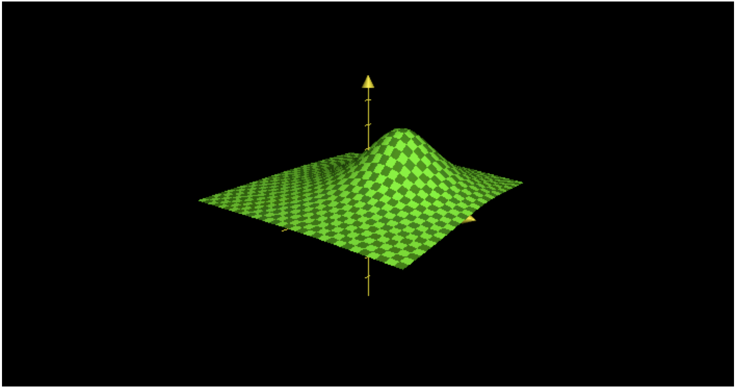


Figure: The collapsing hill

These collapsing hills violate (18) and the scaling.

Outline

The Half-Width and Number of Valleys

- Using Hack's law, one can work out the half-width

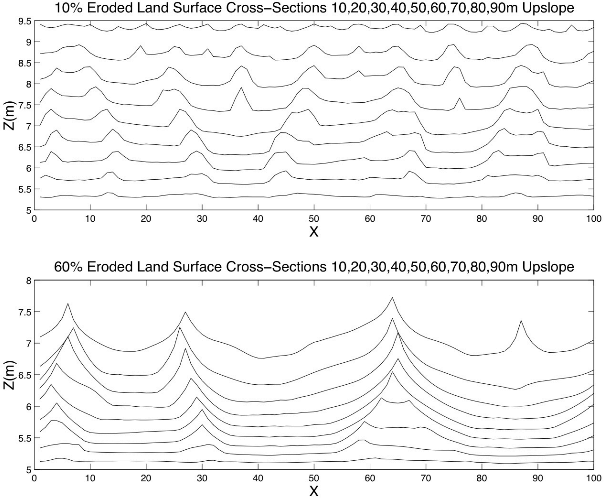
$$\ell \sim 0.40(H_{\max})^{4/3},$$

- and the number of the valleys

$$n \sim \frac{L}{0.80(H_{\max})^{4/3}}$$

- for mature long-lived landscape, where $H_{\max}$ is the maximum height of the mountain (upper boundary) and L is the length of the mountain range

Stochastic Theory by Linearization



The Scalings of a Landsurface

- By linearizing around young, adolescent and mature surfaces, we get a stochastic theory of erosion
- There are actually three ranges in "Hack's law" that have been identified, see B., Hernández and Smith (2007):
- Scaling exponent 1/2, Channelization
- Scaling exponent 2/3, Evolution of hillsides, see Welsh, B. and Bertozzi (2007).
- Scaling of Shocks, Bores, Hydraulic Jumps
- River turbulence has scaling exponent 3/4, Hack's law
- This is the largest range by far !

The Statistical Theory is Determined by the Invariant Measure

- The linearized PDEs driven by noise allows one to compute an invariant measure
- This measure determines all the deterministic statistical quantities, variogram, structure functions PDFs, etc.
- These quantities along with the mountains and ridges are the only deterministic quantities that exist
- The noise is generic, white in time and close to white in space
- Is is possible the the land surface evolutions is driven by the turbulent noise in the water flow?

Theorem (5)

In one dimension the Navier-Stokes equation (with pressure) driven by generic noise, has a unique solution if U is sufficiently large. Moreover, there exists a unique measure left invariant by the flow. The flow is ergodic and strongly mixing and the second structure function (variogram) scales with roughness exponent $\chi = 3/4$ in the statistically stationary state

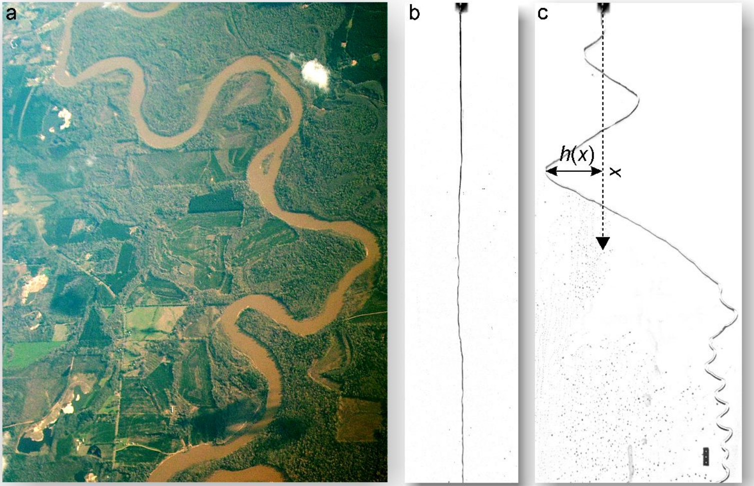
$$S_2(x) \sim |x|^{3/2}$$

All the statistical properties of the solution are determined by the invariant measure.

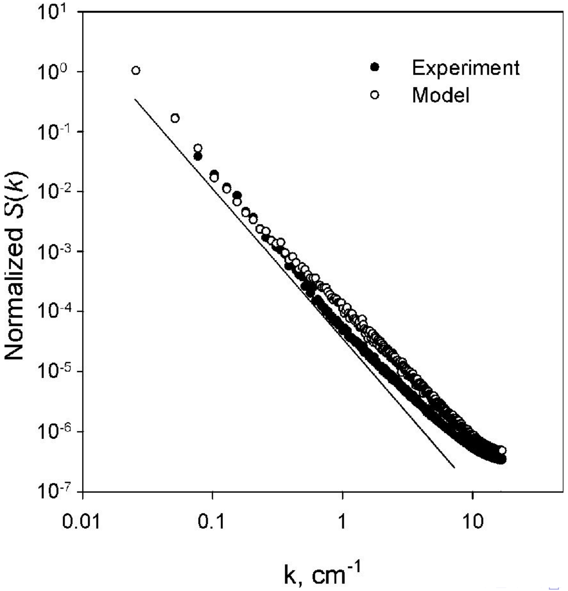
B. Turbulent Rivers (2007).

Meanderings of the Mississippi

B., K. Mertens, V. Putkaradze, P. Vorobieff (2008)



The Meandering Exponent (in the lab) is Determined by Turbulent Flow, $S(k) \sim k^{-5/2}$



Conclusions

- All the scaling laws of landsurface theory and river meanderings are analogous to the roughness coefficient of turbulent flow in rivers
- The theory gives an invariant measure that determines all the statistics of landsurface evolution
- This only holds up to the upper cut-off of the scaling where tectonic forces etc. must be taken into account
- The theory applies to mature surfaces, for young and channelizing surfaces the theory still exists but the scaling laws are different
- The mature surfaces possesses a scaling corresponding to Hölder continues functions of order 3/4. This scaling produces Hack's law

Speculation

- Kantorovich devised a notion of distance between probability measures.
- This distance is the optimal transport cost from one measure to the other; it is called Kantorovich-Rubinstein or Wasserstein distance, see F. Otto (2001).
- The invariant measure of the surface evolution should move towards the measure of the optimal surface in time, if we measure the distance in the Kantorovich-Rubinstein or Wasserstein metric.
- One should also be able to understand the surface itself as a measure and see it move towards the optimal measure in time, using a similar metric.

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