Stability of river bed forms

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Don't press this button!

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Thank you for your attention!

Stability people @ Unige

Giovanni Seminara

70s PhD at the Imperial College on hydrodynamic stability

80s → today: dynamics of free and forced bars, meanders, tidal morphodynamics

Paolo Blondeaux

80s → today: bar-bend theory of river meanders, sea ripples, sand banks, coastal morphodynamics

Marco Tubino

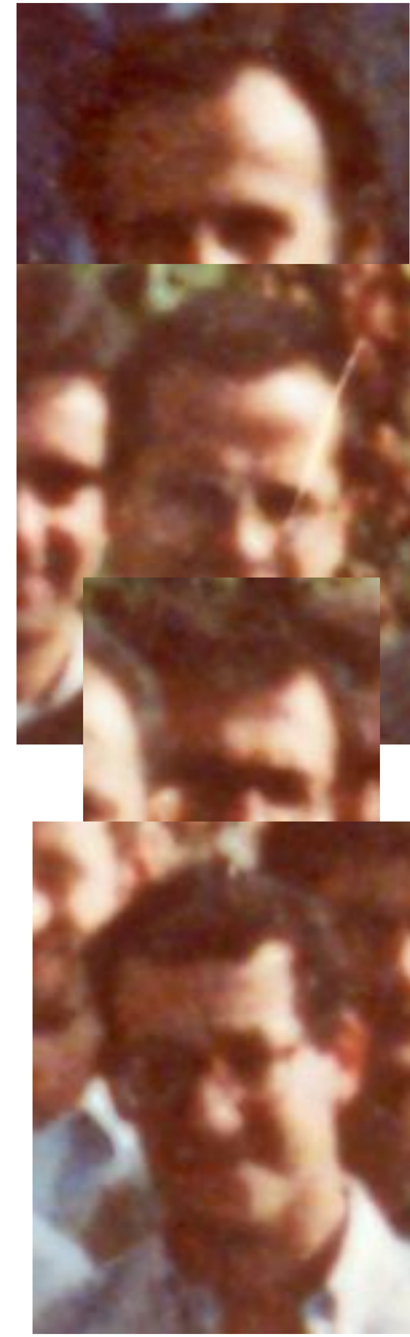
90s → today: fluvial bars, meanders, bifurcations, braiding

Marco Colombini

90s → today: fluvial bars, sand ribbons, streaks, dunes, antidunes, ripples, 3D dunes

XXI cen.: **Repetto, Zolezzi, Solari, Stocchino, Bolla, Besio…many others**

bars, meanders, width variations, dunes, river, coastal and tidal morphodynamics…



The origins …

Perugia – La Colombella – 1990

1st Summer School on

“Stability of Rivers and Coastal Forms”

E-book

1D SEDIMENT TRANSPORT MORPHODYNAMICS
with applications to
RIVERS AND TURBIDITY CURRENTS

http://hydrolab.illinois.edu/people/parkerg//morphodynamics_e-book.htm

Open questions

geoflows13 – Open Questions

geoflows13.wikispaces.com/Open+Questions

C) Morphodynamics

C.I) Are there common morphodynamic organizing principles active across the entire range of particle/fluid density ratios and particle volume fractions, and between gas- and liquid-mediated flows? If so, what are these principles?

C.II) How is a turbulent shear flow modified by the presence of bed forms in gases and in liquids?

C.III) Which mechanisms dominate the wavelength selection of bed forms in different parameter regimes?

C.IV) What can linear and nonlinear stability theory based on continuum theory teach us about wavelength selection?

C.V) What are the mechanisms that govern relaxation times and saturation lengths?

C.VI) What types of additional field measurements should be conducted in order to allow for the formulation and testing of simplified models?

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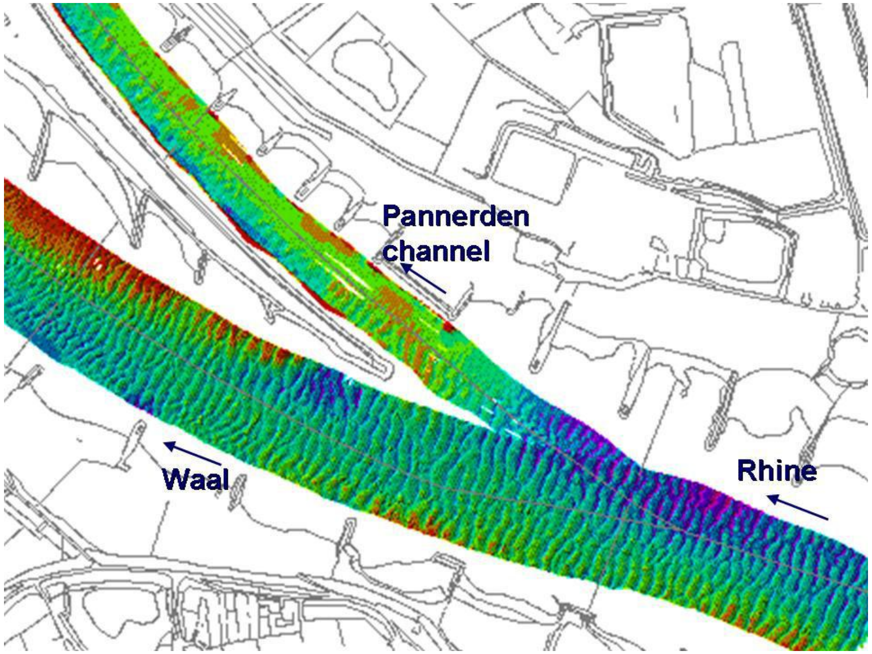
CHAPTER 8:
FLUVIAL BEDFORMS

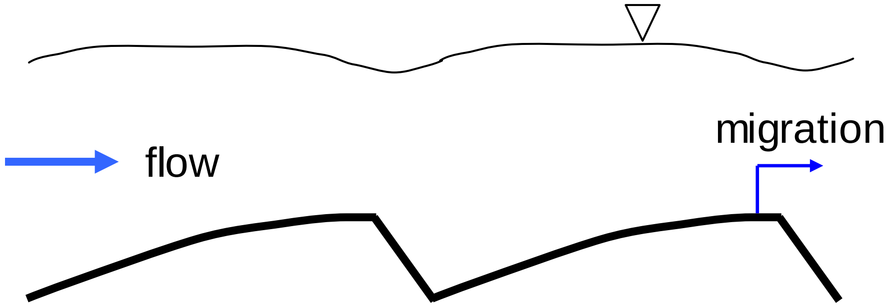
The interaction of flow and sediment transport often creates bedforms such as ripples, dunes, antidunes, and bars. These bedforms in turn can interact with the flow to modify the rate of sediment transport.

Dunes in the North Loup River, Nebraska, USA; image courtesy D. Mohrig

TOUR OF BEDFORMS IN RIVERS: DUNES

Dunes are characteristic of subcritical flows (**Fr** sufficiently below 1) and of sand-bed or rivers. Typical wavelengths can range up to 100 m, and wave height can range up to 5 m or more in large rivers. Dunes migrate downstream and are usually asymmetric, with a gentle stoss (upstream) side and a steep lee (downstream) side. They interact weakly with the water surface, such that the flow accelerates over the crests, where water surface elevation is slightly reduced. (That is, the water surface is out of phase with the bed.)

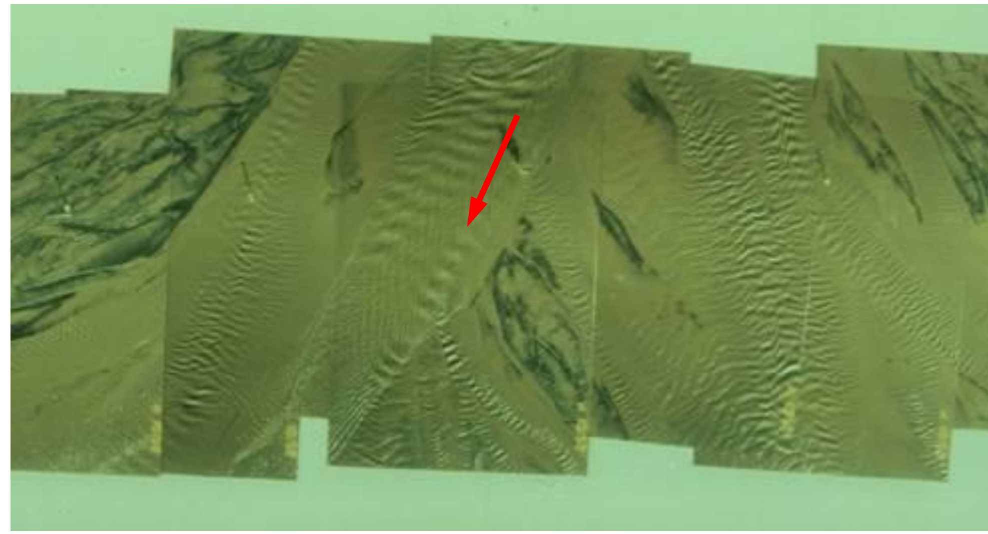


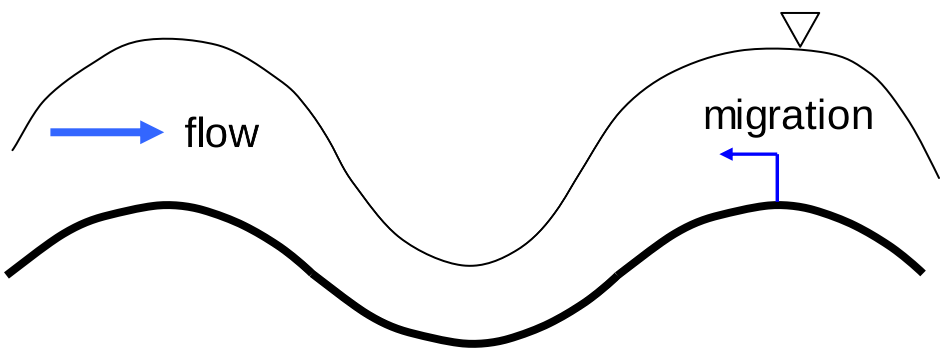


Dunes in the Rhine Delta, The Netherlands.
Image courtesy A. Wilburs and A. Blom

TOUR OF BEDFORMS IN RIVERS: ANTIDUNES

Antidunes occur in rivers with sufficiently high (but not necessarily supercritical) Froude numbers. They can occur in sand-bed and gravel-bed rivers. The most common type of antidune migrates upstream, and shows little asymmetry. The water surface is strongly in phase with the bed. A train of symmetrical surface waves is usually indicative of the presence of antidunes.





Trains of surface waves indicating the presence of antidunes in braided channels of the tailings basin of the Hibbing Taconite Mine, Minnesota, USA. Flow is from top to bottom.

How do you tell a dune from an antidune?

Bedform classification

- By shape: yes, dunes are asymmetric, antidunes are not; but for a linear analysis they are both sinusoidal.
- By free surface phase: yes, dunes are out of phase, antidunes are in phase.
- By direction of migration: yes, downstream for dunes, upstream for antidunes; but you can have downstream-migrating antidunes.
- By wavelength: no, they have almost the same.
- By Froude number: yes, subcritical for dunes, supercritical for antidunes; but antidunes can become unstable for Froude numbers below one.

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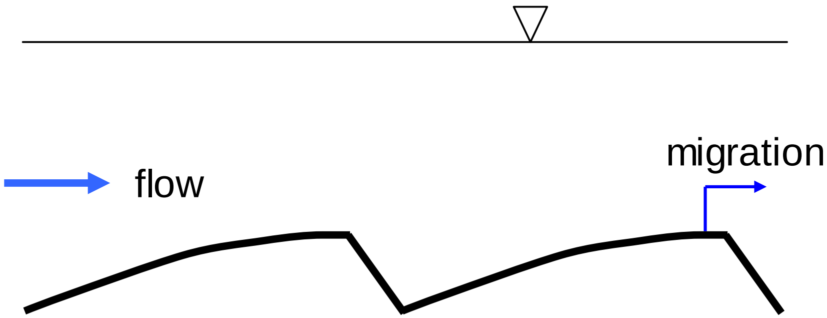
TOUR OF BEDFORMS IN RIVERS: RIPPLES

Ripples are characteristic of very low transport rates in rivers with fine sediments (less than 0.6 mm). Typical wavelengths are on the order of 10 cm and and wave heights are on the order of cm. Ripples migrate downstream and are asymmetric with a gentle stoss (upstream) side and a steep lee (downstream) side. Ripples do not interact with the water surface.



View of the Rum River, Minnesota USA

Ripples in the Rum River at very low flow; L ~ 10 - 20 cm.

TOUR OF BEDFORMS IN RIVERS: DUNES

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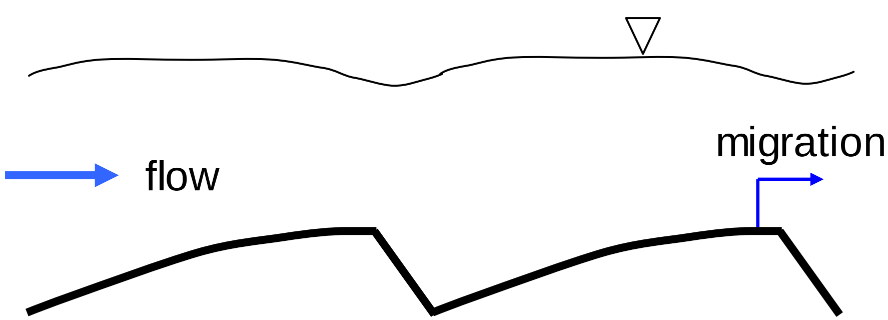

Dunes in a flume in Tsukuba University, Japan: flow turned off. Image courtesy H. Ikeda.

How do you tell a ripple from a dune?

- By shape: no, they are both asymmetric.
- By free surface phase: no, free surface is flat for ripples.
- By direction of migration: no, they both migrate downstream.
- By wavelength: yes, dunes are about ten times longer than ripples.
- By Froude number: no, they are both subcritical bedforms.
- By flow regime: yes, ripples only appear in the smooth (or transitional) regime.

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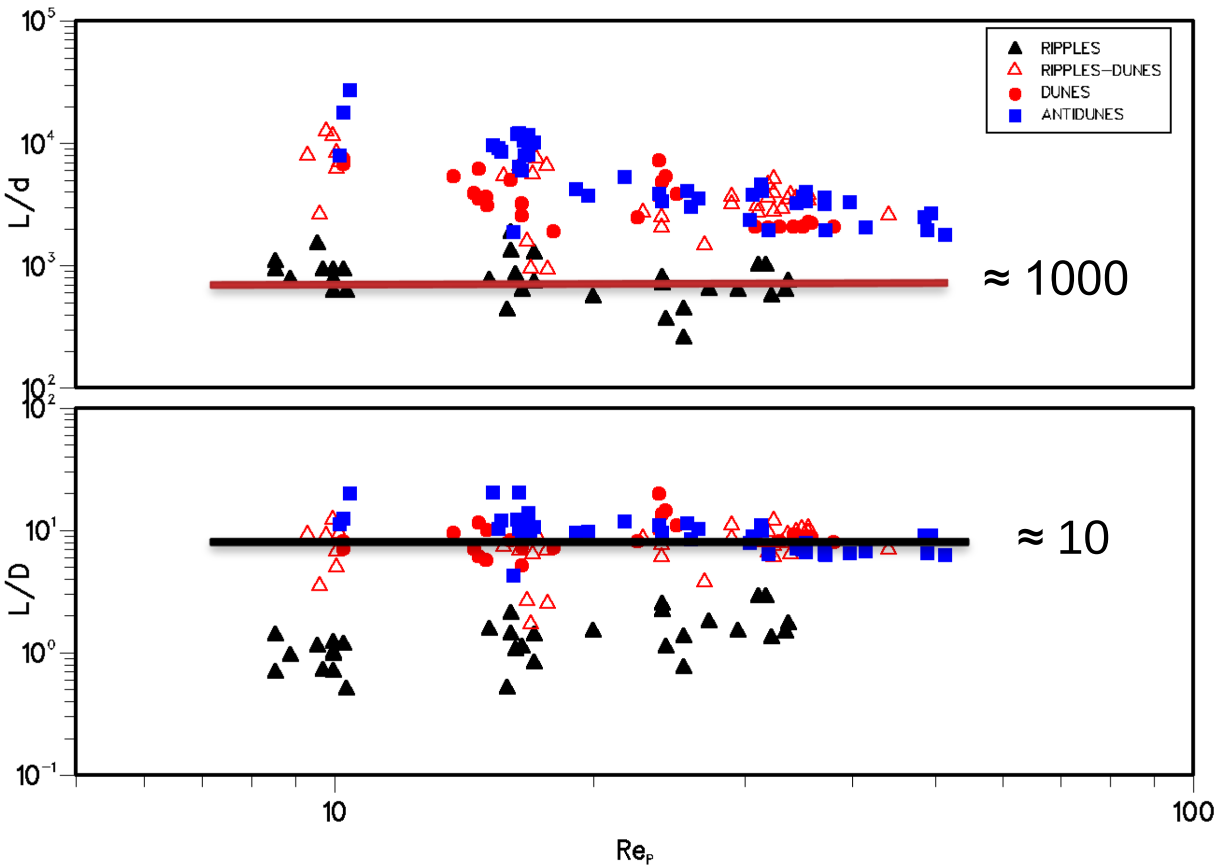
Longitudinal scaling: ripples & dunes

Experiments by Guy, Simons & Richardson (1966)



$$\mathrm{Re}_P = \frac{\sqrt{(s-1)gd^*}\,d^*}{n}$$

What is linear stability all about?

- Is about perturbing a base state with small, strictly infinitesimal, perturbations and looking at their time evolution.
- Base state is a steady uniform flow in an infinitely wide open channel with active sediment transport.
- Perturbations evolve exponentially in time. They either decay (negative growth rate) or amplify (positive growth rate).
- Perturbations can also migrate downstream (positive celerity) or upstream (negative celerity).

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EXPANSION OF A GENERIC VARIABLE:

Base state — Perturbation

$$G(x,\eta,t) = G_0(\eta) + \varepsilon\left[G_1(x,\eta,t) + c.c.\right] \qquad \eta = \frac{y - R}{D}$$

Small parameter

- **LEADING ORDER** - $O(\varepsilon^0)$: uniform flow

$$G_1(x,\eta,t) = g_1(\eta)\exp\left[ik_X(x - \omega t)\right] \qquad k_X = \frac{2\pi D^*}{L_X^*}$$

- **LINEAR LEVEL** - $O(\varepsilon^1)$: eigenvalue problem

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$$\Omega = k_X\omega^i = \Omega(k_X; parameters) \qquad c = \omega^r = c(k_X; parameters)$$

Growth rate — Celerity

A smorgasbord of parameters ...

ROUGH CASE (no Reynolds dependence)

Characteristic scales

- D^* Uniform flow depth
- U^* Uniform flow velocity (avgd)
- t^* Bed shear stress
- S Channel slope (avgd)
- d^* Grain diameter
- z_R^* Bed roughness
- L_X^* Longitudinal wavelength

Nondimensional parameters

- J Shields parameter
- Fr Froude number
- C Chézy coefficient
- d Grain size (non dim.)
- z_R Bed roughness (non dim.)
- k_X Longitudinal wavenumber

$$J = \frac{t^*}{\rho (s-1) g d^*}, \quad Fr = \frac{U^*}{\sqrt{g D^*}}, \quad C = \frac{1}{k}\ln\left(\frac{11.09 D^*}{z_R^*}\right), \quad d = \frac{d^*}{D^*}, \quad z_R = \frac{z_R^*}{D^*}, \quad k_X = \frac{2\pi D^*}{L_X^*}$$

Nondimensional parameters

ϑ Shields parameter

Fr Froude number

C Chézy coefficient

d Grain size (non dim.)

z_R Bed Roughness (non dim.)

k_X Longitudinal Wavenumber

… but two are enough!

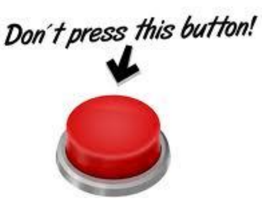
1) Either *C* or *d*

$$C = \frac{1}{k} \ln\left(\frac{11.09}{2.5d}\right)$$

2) Either ϑ or *Fr*

$$\vartheta = \frac{Fr^2}{C^2 (s-1) d}$$

A model problem: roll-wave instability



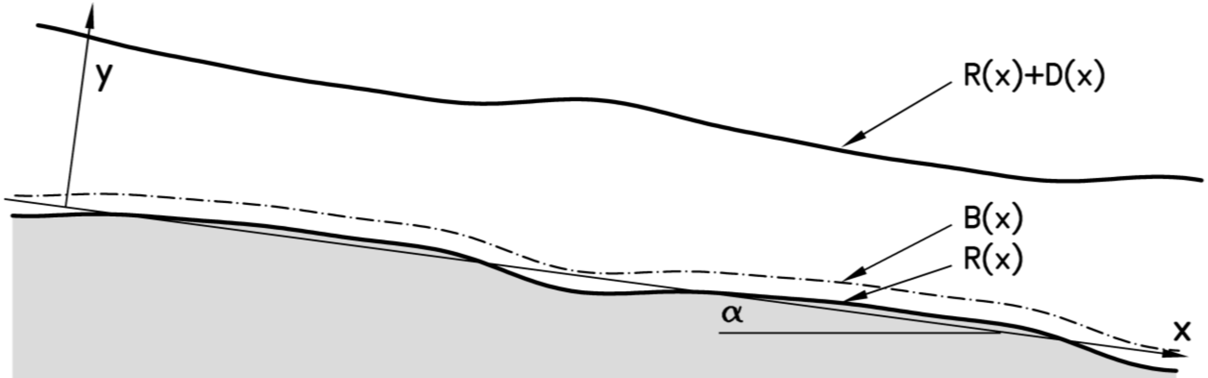
A model problem: roll waves instability

Roll waves in a concrete channel in Lions Bay, Brithish Columbia, Canada

Video courtesy of N. Balmforth

Roll waves are long free surface waves that propagate very fast downstream over a flat non erodible bed. The roll-wave hydrodynamic stability has been studied by Jeffreys (1925), Dressler (1949), Needham & Merkin (1984) and many others. Recently by Balmforth & Vakil (2012) who studied roll-wave morphodynamic instability

Depht-averaged flow model



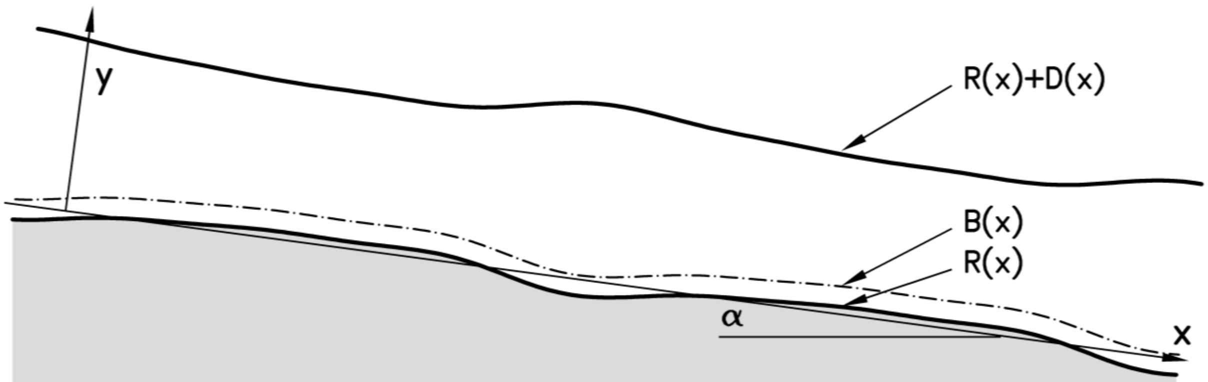
$$h = \frac{y - R}{D}$$

1D SHALLOW-WATER EQUATION + CONTINUITY

$$U_{,t} + UU_{,x} - \frac{S}{Fr^2} + \frac{1}{Fr^2}\left(R + D\right)_{,x} + \frac{T_R}{D} - \frac{1}{D}\left[D\left(T_{xx} - T_D\right)\right]_{,x} = 0$$

$$D_{,t} + UD_{,x} + DU_{,x} = 0$$

Depht-averaged flow model



$$\eta=\frac{y-R}{D}$$

1D SHALLOW-WATER EQUATION + CONTINUITY

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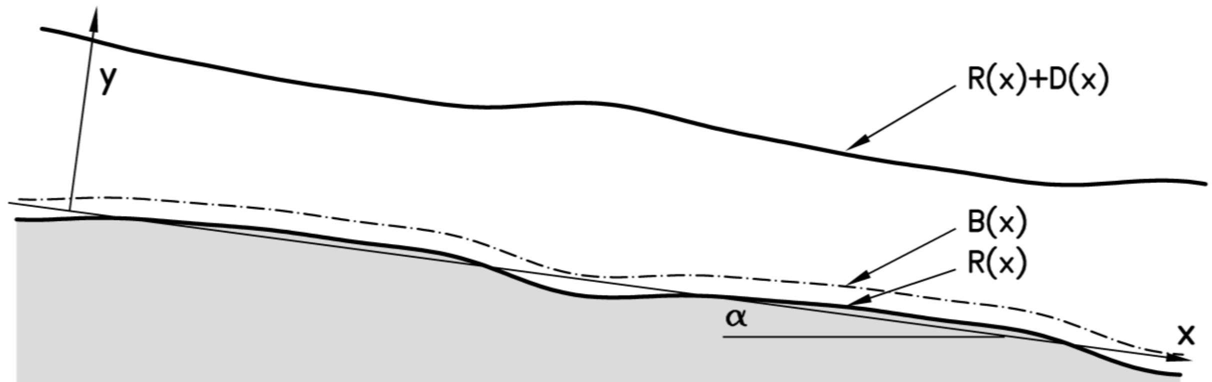
where

$$T_R=\frac{1}{D}\left[\nu_T u_{,\eta}\right]_{\eta=0} \qquad T_{xx}=2\int_0^1 \nu_T u_{,x}\,d\eta \qquad T_D=\int_0^1 (u-U)^2\,d\eta$$



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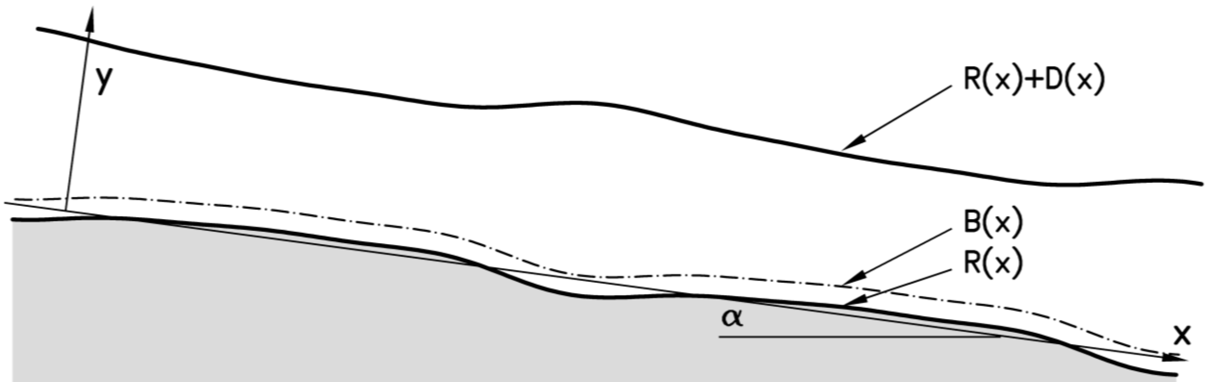
Equations have been made dimensionless with $U_0^*,\ D_0^*$

where

$$U_0^*=C\sqrt{gD_0^*S} \qquad C=\frac{1}{k}\ln\left(\frac{11.09}{2.5d}\right) \qquad \Rightarrow \qquad S=\frac{Fr^2}{C^2}=\tan\alpha$$

Depht-averaged flow model



1D SHALLOW-WATER EQUATION + CONTINUITY

$$U_{,t} + UU_{,x} - \frac{S}{Fr^2} + \frac{1}{Fr^2}(R+D)_{,x} + \frac{T_R}{D} - \frac{1}{D}\left[D\left(T_{xx} - T_D\right)\right]_{,x} = 0$$

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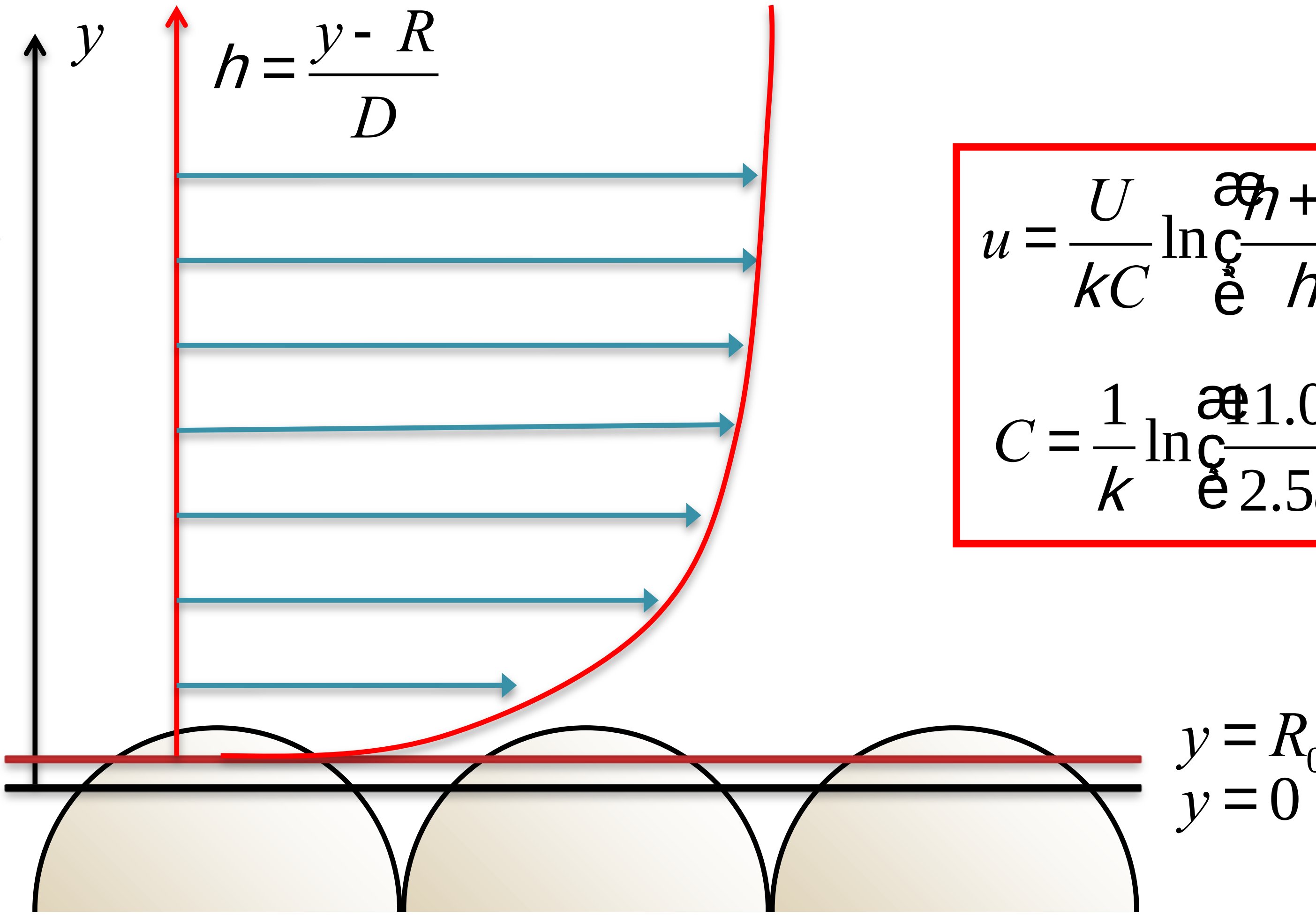
A ‘closure’ is required for the stresses, which is built upon the following vertical profiles for the velocity and the eddy viscosity

$$u = \frac{U}{\kappa C}\ln\left(\frac{\eta+\eta_0}{\eta_0}\right) \qquad \nu_T = \frac{\kappa}{C}UD\left(\eta+\eta_0\right)\left(1-\eta\right)$$

Vertical profile of velocity



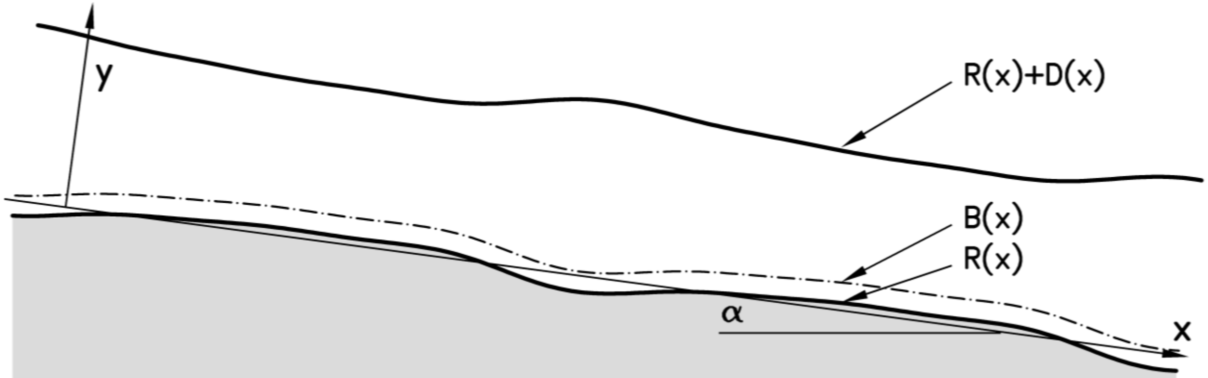
$$u = \frac{U}{\kappa C} \ln\left(\frac{\eta + \eta_0}{\eta_0}\right)$$

$$C = \frac{1}{\kappa} \ln\left(\frac{11.09}{2.5d}\right)$$

$$U = \int_0^1 u\, d\eta = \frac{U}{\kappa C} \int_0^1 \ln\left(\frac{\eta + \eta_0}{\eta_0}\right) d\eta = -\frac{U}{\kappa C}\left[1 + \ln(\eta_0)\right]$$

$$\eta_0 = \exp\left(-\kappa C - 1\right) = \frac{d}{12} = \frac{2.5d}{30} = \frac{y_R}{30} = R_0$$

Depht-averaged flow model



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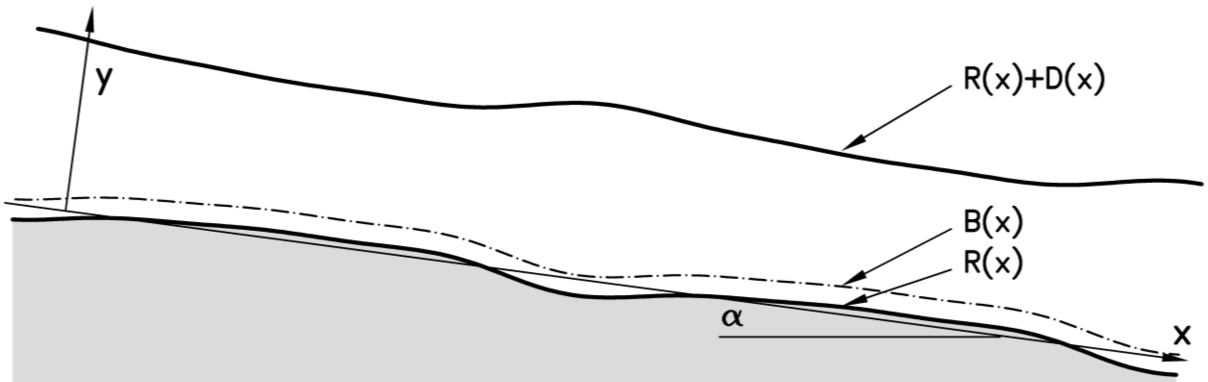
$$D_{,t} + UD_{,x} + DU_{,x} = 0$$

where

$$T_R = \frac{U^2}{C^2} \qquad T_{xx} = \frac{1}{C^2} NDUU_{,x} \qquad T_D = \frac{U^2}{k^2 C^2}$$

and

$$N = \frac{1}{3}\left(kC + \frac{1}{6}\right)$$



Linearization

LINEARIZATION

$$G(x,t) = G_0 + \epsilon G_1(x,t)$$

BASE STATE

$$U_0 = 1 \qquad D_0 = 1 \qquad T_{R0} = \frac{1}{C^2} \qquad T_{xx0} = 0 \qquad T_{D0} = \frac{1}{k^2 C^2}$$

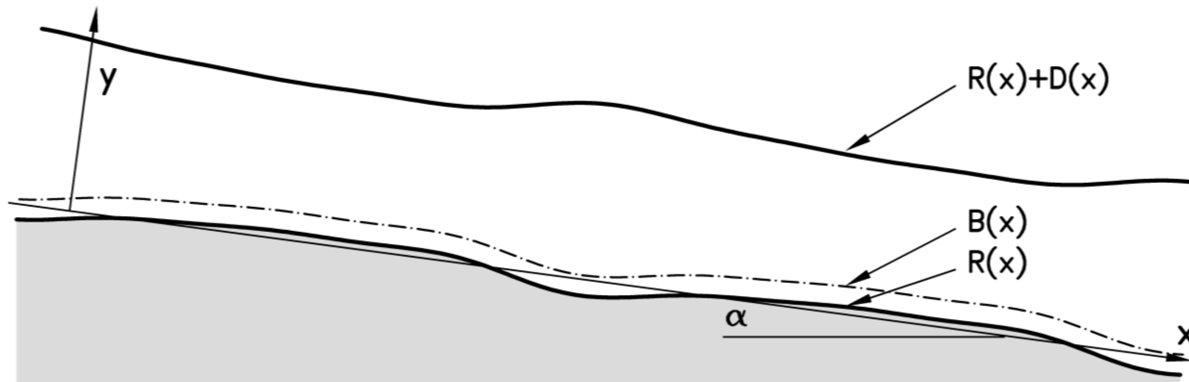
LINEAR LEVEL

$$U_{1,t} + U_{1,x} + \frac{1}{Fr^2}\left(R_1 + D_1\right)_{,x} + T_{R1} - T_{R0} D_1 - \left(T_{xx1} + T_{D1}\right)_{,x} + T_{D0} D_{1,x} = 0$$

$$D_{1,t} + D_{1,x} + U_{1,x} = 0$$

$$T_{R1} = 2 T_{R0} U_1 \qquad T_{xx1} = T_{R0} N U_{1,x} \qquad T_{D1} = 2 T_{D0} U_1$$

Linearization



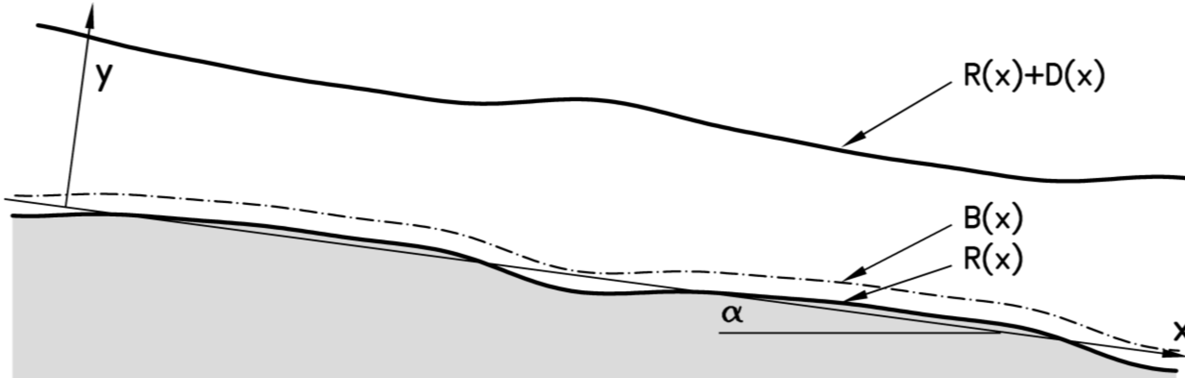
LINEAR LEVEL

$$G_1(x,t) = g_1 \exp[ik(x - \omega t)] + c.c.$$

$$\left(A - \omega I\right) \cdot \vec{x} = -r_1 \vec{f}$$

$$\begin{bmatrix} a_{11} - \omega & a_{12} \\ 1 & 1 - \omega \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ d_1 \end{bmatrix} = -r_1 \begin{bmatrix} 1/Fr^2 \\ 0 \end{bmatrix}$$

$$a_{11} = 1 + 2T_{D0} - i\frac{T_{R0}}{k_x}\left(2 + Nk_x^2\right) \qquad a_{12} = \frac{1}{Fr^2} - T_{D0} + i\frac{T_{R0}}{k_x}$$



EIGENVALUE PROBLEM

$$\left(A-\omega I\right)\cdot\vec{x}=\{0\}$$

$$\begin{bmatrix} a_{11}-\omega & a_{12} \\ 1 & 1-\omega \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

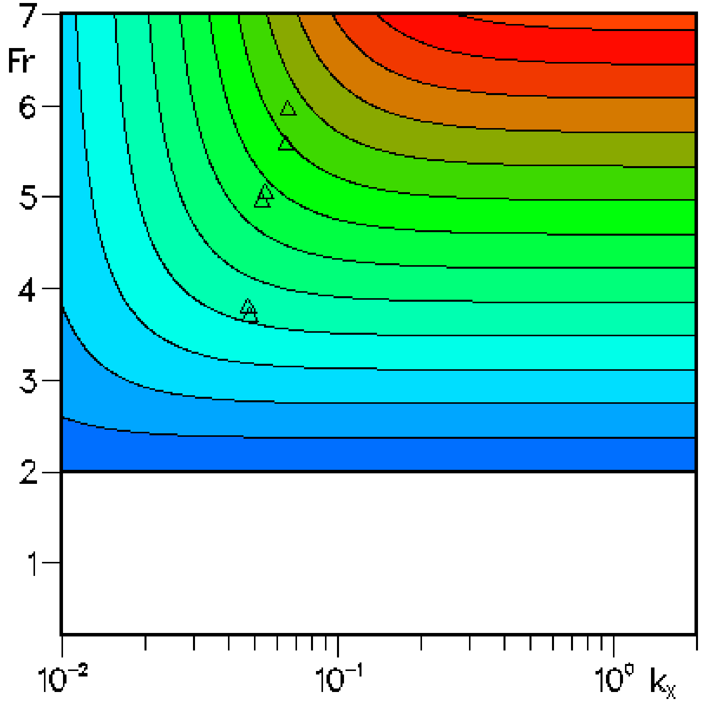
$$\det\left(A-\omega I\right)=0$$

Two eigenvalues: ‘fast’ and ‘slow’

Growth rate of flow fast eigenvalue

Without

Dispersive and Reynolds stresses



Experiments by Brock (1969)

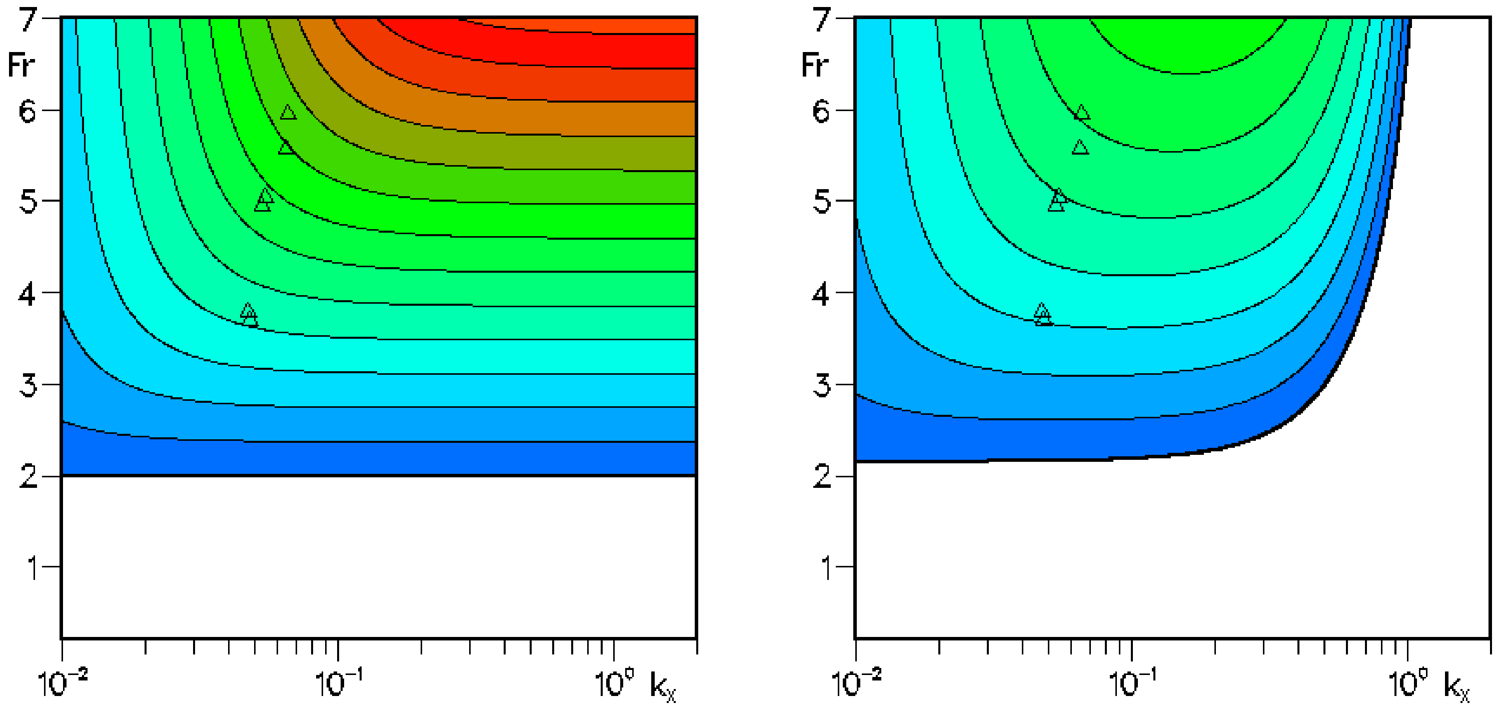
Roll wave instability

Flow fast eigenvalue

Roll wave instability

Without Dispersive and Reynolds stresses

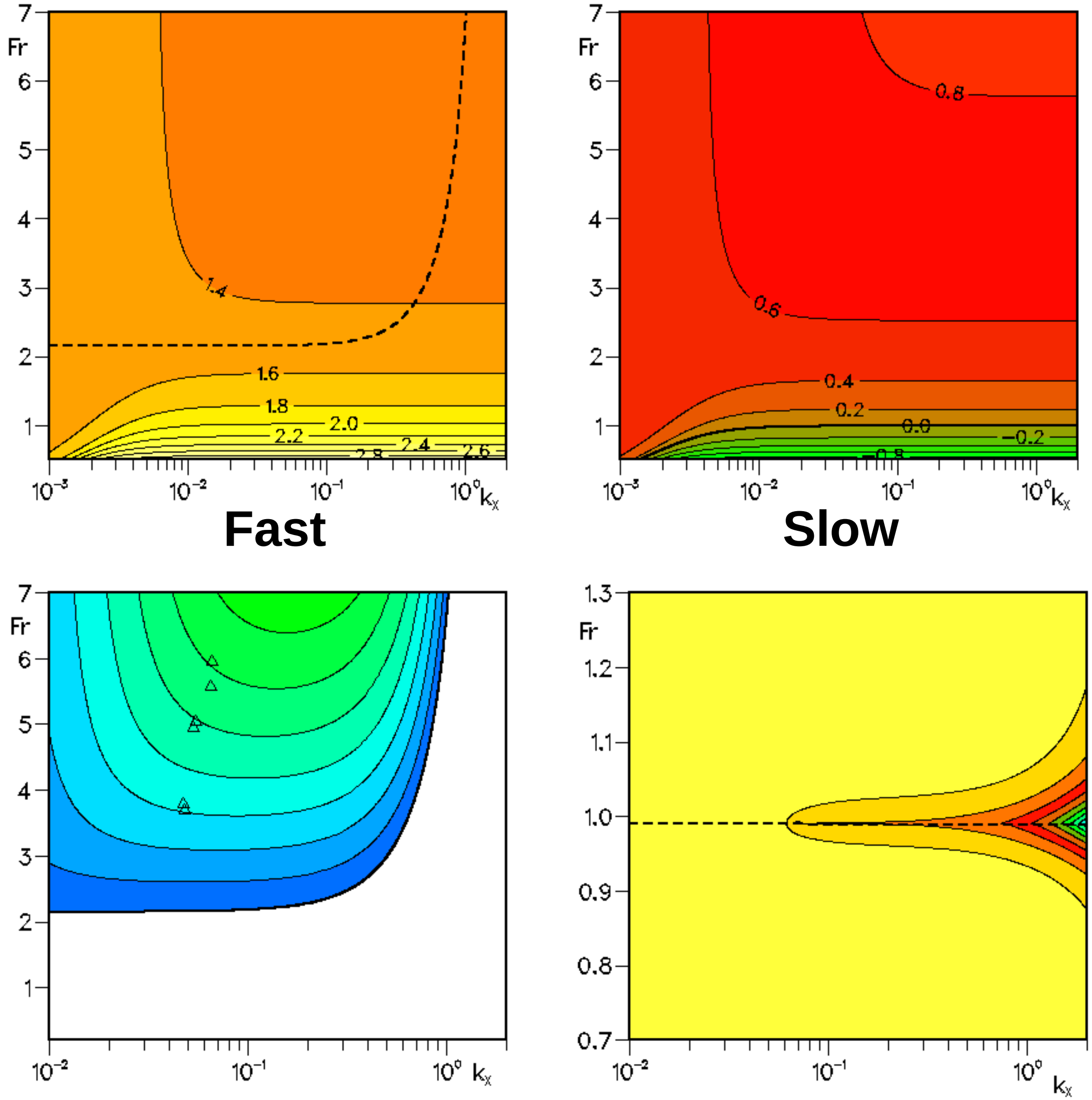
With Dispersive and Reynolds stresses



Experiments by Brock (1969)

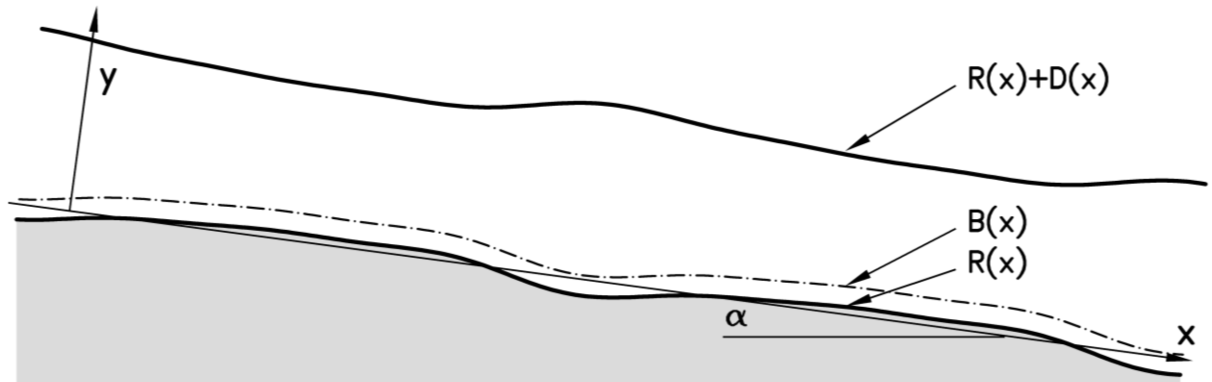
Fast & slow flow eigenvalues

Celerity



Growth rate

Sediment transport model



SEDIMENT CONTINUITY EQUATION (EXNER)

$$R_{,t} + Q_S \mathrm{F}_{,x} = 0 \qquad Q_S = \frac{1}{F}\frac{\sqrt{(s-1)d^3}}{1-p} << 1$$

BEDLOAD FUNCTION (MPM after Wong & Parker (2006))

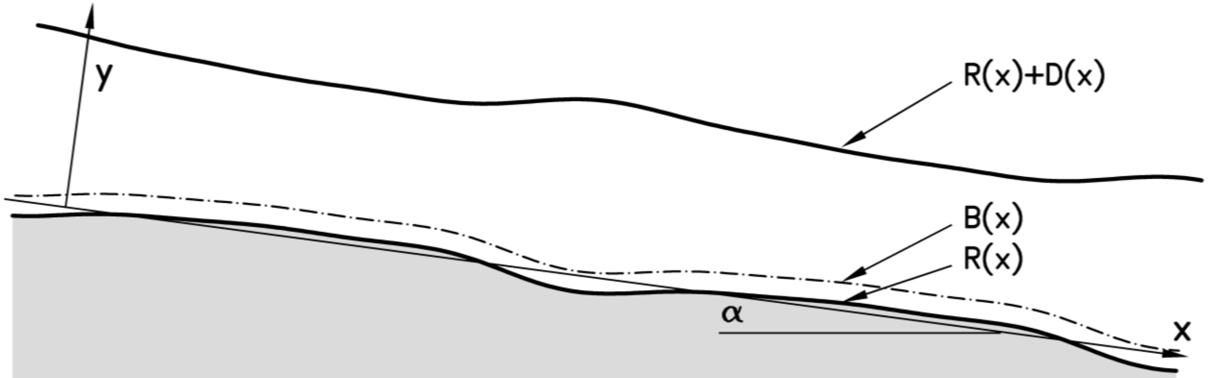
$$\mathrm{F} = 3.97\left(J - J_C\right)^{3/2}$$

CORRECTION FOR SEDIMENT WEIGHT

$$J_C = 0.0495 - m_x\left(S - R_{,x}\right)$$

$m_x = 0.1$ Fredsøe (1974)

$m_x = 2.8J$ Richards (1980)



LINEARIZATION

$$G(x,t) = G_0 + \varepsilon G_1(x,t)$$

BASE STATE

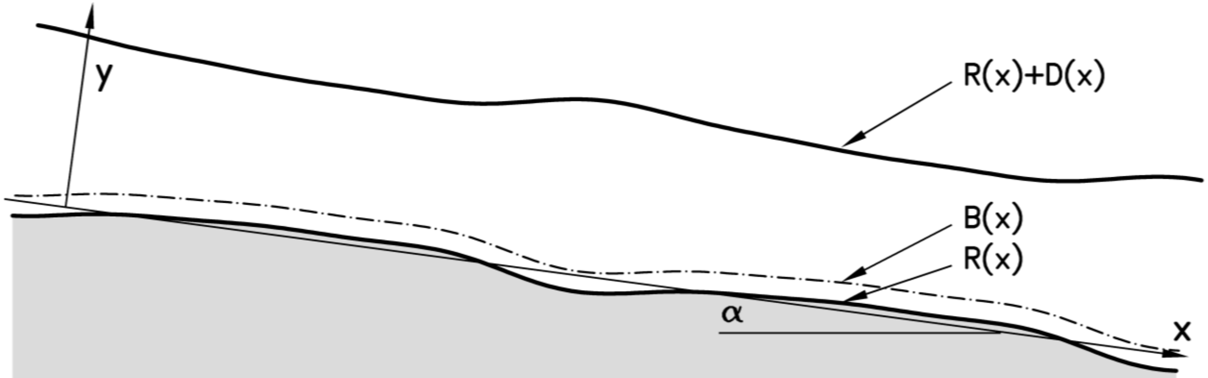
$$\mathrm{F}_0 = 3.97\left(J_0 - J_{C0}\right)^{3/2} \qquad J_0 = \frac{T_{R0} Fr^2}{(s-1)d} \qquad J_{C0} = 0.0495 - m_x S$$

LINEAR LEVEL

$$R_{1,t} + Q_S \mathrm{F}_{1,x} = 0$$

$$\mathrm{F}_1 = \frac{3}{2}\frac{\mathrm{F}_0}{J_0 - J_{C0}}\left(J_1 - J_{C1}\right) \qquad J_1 = T_{R1} J_0 / T_{R0} \qquad J_{C1} = m_x R_{1,x}$$

Linearization



LINEAR LEVEL

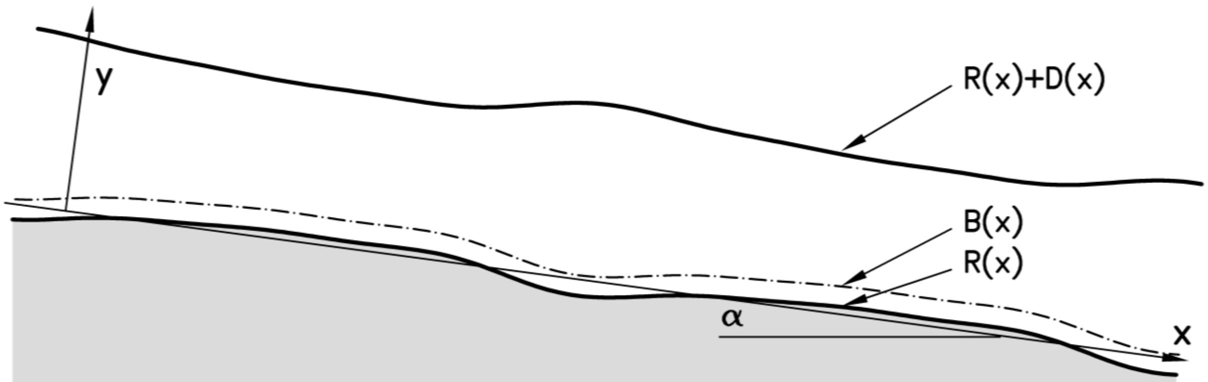
$$G_1(x,t) = g_1 \exp[ik(x - \omega t)] + c.c.$$

$$\left(B - \omega I\right) \cdot \vec{x} = \{0\}$$

SW Equation
Continuity
Exner

$$\begin{bmatrix} a_{11} - \omega & a_{12} & 1/Fr^2 \\ 1 & 1 - \omega & 0 \\ b_{31} & 0 & b_{33} - \omega \end{bmatrix} \times \begin{bmatrix} u_1 \\ d_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b_{31} = 3Q_S \frac{\Phi_0 \vartheta_0}{\vartheta_0 - \vartheta_{C0}} \qquad b_{33} = -\frac{3}{2} Q_S i k_x \frac{\Phi_0 m_x}{\vartheta_0 - \vartheta_{C0}}$$



EIGENVALUE PROBLEM

$$\left(B-\omega I\right)\cdot\vec{x}=\left\{0\right\}$$

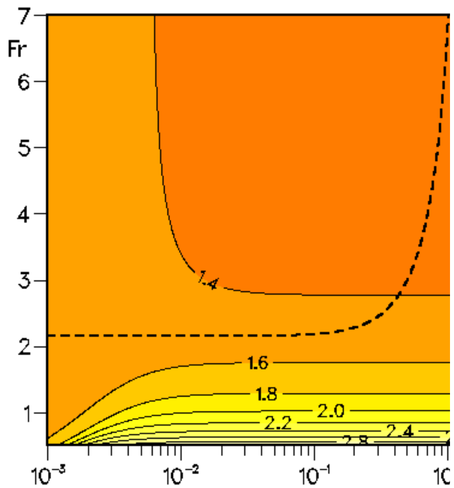
$$\begin{bmatrix} a_{11}-\omega & a_{12} & 1/Fr^2 \\ 1 & 1-\omega & 0 \\ b_{31} & 0 & b_{33}-\omega \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ d_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

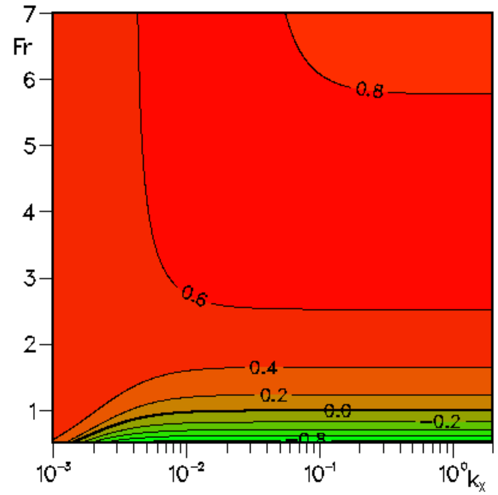
$$\det\left(B-\omega I\right)=0$$

Three eigenvalues: ‘fast’, ‘slow’ and ’morpho’

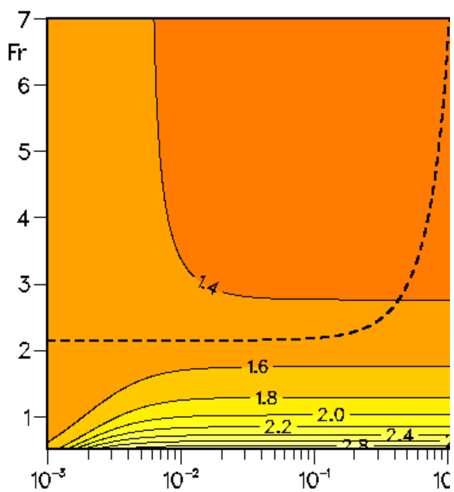
Flow & morphodynamic instabilities

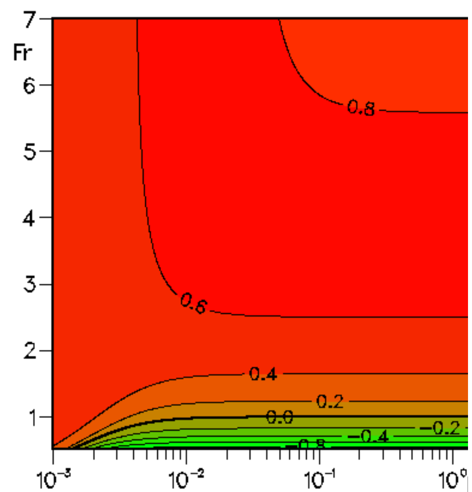
Flow instability

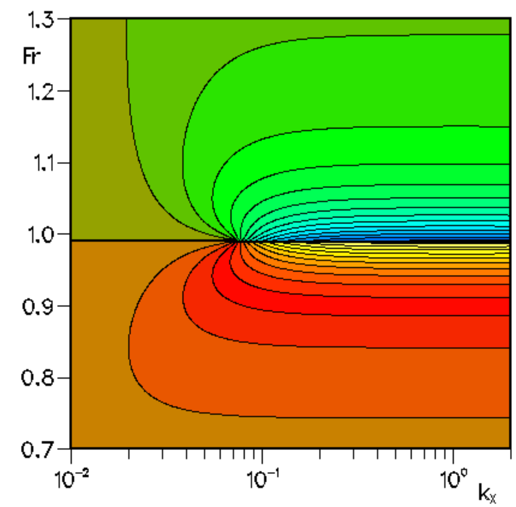




Fast **Slow** **Morpho**

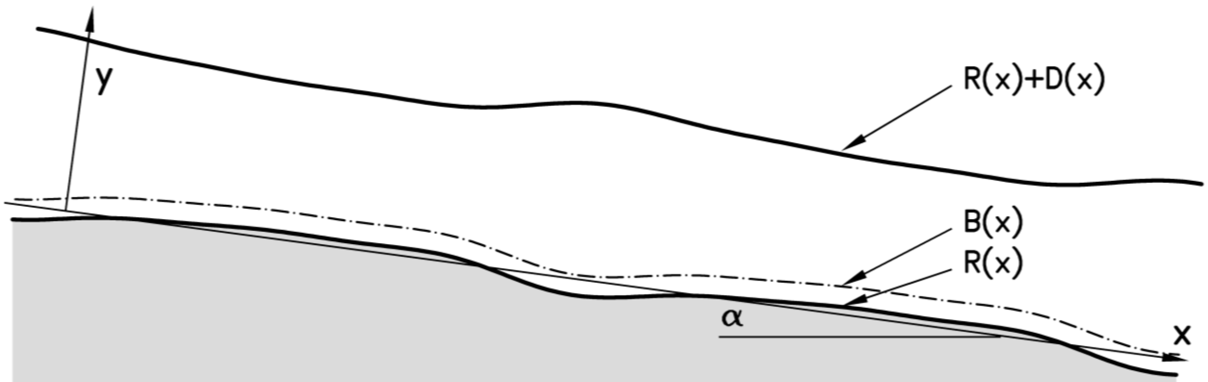






Morphodynamic instability

Quasi-steady approximation



1D SHALLOW-WATER EQUATION + CONTINUITY

$$\cancel{U_{,t}} + UU_{,x} - \frac{S}{Fr^2} + \frac{1}{Fr^2}(R+D)_{,x} + \frac{T_R}{D} - \frac{1}{D}\left[D\left(T_{xx} - T_D\right)\right]_{,x} = 0$$

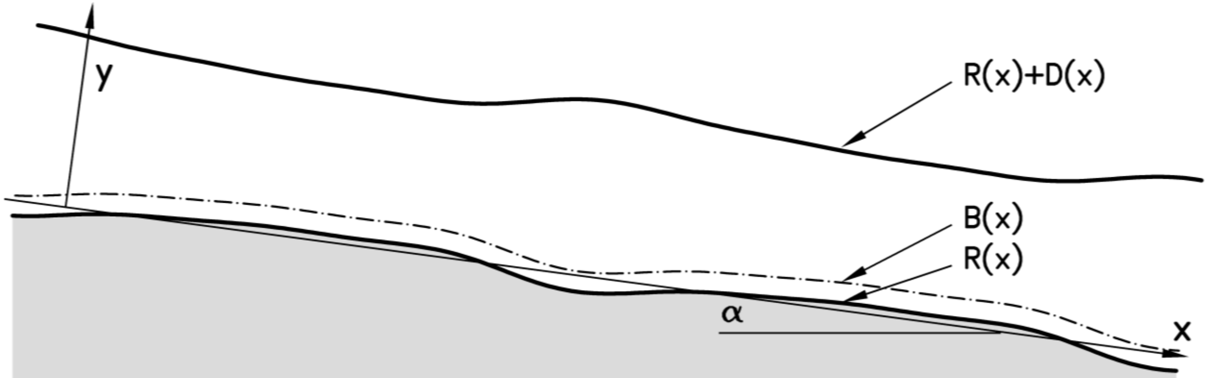
$$\cancel{D_{,t}} + UD_{,x} + DU_{,x} = 0$$

SEDIMENT CONTINUITY EQUATION (EXNER)

$$R_{,t} + Q_S \Phi_{,x} = 0$$

$$\begin{bmatrix} a_{11} - \cancel{\omega} & a_{12} & 1/Fr^2 \\ 1 & 1 - \cancel{\omega} & 0 \\ b_{31} & 0 & b_{33} - \omega \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ d_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Quasi-steady approximation



EIGENVALUE PROBLEM

$$\begin{bmatrix} a_{11} & a_{12} & 1/Fr^2 \\ 1 & 1 & 0 \\ b_{31} & 0 & b_{33} - \omega \end{bmatrix} \times \begin{bmatrix} u_1 \\ d_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

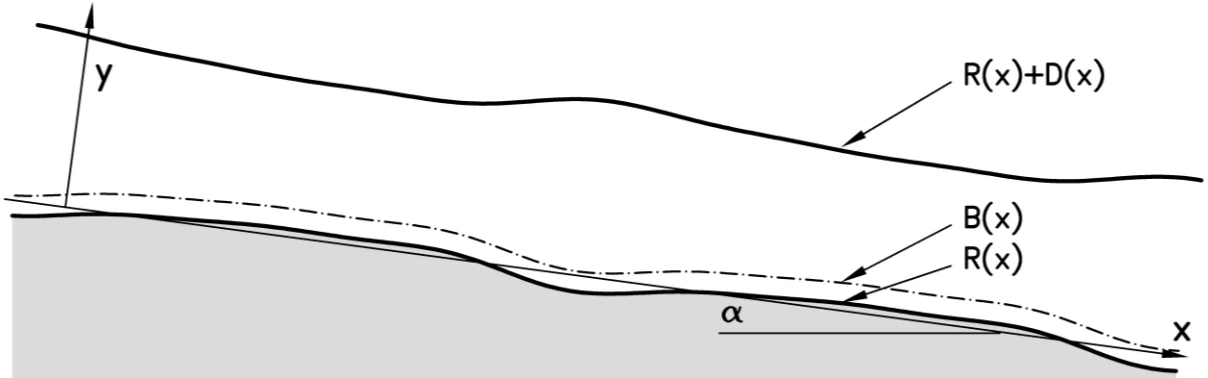
$$\omega = b_{33} - \frac{b_{31}}{Fr^2\left(a_{11} - a_{12}\right)}$$

One eigenvalue: 'morpho'

$$\Omega = k_x^2 \Omega_0 (T - W)$$

$$T = \frac{t_{R1}^i}{k_x} \qquad W = \frac{m_x}{J_0}$$

Quasi-steady approximation



EIGENVALUE PROBLEM

$$\begin{bmatrix} a_{11} & a_{12} & 1/Fr^2 \\ 1 & 1 & 0 \\ b_{31} & 0 & b_{33} - \omega \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ d_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega = b_{33} - \frac{b_{31}}{Fr^2\left(a_{11} - a_{12}\right)}$$

One eigenvalue: 'morpho'

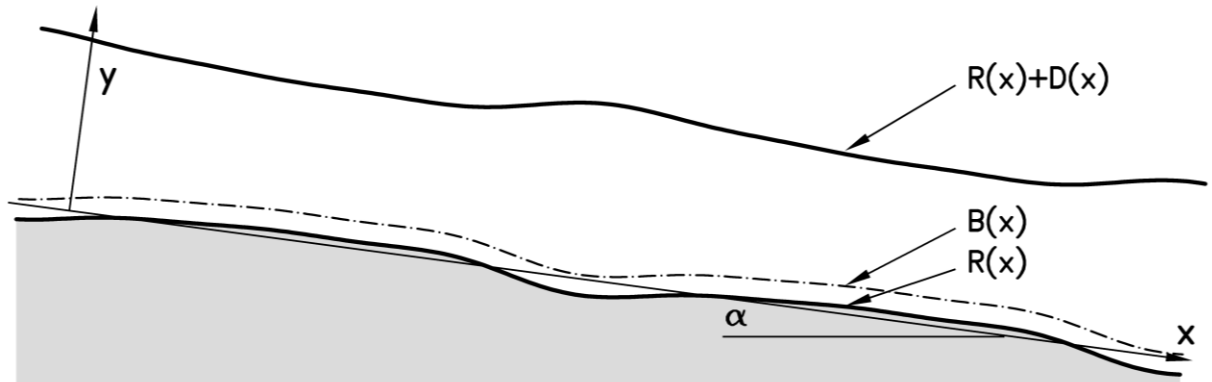
$$\Omega = k_x^2 \Omega_0 (T - W)$$

$$T = \frac{t_{R1}^i}{k_x} \qquad W = \frac{m_x}{J_0}$$

Either stabilizing or destabilzing

Always stabilizing

Quasi-steady approximation



EIGENVALUE PROBLEM

$$\begin{bmatrix} a_{11} & a_{12} & 1/Fr^2 \\ 1 & 1 & 0 \\ b_{31} & 0 & b_{33} - w \end{bmatrix} \times \begin{bmatrix} u_1 \\ d_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

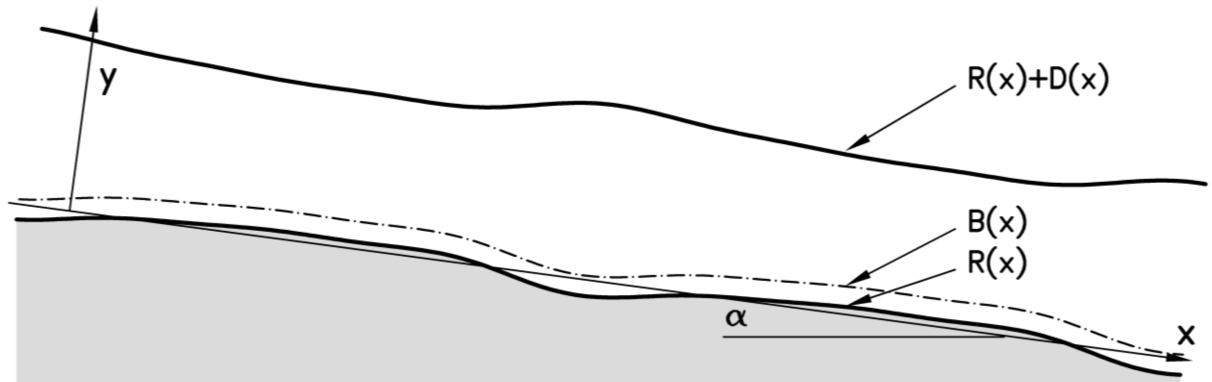
1) Determine linear flow response to a unitary bed perturbation

$$\begin{bmatrix} a_{11} & a_{12} \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} \widehat{u}_1 \\ \widehat{d}_1 \end{bmatrix} = - \begin{bmatrix} 1/Fr^2 \\ 0 \end{bmatrix}$$

2) Substitute into Exner's equation

$$W = k_x^2 W_0 (T - W) \qquad T = \frac{\widehat{t}^{\,i}_{R1}}{k_x} \qquad W = \frac{m_x}{J_0}$$

Resonance



EIGENVALUE PROBLEM

$$\begin{bmatrix} a_{11} & a_{12} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \hat{u}_1 \\ \hat{d}_1 \end{bmatrix} = -\begin{bmatrix} 1/Fr^2 \\ 0 \end{bmatrix}$$

FLOW RESPONSE to a unitary bed perturbation

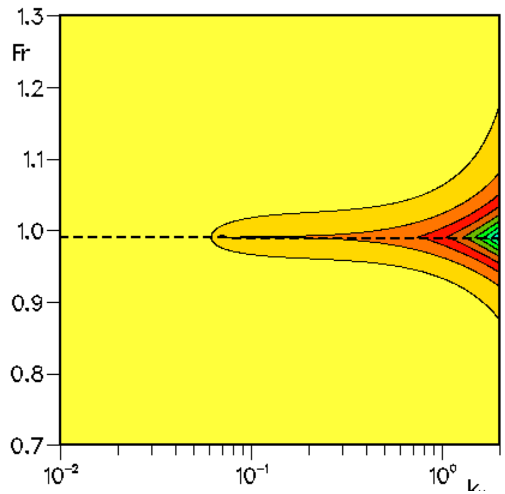
$$\begin{bmatrix} a_{11}-\omega & a_{12} \\ 1 & 1-\omega \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

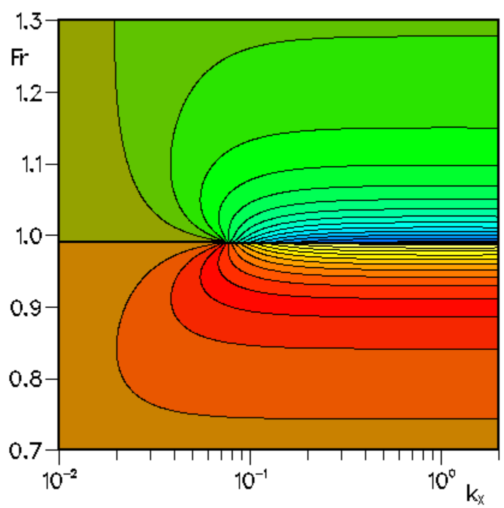
FLOW INSTABILITY

$\omega \cong 0 \rightarrow$ **RESONANCE**

Morphodynamic eigenvalue

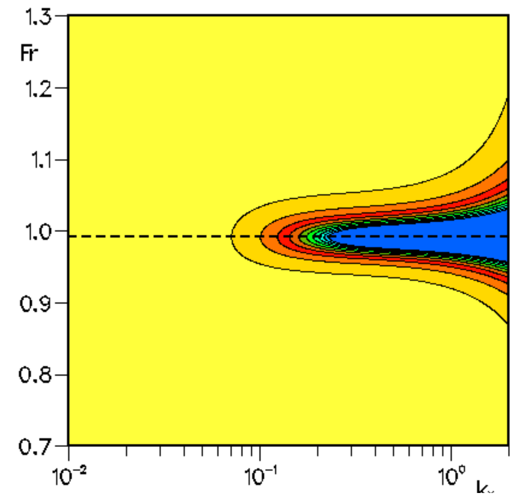
Fully unsteady (coupled)

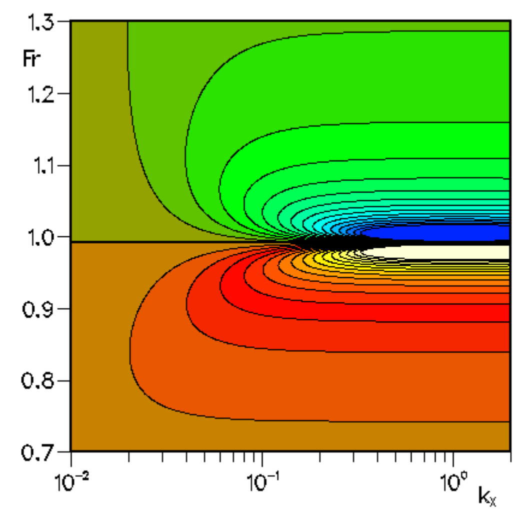




NO RESONANCE

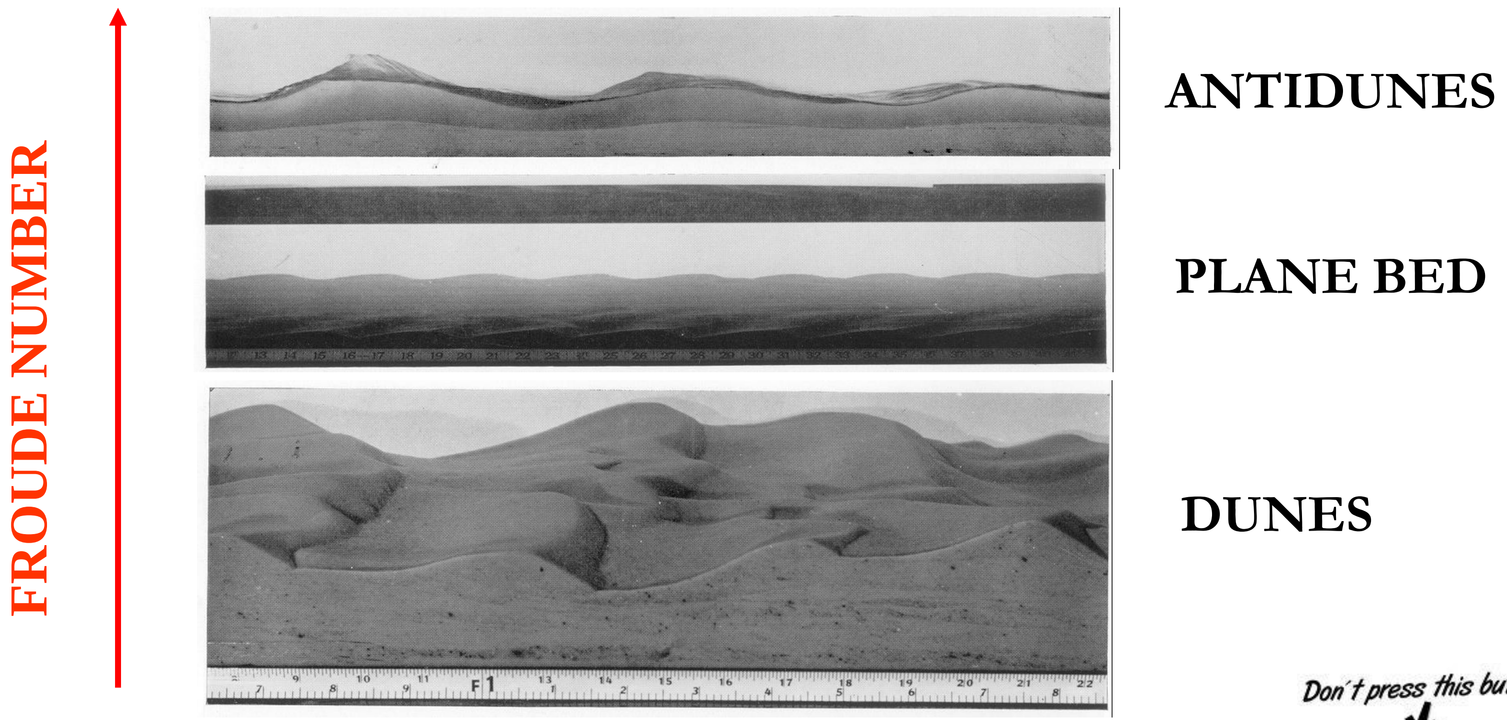
Quasi-steady (decoupled)





RESONANCE

2D Dunes and Antidunes



Laboratory experiments by Kennedy (1963)

- **Kennedy (1963,1969) - Irrotational** D A R
- **Engelund (1970), Fredsøe (1974) - Slip velocity** D A R
- **Parker (1975) - Irrotational** D A R

The neverending story

The neverending story

- **Kennedy (1963,1969) - Irrotational** D A R
- **Engelund (1970), Fredsøe (1974) - Slip velocity** D A R
- **Parker (1975) - Irrotational** D A R

A.J. Reynolds (1976) – Nordic Hydrology (after Euromech 48, 1974, Technical University of Denmark) – *A Decade's Investigation of the Stability of Erodible Stream Beds.*

"The best understood bed features are the least important – antidunes.

We have a good understanding of the mechanism of instability – namely, the finite time required for the suspended sediment to adapt to changed conditions – and of the role of the free surface …

The general nature of the transition between dunes and antidunes is understood, but the stability boundaries are critically dependent on the balance between suspended load and bed load, among other factors."

- **Kennedy (1963,1969) - Irrotational** D A R
- **Engelund (1970), Fredsøe (1974) - Slip velocity** D A R
- **Parker (1975) - Irrotational** D A R

A.J. Reynolds (1976) – Nordic Hydrology (after Euromech 48, 1974, Technical University of Denmark) – *A Decade's Investigation of the Stability of Erodible Stream Beds.*

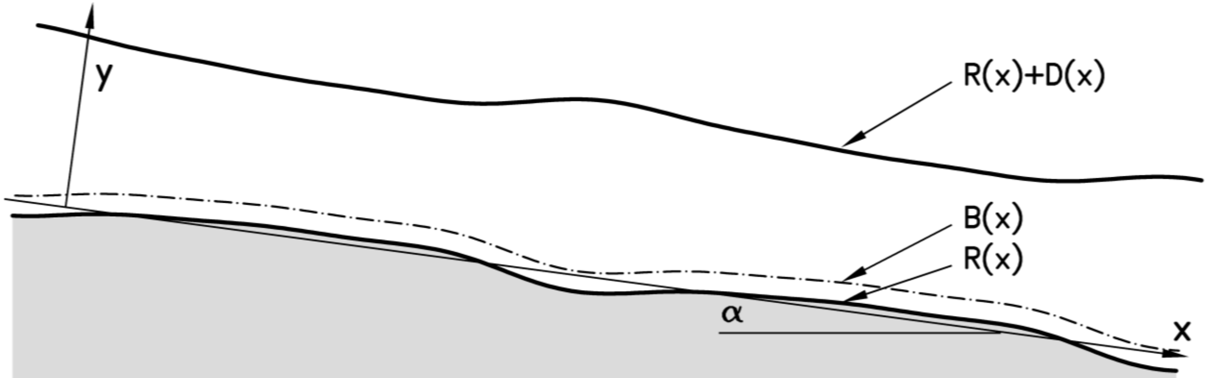
"It is widely conceded that ripples – dune-like features much smaller than the channel dimensions – are associated with the boundary layer on the perturbed channel bed, that their size is determined by the dimensions of the region of rapid velocity variation, and that the mechanism of instability presumably involves this variation. However, no analysis based on these has been advanced. The case of wind ripples has been treated, it is true, but in the hydraulic environment the transport of particles is predominantly as bed load, unlike the aeolian situation where saltation is of great importance.
… it is apparent that our understanding decreases as the role of bed load becomes more dominant …

The neverending story

- Kennedy (1963,1969) - Irrotational — D A R
- Engelund (1970), Fredsøe (1974) - Slip velocity — D A R
- Parker (1975) - Irrotational — D A R

- Richards (1980) - Rotational — D A R
- Sumer & Backioglu (1984) - Rotational — D A R
- Coleman & Fenton (2000) - Irrotational — D A R
- Colombini (2004) - Rotational — D A R
- Fourriére, Claudin & Andreotti (2010) - Rotational — D A R
- Colombini & Stocchino (2011) - Rotational — D A R

Flow model - I



2D REYNOLDS EQUATIONS + CONTINUITY

$$u_{,t} + uu_{,x} + vu_{,y} + p_{,x} - S/Fr^2 - T_{xx,x} - T_{xy,y} = 0$$

$$v_{,t} + uv_{,x} + vv_{,y} + p_{,y} + 1/Fr^2 - T_{xy,x} - T_{yy,y} = 0$$

$$u_{,x} + v_{,y} = 0$$

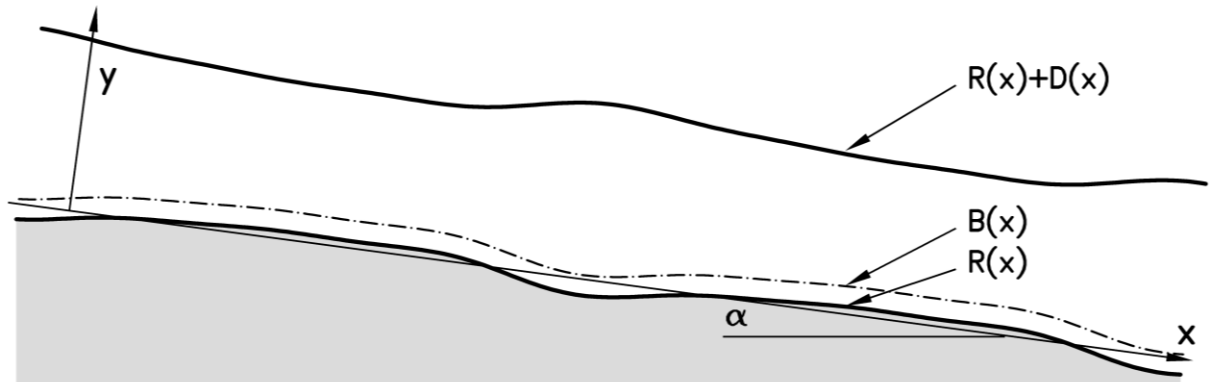
Equation are made dimensionless with:

$$r,\ U_0^*,\ D_0^* \quad \Rightarrow \quad S = \frac{Fr^2}{C^2}$$

Vertical transformation is introduced:

$$h = \frac{y - R}{D}$$

Flow model – II



2D REYNOLDS EQUATIONS + CONTINUITY

$$u_{,t} + u u_{,x} + v u_{,y} + p_{,x} - S/Fr^2 - T_{xx,x} - T_{xy,y} = 0$$

$$v_{,t} + u v_{,x} + v v_{,y} + p_{,y} + 1/Fr^2 - T_{xy,x} - T_{yy,y} = 0$$

$$u_{,x} + v_{,y} = 0$$

BOUSSINNESQ’s TURBULENCE CLOSURE

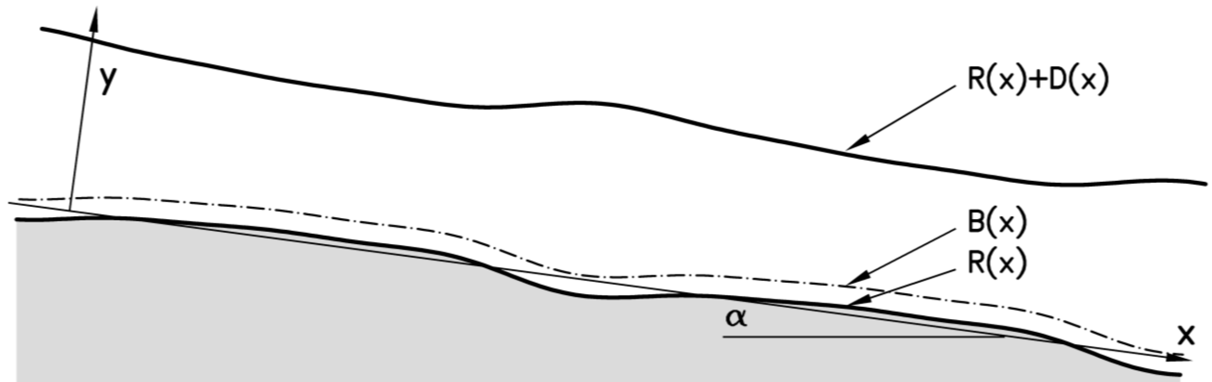
$$T_{xx} = 2 n_T u_{,x} \qquad T_{yy} = 2 n_T v_{,y} \qquad T_{xy} = n_T \left(u_{,y} + v_{,x}\right)$$

EDDY VISCOSITY & MIXING LENGTH

$$n_T = l^2 \sqrt{\left(u_{,x} - v_{,y}\right)^2 + \left(u_{,y} + v_{,x}\right)^2}$$

$$l = kD\left(h + h_0\right)\left(1 - h\right)^{1/2}$$

Sediment transport model



SEDIMENT CONTINUITY EQUATION (EXNER)

$$R_{,t} + Q_S F_{,x} = 0 \qquad Q_S = \frac{1}{F}\frac{\sqrt{(s-1)d^3}}{1-p} << 1$$

BEDLOAD FUNCTION (MPM after Wong & Parker (2006))

$$F = 3.97\left(J_B - J_C\right)^{3/2}$$

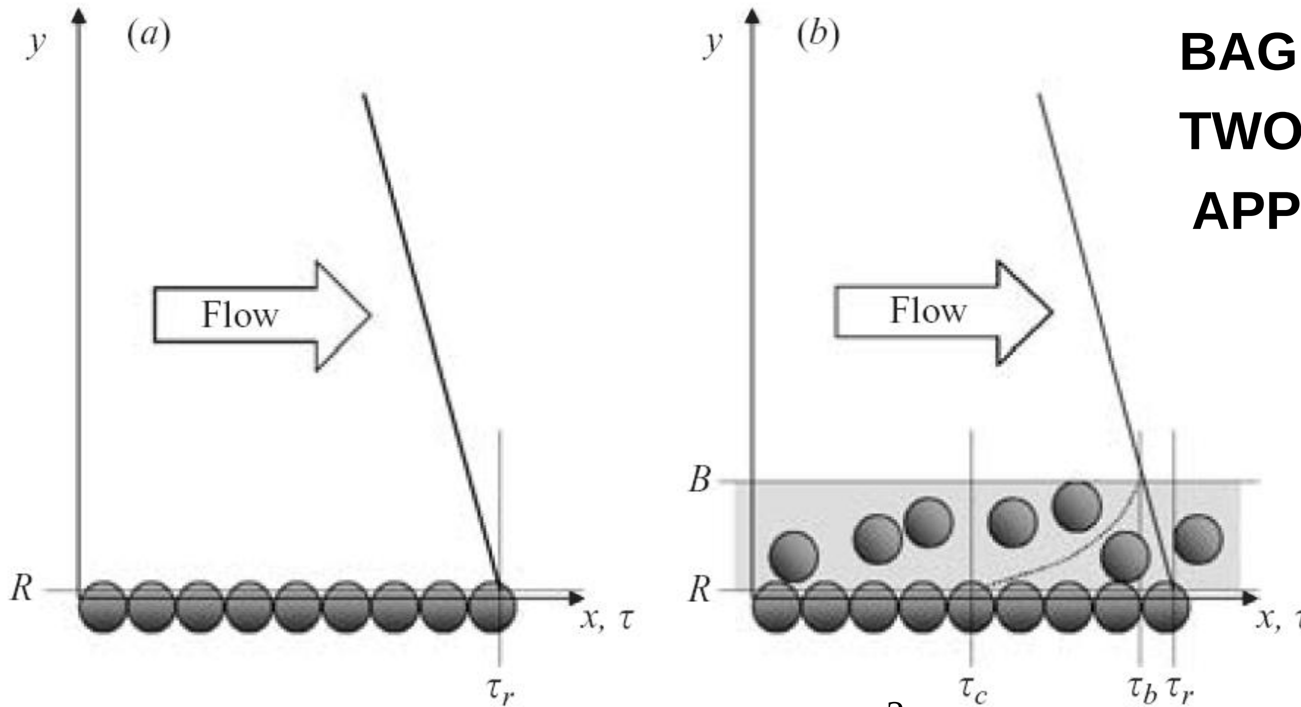
CORRECTION FOR SEDIMENT WEIGHT

$$J_C = 0.0495 - m_x\left(S - R_{,x}\right)$$

$m = 0.1$ Fredsøe (1974)

$m = 2.8J$ Richards (1980)

BAGNOLD'S TWO-LAYER APPROACH



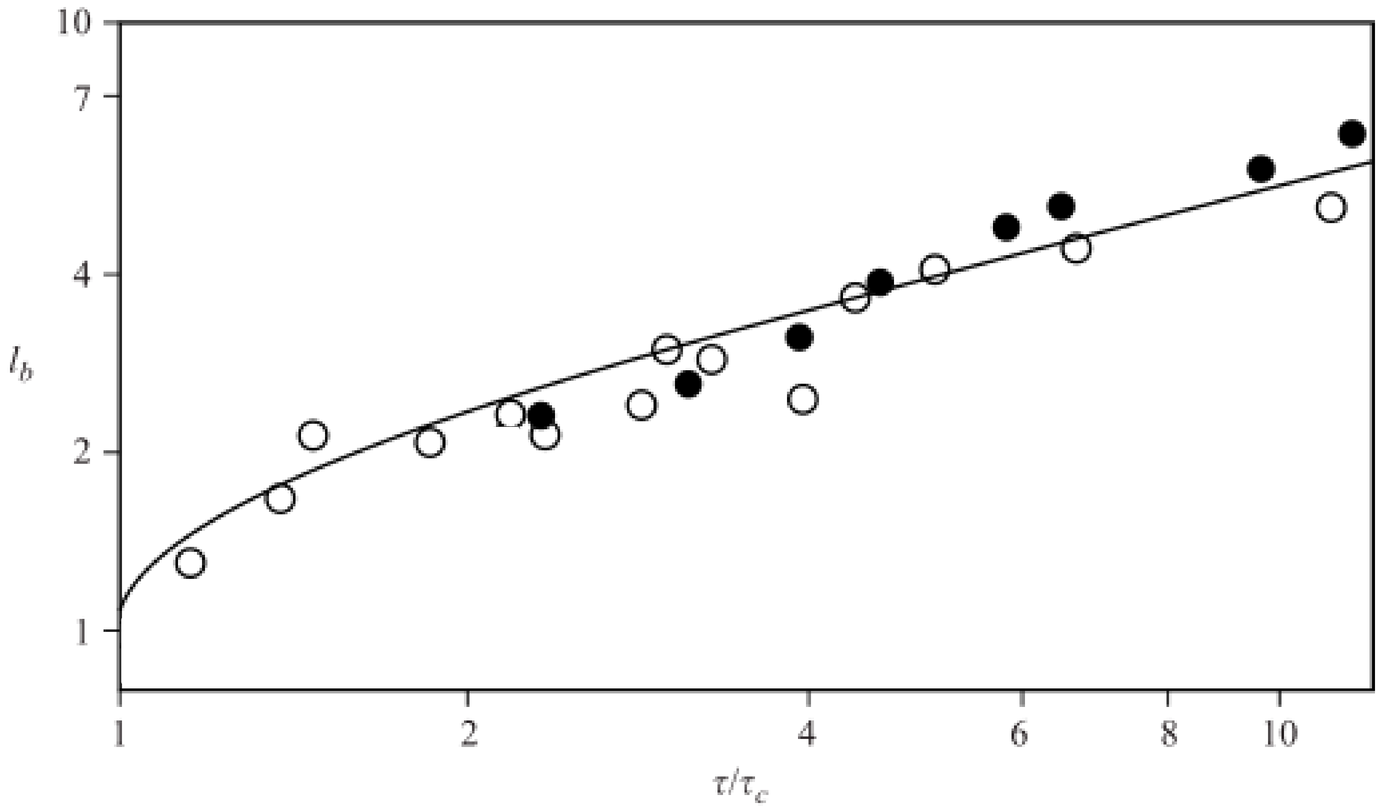
$$J_B = J_0 T_B = T_{xy}\Big|_{y=B} \qquad J_0 = \frac{F^2}{C^2(s-1)d}$$

$$B = R + h_B D \qquad h_B = l_B d$$

BEDLOAD LAYER THICKNESS (scaled with grain size)

$$l_B = 1 + 1.3\left(\frac{J_0 - J_{C0}}{J_{C0}}\right)^{0.55}$$

Bedload layer thickness



Dimensionless (with grain size) maximum saltating height versus t/t_c. Experiments of Sekine & Kikkawa (1992) (hollow) and Lee & Hsu (1994) (solid).

Solid line is:

$$l_B = 1 + 1.3\left(\frac{\vartheta_0 - \vartheta_{C0}}{\vartheta_{C0}}\right)^{0.55}$$

LINEARIZATION

$$G(x,h,t)=G_0(h)+e\,\mathrm{G}_1(x,h,t)$$
$$G_1(x,h,t)=\ g_1(h)\exp[ik_x(x-\ wt)]+c.c.$$

BASIC LEVEL: UNIFORM FLOW

LINEAR LEVEL: EIGENVALUE PROBLEM (shooting)

LINEARIZATION

$$G(x,\eta,t) = G_0(\eta) + \varepsilon\, \mathrm{G}_1(x,\eta,t)$$
$$G_1(x,\eta,t) = g_1(\eta)\exp[ik_x(x - \omega t)] + c.c.$$

BASIC LEVEL: UNIFORM FLOW

LINEAR LEVEL: EIGENVALUE PROBLEM (shooting)

… we always shoot first, and only then relax.

Extracted from “Numerical recipes”

LINEARIZATION

$$G(x,h,t) = G_0(h) + e\,\mathrm{G}_1(x,h,t)$$
$$G_1(x,h,t) = g_1(h)\exp[ik_x(x - wt)] + c.c.$$

BASIC LEVEL: UNIFORM FLOW

LINEAR LEVEL: EIGENVALUE PROBLEM (shooting)

… we always shoot first, and only then relax.

Extracted from “Numerical recipes”

$$\mathrm{G}(w) \cdot X = \{0\}$$
$$\det(\mathrm{G}(w)) = 0$$

$$X = (c_1, c_2, d, r)^T$$

Stress perturbations at the reference level (tangential and normal)

Depth perturbation

Bed perturbation

Decoupled (quasi-steady) solution

$$\begin{matrix} & G & & X \end{matrix}$$

BCs on free surface / **EXNER**

$$\left.\begin{matrix}\text{BCs on free surface}\\ \\ \text{EXNER}\end{matrix}\right. \quad \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} - g_{44} w \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ d \\ r \end{pmatrix} = 0$$

1) Determine linear flow response to a unitary bed perturbation

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ d \end{pmatrix} = - \begin{pmatrix} g_{14} \\ g_{24} \\ g_{34} \end{pmatrix}$$

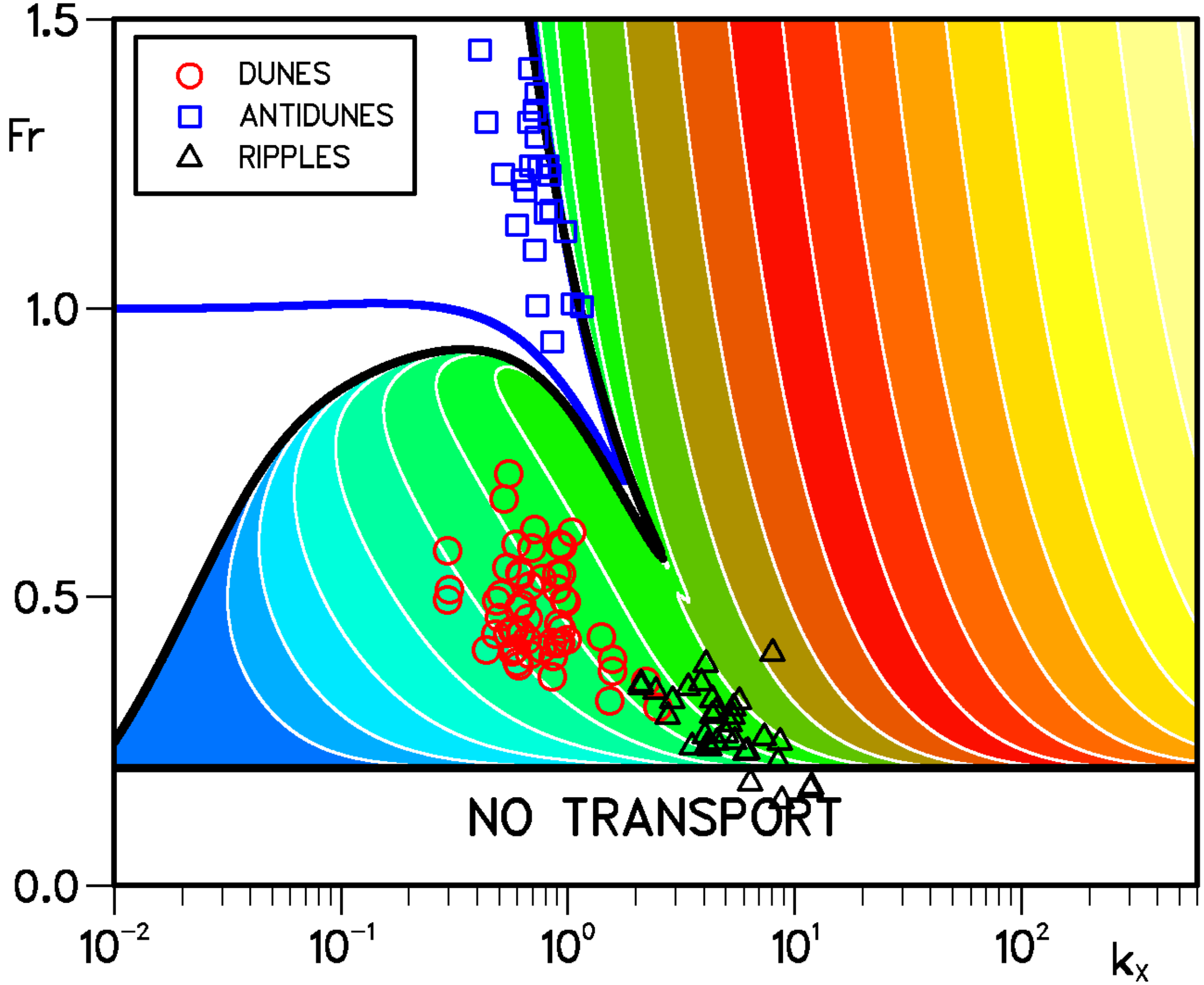
2) Substitute into Exner's equation

$$W = k_x^2 W_0 (T + P - W) \qquad T = \frac{t_R^i}{k_x} \qquad P = \frac{t_B^i - t_R^i}{k_x} \qquad W = \frac{m_x}{J_0}$$

2D linear stability: dunes & antidunes

$$\Omega = k_x^2 \Omega_0 (T + P - W)$$

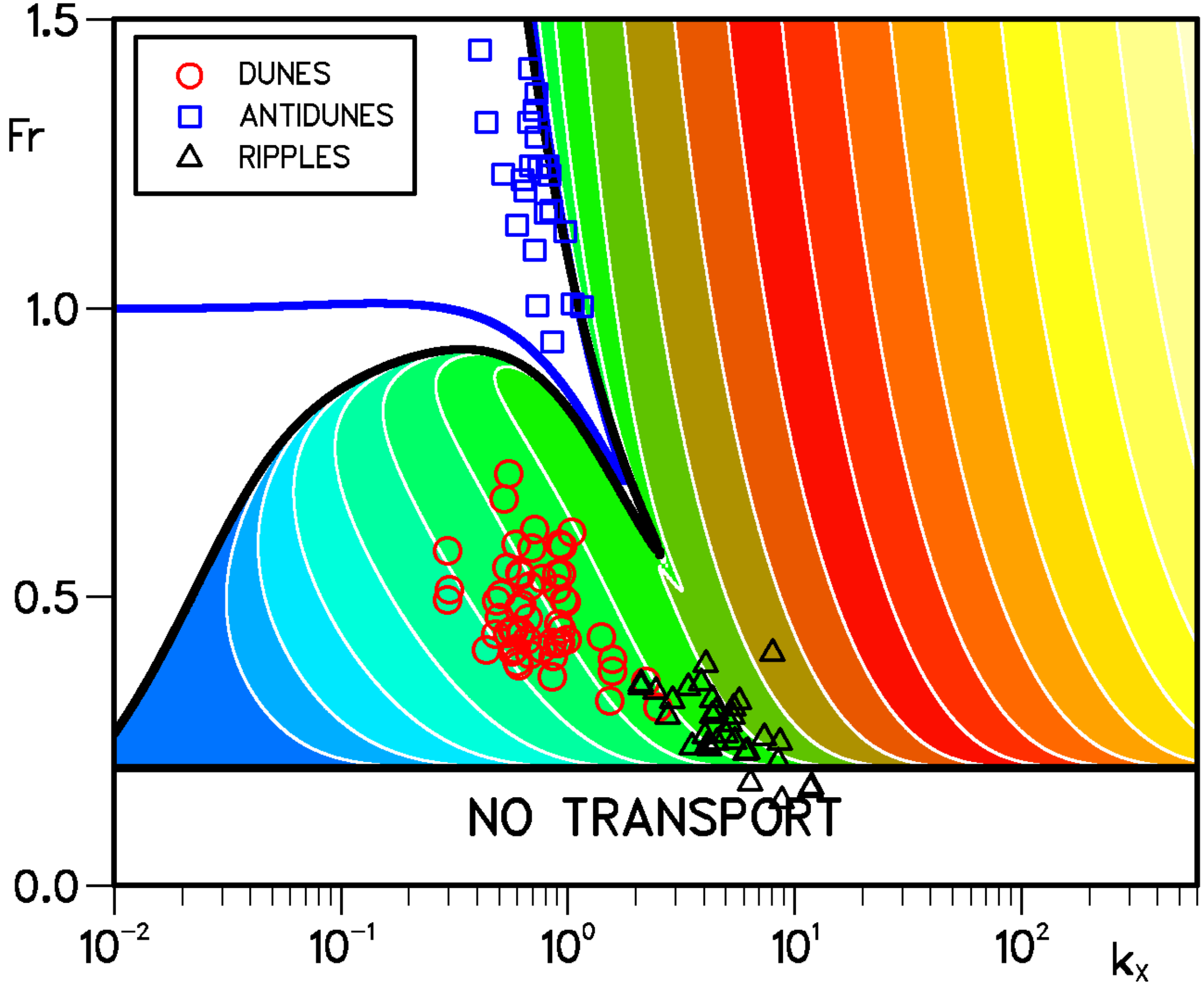
$$T = \frac{t_R^i}{k_x} \qquad P = 0 \qquad W = 0$$



2D linear stability: dunes & antidunes

$$\Omega = k_x^2 \Omega_0 (T + P - W)$$

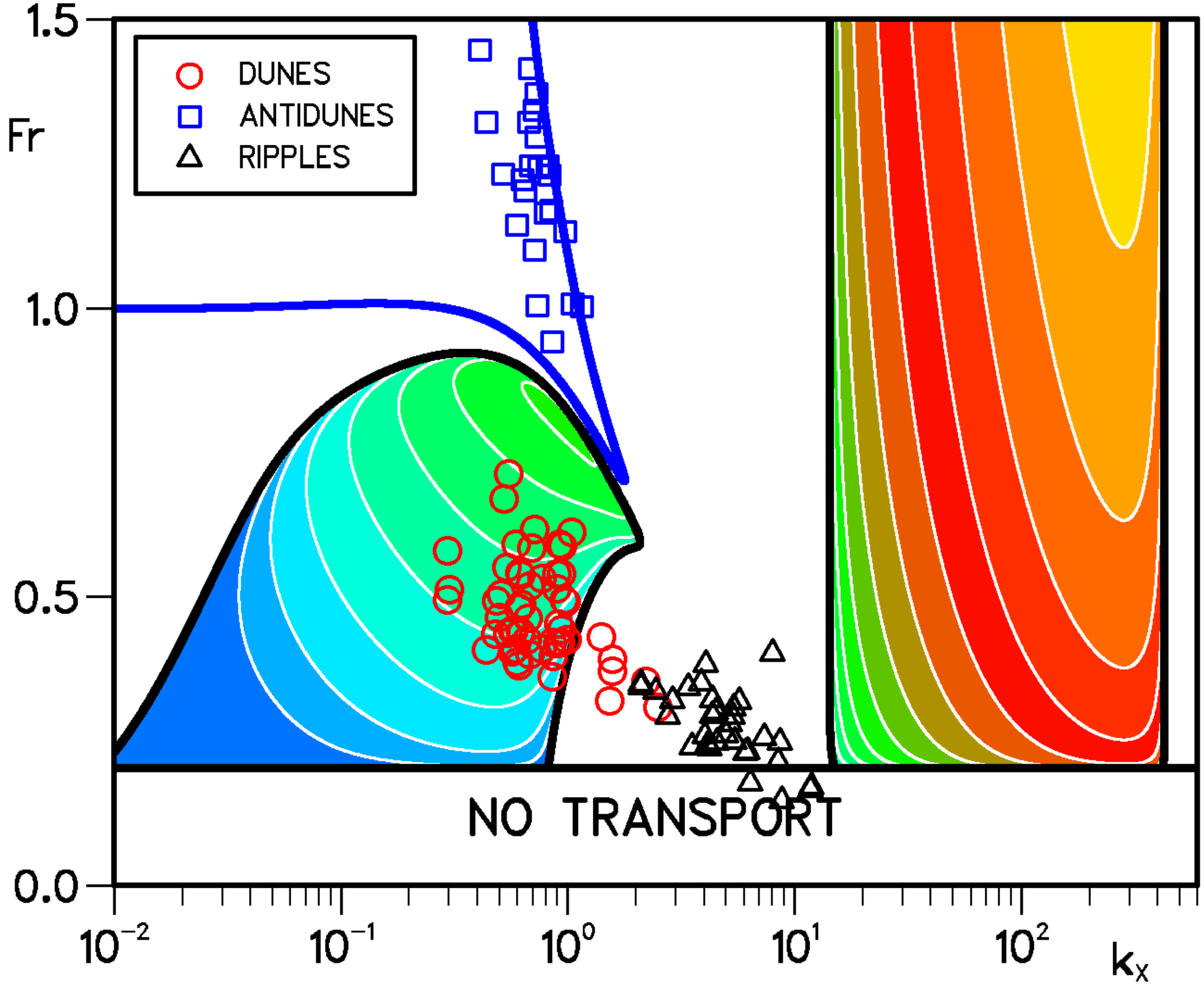
$$T = \frac{t_R^i}{k_x} \qquad P = 0 \qquad W = \frac{m_x}{J_0}$$



2D linear stability: dunes & antidunes

$$\mathrm{W} = k_x^2 \mathrm{W}_0 (T + P - W)$$

$$T = \frac{t_R^i}{k_x} \qquad P = 0 \qquad W = b$$



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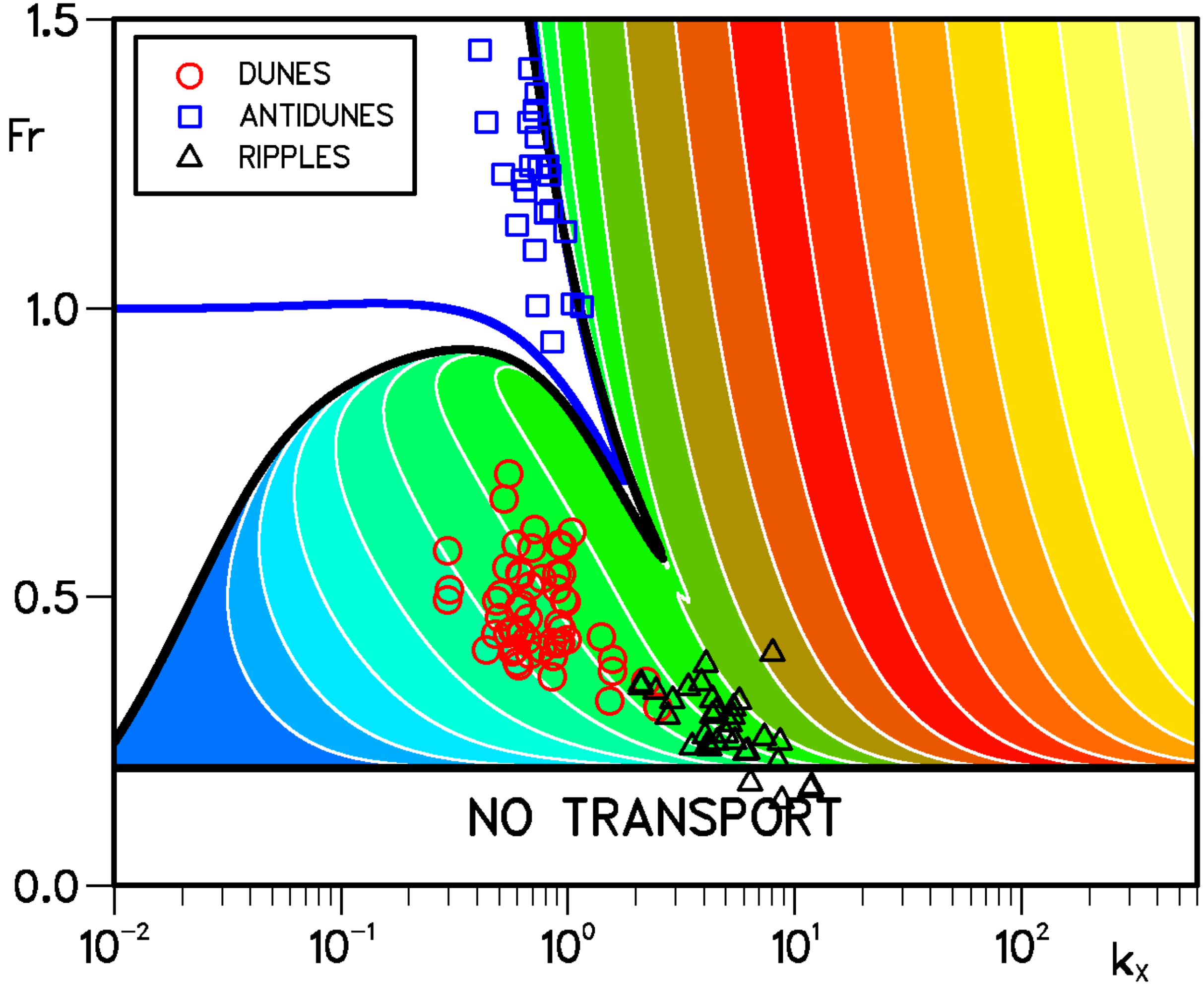
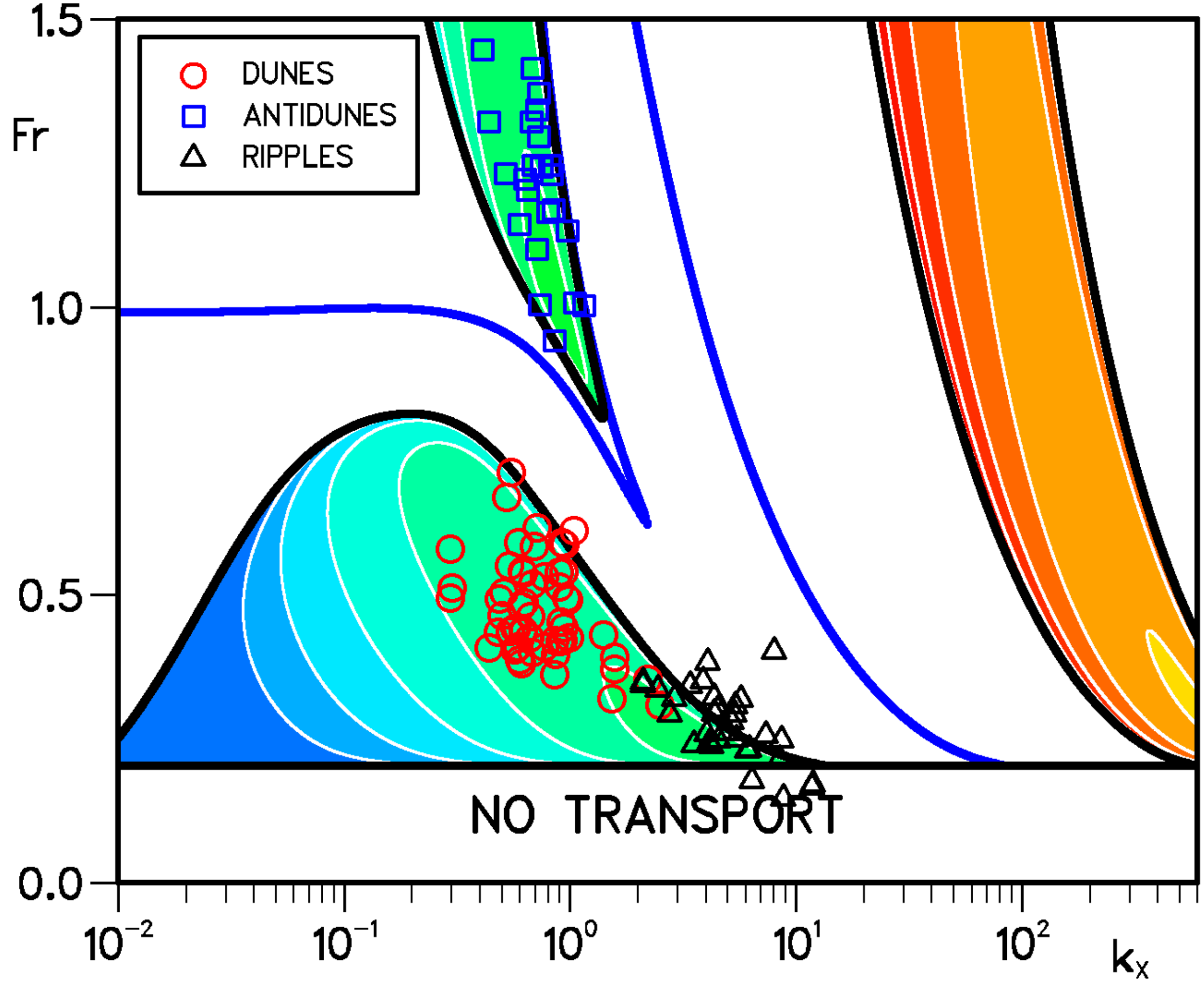


2D linear stability: dunes & antidunes

$$W = k_x^2 W_0 (T + P - W)$$

$$T = \frac{t_R^i}{k_x} \qquad P = \frac{t_B^i - t_R^i}{k_x} \qquad W = 0$$



2D linear stability: dunes & antidunes

$$W = k_x^2 W_0 (T + P - W)$$

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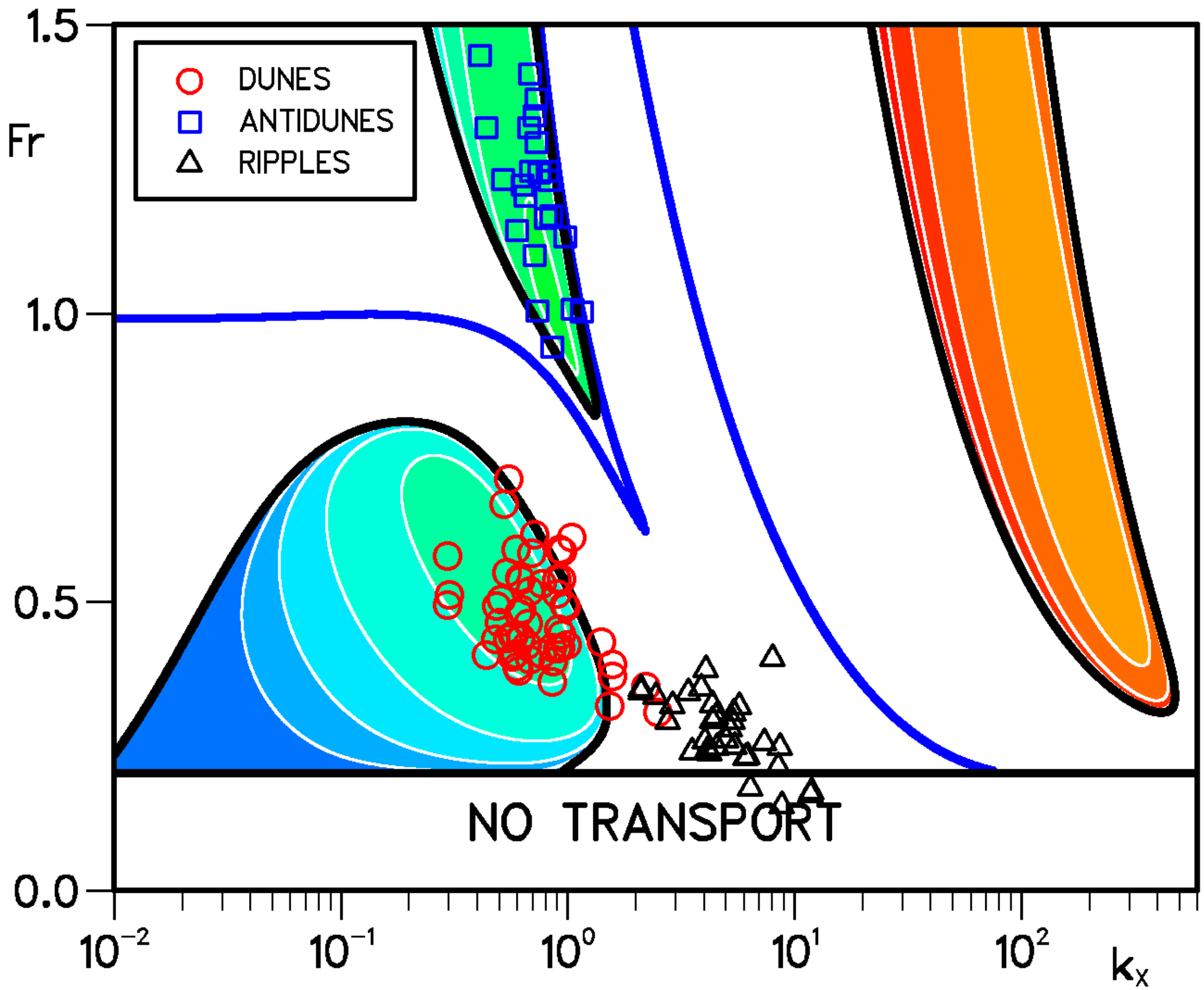
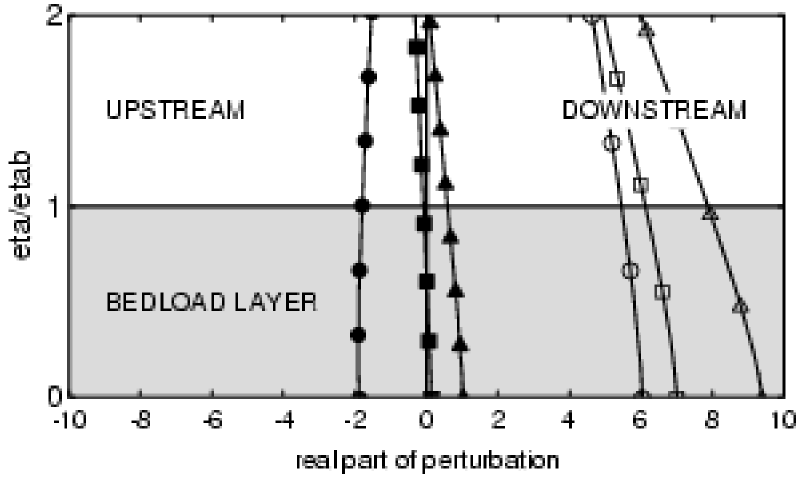
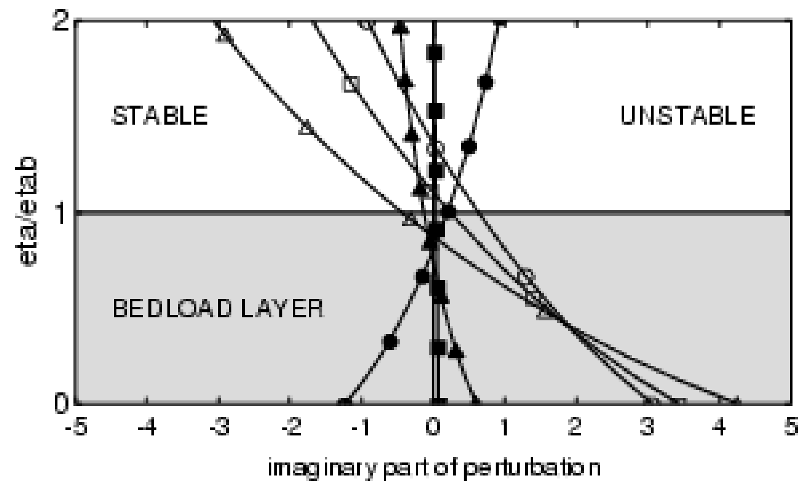


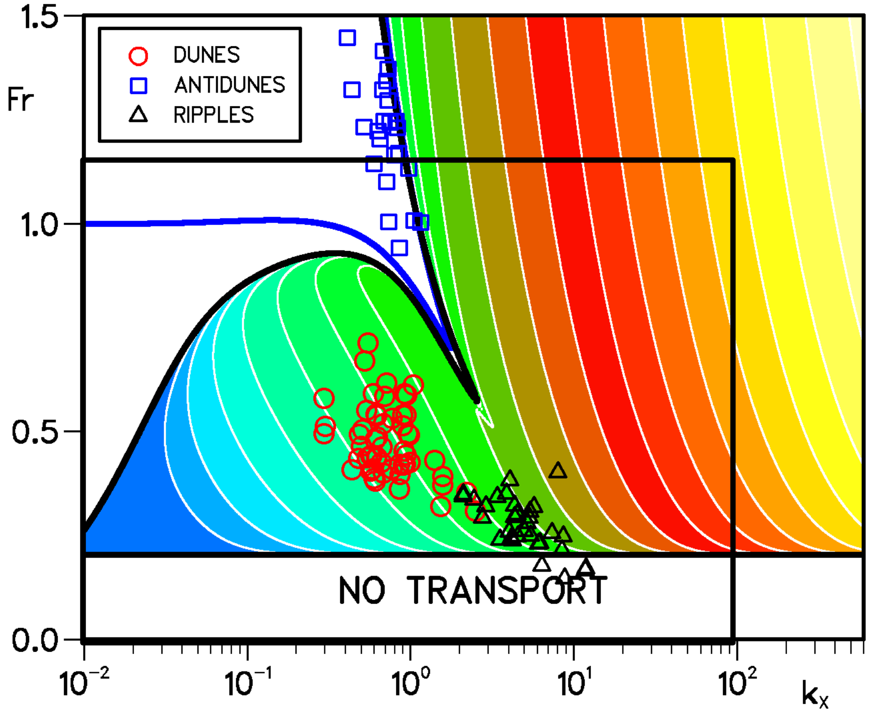
Variation of real and imaginary part of perturbed tangential shear stress with distance from reference level

k_x=1

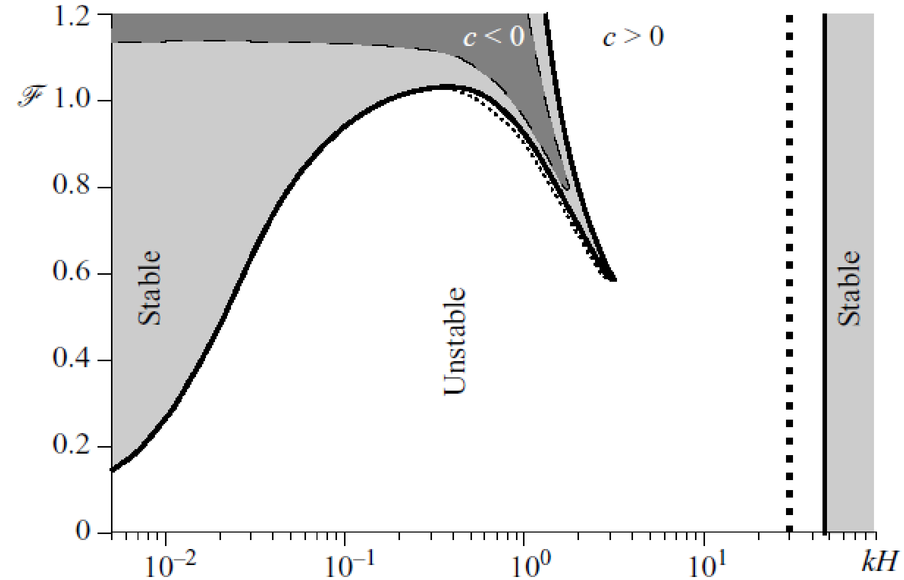




2D linear stability: saturation length?

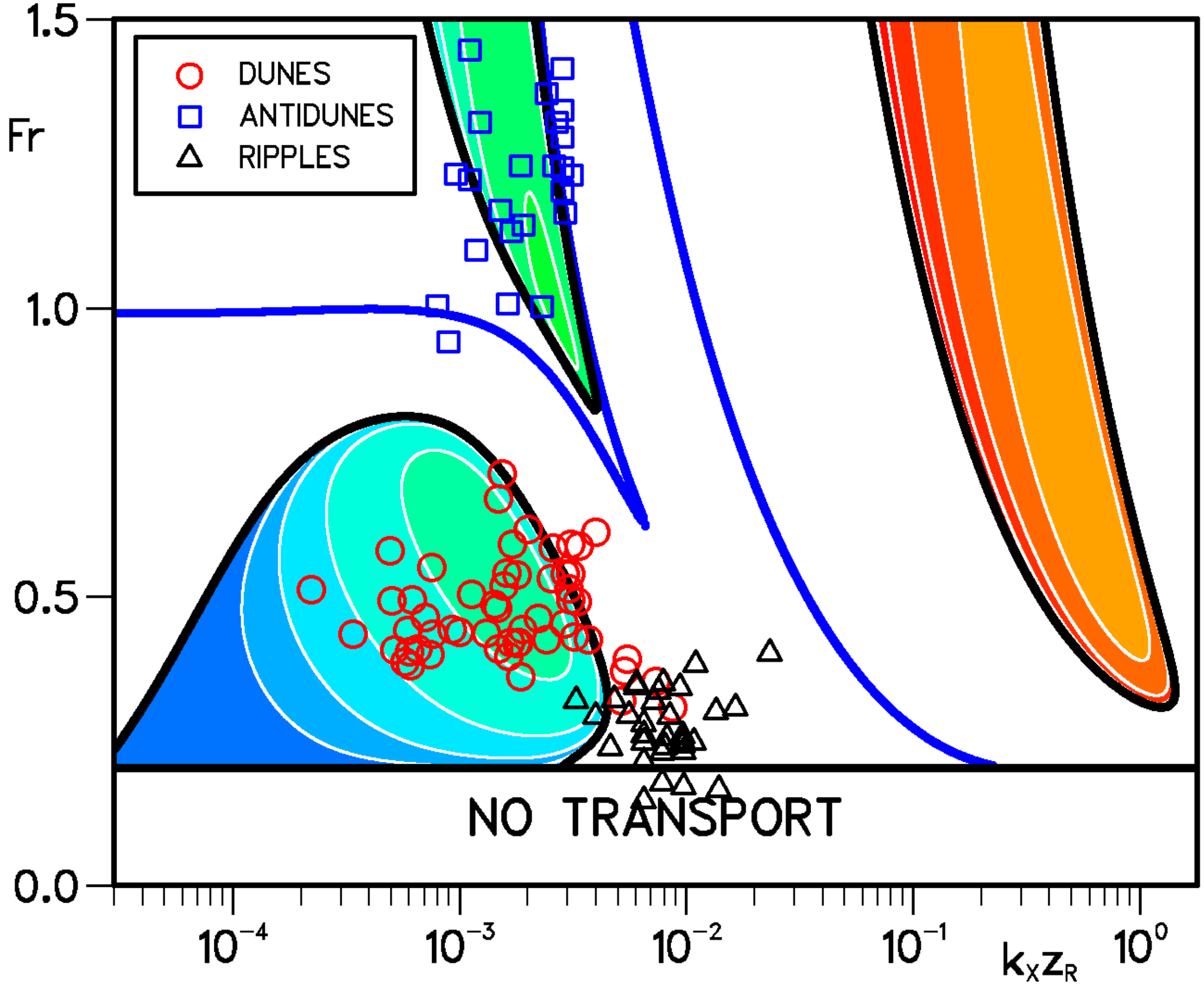


Fourriére, Claudin & Andreotti (2010)



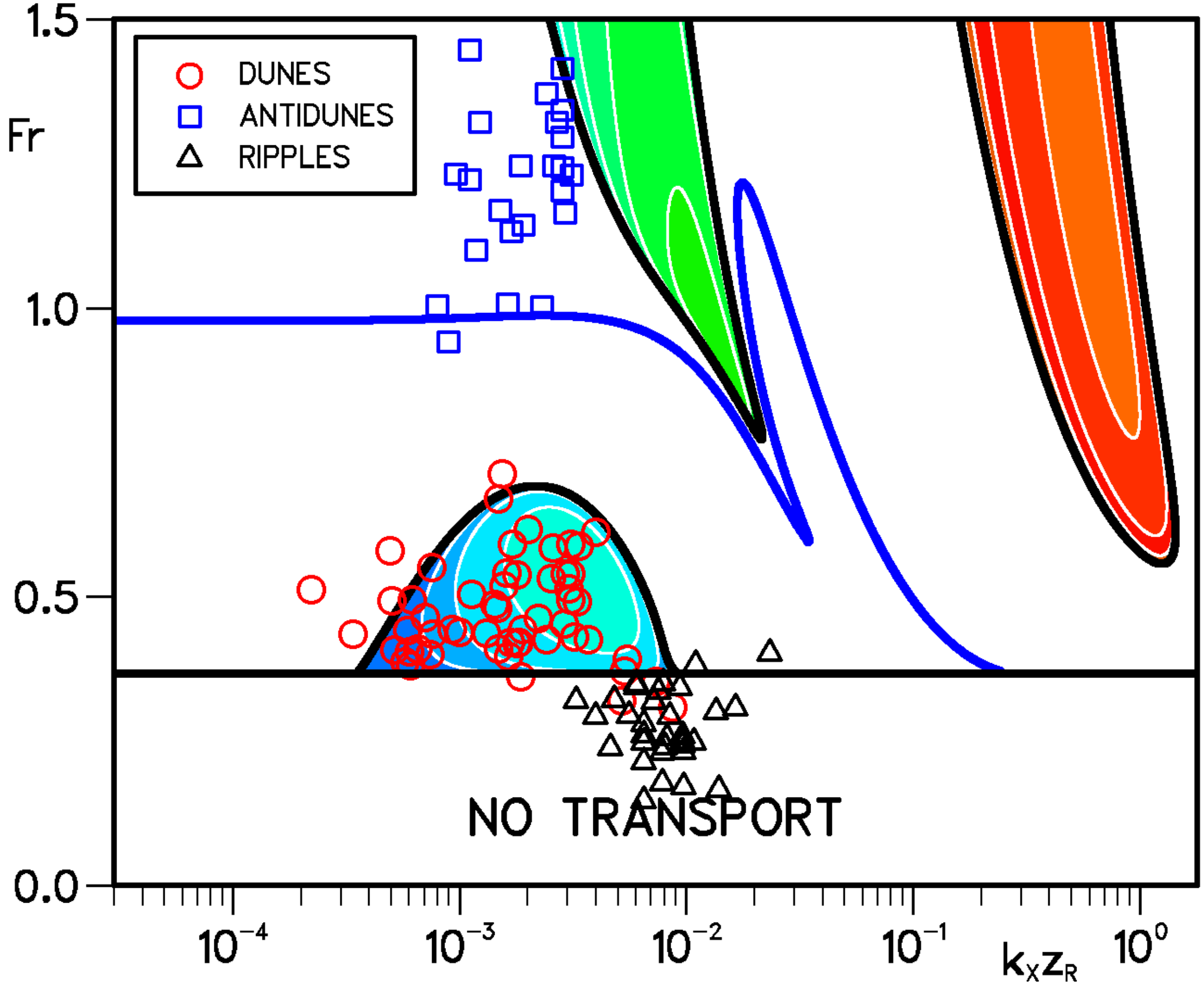
2D linear stability: spurious instability

$z_R=0.003$



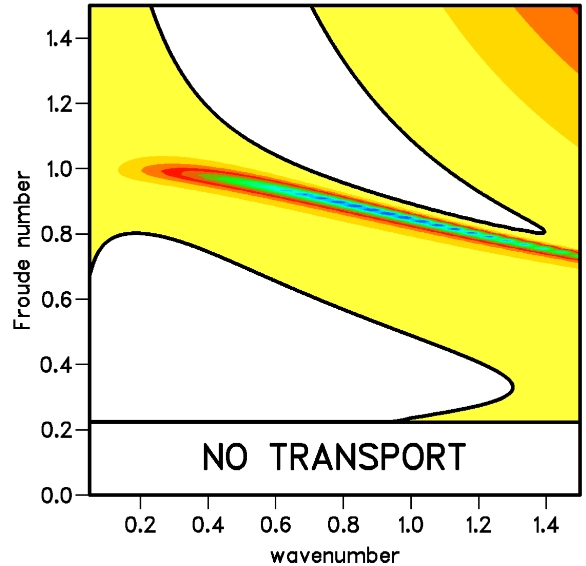
2D linear stability: spurious instability

$z_R = 0.015$



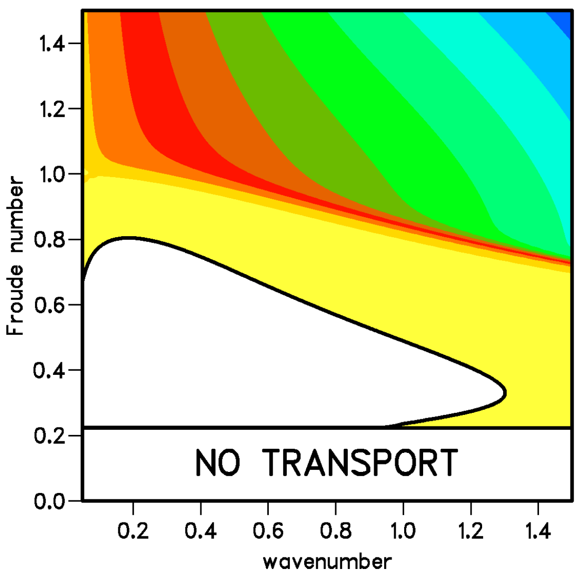
Coupling, decoupling and resonance

Quasi-steady (decoupled)



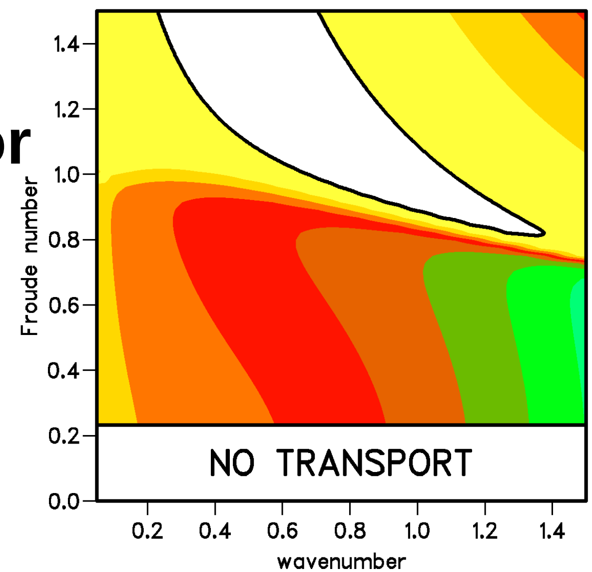
Single eigenvalue for dunes and antidunes

RESONANCE



Separate eigenvalues for dunes and antidunes

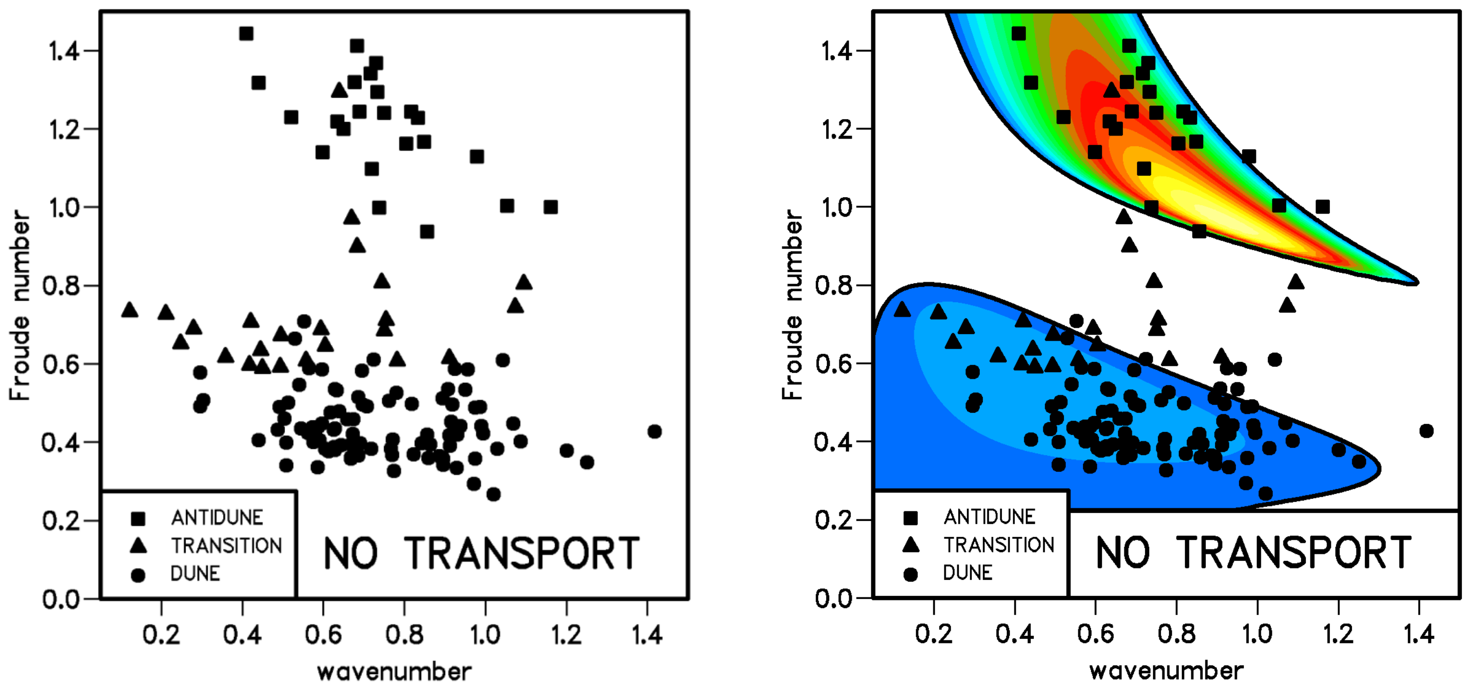
NO RESONANCE



Fully unsteady (coupled)

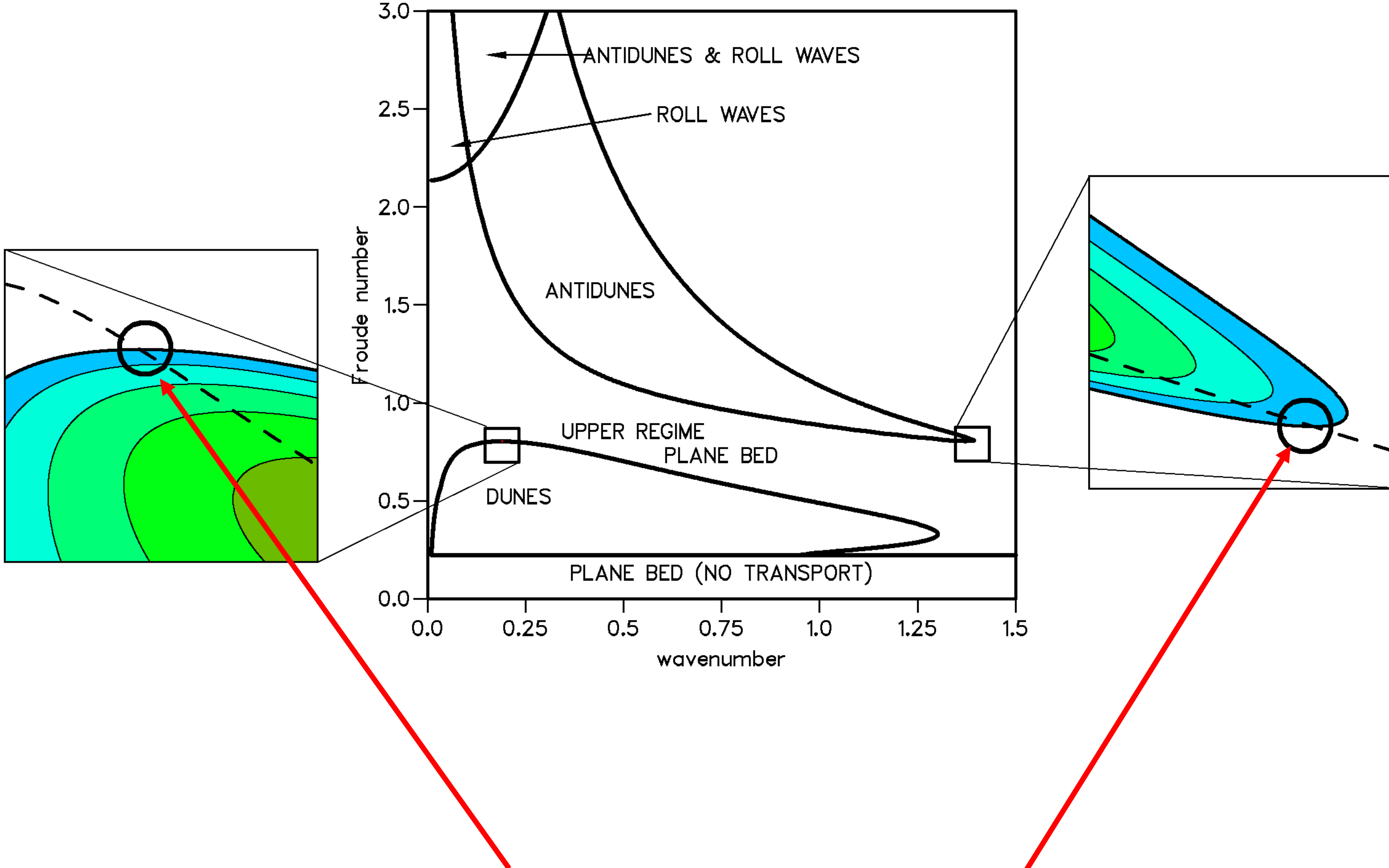
2D linear stability: results

Experiments by Guy, Simons & Richardson (1966)



- **Froude number (or Shields stress) is the parameter governing instability**
- **Wavelengths scale with flow depth (k_x=O(1))**

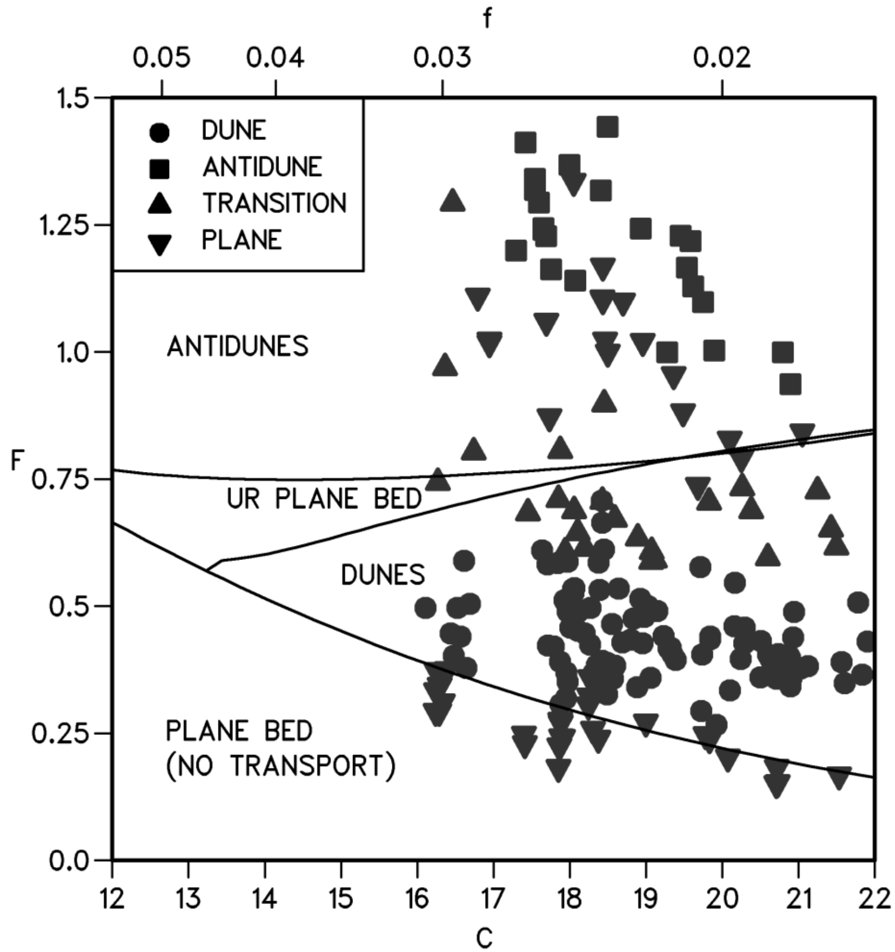
2D linear stability: results



CRITICAL POINTS

2D linear stability: results

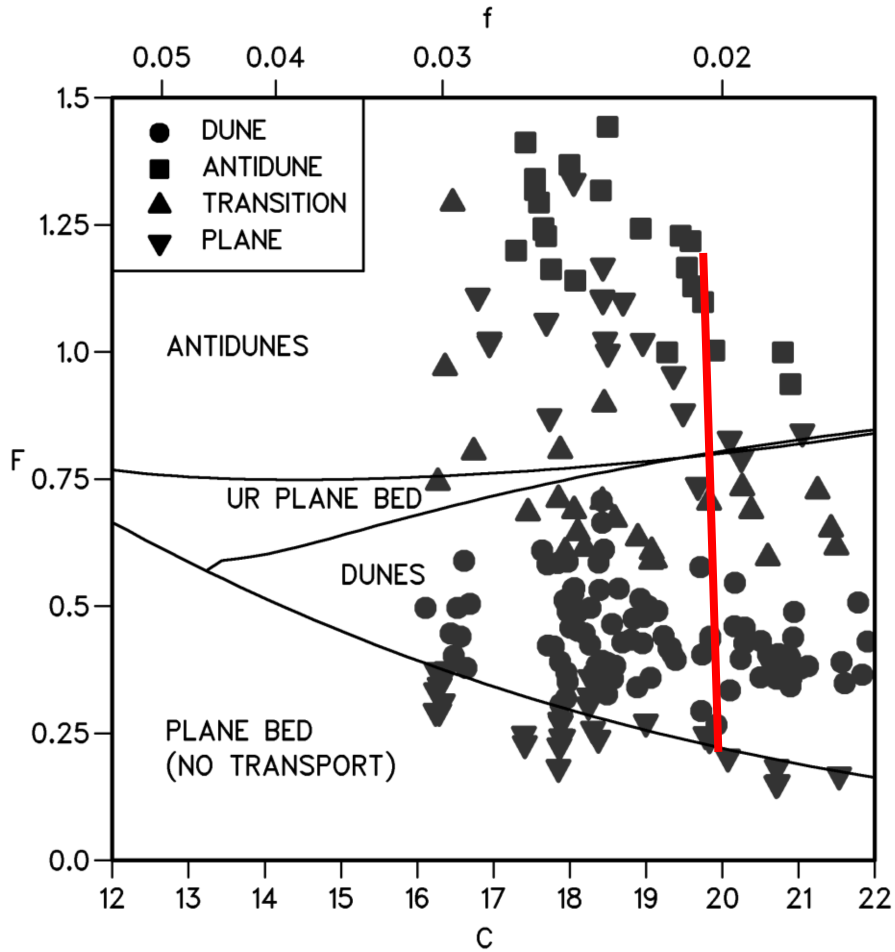
Experiments by Guy, Simons & Richardson (1966)



$$C = \sqrt{\frac{8}{f}} = \frac{1}{k}\ln\left(\frac{1.09}{2.5d}\right)$$

2D linear stability: results

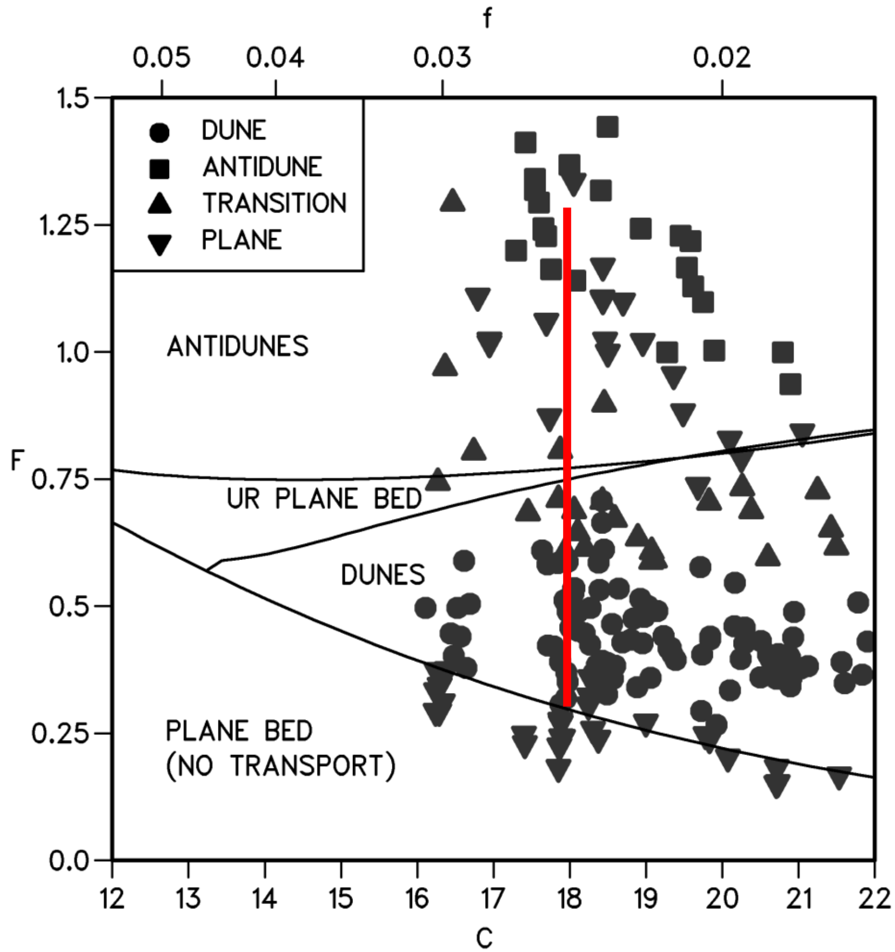
Experiments by Guy, Simons & Richardson (1966)



$$C = \sqrt{\frac{8}{f}} = \frac{1}{k}\ln\left(\frac{11.09}{2.5d}\right)$$

Experiments by Guy, Simons & Richardson (1966)

2D linear stability: results



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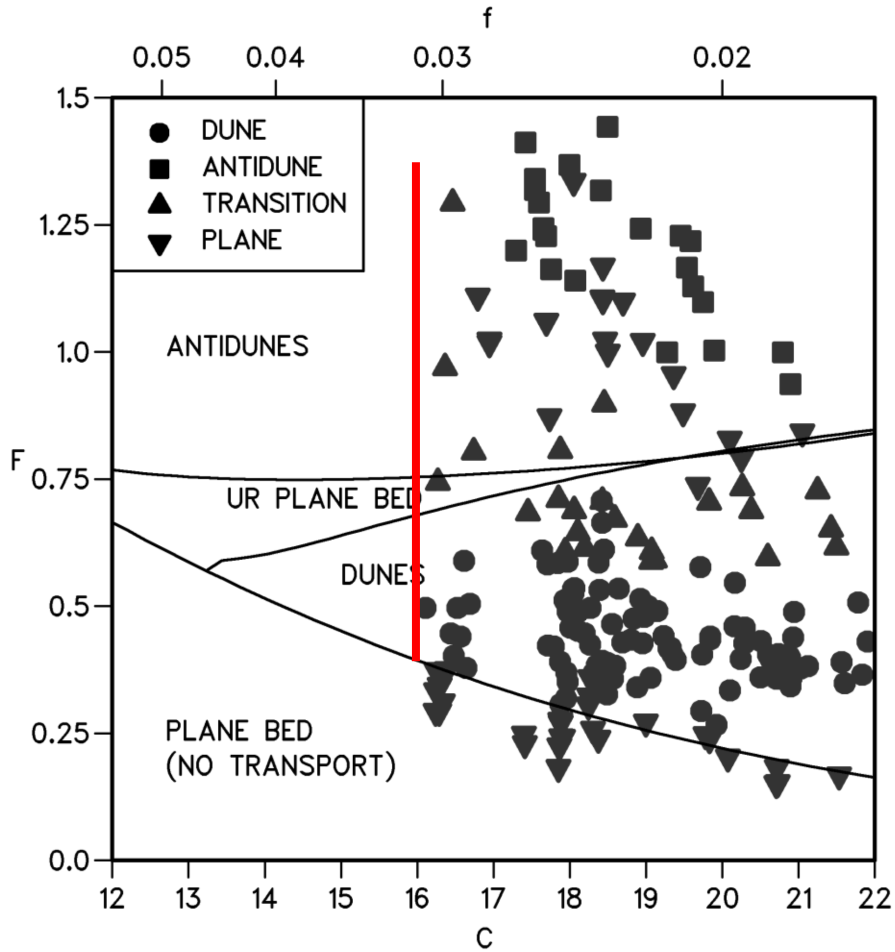

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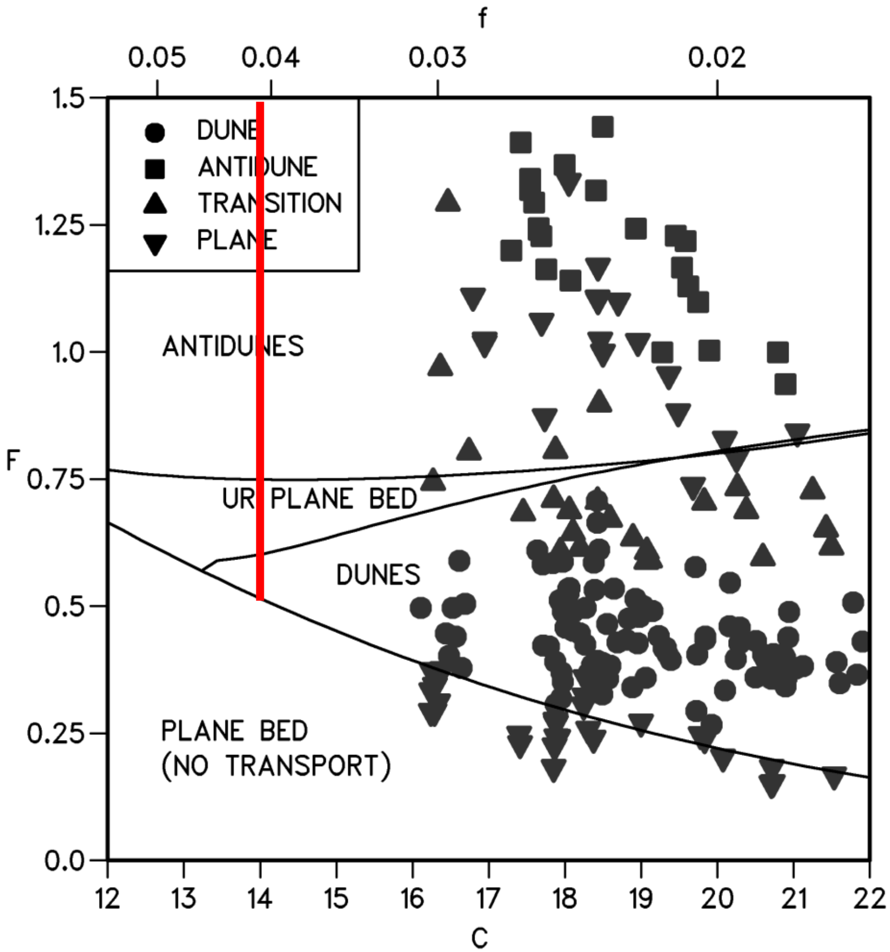
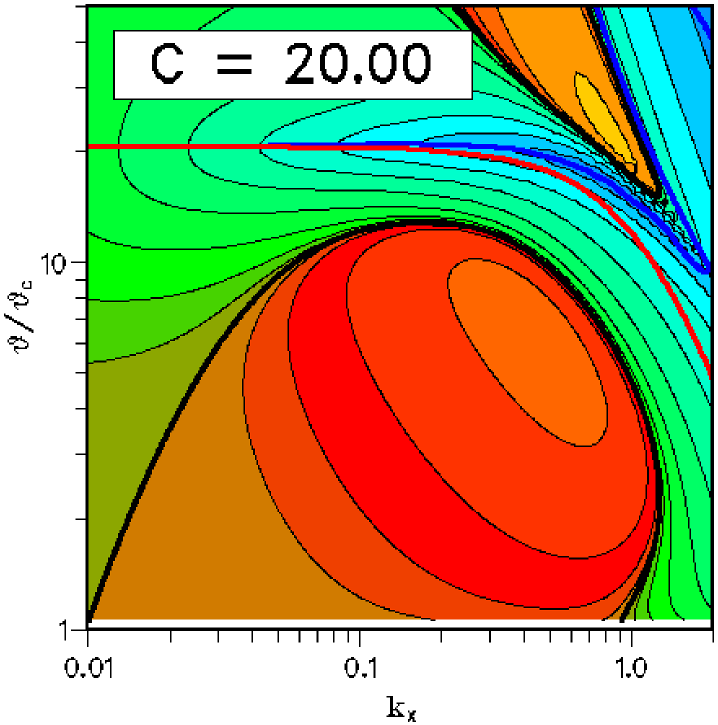


$$C = \sqrt{\frac{8}{f}} = \frac{1}{k}\ln\left(\frac{11.09}{2.5d}\right)$$

Results: effect of C



- Froude number is the parameter governing instability, but, for any given C, □□is proportional to Fr^2 through:

$$J @ 0.14 \frac{Fr^2 e^{kC}}{C^2}$$

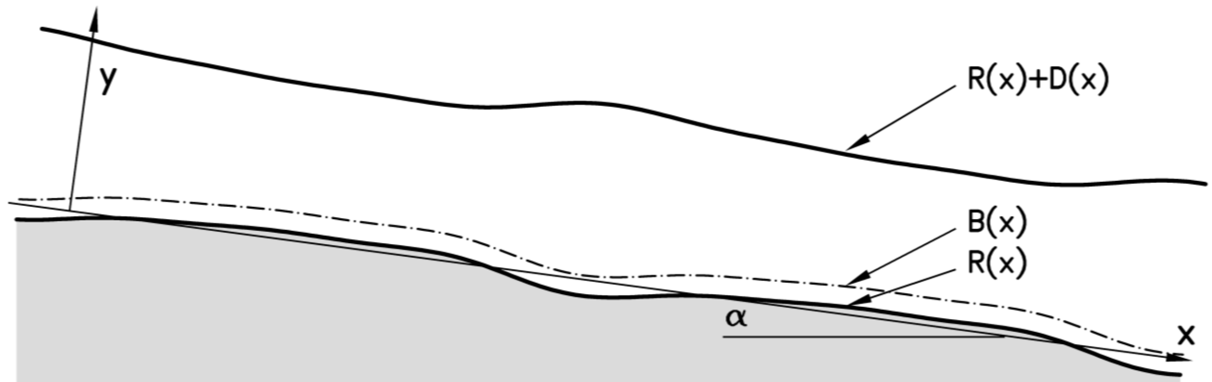
- Red line - c=0 for potential flow theory. $Fr^2 = Tanh(k_X)/k_X$
- Blue line – c=0 for rotational flow theory. $t_B^r(k_x, Fr, C) = 0$
- C decreasing □ d increasing □ either coarser sediment (same flow depth) or shallower flow (same sediment)

Ripples and Dunes

Ripples in the Hunter River, New South Wales, Australia

Image courtesy of M.C. Rygel

Flow model – II



2D REYNOLDS EQUATIONS + CONTINUITY

$$u_{,t}+uu_{,x}+vu_{,y}+p_{,x}-S/Fr^2-T_{xx,x}-T_{xy,y}=0$$

$$v_{,t}+uv_{,x}+vv_{,y}+p_{,y}+1/Fr^2-T_{xy,x}-T_{yy,y}=0$$

$$u_{,x}+v_{,y}=0$$

BOUSSINNESQ's TURBULENCE CLOSURE

$$T_{xx}=2n_T u_{,x} \qquad T_{yy}=2n_T v_{,y} \qquad T_{xy}=n_T\left(u_{,y}+v_{,x}\right)$$

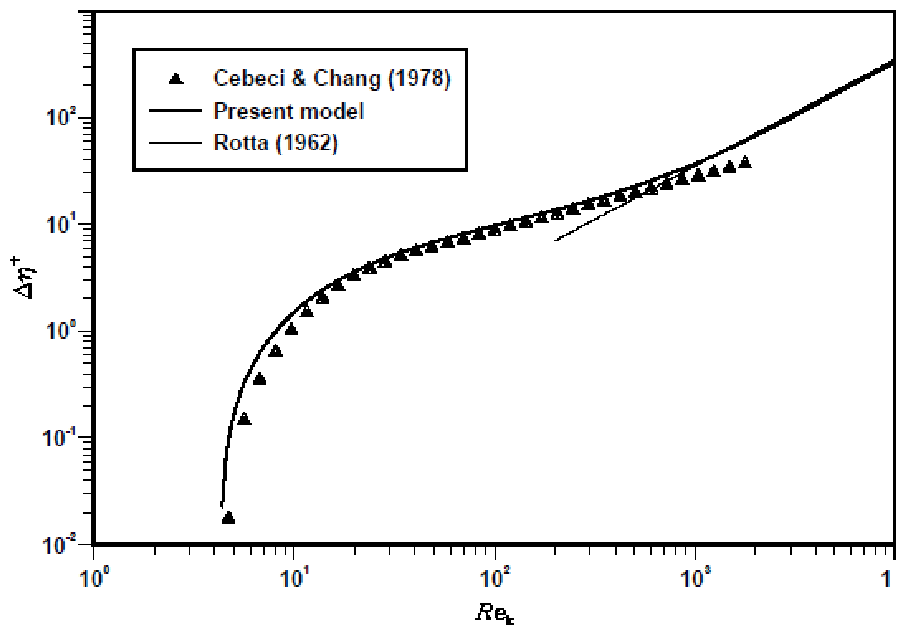
EDDY VISCOSITY & MIXING LENGTH

$$n_T=l^2\sqrt{\left(u_{,x}-v_{,y}\right)^2+\left(u_{,y}+v_{,x}\right)^2}$$

$$l=kD\left(h+Dh\right)\left[1-\exp\left(-\frac{h+Dh}{A}D\frac{\mathrm{Re}_k}{2.5d}\right)\right]\left(1-h\right)^{1/2}$$

Formulation of the problem

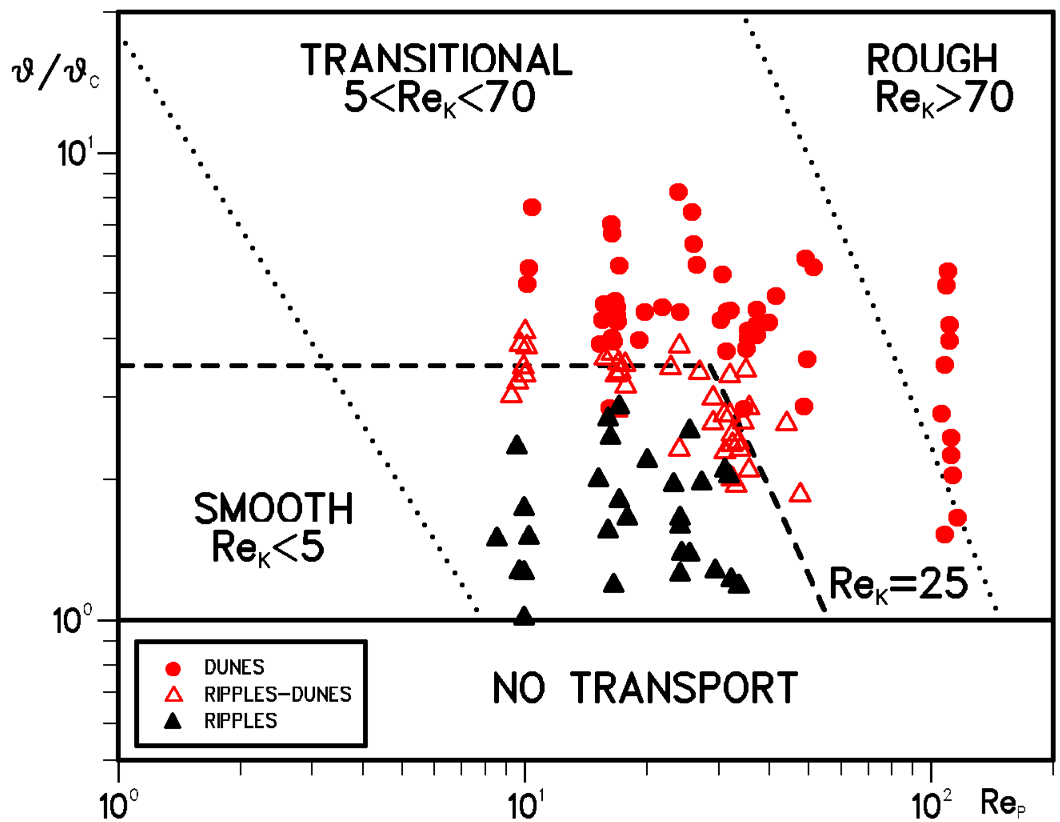
$$\Delta\eta^+ = 0.9\left[\sqrt{Re_k} - Re_k \exp\left(-\frac{Re_k}{6}\right)\right]$$



Conductance Coefficient (smooth-transitional-rough regimes)

$$C = \frac{U^*}{u_b^*} = \sqrt{\frac{8}{f_b}} = \frac{1}{\kappa}\ln\left[\left(\frac{a_r r_b^*}{k_s^*}\right)^{1-\beta}\left(\frac{Re\sqrt{f_b}}{a_s}\right)^{\beta}\right]$$

Experiments by Guy, Simons & Richardson (1966)



Roughness Reynolds number

$$Re_k = \frac{u_b^* k_s^*}{\nu}$$

Particle Reynolds number

$$Re_p = \frac{\sqrt{(s-1)gd^*}\,d^*}{\nu}$$

$$Re_k = \sqrt{\theta}\,Re_p \frac{k_s^*}{d^*}$$

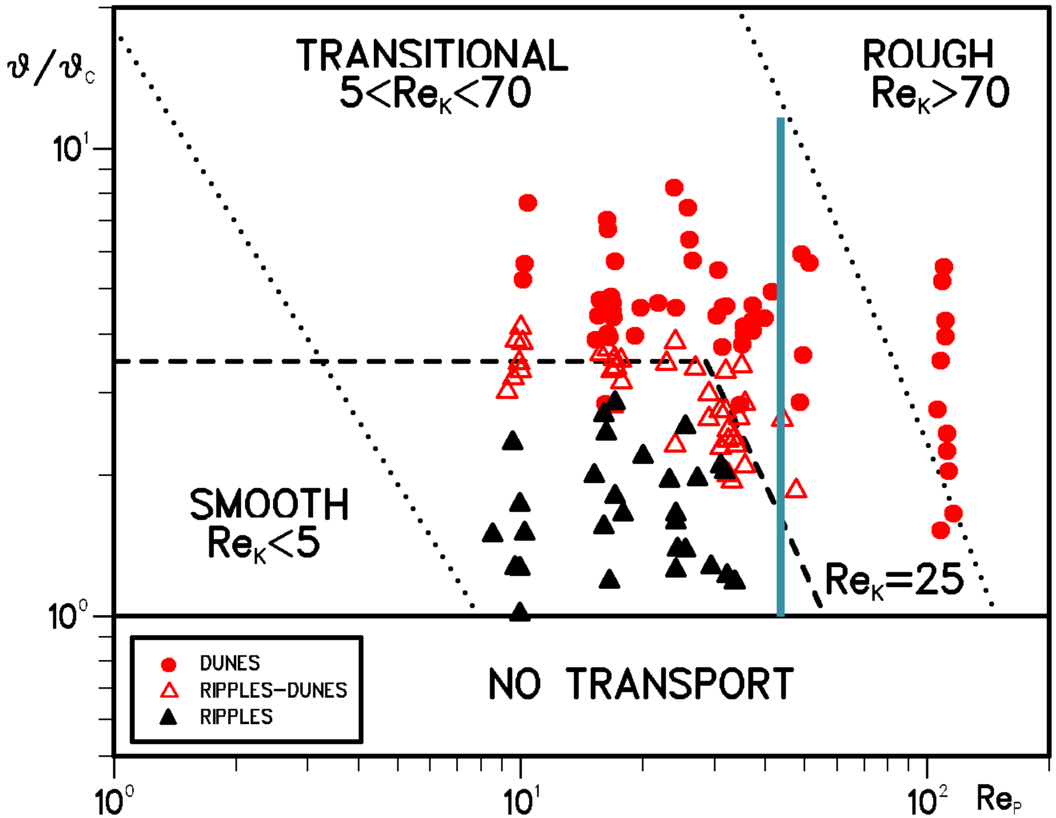
$$\theta_c = 0.22 Re_p^{-0.6} + 0.06 \exp\left(-17.73 Re_p^{-0.6}\right)$$

Critical Shields Parameter

Experimental Observations

Experimental Observations

Experiments by Guy, Simons & Richardson (1966)



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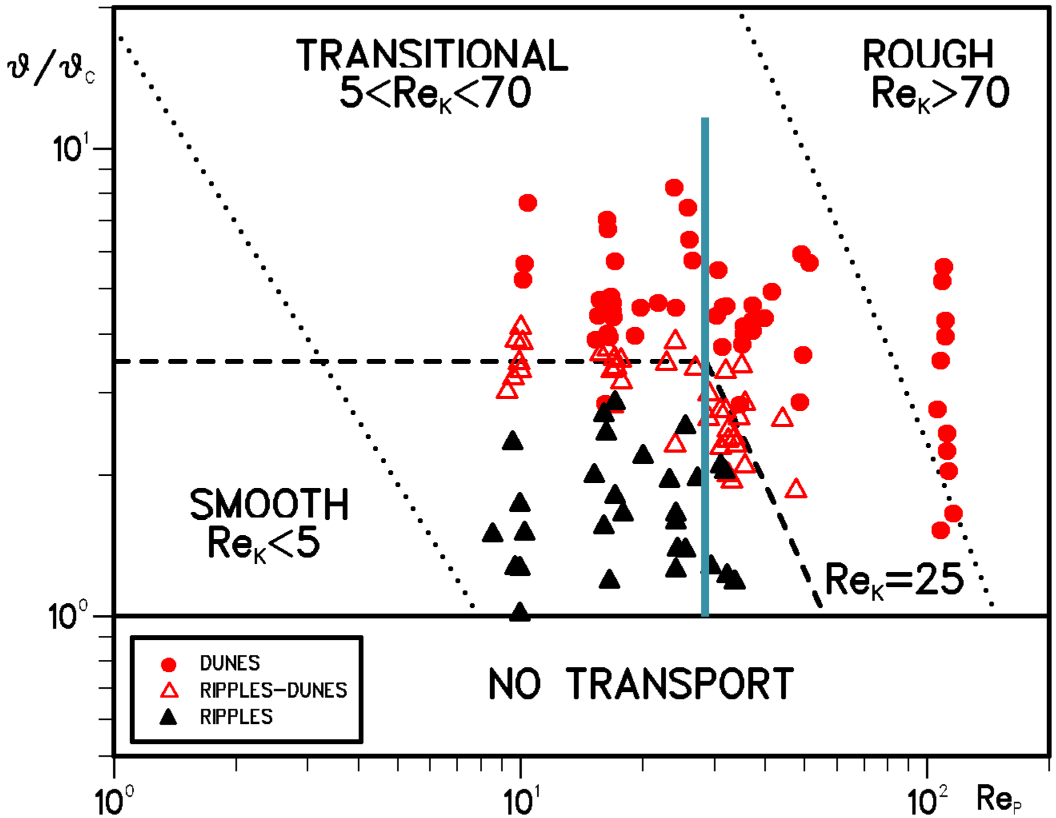

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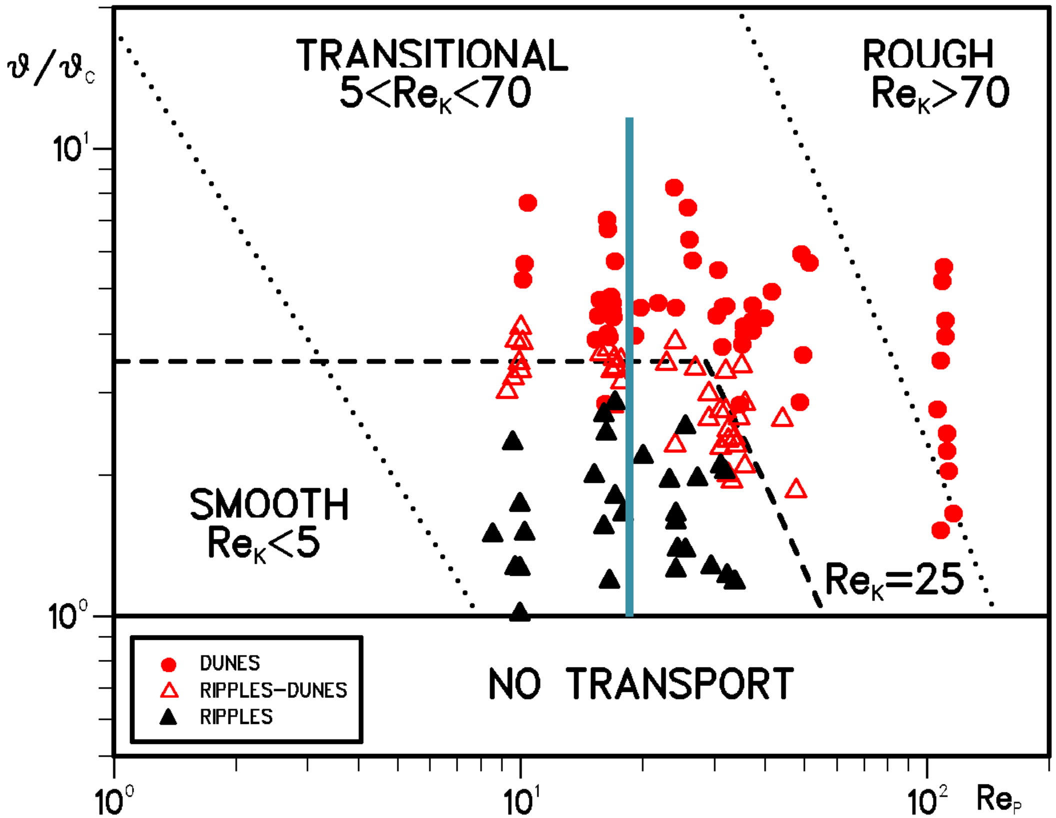

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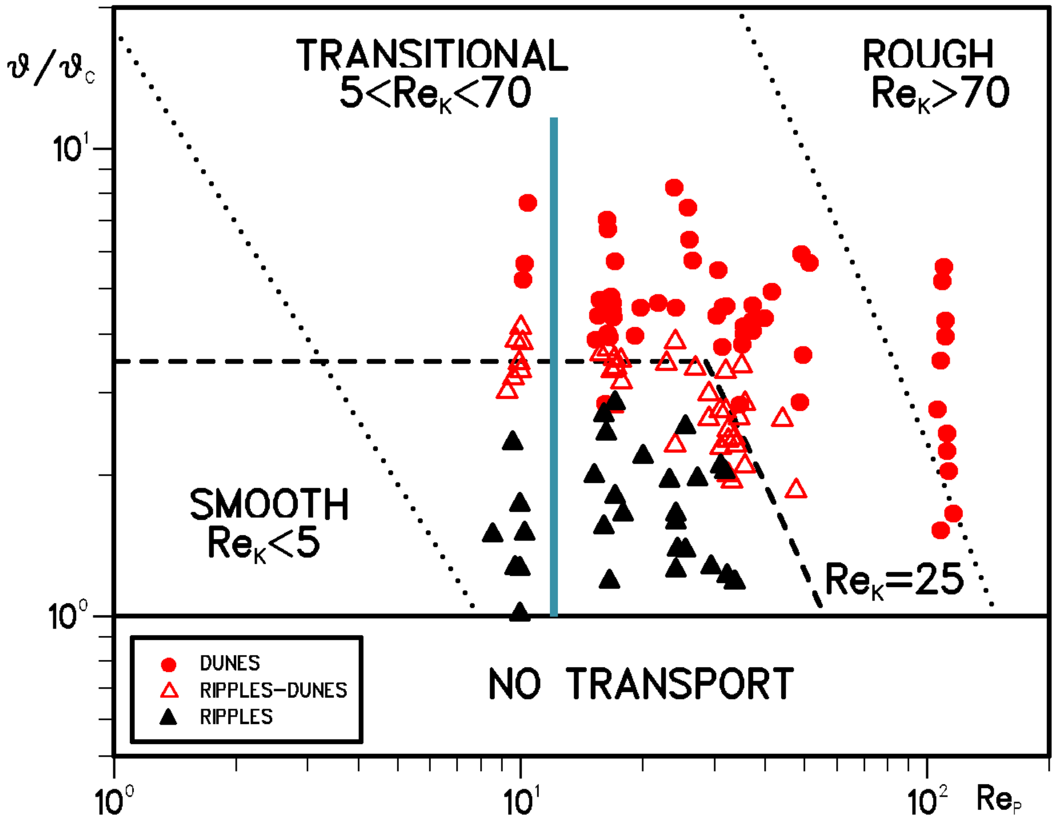

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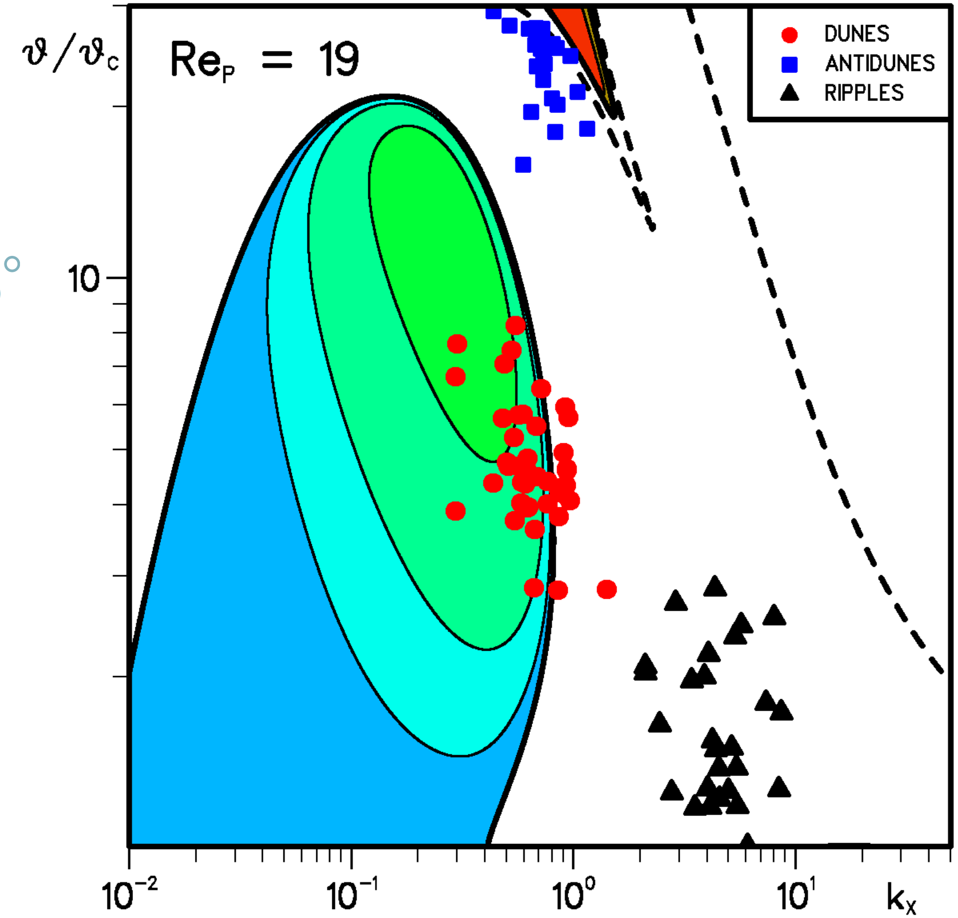
Critical Shields Parameter

Results: effect of C



Re_P is the parameter that controls the transition between ripples and dunes

Re_P decreasing □ d^* decreasing □ finer sediment

C, d = constant □ d^* decreasing □ shallower flow

Stability of river bed forms

Marco Colombini
Dipartimento di Ingegneria Civile, Chimica e Ambientale
University of Genova, Italy

Thank you for your attention!