Growth of a drainage network

KITP, October 28th, 2013

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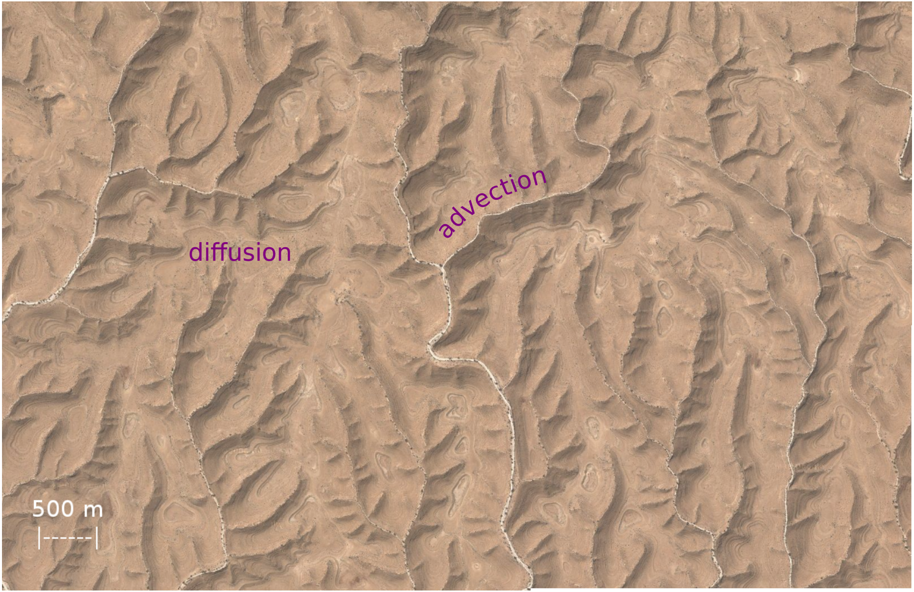
Massachusetts Institute of Technology

Diffusion around a network

Puntland, Somalia

Diffusion around a network

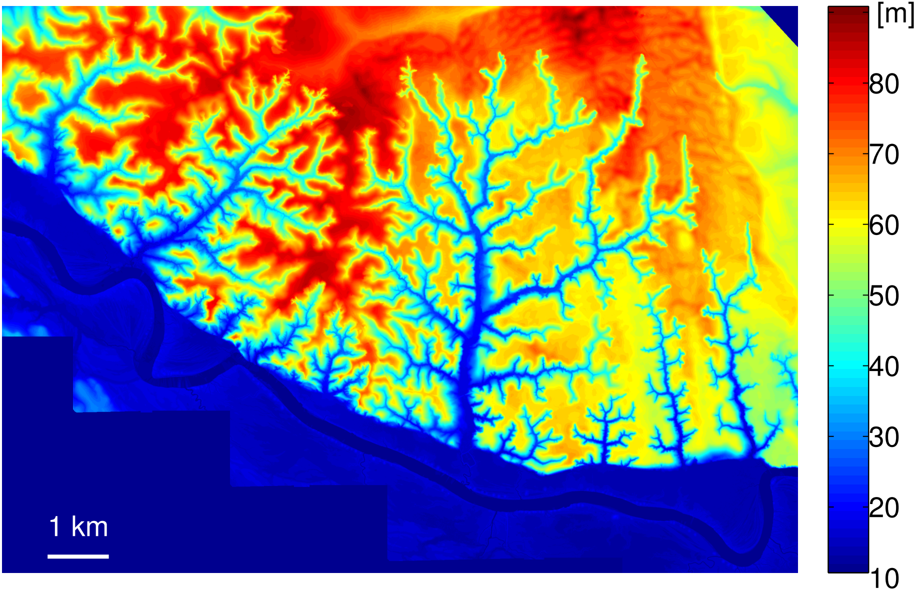
Diffusion around a network



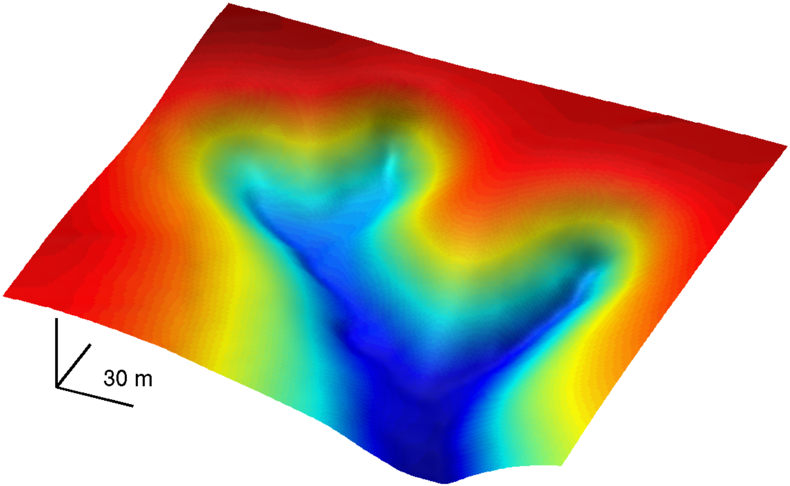
Seepage channels in Florida

Dunne 1980, Abrams et al. 2009

Seepage channels in Florida



Steepheads



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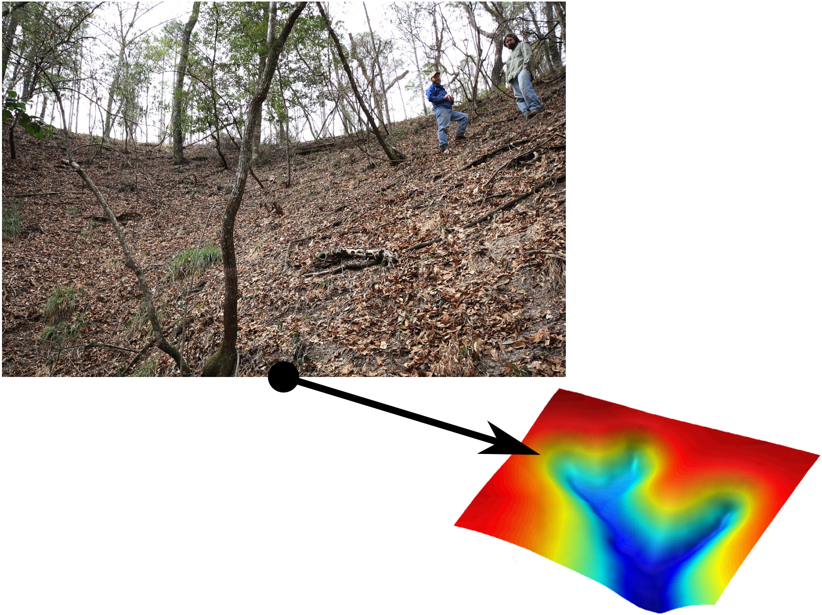

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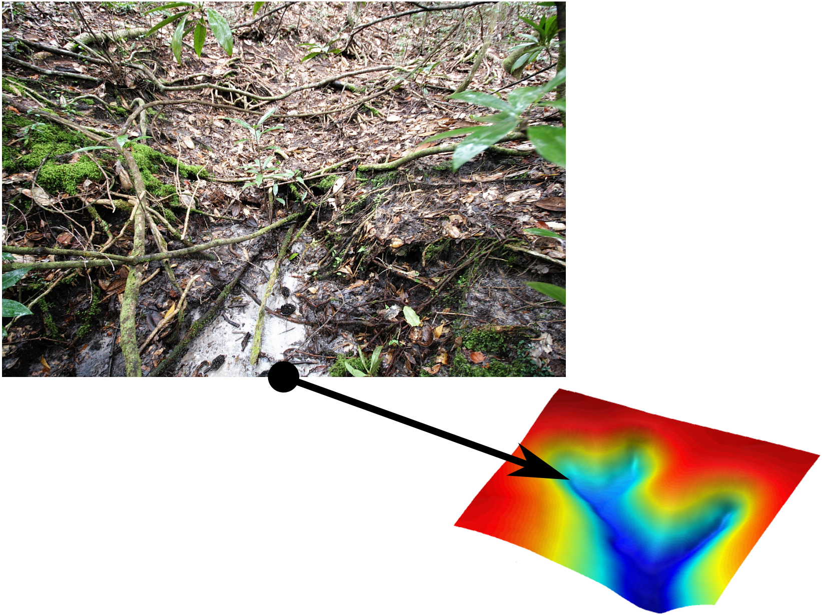

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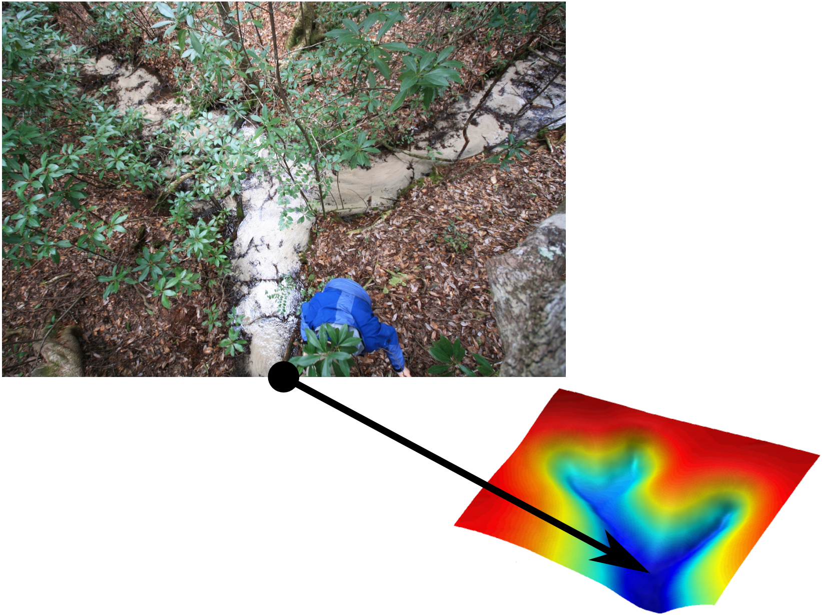
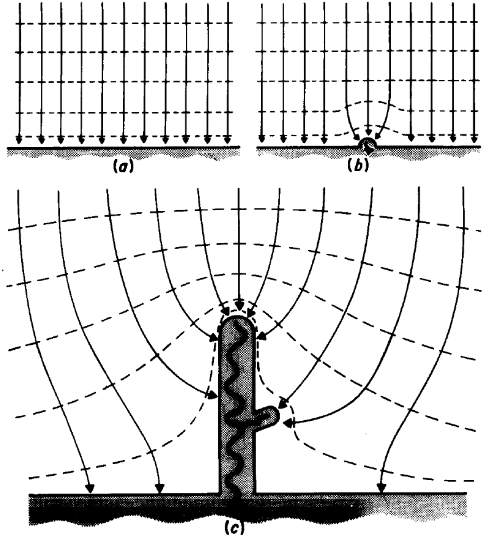


Seepage erosion

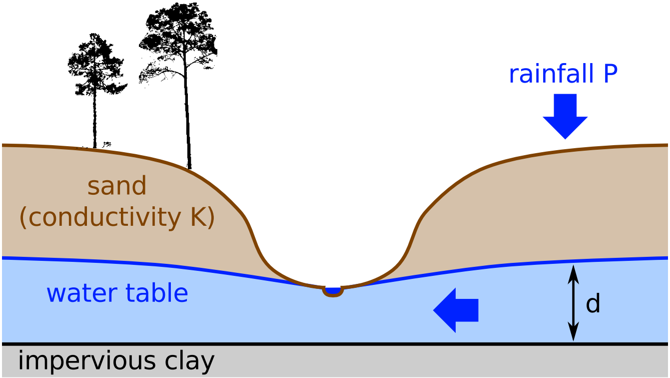


Dunne 1980

Saffman-Taylor instability

J. Ignés-Mullol

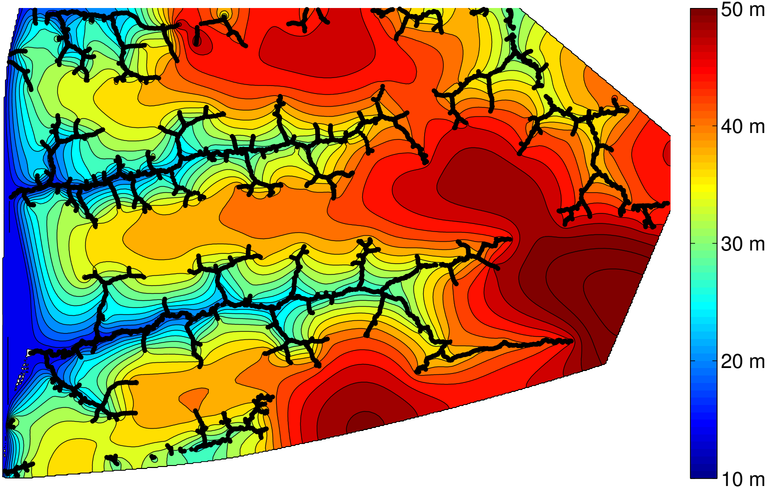
Groundwater



Darcy's law + Dupuit's approximation

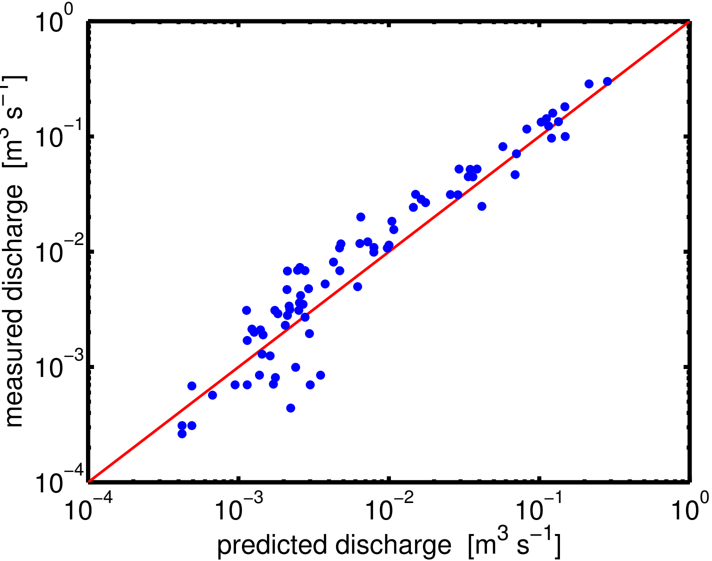
➔ Poisson's equation $P = -\frac{K}{2}\nabla^2 d^2$

Water table elevation

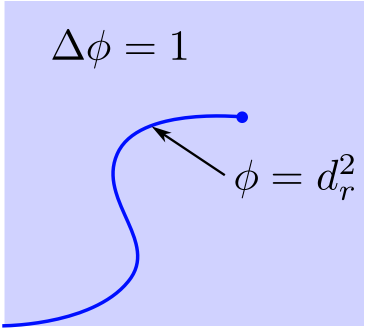


Finite elements with FreeFem++
Petroff et al. 2012

River discharge



Near the stream's tip



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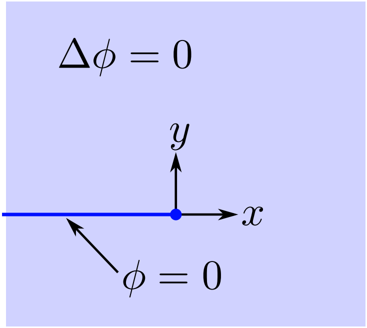

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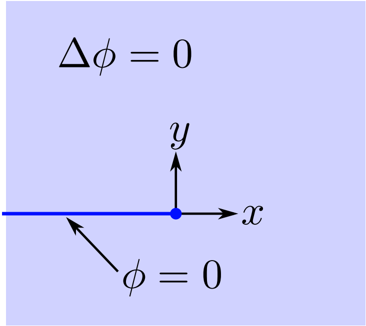

Analytical function :

$$\phi(x, y) = \mathrm{Re}(\Phi(z)) \quad \text{with} \quad z = x + iy$$

Growth rules

$$\Phi = a\, z^{1/2} + i\, b\, z + c\, z^{3/2} + \mathcal{O}\left(z^2\right)$$

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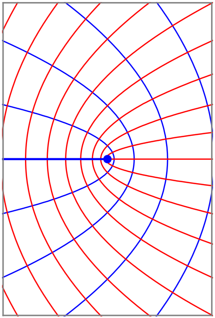

$z^{1/2}$

tip velocity

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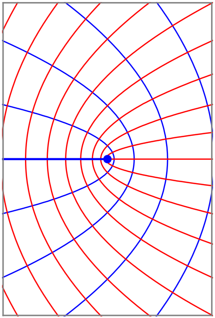

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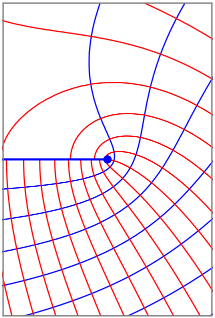

$z^{1/2} + 0.5\, i\, z$

direction

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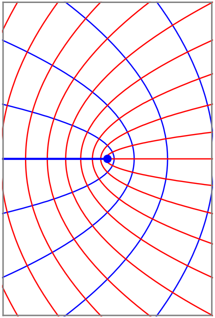

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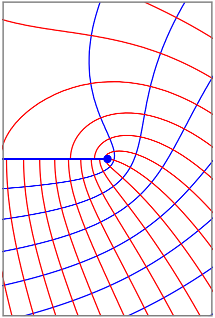

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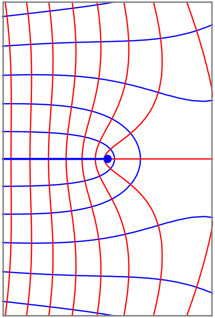
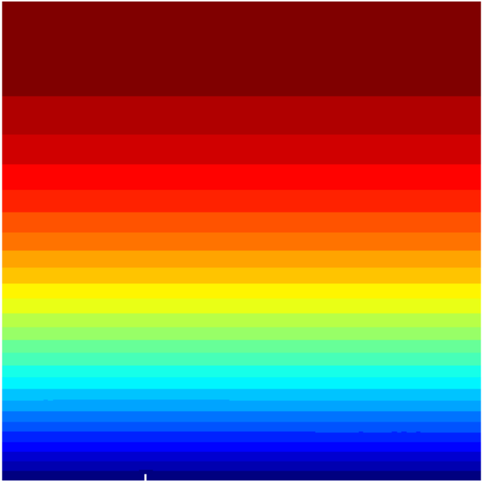


$z^{1/2} - 0.5\, z^{3/2}$

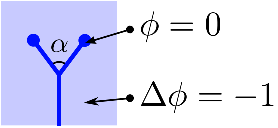
bifurcation

Network growth

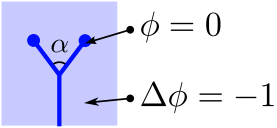


Petroff et al. 2013

Bifurcations

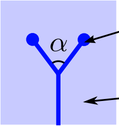


Bifurcations



Locally $\Delta\phi \approx 0$

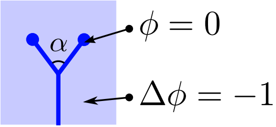
Bifurcations



Locally $\Delta\phi \approx 0$

$$\phi = \text{Re}\left((-z)^{\alpha/\pi} \left(-z^2 + \frac{2\pi}{\alpha} \right)^{\frac{2\pi - \alpha}{2\pi}} \right)$$

Bifurcations



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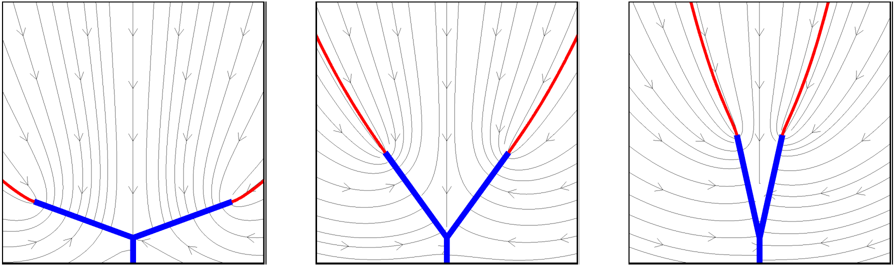

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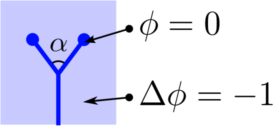

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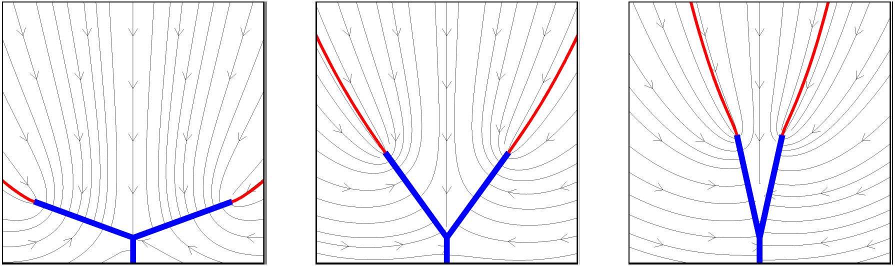
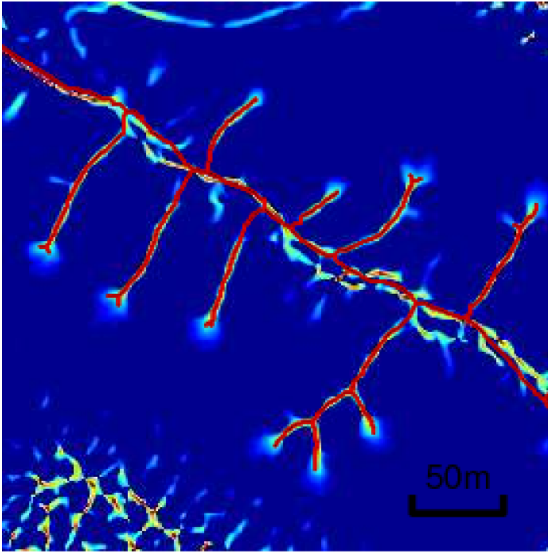


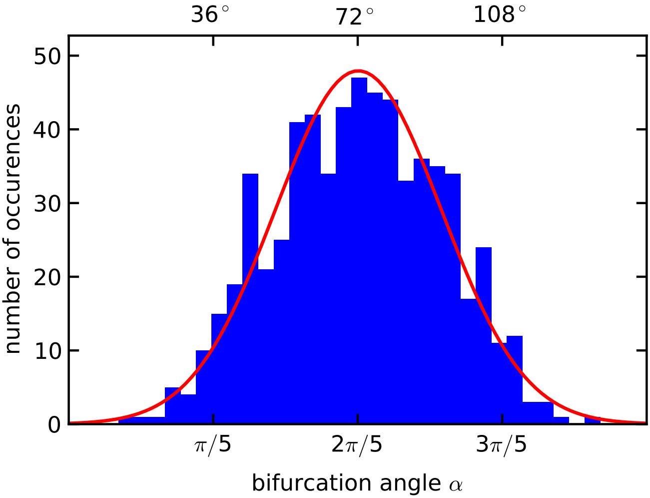
Only self-similar bifurcation : $\alpha = \frac{2\pi}{5} = 72°$

Bifurcations



Contour curvature of the lidar map

Bifurcations



Devauchelle et al. 2012

Geometry controls the shape of the bifurcation

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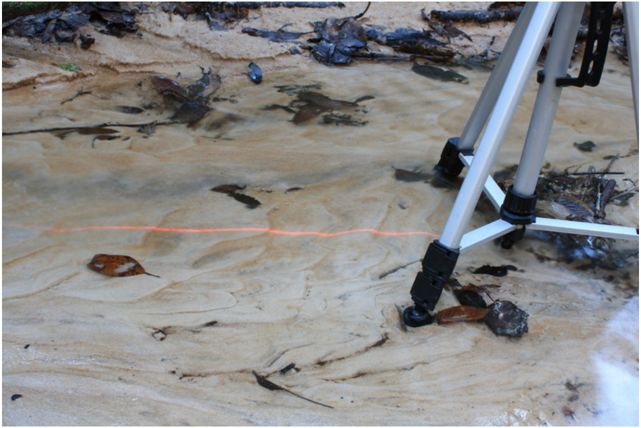
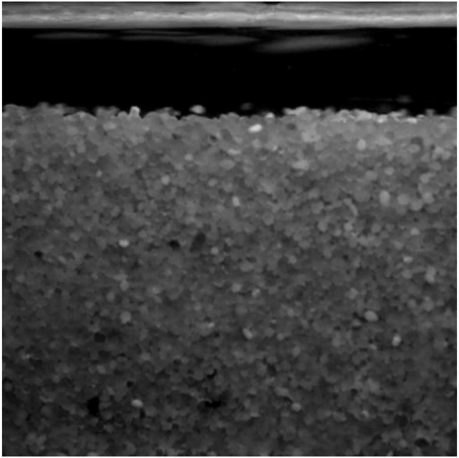
What about the growth dynamics ?

Bedload transport

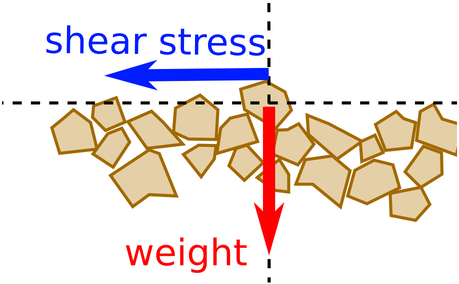


Bedload transport



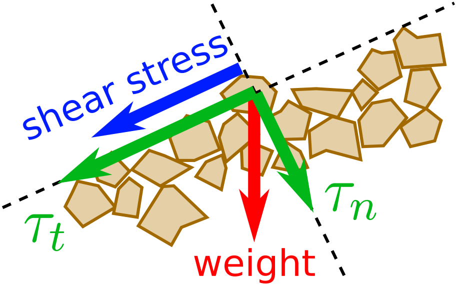
Lajeunesse et al. 2010

Bedload transport



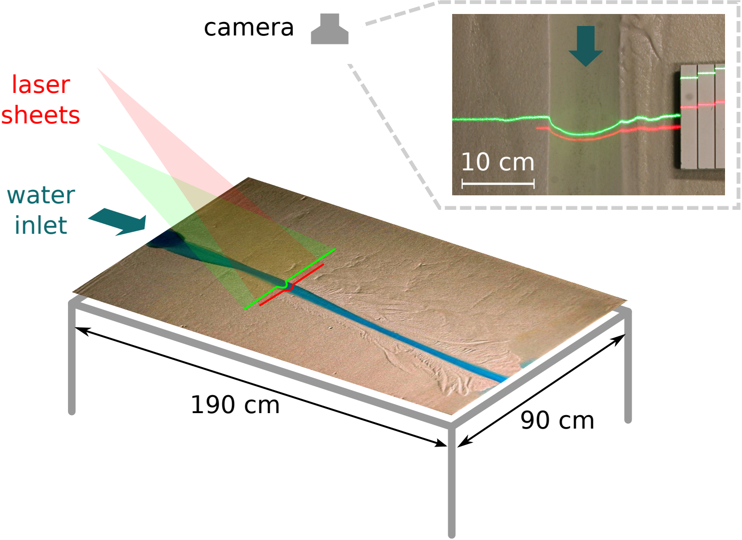
Shields parameter $\theta = \frac{\text{shear stress}}{\text{weight}} = \frac{\tau}{(\rho_s - \rho_w) g d_s}$

Bedload transport



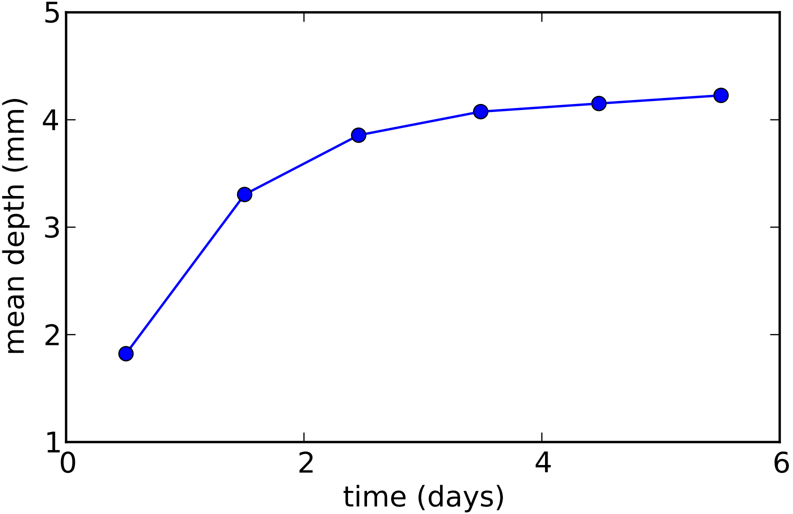
Coulomb friction $\mu = \frac{\tau_t}{\tau_n}$

Laboratory river

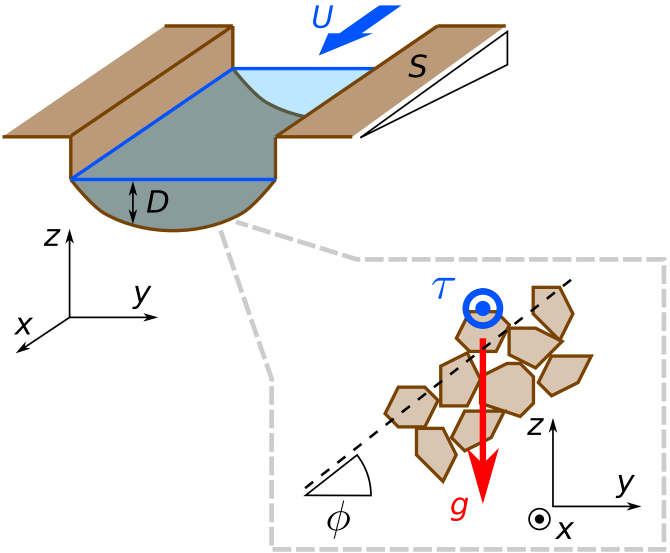


Seizilles et al. 2013

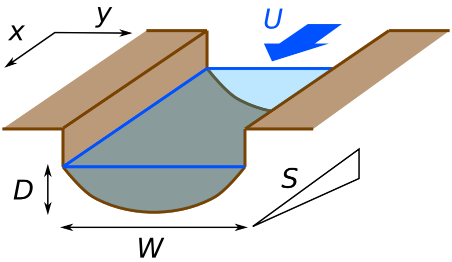
Laboratory river



Threshold theory



Threshold theory

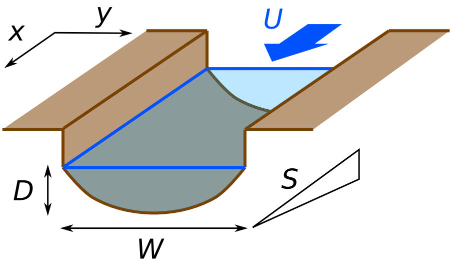


$$\frac{\text{tangential force}}{\text{normal force}} = \mu$$

μ Coulomb friction coefficient

Glover et al. 1951, Henderson 1961

Threshold theory

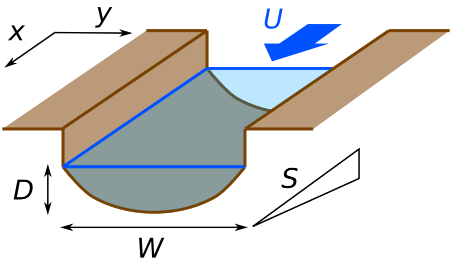


$$L = \frac{\mu\,\rho_w\,d_s}{(\rho_s - \rho_w)\,\theta_t\,S}$$

$$\underbrace{\left(\frac{\partial D}{\partial y}\right)^2}_{\text{gravity}} + \underbrace{\left(\frac{D}{L}\right)^2}_{\text{fluid force}} = \mu^2$$

μ Coulomb friction coefficient
θ_t critical Shields parameter
d_s sediment size
ρ_s, ρ_w sediment and water density

Threshold theory



$$L = \frac{\mu\,\rho_w\,d_s}{(\rho_s - \rho_w)\,\theta_t\,S}$$

$$D(y) = \mu L \cos\left(\frac{y}{L}\right)$$

μ Coulomb friction coefficient
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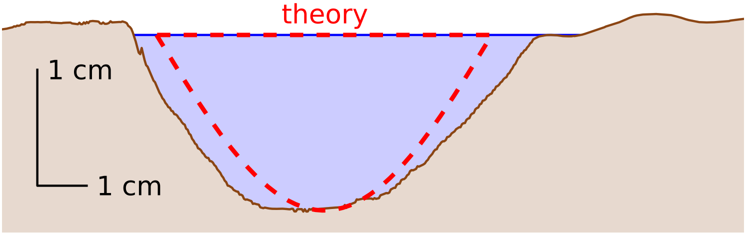
Experimental results



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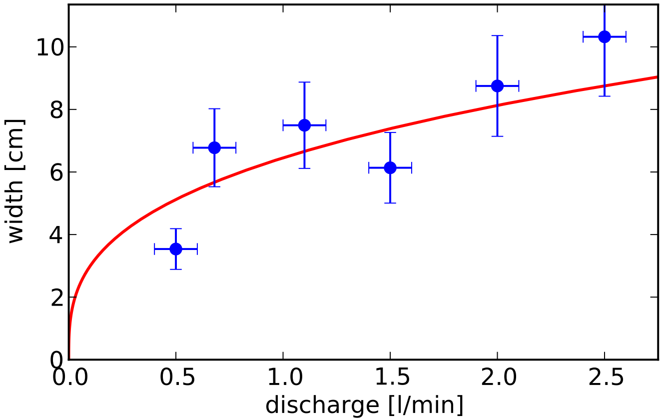

$$\text{width} = \frac{\pi}{\mu^{2/3}} \left(\frac{9\,\nu\,\rho_f\,\theta_t}{4\,g\,d_s(\rho_s - \rho_f)} \right)^{1/3} \text{discharge}^{1/3}$$

Discharge controls slope

$$\text{discharge} \times \text{slope}^3 = \left(\frac{\theta_t \, (\rho_s - \rho_f) \, d_s}{\rho_f} \right)^4 \frac{4 \, g}{9 \, \mu \, \nu}$$

Discharge controls slope

For turbulent flows :

$$\text{discharge} \times \text{slope}^2 = \frac{2\,\mathcal{K}(1/2)}{3\,C_f}\,\sqrt{2\theta_t^3 g \left(\frac{(\rho_s - \rho_f)\,d_s}{\rho_f}\right)^5}$$

C_f is the Chézy friction coefficient

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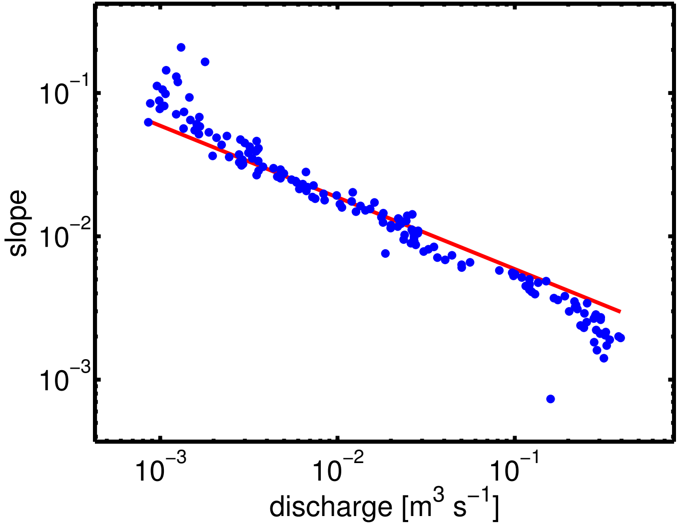
Discharge controls slope



Field data from Bristol, Florida

Discharge and slope are linked through the mechanical equilibrium of a grain

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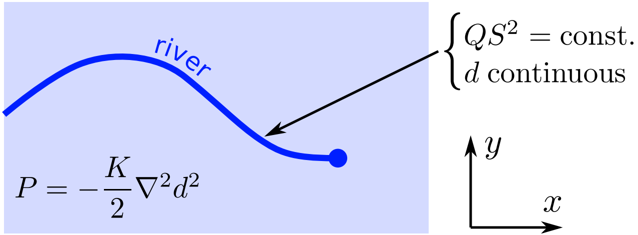
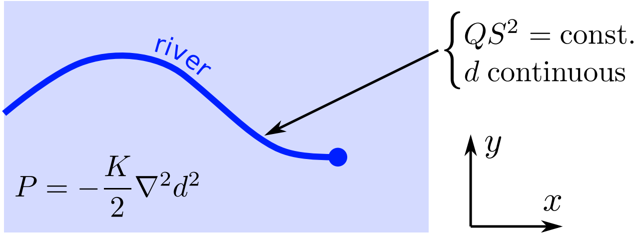
How is this connected to groundwater flow ?

Coupling river and groundwater



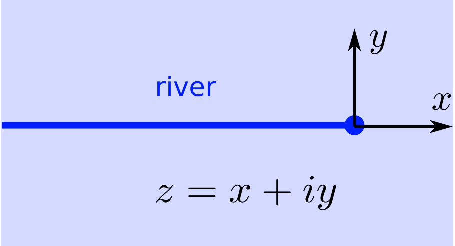
Coupling river and groundwater



$$Q = \frac{K}{2}\int_0^s \left[\frac{\partial d^2}{\partial n}\right] \mathrm{d}s$$

$$S = \frac{\partial d}{\partial s}$$

Near the tip

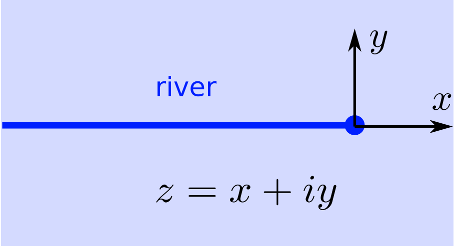


Analitical function $\Phi = \phi + i\psi$

After rescaling :

$$\begin{cases} [\psi] \left(\dfrac{\partial \phi}{\partial x}\right)^2 & = & 1 \\ [\phi] & = & 0 \end{cases}$$

Near the tip



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Unique solution ?

$$\Phi = \sqrt{3}\, z^{2/3}$$

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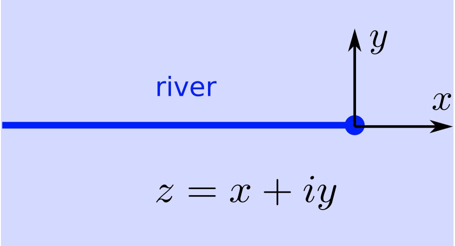

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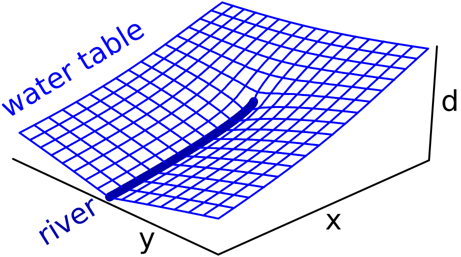
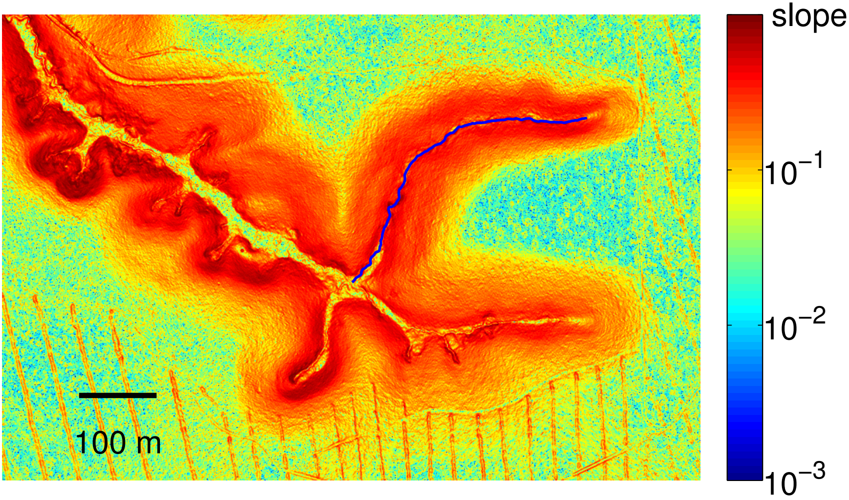


Longitudinal profile of a stream



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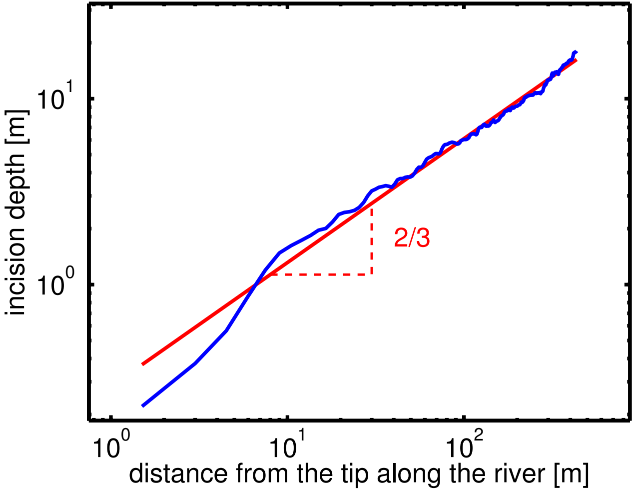
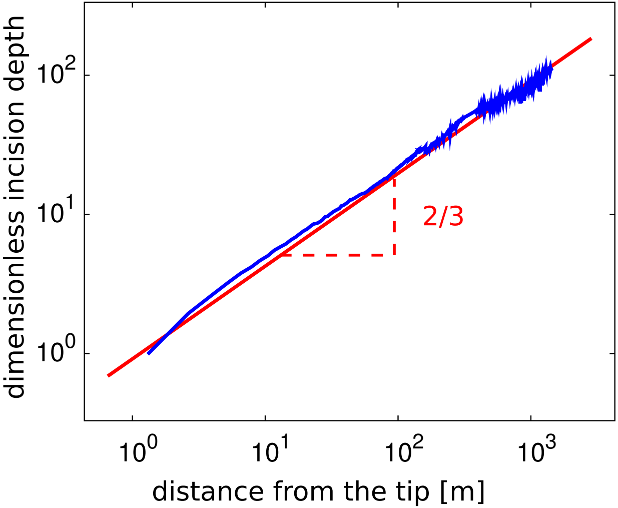


Devauchelle et al. 2011

Longitudinal profile of a stream



Average over 166 streams

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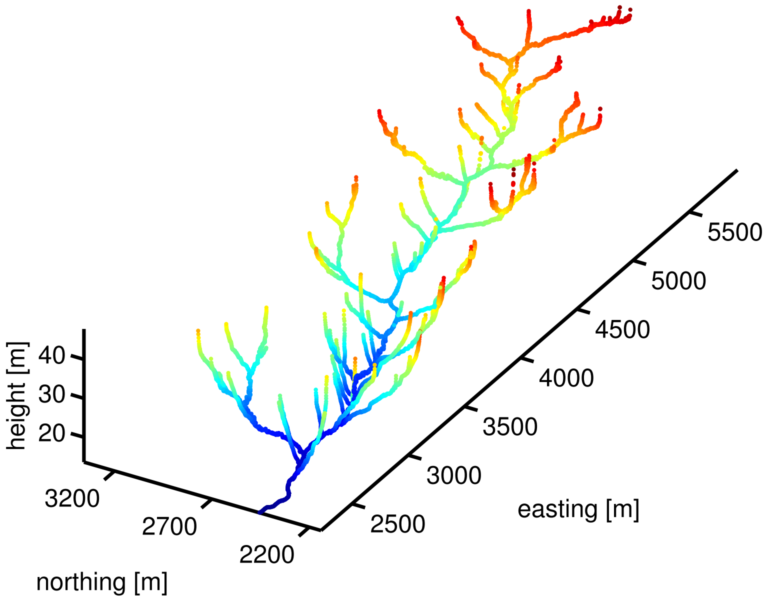
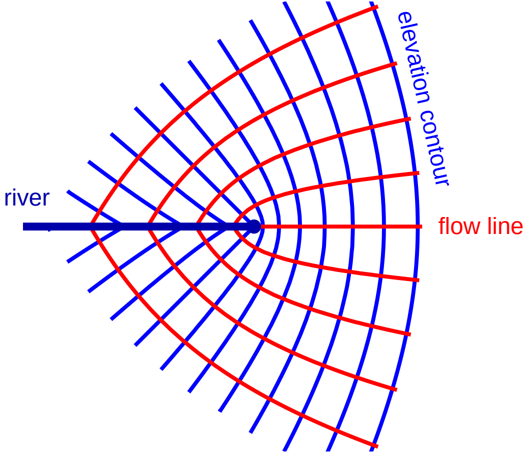
Longitudinal profile of a stream



How unique is the solution ?



$$\Phi = z^{2/3}$$

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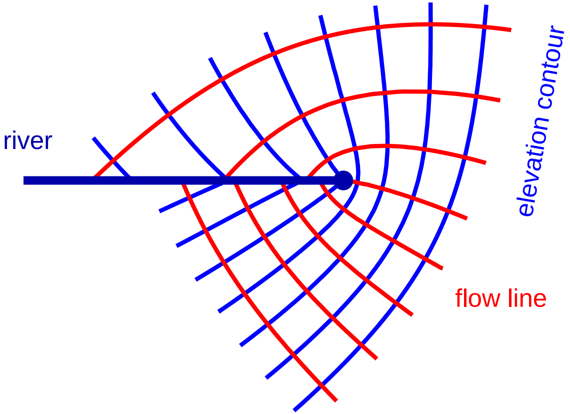
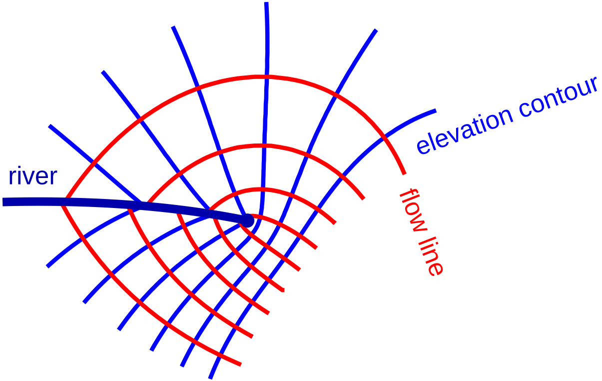


$$\phi = z^{2/3} + f(z) \quad \text{with} \quad f(z^*) = -f(z)^*$$

How unique is the solution ?



$\phi = f(z)^{2/3}$ with $|f'(z)| = 1$ at the river

Conclusion

- The 3D structure of the network shows it is near equilibrium

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- We must add sediment transport to the stream model