Particle-size segregation and spontaneous levee formation II

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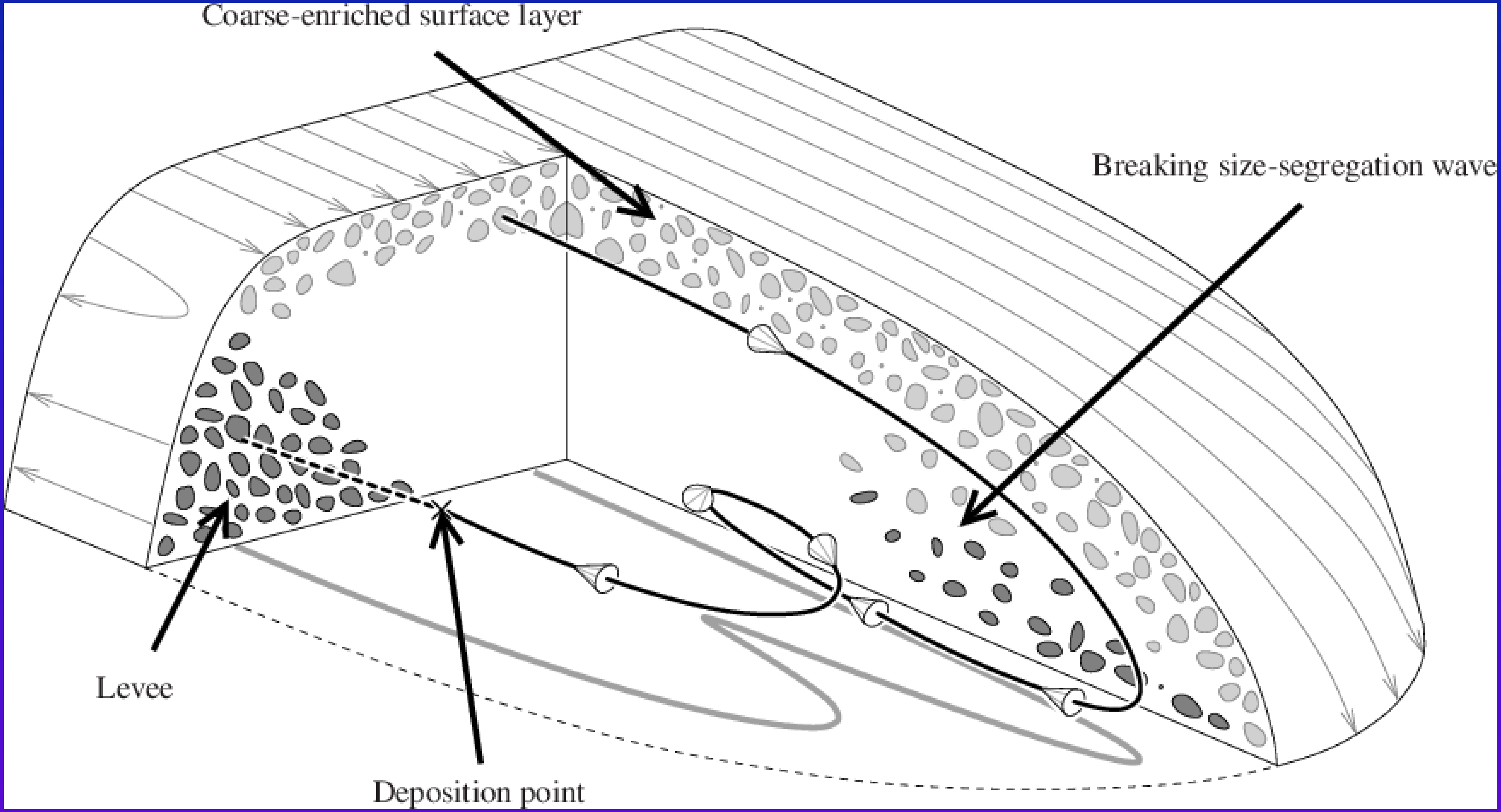
MANCHESTER 1824

USGS

Schematic diagram of the levee formation process



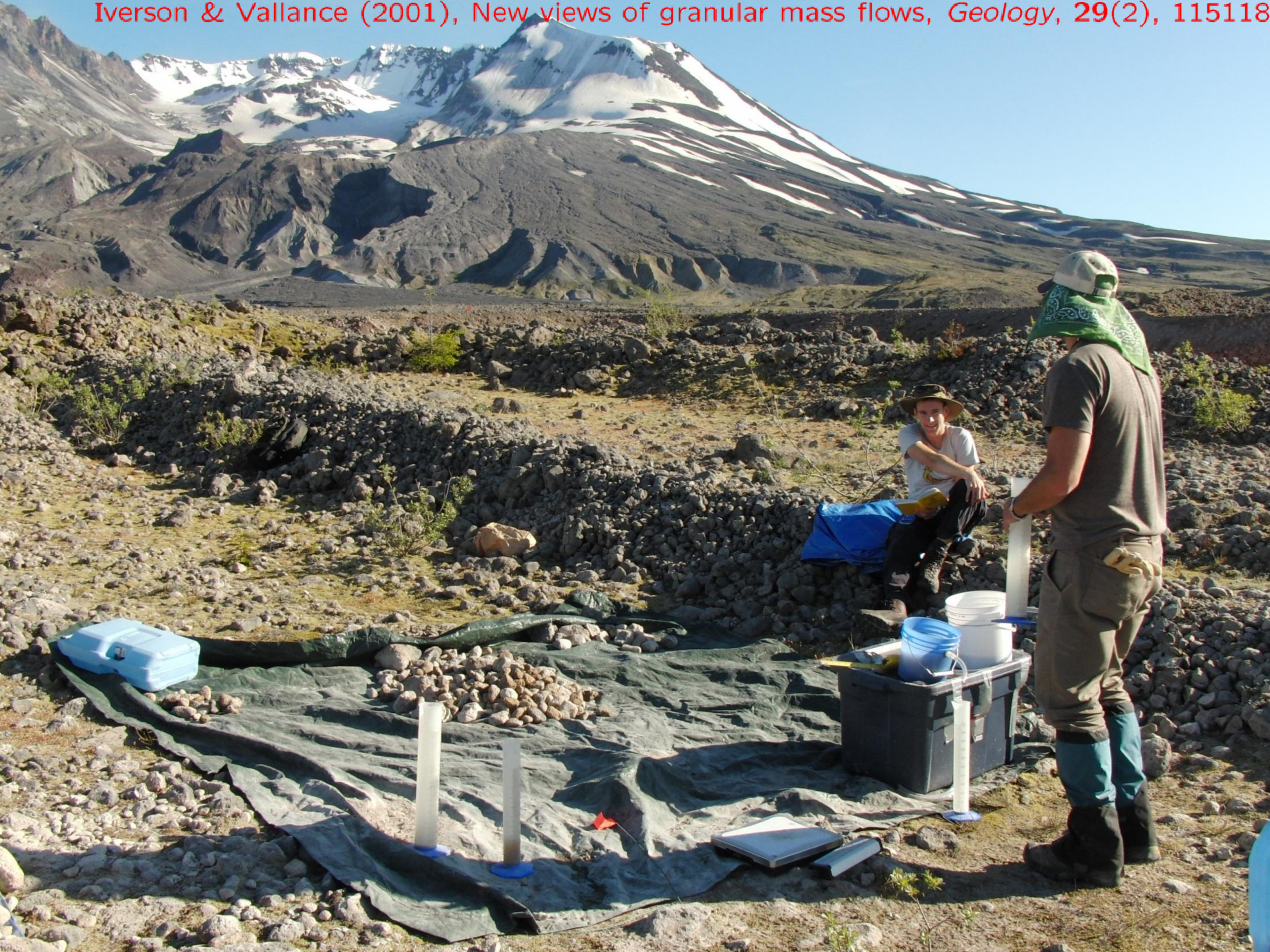
- larger particles are shouldered to the sides to create levees
- this is an example of a segregation-mobility feedback effect

Mount St Helens May 18, 1980


Mt St Helens July 22, 1980

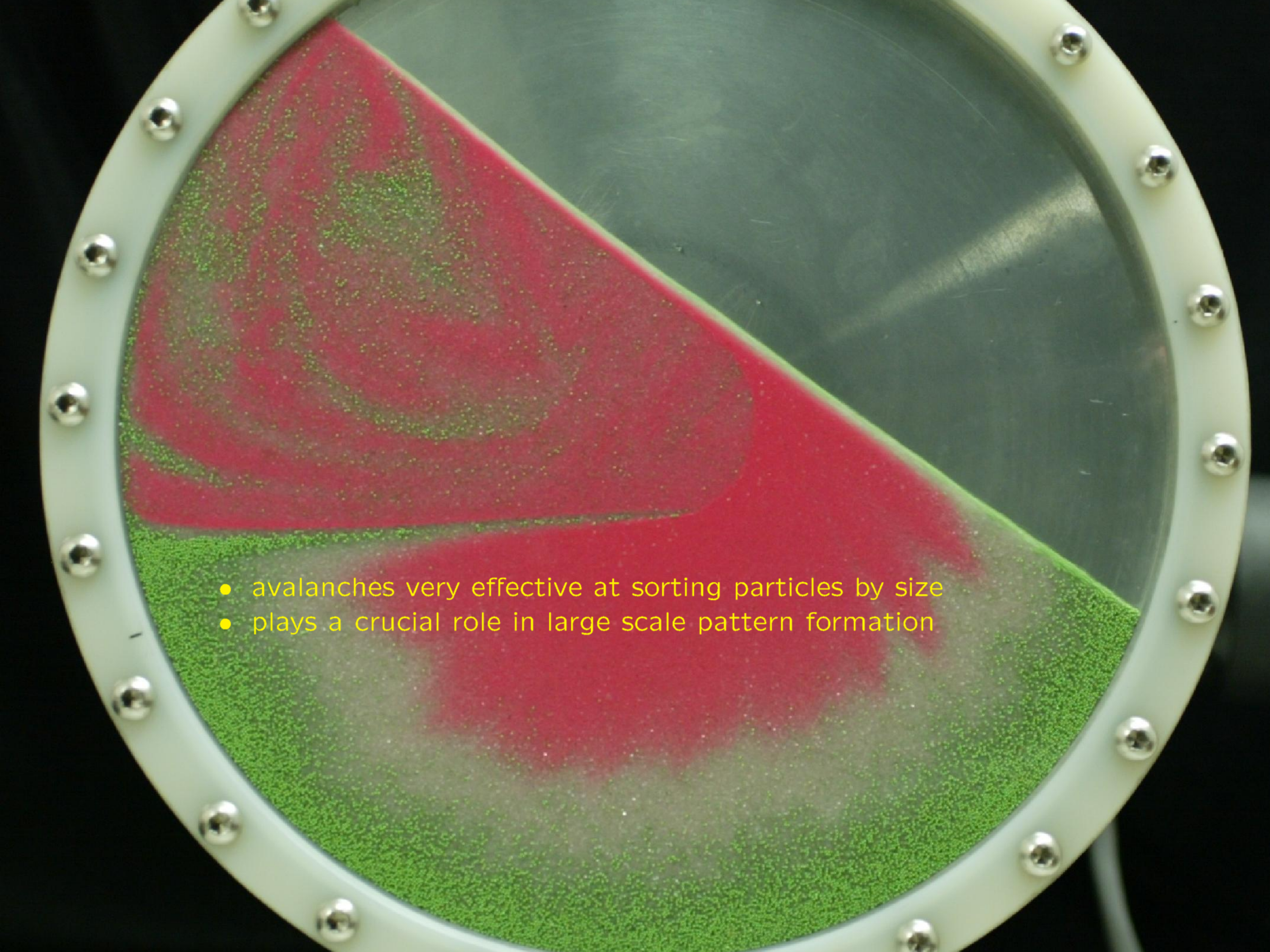

Mount St Helens, USA, 1980
Finer grained interior
Coarse rich levee
• segregation occurs in many hazardous natural flows
– debris-flows, pyroclastic flows & snow avalanches
• and leads to spontaneous flow organization and longer run-out

Iverson & Vallance (2001), New views of granular mass flows, *Geology*, **29**(2), 115118

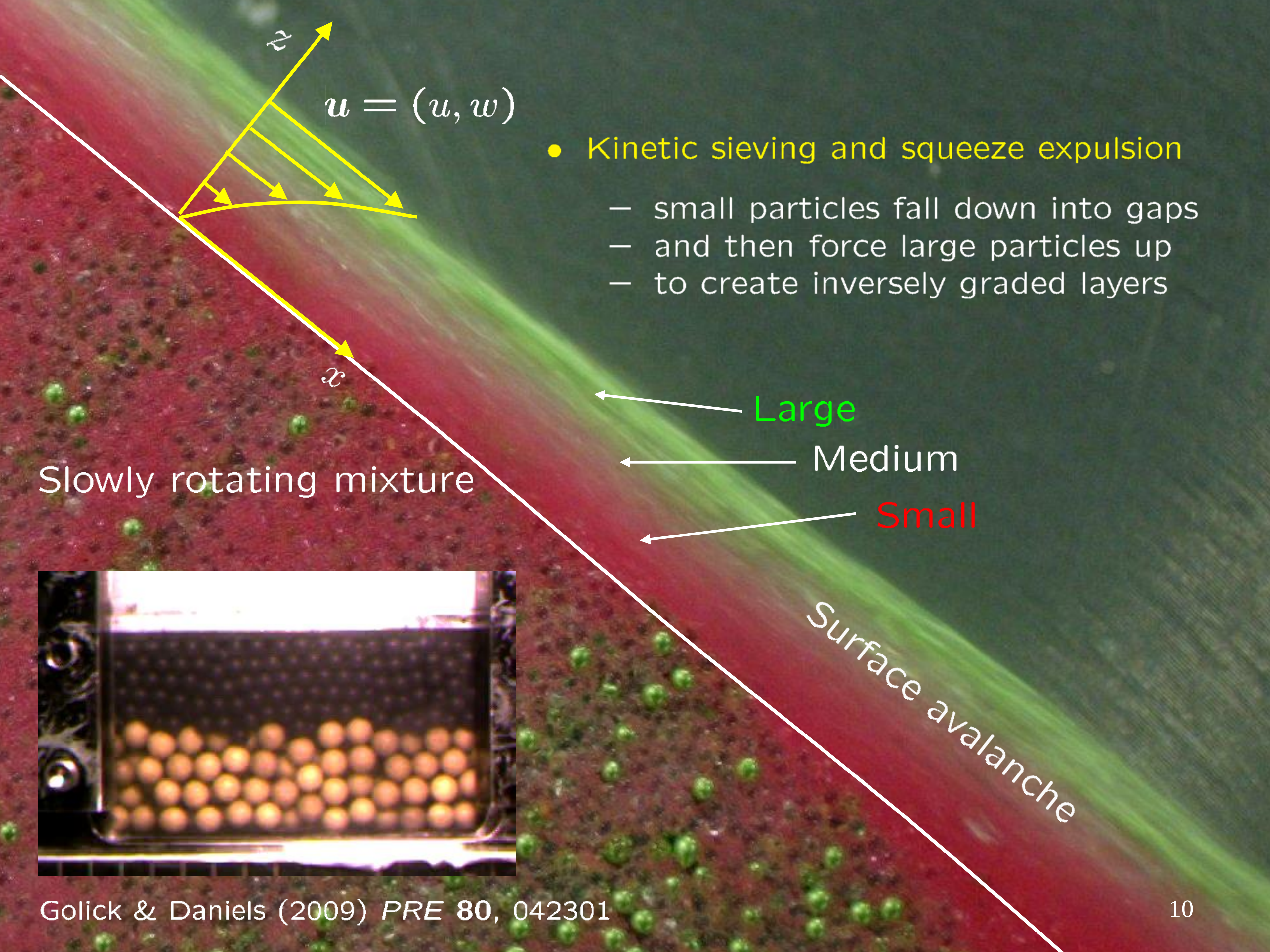


Segregation induced finger formation …

Pouliquen, Delours & Savage (1997), Nature. **386**, 816-817.
Woodhouse *et al.* (2012), J. Fluid Mech. **709**, 543-580.



- avalanches very effective at sorting particles by size
- plays a crucial role in large scale pattern formation



$\boldsymbol{u} = (u, w)$
z
x
• Kinetic sieving and squeeze expulsion
– small particles fall down into gaps
– and then force large particles up
– to create inversely graded layers
Large
Medium
Small
Slowly rotating mixture
Surface avalanche
Golick & Daniels (2009) *PRE* **80**, 042301

Mixture framework

- the volume fraction ϕ^ν of constituent ν, per unit volume of mixture, lies in the range

$$0 \leq \phi^\nu \leq 1.$$

and their sum

$$\sum_{\forall \nu} \phi^\nu = 1.$$

- In standard mixture theory the partial and intrinsic density, stress, pressure and velocity fields satisfy

$$\rho^\nu = \phi^\nu \rho^{\nu *}, \quad \boldsymbol{\sigma}^\nu = \phi^\nu \boldsymbol{\sigma}^{\nu *}, \quad p^\nu = \phi^\nu p^{\nu *}, \quad \boldsymbol{u}^\nu = \boldsymbol{u}^{\nu *}$$

- The bulk density, pressure and velocity are

$$\rho = \sum_{\forall \nu} \rho^\nu, \quad p = \sum_{\forall \nu} p^\nu, \quad \rho \boldsymbol{u} = \sum_{\forall \nu} \rho^\nu \boldsymbol{u}^\nu$$

Mass and momentum balances for each constituent

- Each constituent satisfies individual mass

$$\frac{\partial \rho^\nu}{\partial t} + \nabla \cdot (\rho^\nu \boldsymbol{u}^\nu) = 0,$$

and momentum balances

$$\frac{\partial}{\partial t}(\rho^\nu \boldsymbol{u}^\nu) + \nabla \cdot (\rho^\nu \boldsymbol{u}^\nu \otimes \boldsymbol{u}^\nu) = \nabla \cdot \boldsymbol{\sigma}^\nu + \rho^\nu \boldsymbol{g} + \boldsymbol{\beta}^\nu,$$

where $\otimes$ is the dyadic product and $\boldsymbol{g}$ is the gravitational acceleration vector.

- The interaction drag $\boldsymbol{\beta}^\nu$ consists of three terms

$$\boldsymbol{\beta}^\nu = p \nabla f^\nu - \rho^\nu c(\boldsymbol{u}^\nu - \boldsymbol{u}) - \rho d \nabla \phi^\nu,$$

c is the coefficient of inter-particle drag,
d is the coefficient of diffusive remixing

- Acceleration negligible in the z direction.
- Bulk momentum balance $\Rightarrow$ pressure lithostatic

$$p = \rho g(s - z)\cos\zeta.$$

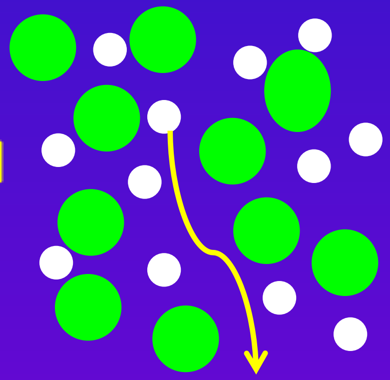
- If pressure is shared in proportion

$$p^\nu = \phi^\nu p, \quad \Rightarrow \quad \text{NO SEGREGATION}$$

- Use non-standard partial/intrinsic pressure relation

$$p^\nu = f^\nu p,$$

where f^ν determines the proportion load



$$0 \le f^\nu \le 1, \quad \sum_{\forall \nu} f^\nu = 1.$$

- Assuming normal accelerations are negligible

$$\phi^\nu w^\nu = \phi^\nu w + (f^\nu - \phi^\nu)(g/c)\cos\zeta - (d/c)\frac{\partial \phi^\nu}{\partial z},$$

$$\begin{aligned} f^\nu - \phi^\nu > 0 &\quad \text{particles rise} \\ f^\nu - \phi^\nu = 0 &\quad \text{no relative motion} \\ f^\nu - \phi^\nu < 0 &\quad \text{particles percolate downwards} \end{aligned}$$

- Particles in a pure phase carry all of the load

$$f^\nu = 1, \quad \text{when} \quad \phi^\nu = 1,$$

- When no particles, they cannot carry any load

$$f^\nu = 0, \quad \text{when} \quad \phi^\nu = 0.$$

- Large particles carry more load than small

$$f^l = \phi^l + B_{ls}\phi^l\phi^s \quad \text{(bi-disperse)}$$

The multi-component segregation remixing equation

- f^ν must reduce to bidisperse case in any submixture
- suggests additive decomposition

$$f^\nu = \phi^\nu + \sum_{\forall\mu} B_{\nu\mu}\phi^\nu\phi^\mu, \qquad \text{(polydisperse case)}$$

where $B_{\nu\mu} = -B_{\mu\nu}$ and $B_{(\nu\nu)} = 0$.

- Scaling on thickness H, length L and velocity U implies segregation remixing equation (phase ν) is

$$\frac{\partial\phi^\nu}{\partial t} + \nabla\cdot(\phi^\nu \boldsymbol{u}) + \frac{\partial}{\partial z}\left(\sum_{\forall\mu} S_{\nu\mu}\phi^\nu\phi^\mu\right) = \frac{\partial}{\partial z}\left(D_r\frac{\partial\phi^\nu}{\partial z}\right),$$

where

$$S_{\nu\mu} = \frac{Lg\cos\zeta}{HUc}B_{\nu\mu}, \qquad D_r = \frac{Ld}{H^2Uc}.$$

Gray & Ancey (2011) *J. Fluid Mech.* **678**, 535-588

Consistent with experiments of Savage & Lun (1988) and Vallance & Savage (2000)

Bi-disperse mixtures

- Yields two equations for large and small particles

$$\frac{\partial \phi^l}{\partial t} + \nabla \cdot (\phi^l \boldsymbol{u}) + \frac{\partial}{\partial z}(S_{ls}\phi^l\phi^s) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^l}{\partial z}\right),$$
$$\frac{\partial \phi^s}{\partial t} + \nabla \cdot (\phi^s \boldsymbol{u}) - \frac{\partial}{\partial z}(S_{ls}\phi^s\phi^l) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right).$$

- The summation condition $\sum \phi^\nu = 1$ implies

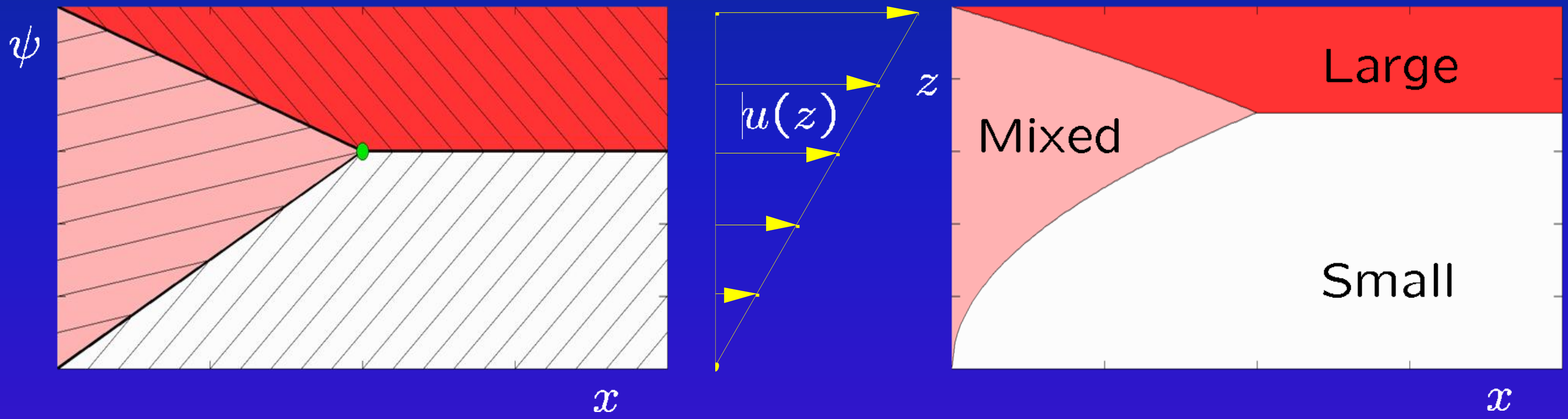
$$\phi^l + \phi^s = 1,$$

- Large particle concentration can be eliminated

$$\frac{\partial \phi^s}{\partial t} + \nabla \cdot (\phi^s \boldsymbol{u}) - \frac{\partial}{\partial z}(S_{ls}\phi^s(1-\phi^s)) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right).$$

Bridgwater, Foo & Stephens (1985), *Powder Technol.* **41**, 147-158
Savage & Lun (1988) *J. Fluid Mech.* **189**, 311-335
Dolgunin & Ukolov (1995) *Powder Technol.* **83**, 95-103
Gray & Thornton (2005) *Proc. Roy. Soc. A.* **461**, 1447-1473.
Thornton, Gray & Hogg (2006) *J. Fluid Mech.* **550**, 1-25.
Gray & Chugunov (2006) *J. Fluid Mech.* **569**, 365-398.

Steady-state concentration shocks in absence of diffusive-remixing



- shock height $s(x)$ satisfies the jump condition

$$\left[\!\left[\phi u \frac{ds}{dx} + S_{ls}\phi(1-\phi) \right]\!\right] = 0 \quad \Rightarrow \quad u\frac{ds}{dx} = S_{ls}(\phi^+ + \phi^- - 1) = \frac{d\psi}{dx}$$

- Using depth-integrated velocity coordinates

$$\psi = \int_0^z u(z')\, dz'$$

- this can be integrated to show there are three intersecting shocks for a homogeneous inflow with $\phi = \phi_0$

$$\psi_1 = S_{ls}\phi_0 x, \quad \psi_2 = 1 - S_{ls}(1-\phi_0)x, \quad \psi_3 = \phi_0$$

A ternary mixture of large medium and small particles

- Theory yields three equations, but one can be eliminated since

$$\phi^m = 1 - \phi^s - \phi^l$$

- To give two equations for the large and small particles

$$\frac{\partial \phi^l}{\partial t} + \nabla \cdot (\phi^l \boldsymbol{u}) + \frac{\partial}{\partial z}\left(S_{lm}\phi^l(1-\phi^l-\phi^s) + S_{ls}\phi^l\phi^s\right) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^l}{\partial z}\right)$$

$$\frac{\partial \phi^s}{\partial t} + \nabla \cdot (\phi^s \boldsymbol{u}) + \frac{\partial}{\partial z}\left(-S_{ls}\phi^s\phi^l - S_{ms}\phi^s(1-\phi^l-\phi^s)\right) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi^s}{\partial z}\right)$$

- A steady-state solution for a homogeneous inflow at $x = 0$

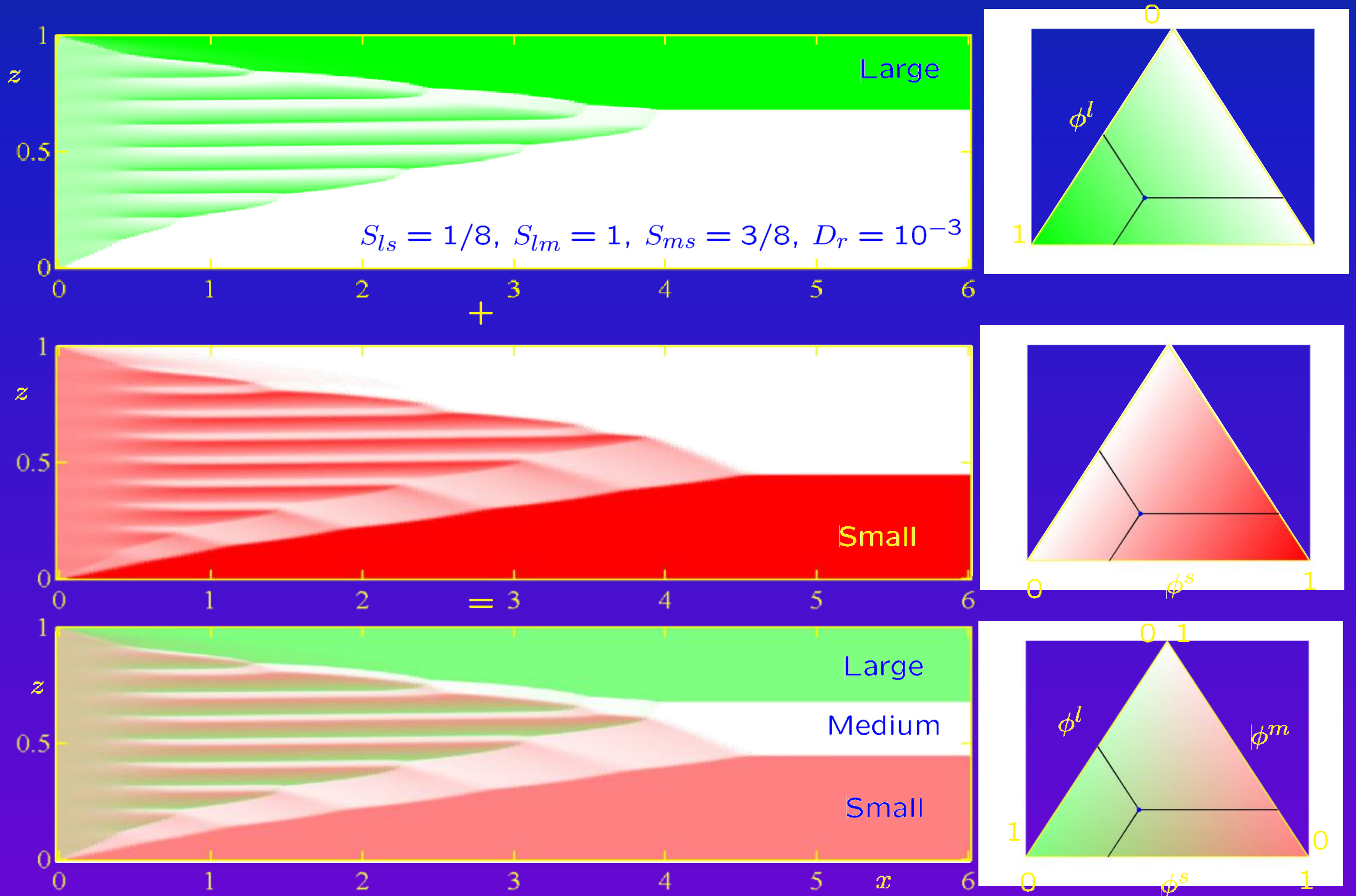
$$\phi^\nu(0, z) = \phi^\nu_0$$

- and with prescribed velocity field

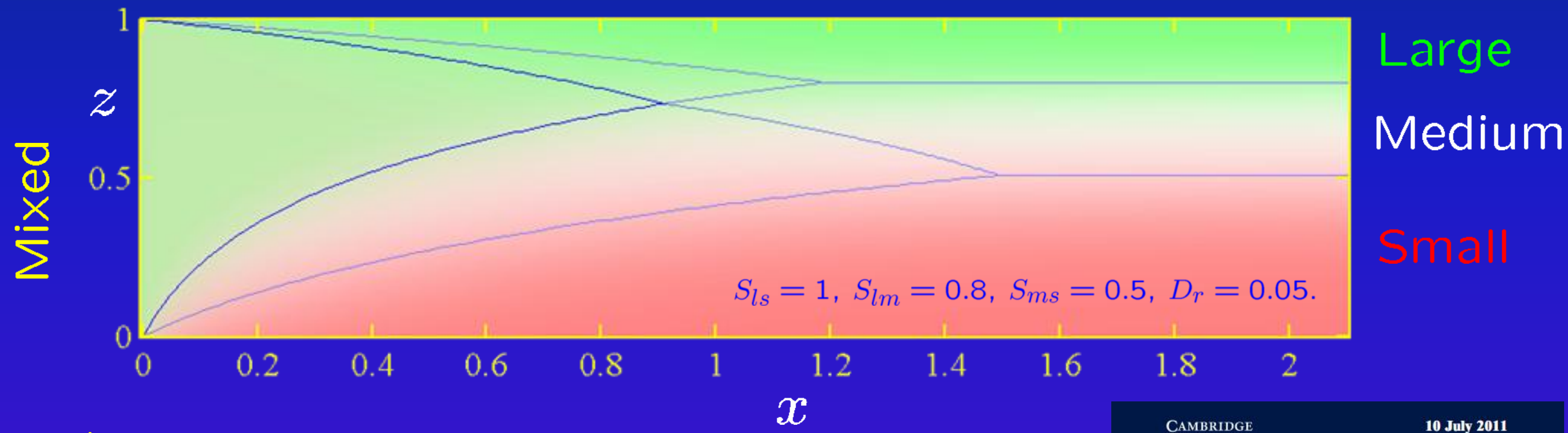
$$u = u(z), \quad w = 0$$

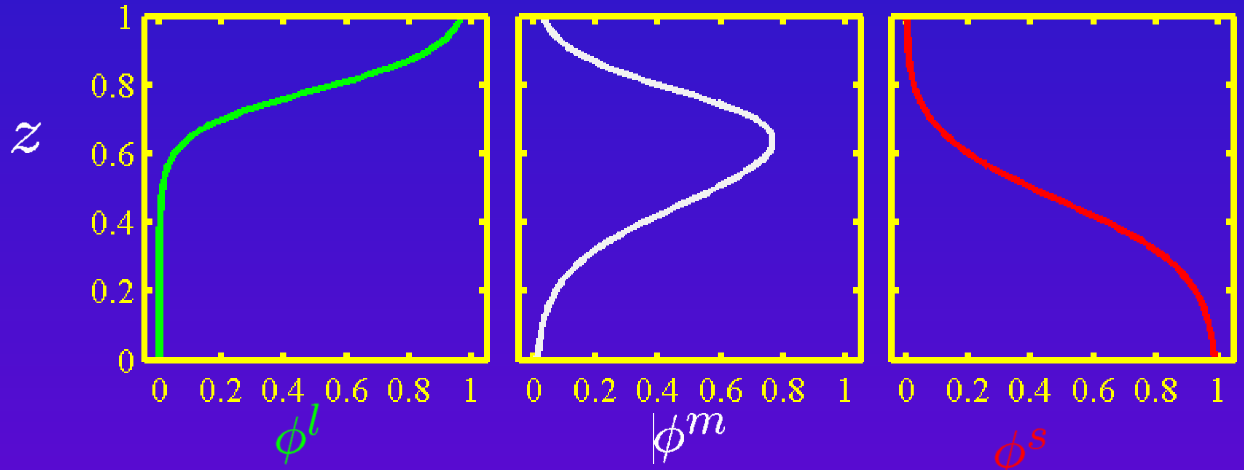
- subject to no-flux conditions at $z = 0, 1$
- can be computed using Matlab function pdepe (Galerkin Method)

For non-monotonic segregation rates there can be linear instabilities!



Reverse distribution grading



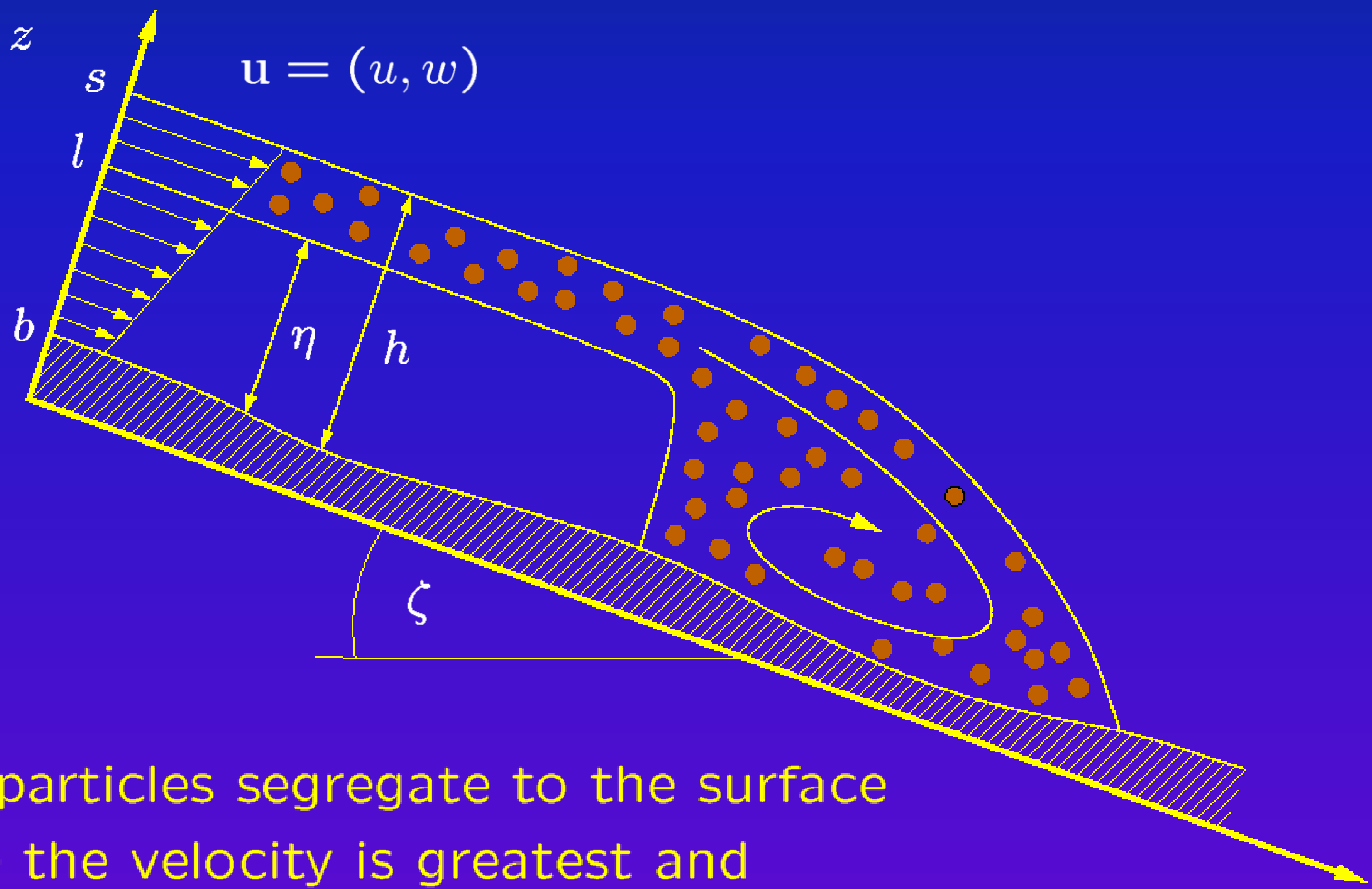


- comparable to experiments ...

  Gray & Ancey (2011) *J. Fluid Mech.* **678**, 535-558
  Wiederseiner *et al.* (2011) *Phys. Fluids* **23**, 013301

- and DEM simulations (Jim McElwaine ...)

Transport and accumulation of large particles



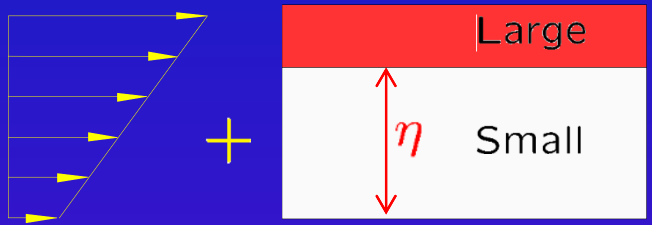
- large particles segregate to the surface
- where the velocity is greatest and
- are transported to the flow front where they are
- over run and recirculated by particle size segregation

A depth averaged theory for particle size segregation

- Integrating the segregation-remixing equation w.r.t z
- subject to the no flux and kinematic boundary conditions gives

$$\frac{\partial}{\partial t}(h\bar{\phi}) + \frac{\partial}{\partial x}(h\overline{\phi u}) = 0$$

- where the integrals evaluated assuming



i.e. linear velocity with basal slip and sharp segregation

$$h\bar{\phi} = \int_b^s \phi^s \, dz = \eta$$

$$h\overline{\phi u} = \int_b^s \phi^s u \, dz = \eta\bar{u} - (1-\alpha)\bar{u}\eta\left(1 - \frac{\eta}{h}\right)$$

- This yields the large particle transport equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta\bar{u}) - \frac{\partial}{\partial x}\left((1-\alpha)\bar{u}\eta\left(1-\frac{\eta}{h}\right)\right) = 0.$$

- for the evolution of the inversely graded shock interface η.

Gray & Kokelaar (2010) *J. Fluid Mech.* **652**, 105–137

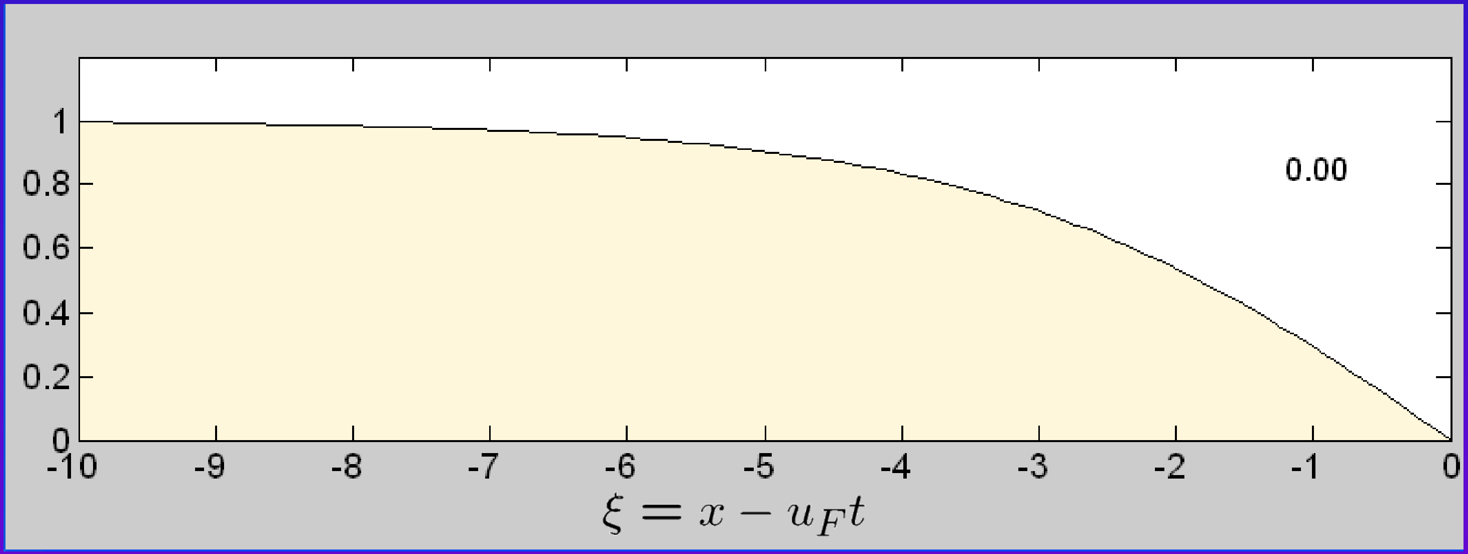
- Using $\eta = h\bar{\phi}$ this can also be rewritten as

$$\frac{\partial}{\partial t}(h\bar{\phi}) + \frac{\partial}{\partial x}(h\bar{\phi}\bar{u}) - \frac{\partial}{\partial x}\left((1-\alpha)h\bar{u}\bar{\phi}\left(1-\bar{\phi}\right)\right) = 0.$$

- Remarkably similar to the segregation equation ...

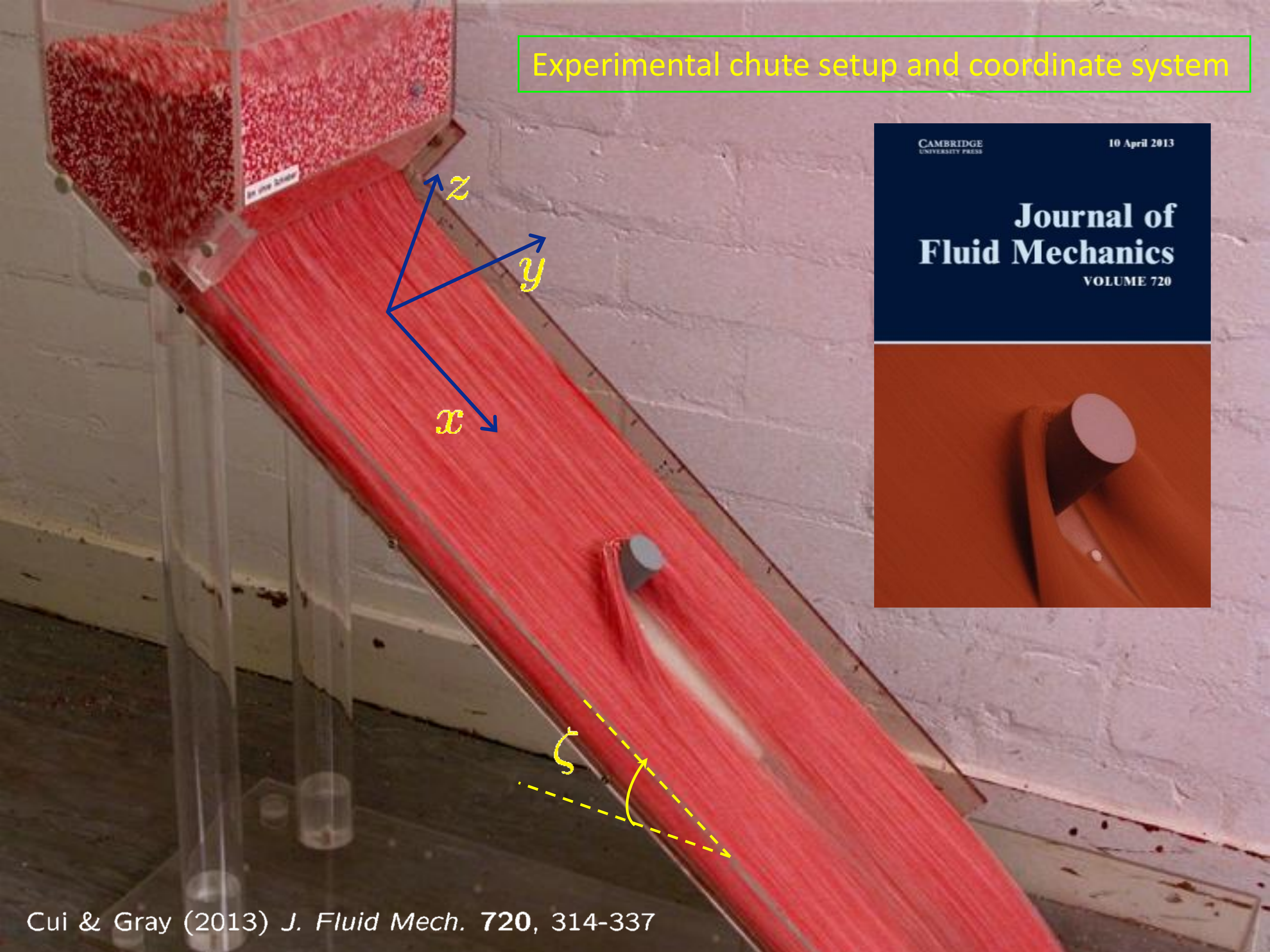
$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}(\phi u) + \frac{\partial}{\partial z}(\phi w) - S_{ls}\frac{\partial}{\partial z}\left(\phi(1-\phi)\right) = \frac{\partial}{\partial z}\left(D_r \frac{\partial \phi}{\partial z}\right)$$

- Large grains transported forwards to form bouldery flow front



- more RESISTIVE larger particles ⇒ feedback on bulk flow

Experimental chute setup and coordinate system



Cui & Gray (2013) *J. Fluid Mech.* **720**, 314-337

- For avalanche thickness h and mean velocity $\bar{\boldsymbol{u}} = (\bar{u}, \bar{v})$ in the downslope x and cross-slope y directions.

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) &= 0, \\ \frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(h\bar{u}^2) + \frac{\partial}{\partial y}(h\bar{u}\bar{v}) + \frac{\partial}{\partial x}\left(\frac{1}{2}gh^2\cos\zeta\right) &= hgS_{(x)}, \\ \frac{\partial}{\partial t}(h\bar{v}) + \frac{\partial}{\partial x}(h\bar{u}\bar{v}) + \frac{\partial}{\partial y}(h\bar{v}^2) + \frac{\partial}{\partial y}\left(\frac{1}{2}gh^2\cos\zeta\right) &= hgS_{(y)}, \end{aligned}$$

- source terms composed of gravity, basal friction μ and gradients of the basal topography b

$$\begin{aligned} S_{(x)} &= \sin\zeta - \mu(\bar{u}/|\bar{\boldsymbol{u}}|)\cos\zeta - \frac{\partial b}{\partial x}\cos\zeta, \\ S_{(y)} &= -\mu(\bar{v}/|\bar{\boldsymbol{u}}|)\cos\zeta - \frac{\partial b}{\partial y}\cos\zeta, \end{aligned}$$

- system is hyperbolic, Froude number $\mathrm{Fr} = |\bar{\boldsymbol{u}}|/\sqrt{gh\cos\zeta}$

Grigorian *et al.* 1967; Savage & Hutter (1989), Gray *et al.* P. Roy. Soc. 1999, JFM 2003

Granular jets and hydraulic jumps on an inclined plane

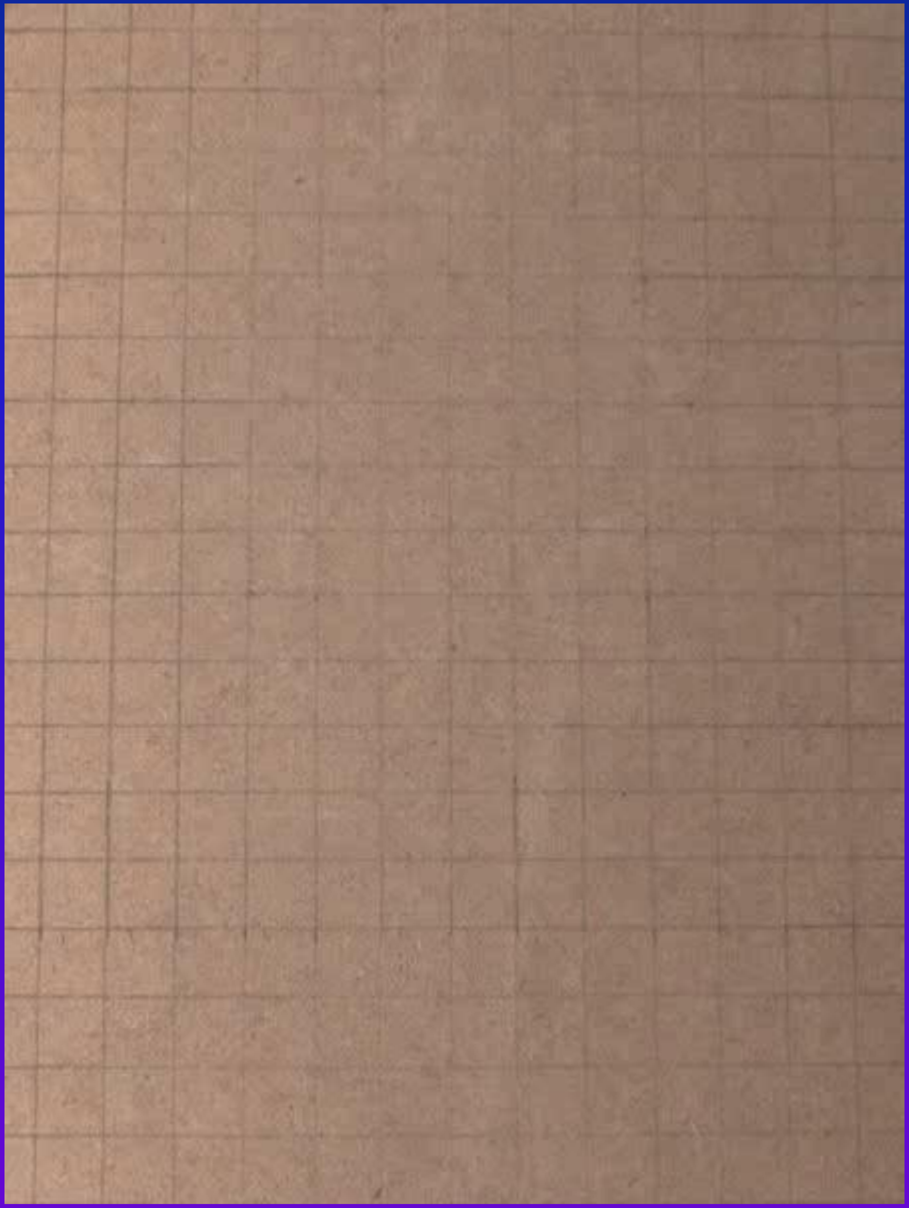
- Oblique impingement of an inviscid jet (Hasson & Peck 1964)

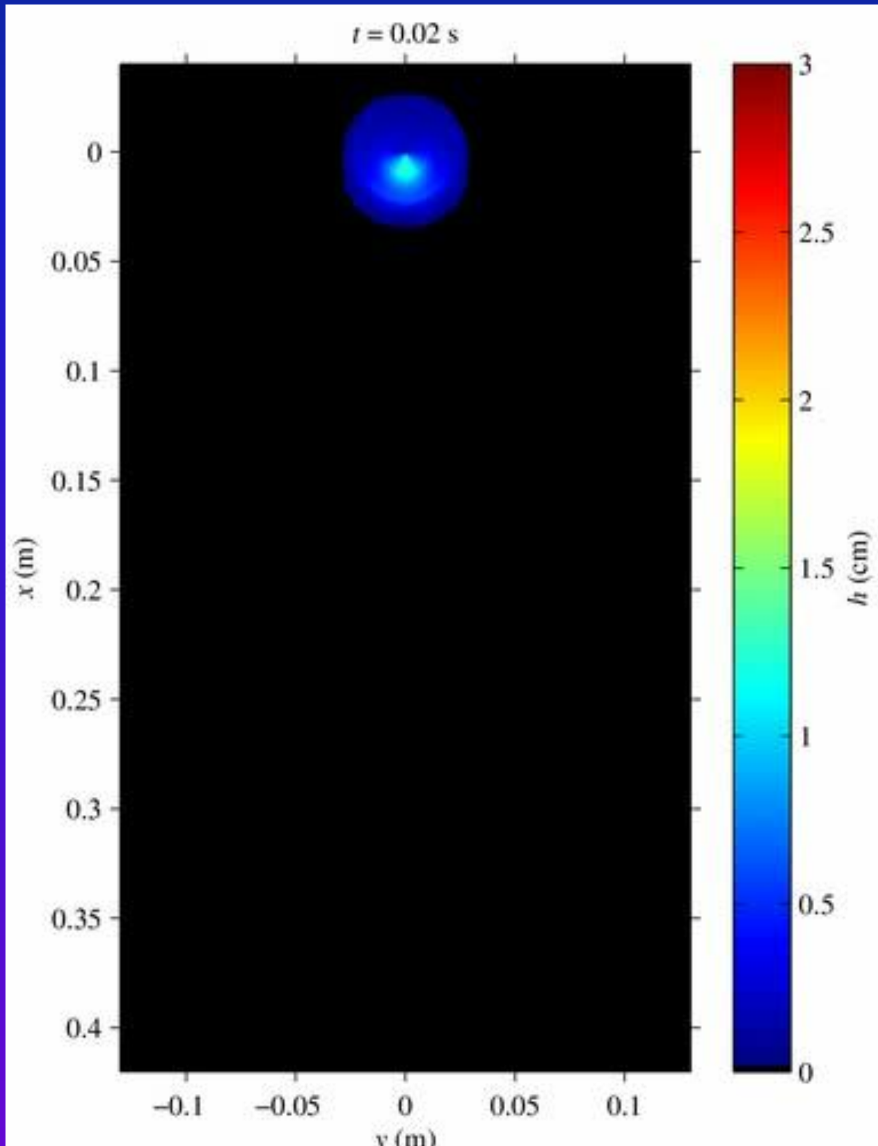
- Friction law for rough beds

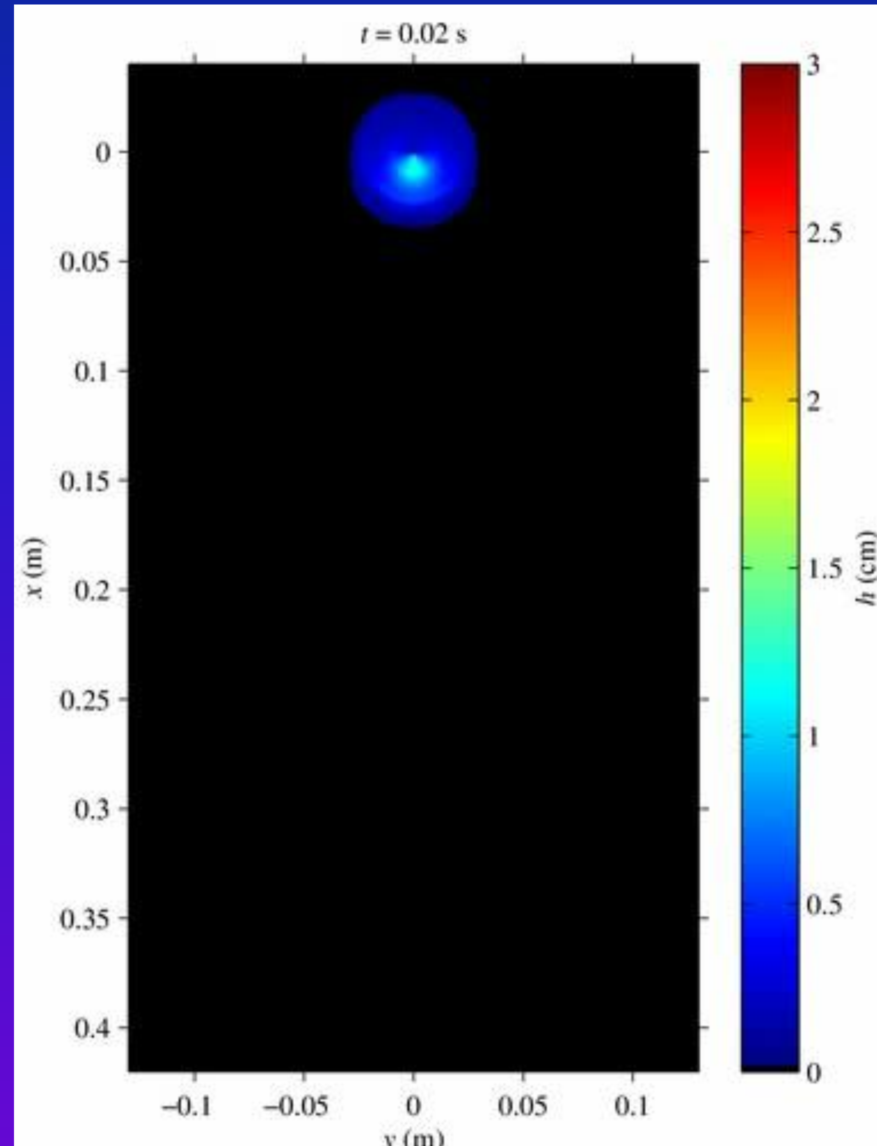
$$\mu = \tan\zeta_1 + \frac{\tan\zeta_2 - \tan\zeta_1}{1 + \beta h/(\mathcal{L}\mathrm{Fr})},$$

- including treatment of static material for $0 < \mathrm{Fr} < \beta$ (Pouliquen & Forterre 2002)

Johnson & Gray (2011) *J. Fluid Mech.* **675**, 87-116







Johnson & Gray (2011) *J. Fluid Mech.* **675**, 87-116

A simple two-dimensional fully coupled segregation model

- For avalanche thickness h, small particle thickness η and depth-averaged velocity $\overline{\boldsymbol{u}}$ the 2D coupled model is

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \mathrm{div}(h\overline{\boldsymbol{u}}) &= 0, \\
\frac{\partial \eta}{\partial t} + \mathrm{div}\left(\eta\overline{\boldsymbol{u}} - (1-\alpha)\eta\left(1-\frac{\eta}{h}\right)\overline{\boldsymbol{u}}\right) &= 0, \\
\frac{\partial}{\partial t}(h\bar{u}) + \mathrm{div}(h\overline{\boldsymbol{u}} \otimes \overline{\boldsymbol{u}}) + \mathrm{grad}\left(\frac{1}{2}h^2\cos\zeta\right) &= h\boldsymbol{S},
\end{aligned}
$$

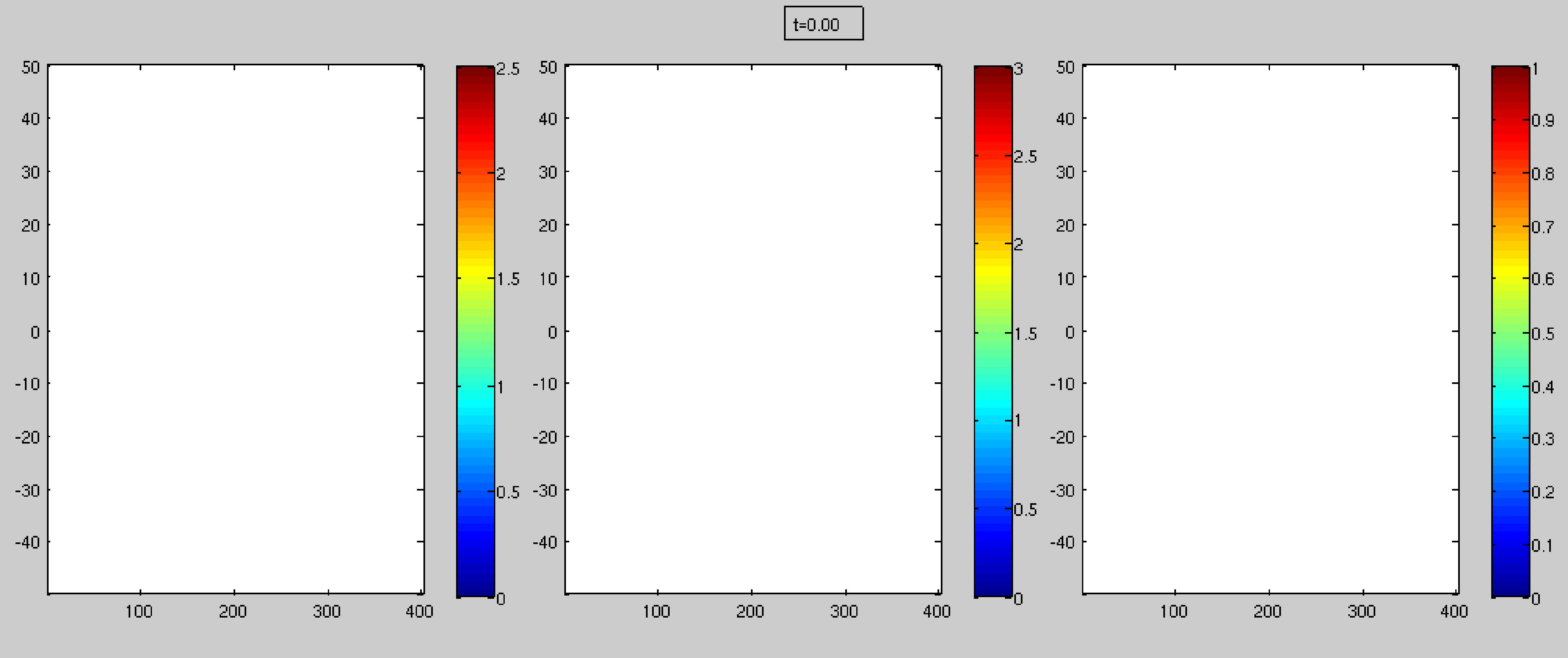
- source terms composed of gravity and basal friction

$$
\boldsymbol{S} = \begin{pmatrix} \sin\zeta - \mu(\overline{u}/|\overline{\boldsymbol{u}}|)\cos\zeta, \\ -\mu(\overline{v}/|\overline{\boldsymbol{u}}|)\cos\zeta, \end{pmatrix}
$$

- coupling through $\bar{\phi} = \eta/h$ dependent friction coefficient

$$
\mu = \left(1-\bar{\phi}\right)\mu^L + \bar{\phi}\mu^S, \qquad \mu^L > \mu^S
$$

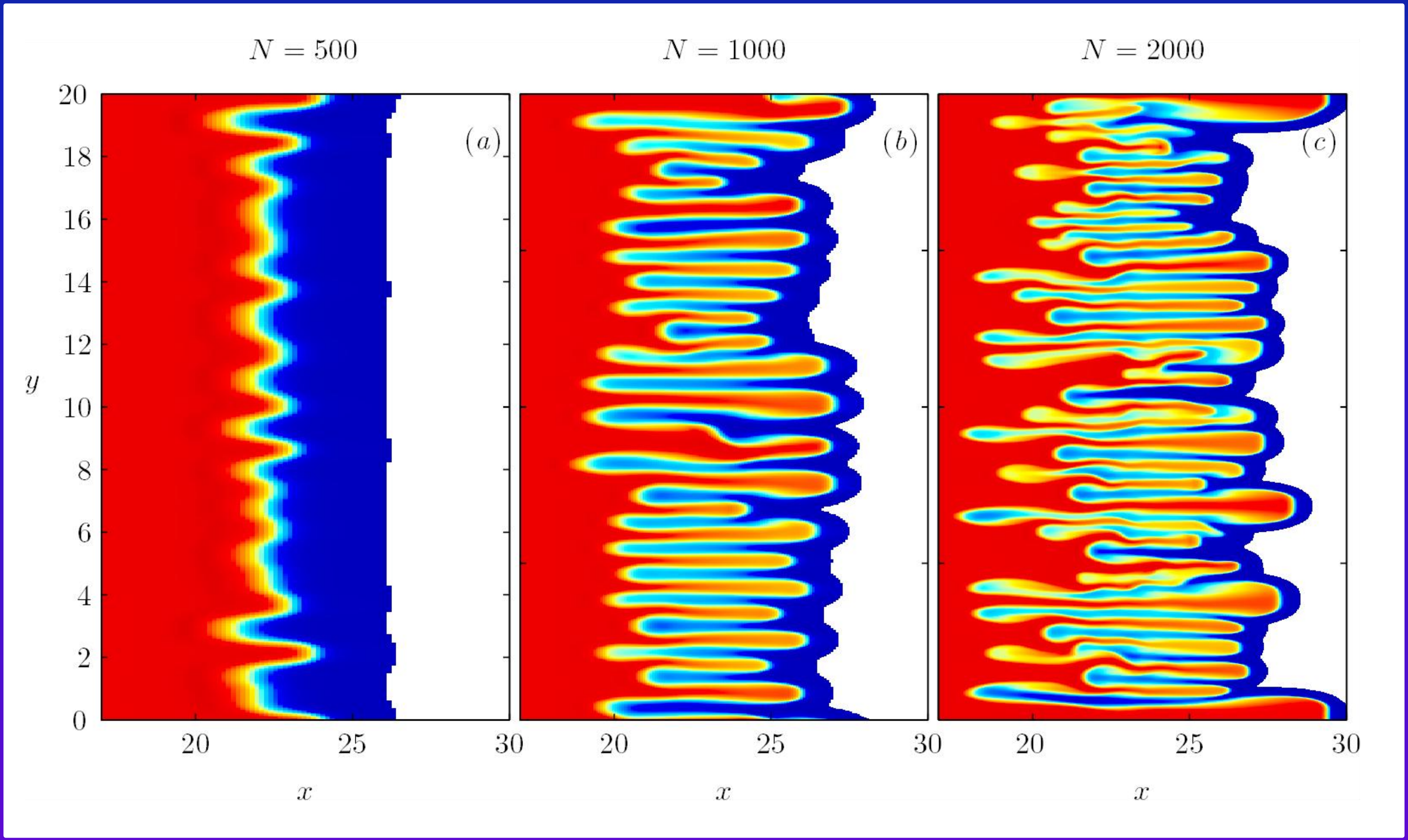
Woodhouse *et al.* (2012), J. Fluid Mech. **709**, 543-580.



h $|\overline{\boldsymbol{u}}|$ $\overline{\phi}$

- The model is hyperbolic
- captures the instability mechanism
- and forms large rich lateral levees, BUT ....

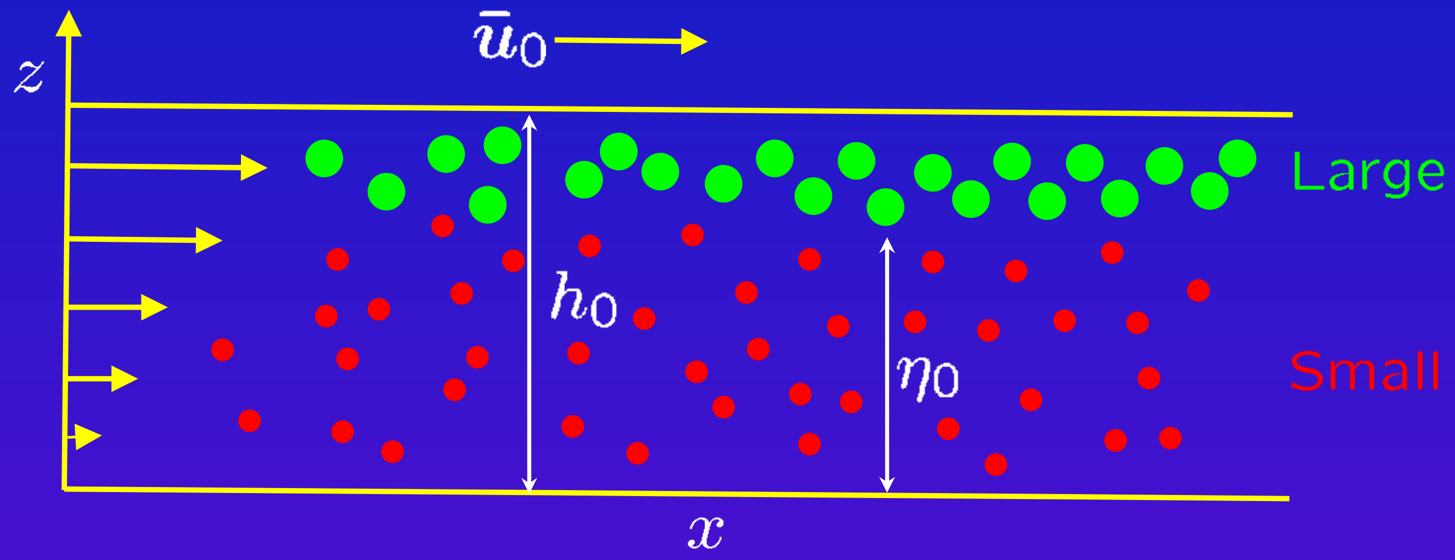
Numerical solutions are grid dependent …!



- this indicates that there is some important physics missing

- Linear stability about a steady uniform base state

$$\bar{u} = \bar{u}_0, \ \bar{v} = 0, \ h = h_0, \ \eta = \eta_0$$

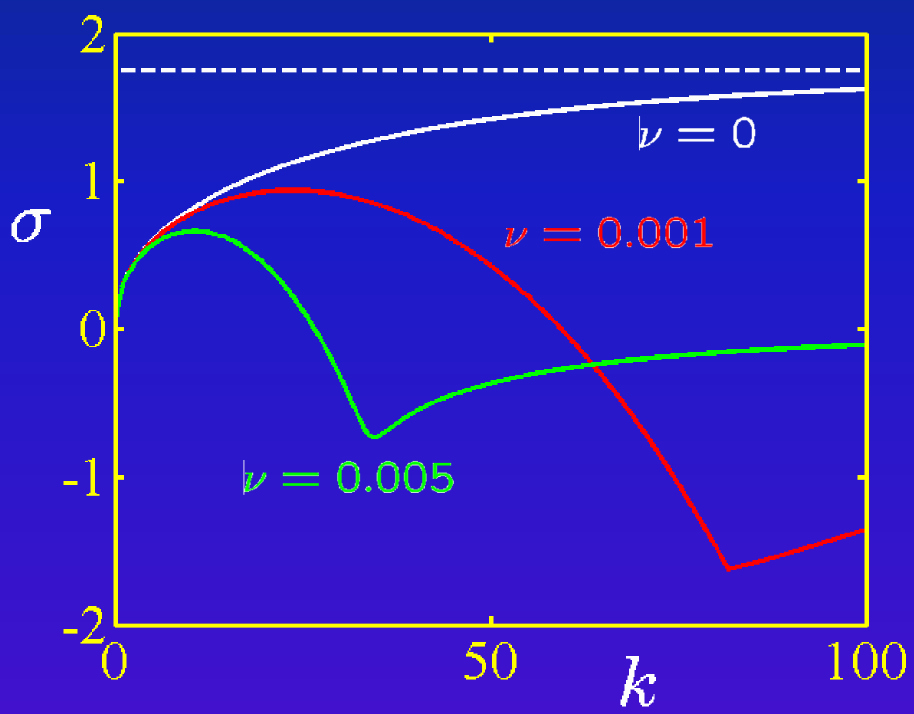


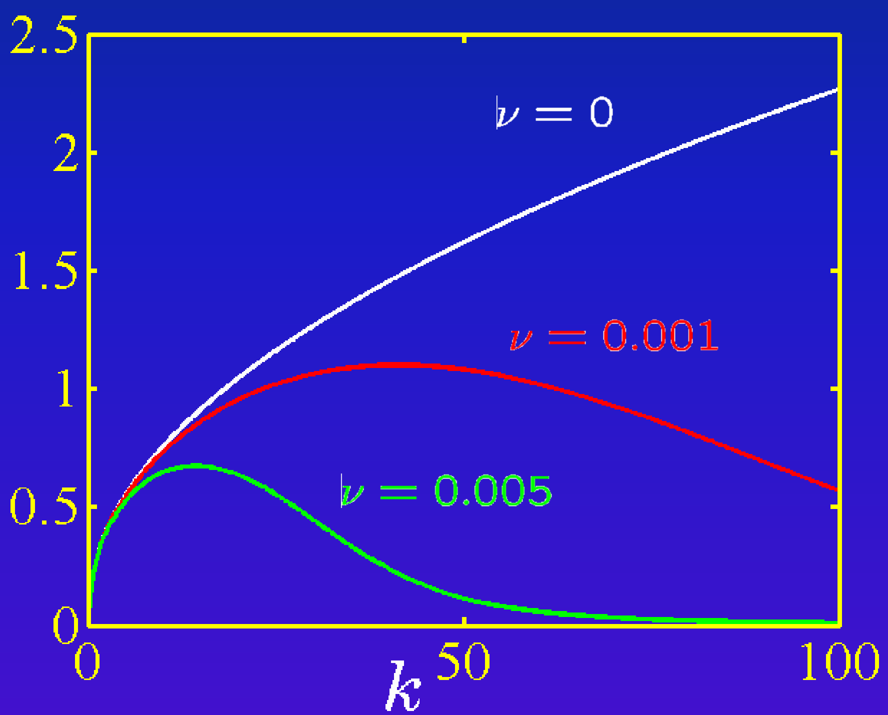
- Predicts unbounded short wavelength instability when

$$\mathrm{Fr} = \mathrm{Fr}_c = \frac{1}{(1-\alpha)|2\eta_0 - 1|}$$

- This is when the characteristics coincide

$\mathsf{Fr} \neq \mathsf{Fr}_c$ $\qquad$ $\mathsf{Fr} = \mathsf{Fr}_c$





- Depth averaging the $\mu(I)$ rheology suggests
- adding a diffusive term to the righthand side of the form

$$\frac{\partial}{\partial x}\left(\nu h^{\frac{3}{2}} \frac{\partial \bar{u}}{\partial x}\right)$$

- which gives cut-off ($\mathsf{Fr} = \mathsf{Fr}_c$) and boundedness ($\mathsf{Fr} \neq \mathsf{Fr}_c$)

A two-dimensional fully coupled model including rheology

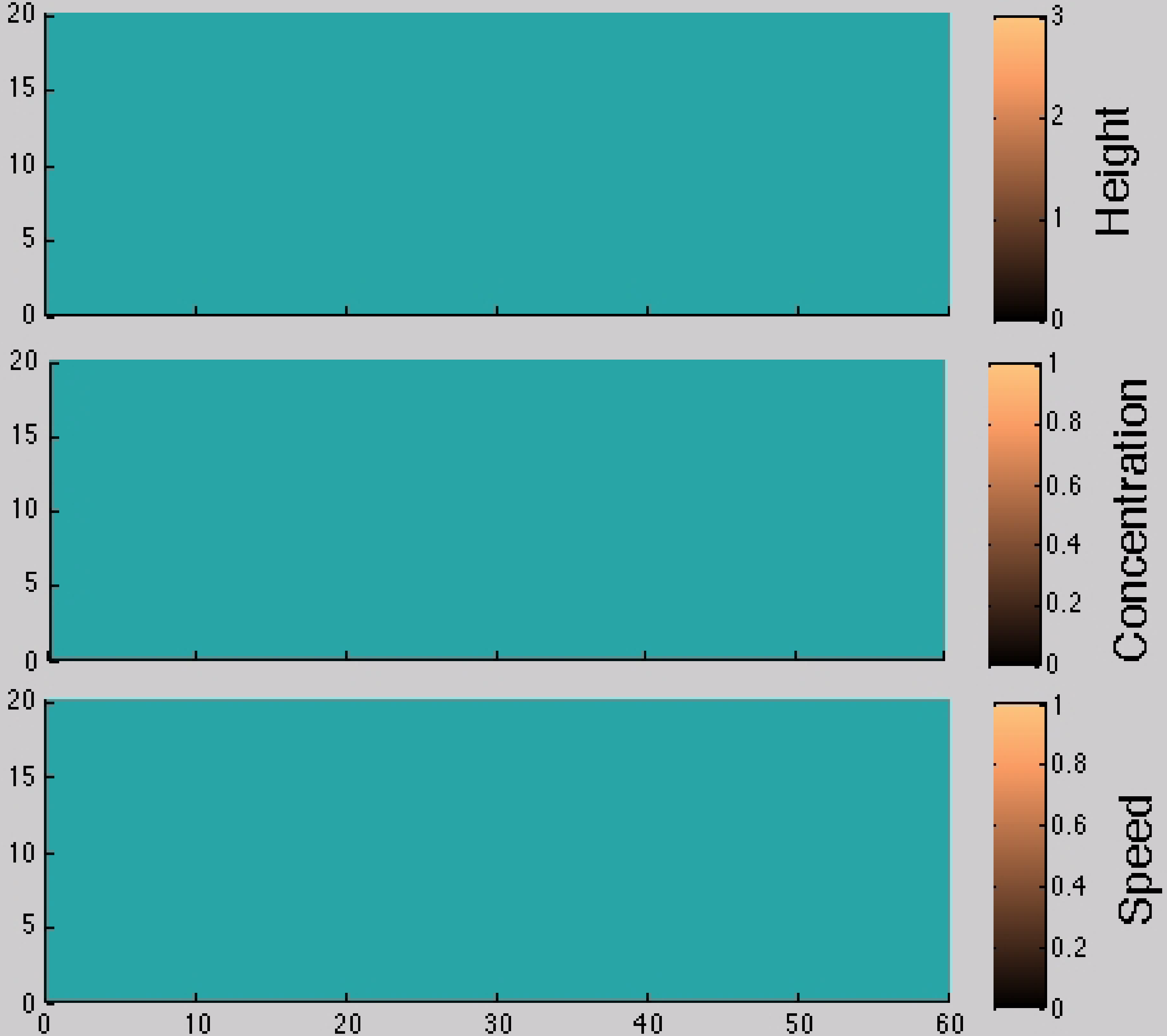
- When the depth-averaged $\mu(I)$-rheology is generalized to 2D it suggests a system of conservation laws of the form

$$\begin{aligned}
\frac{\partial h}{\partial t} + \mathrm{div}(h\overline{\boldsymbol{u}}) &= 0, \\
\frac{\partial \eta}{\partial t} + \mathrm{div}\left(\eta\overline{\boldsymbol{u}} - (1-\alpha)\eta\left(1-\frac{\eta}{h}\right)\overline{\boldsymbol{u}}\right) &= 0, \\
\frac{\partial}{\partial t}(h\bar{u}) + \mathrm{div}(h\overline{\boldsymbol{u}}\otimes\overline{\boldsymbol{u}}) + \mathrm{grad}\left(\frac{1}{2}h^2\cos\zeta\right) &= h\boldsymbol{S} + \mathrm{div}\left(\nu h^{\frac{3}{2}}\boldsymbol{D}\right),
\end{aligned}$$

- where the two-dimensional strain-rate tensor is

$$\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right)$$

- and $\boldsymbol{L} = \mathrm{grad}(\bar{\boldsymbol{u}})$ is the depth-averaged velocity gradient
- Numerics converges ... (Baker, Johnson & Gray in prep)



Height
Concentration
Speed